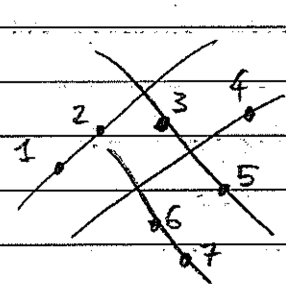
ON THE CONE OF EFFECTIVE 2-CYCLES ON $\overline{M}_{0,7}$.

$n \geq 3$, $k = \overline{k}$, any characteristic. I'll point out when I have to be careful about the characteristic.

$\overline{M}_{0,n}$ smooth, proj, conn.
$n-3$ dim. fine moduli space



$0 \leq k \leq n-3$

elements of this set are called k-cycles

$$N_k(\overline{M}_{0,n}) = \left\{ \sum_{finite} a_i Z_i \;\middle|\; a_i \in \mathbb{R},\ Z_i \subseteq \overline{M}_{0,n} \text{ closed, irr., red. subscheme}, \dim(Z_i) = k \right\} \Big/ \text{num equi}$$

finite dimensional vector space.

The cone of effective k-cycles:

$$\mathrm{Eff}_k(\overline{M}_{0,n}) = \left\{ \sum_{finite} a_i Z_i \;\middle|\; a_i \in \mathbb{R}_{\geq 0}, \ldots \right\} \Big/ \text{num. equiv.}$$

Sitting inside there's a special subcone generated by very special k-cycles.

Def: The locus of points on $\overline{M}_{0,n}$ parametrizing curves with at least $n-3-k$ nodes has pure dimension k. The irr components are called k-boundary strata.
(Say about boundary divisors).

Def: $V_\kappa(\overline{M}_{0,n})$ cone generated by the κ-boundary strata.

Fulton's question (Keel-McKernan):

Is $V_\kappa(\overline{M}_{0,n}) = \mathrm{Eff}_\kappa(\overline{M}_{0,n})$?

$\kappa = 0, n-3$, trivially true (explain)

$\boxed{n=5}$ $V_1(\overline{M}_{0,5}) \underset{\substack{\text{Kapranov} \\ \text{blow up} \\ \text{construction}}}{=} \mathrm{Eff}_1(\overline{M}_{0,5})$

$\boxed{n=6}$ $V_1(\overline{M}_{0,6}) \overset{\text{Keel-McKernan}}{=} \mathrm{Eff}_1(\overline{M}_{0,6})$

$V_2(\overline{M}_{0,6}) \subsetneq \mathrm{Eff}_2(\overline{M}_{0,6})$
Keel-Vermeire divisors δ^{KV}

Mention that pullbacks ~~give counterexamples~~ $\forall n$, $\kappa = n-4$

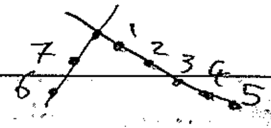
Hassett, Tschinkel, Castravet: $\mathrm{Eff}_2(\overline{M}_{0,6})$ is generated by boundary divisors and δ^{KV}.

$\boxed{n=7}$ $V_1(M_{0,7}) \overset{\text{Keel-McKernan}}{=} \mathrm{Eff}_1(\overline{M}_{0,7})$

$\boxed{V_2(\overline{M}_{0,7}) \subsetneq \mathrm{Eff}_2(\overline{M}_{0,7})}$ by means of a lifting lemma.

$V_3(M_{0,7}) \subsetneq \mathrm{Eff}_3(M_{0,7})$

Castravet-Tevelev, Doran, Giansiracusa, Jensen, Opie.



Lifting lemma

$$\begin{tikzcd} \overline{M}_{0,6} = D_{67} \arrow[r, hook, "i"] \arrow[dr, "id"'] & \overline{M}_{0,7} \arrow[d, "\pi"] \\ & \overline{M}_{0,6} \end{tikzcd}$$

$\delta^{KV} \in \mathrm{Eff}_2(\overline{M}_{0,6}) \setminus V_2(\overline{M}_{0,6})$.

$i_* \delta^{KV} \in \mathrm{Eff}_2(\overline{M}_{0,7})$

$i_* \delta^{KV} = \sum a_i Z_i \Rightarrow \delta^{KV} = \sum a_i \pi_* Z_i$ $\ast$

a_i: $\forall i \geq 0$

Z_i: 2-boundary strata

$\pi_* Z_i$: 0 or boundary divisor (explain if someone asks) $\square$

$\Rightarrow i_* \delta^{KV} \notin V_2(\overline{M}_{0,7})$

Remark: $1 < k < n-3 \Rightarrow V_k(\overline{M}_{0,n}) \subsetneq \mathrm{Eff}_k(\overline{M}_{0,n})$
We are left with $n \geq 8$, $k=1$. Open problem (F-conjecture)

$$V_2(\overline{M}_{0,7}) \subsetneq V_2^{KV}(\overline{M}_{0,7}) \subsetneq \mathrm{Eff}_2(\overline{M}_{0,7})$$

V_2^{KV}: cone generated by $V_2(\overline{M}_{0,7})$ and the lifts of the KV divisors

Description of $V_2(\overline{M}_{0,7})$

2-boundary stratum $\leftrightarrow$ $\{$  $\}$ $=: S_{I,J,K} = S_{K,J,I}$

$I \amalg J \amalg K = [7]$

Stability condition: $2 \leq |I|, |K| \leq 4$, $1 \leq |K| \leq 3$.

$\sigma_{I,J,K} := [S_{I,J,K}]$

We have an intersection formula between them.

420 distinct $\sigma_{I,J,K}$ span distinct rays of $\mathrm{Eff}_2(\overline{M}_{0,7})$. Those are extremal (Chen - Coskun).

