INTRODUCTION TO ENRIQUES SURFACES AND THEIR MODULI.

Plan for the talk

- Preliminaries about surfaces;
- History, definition and properties of Enriques surfaces;
- Lattice theory;
- Moduli of Enriques.

<u>Def</u>: A surface S is a 2-dimensional irreducible smooth and projective scheme over $\mathbb{C}$.

<u>Example</u>: $\mathbb{P}^2$, $C_1 \times C_2$

Surfaces are classified in terms of their numerical invariants:

- geometric genus: $p_g(S) := h^0(S, \omega_S)$ / $p_g(\mathbb{P}^2) = 0$
- irregularity: $q(S) := h^1(S, \mathcal{O}_S)$ / $q(\mathbb{P}^2) = 0$
- Kodaira dimension: $\kappa(S) = \max_{m \geq 1}\{\dim \operatorname{Im} \varphi_{\omega_S^{\otimes m}}\}$ / $\kappa(\mathbb{P}^2) = -\infty$

$$S \xrightarrow{\varphi_{\omega_S^{\otimes n}}} \mathbb{P}(H^0(S, \omega_S^{\otimes n}))$$

- Betti numbers: $b_0, b_1, b_2, b_3 = b_1, b_4 = b_0$ / $1, 0, 1, 0, 1$ for $\mathbb{P}^2$
- Plurigenera: $P_n(S) := h^0(S, \omega_S^{\otimes n})$, $n \geq 1$ / $P_n(\mathbb{P}^2) = 0$ $\forall n$.

$$P_1(S) = p_g(S)$$

$\mathbb{P}^2$ is called a rational surface. Rational surfaces are surfaces which are birationally equivalent to $\mathbb{P}^2$. For example $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_n$ Hirzebruch surfaces ($\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$) and blow ups of rational surfaces.

<u>Prop</u>: Let S be a rational surface. Then $0 = p_g(S) = q(S) = P_n(S)$ $\forall n$

<u>Proof</u>: p_g, q and P_n, $n \geq 1$, are birational invariant. $\square$

Question: Is it true that a surface S with $P_n(S)=0$ $\forall n$ and $p_g(S)=q(S)=0$ is rational?

<u>Theorem</u> (Castelnuovo Rationality Criterion): Let S be a surface. Then S is rational $\iff q(S)=P_2(S)=0$ $\iff p_g(S)=q(S)=P_2(S)=0$ (easy: $P_2(S)=0 \Rightarrow p_g(S)=0$)

Castelnuovo: Is it true that $q(S)=p_g(S)=0 \Rightarrow S$ is rational?

Enriques: No!

Enriques example: He found a surface S with $q(S)=p_g(S)=0$, but $2K_S\sim 0$ so $P_2(S)=1$. Therefore S cannot be rational. The following is his example: take a sextic hypersurface in $\mathbb{P}^3$ passing doubly through the edges of the coordinate tetrahedron $X_0X_1X_2X_3=0$ and smooth everywhere else. Take the normalization of this sextic.

<u>Def</u>: An Enriques surface S is a surface with $q=p_g=0$ and $2K_X\sim 0$.

<u>Properties</u>: $K_X \not\sim 0$, $\kappa(S)=0$, $b_1=0$, $b_2=\operatorname{rk} H_2(S;\mathbb{Z})=\operatorname{rk} H^2(S;\mathbb{Z})$

$$\cup : H^2(S;\mathbb{Z})_f \times H^2(S;\mathbb{Z})_f \longrightarrow H^4(S;\mathbb{Z})\cong\mathbb{Z}$$

even unimodular lattice of signature $(1,9)$

<u>Def</u>: A lattice is a pair (L,b) where L is a finitely generated free $\mathbb{Z}$-module with b symmetric bilinear form. If b is non-degenerate, we have an injective morphism:

$$\varphi : L \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(L,\mathbb{Z}) =: L^*$$

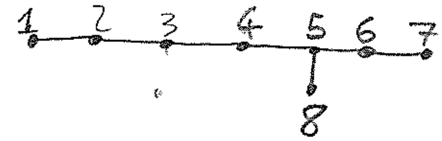
In this case L is unimodular if φ is an isomorphism.

Examples of unimodular lattices:

1) U, hyperbolic lattice $\left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$

2) Let $m \geq 1$, $U(m) = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\right)$

3) E_8, $\left(\mathbb{Z}^8, \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}\right)$ $\leftarrow$ Cartan matrix of the Dynkin diagram E_8.



<u>Theorem</u>: It follows from the classification of unimodular lattices that if S is an Enriques surface, then:

$$H^2(X;\mathbb{Z})_f \cong U \oplus E_8(-1).$$

Moduli of Enriques surfaces: I want to construct a space whose points are in bijection with isomorphism classes of Enriques surfaces.

$$N := U \oplus U(2) \oplus E_8(-2)$$

$$\Omega_N := \{[x] \in \mathbb{P}(N \otimes \mathbb{C}) \mid \langle x, \bar{x} \rangle > 0, \langle x, x \rangle = 0\} = D_N \amalg D_N'$$

Two disjoint connected components

$\uparrow$ Compact symmetric domain of type IV

$\uparrow$ Bounded symmetric domain of type IV

$O^+(N)$ = isometries of N preserving D_N and D_N'.

<u>Theorem</u> (Horikawa, Namikawa):

$$M_{En} := \left(D_N / O^+(N)\right) \setminus \Delta_{-2} \xleftrightarrow{1:1} \{\text{Enriques surfaces}\}/\cong$$

<u>proof</u>: If there's time.

"$\leftarrow$" $[S]$, $\varphi : H^2(S;\mathbb{Z}) \xrightarrow{\cong} U \oplus E_8(-1)$, X K3 $\downarrow 2:1$ S

marking.

Induced marking $\tilde{\varphi} : H^2(X;\mathbb{Z}) \xrightarrow{\cong} \underbrace{U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)}_{= L_{K3}}$

$N \subset L_{K3}$, $\eta_X \in H^0(X, \omega_X)$, $\tilde{\varphi}(\eta_X) \in N \otimes \mathbb{C}$, so

$$[S] \longmapsto [\tilde{\varphi}(\eta_X)] \in (\Omega_N / O^+(N)) \setminus \Delta_{-2} .$$

Δ_{-2} = image of $\mathcal{H}_{-2}$ under
$\Omega_N \to \Omega_N / O^+(N)$

$$\mathcal{H}_{-2} = \bigcup_{\substack{l \in N \\ l^2 = -2}} \{ [x] \in D_N \mid \langle x, l \rangle = 0 \} .$$

Why getting rid of Δ_{-2}? Because it corresponds to K3's without fixed point free involution. $\square$

<u>Theorem</u> (Kondō): M_{En} is rational.