<u>Description of $V_2^{KV}(\overline{M}_{0,7})$</u>:

$\overline{M}_{0,([7]\cup\{x\})\setminus\{a,b\}} = D_{ab} \xhookrightarrow{i} \overline{M}_{0,7}$

$([7]\cup\{x\})\setminus\{a,b\} = \{i,j,k,l,m,x\}$

$\delta^{KV}_{mx,ij} = \delta_{jm} + \delta_{im} + \delta_{kx} + \delta_{lx} + 2\delta_{ijm} - \delta_{mx}$

$\parallel$

$\delta^{KV}_{ij,mx} = \delta^{KV}_{xm,ij} - \delta^{KV}_{mx,ji} = \delta^{KV}_{mx,kl}$

$\sigma^{KV}_{ab,m,ij} := i_* \delta^{KV}_{mx,ij} = \sigma_{jm,ikl,ab} + \ldots$

Lemma: $y \in [7]$, $\pi_y : \overline{M}_{0,7} \to \overline{M}_{0,[7]\setminus\{y\}}$. Then

$$\pi_{y*}\sigma^{KV}_{ab,m,ij} = \begin{cases} \delta^{KV}_{am,ij} & \text{if } y=b \\ \delta^{KV}_{bm,ij} & \text{if } y=a \\ \delta_{ab} & \text{otherwise.} \end{cases}$$

Proposition: $\mathrm{Eff}_2(\overline{M}_{0,7})$ has at least 735 extremal rays:

- 420 $\sigma_{I,J,K}$
- 315 $\sigma_{ab,m,ij}$ KV

Embedded blow ups of $\mathbb{P}^2$ in $\overline{M}_{0,7}$

Thm (Castravet–Tevelev): $p_1, \dots, p_n \in \mathbb{P}^2$, $U := \mathbb{P}^2 \setminus$ (union of lines generated by $p_1, \dots, p_n$).

$$F: U \longrightarrow M_{0,n}$$
$$p \longmapsto [(\mathbb{P}^1; \varphi_p(p_1), \dots, \varphi_p(p_n))]$$

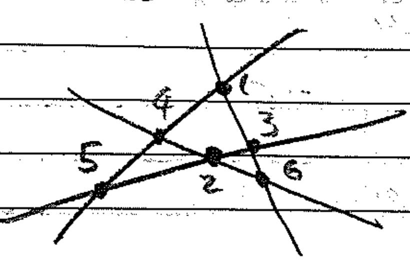
$\varphi_p: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ projection from p.

$$F: Bl_{p_1,\dots,p_n}\mathbb{P}^2 \longrightarrow \overline{M}_{0,n}$$

If $p_1, \dots, p_n$ don't lie on /, X, O, then F is a closed embedding. $F^*\delta_I$ known.

Def: A surface in $\overline{M}_{0,n}$ embedded in this way is called embedded blow up of $\mathbb{P}^2$ in $\overline{M}_{0,n}$ and $p_1, \dots, p_n$ are the associated points.

Remark: KV divisors can be obtained as embedded blow ups of $\mathbb{P}^2$ in $\overline{M}_{0,6}$



Irreducible hypertree on 6 labels.

Def: An embedded blowup of $\mathbb{P}^2$ in $\overline{M}_{0,7}$ is called hypertree surface if $\exists\, i \in [7]$ s.t. $p_1, \dots, \hat{p}_i, \dots, p_n$ realizes an irreducible hypertree on $[7] \setminus \{i\}$. An hypertree surface is special if we can find 3 such i.

Thm: $h \notin V_2^{KV}(\overline{M}_{0,7})$
$\uparrow$ special hyp. surface

Lemma: Let h be a hypertree surface. Then $h \notin V_2(\overline{M}_{0,7})$.

proof: Let $y \in [7]$ be the label that can be forgotten.

Assume $h = \sum_{\text{"}0} \alpha_{I,J,K} \sigma_{I,J,K}$

$$\underbrace{\pi_{y*} h}_{\text{KV-divisor}} = \sum \alpha_{I,J,K} \underbrace{\pi_{y*} \sigma_{I,J,K}}_{\text{0 or a boundary divisor.}} \qquad \cancel{*} \qquad \square$$

proof of the theorem: Let 5, 6, 7 be the labels that can be forgotten.

$$h = \sum_{\text{"}0} \alpha_{I,J,K} \sigma_{I,J,K} + \sum_{\{a,b\} \subset [7]} \sum^{15}_{\text{"}0} \beta_{ab,m,ij} \sigma^{KV}_{ab,m,ij}$$

Claim: $\forall\, \beta_{a'b',m',i'j'} = 0$

At least one between 5, 6, 7 is not in $\{a', \dots$

Assume WLOG $7 \notin \{a', b'\}$.

$$\pi_{7*} h = \sum_{\text{KV divisor}} \alpha_{I,J,K} \underbrace{\pi_{7*} \Gamma_{I,J,K}}_{\text{0 or boundary divisor}} + \sum_{\{a,b\} \subset [6]} \sum^{15} \beta_{ab,m,ij} \overbrace{\pi_{7*} \sigma_{ab,m,ij}}^{\delta_{ab}^{KV}}$$

$$+ \sum_{a \in [6]} \sum^{15} \beta_{a7,m,ij} \underbrace{\pi_{7*} \sigma_{a7,m,ij}}_{\delta^{KV}_{am,ij}}$$

$$\underbrace{\left(\dots + \overset{=0}{\beta_{a'b',m,ij}} + \dots \right)}_{=0} \delta_{a'b'} \qquad \square$$

$\leftarrow$ KV divisors are extremal

~~If there's time:~~

1) Classification of special hyp. surfaces;

2) $\tilde{\sigma}$;

3) $V_2(\tilde{M}_{0,7})$.