

# UMass, Reading Seminar in Algebraic Geometry, $\mathbb{Q}$ -Gorenstein deformations of surface singularities, an overview

Luca Schaffler

October 23, 2017

## Abstract

In this talk we introduce the concept of  $\mathbb{Q}$ -Gorenstein smoothing of surface singularities, and we analyze it in the context of quotient singularities. Important examples of these are cyclic quotient singularities, and we study them in detail. Then we classify the quotient singularities which admit a  $\mathbb{Q}$ -Gorenstein smoothing. To conclude, we introduce the functor  $\text{Def}' X$ , which will be fundamental to understand the deformation space  $\text{Def } X$ .

## 1 Introduction

**Definition 1.1.** Let  $p \in X$  be the germ of a normal surface singularity over  $\mathbb{C}$ . Let  $0 \in \Delta$  be the germ of a smooth curve. A flat family  $\pi: \mathfrak{X} \rightarrow \Delta$  of surfaces is called a  *$\mathbb{Q}$ -Gorenstein smoothing* of  $X$  if

- (i)  $\pi^{-1}(0) = X$ ;
- (ii)  $\pi^{-1}(t)$  is smooth for all  $t \in \Delta \setminus \{0\}$ ;
- (iii)  $K_{\mathfrak{X}/\Delta}$  is  $\mathbb{Q}$ -Cartier, where  $K_{\mathfrak{X}/\Delta} = K_{\mathfrak{X}} - \pi^* K_{\Delta}$ .

**Observation 1.2.** Here are some reasons why it is interesting to study  $\mathbb{Q}$ -Gorenstein smoothings of surface singularities.

- (1) Intrinsic interest in understanding the whole  $\text{Def } X$ . This will be finally settled with Arie's talk next week.
- (2) On the other hand,  $\mathbb{Q}$ -Gorenstein deformation theory can be used to construct interesting smooth surfaces  $\pi^{-1}(t)$ ,  $t \neq 0$ . In this way, Lee and Park [LP06] constructed an example of simply connected minimal surface of general type with  $p_g = 0$  and  $K^2 = 2$  (geography of the moduli space of surfaces of general type).

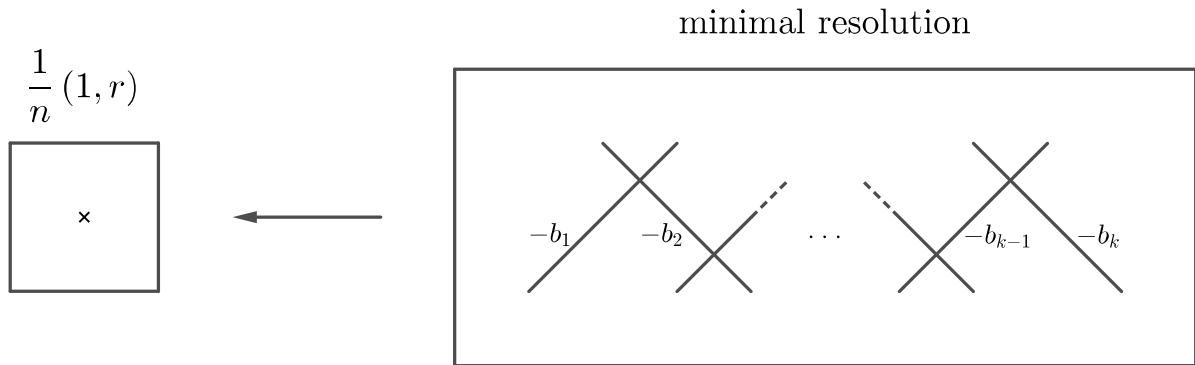


Figure 1: Minimal resolution of a cyclic quotient singularity

(2') Understand the KSBA boundary of the moduli space of surfaces.

We specialize to the case when  $X$  has quotient singularities, i.e., the singularities of the quotient of a smooth projective surface by the action of a finite group  $G$ . Important examples of quotient singularities are *cyclic* quotient singularities.

## 2 Surface cyclic quotient singularities

**Definition 2.1.** A *cyclic* quotient singularity is locally analytically isomorphic to

$$\mathbb{C}^2 / \langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^r \end{pmatrix} \rangle,$$

where  $r < n$  are two coprime positive integers,  $\text{g.c.d.}(r, n) = 1$  and  $\xi$  is a primitive  $n$ -th root of unity. We denote such singularity with  $\frac{1}{n}(1, r)$ .

**Remark 2.2.** The exceptional divisor of the minimal resolution of a cyclic quotient singularity is always a chain of smooth rational curves with negative self-intersection numbers  $-b_1, \dots, -b_k$ . Since a cyclic quotient singularity can be characterized by these numbers, sometimes we denote the cyclic quotient singularity by  $[b_1, \dots, b_k]$  (see Figure 1). The numbers  $b_1, \dots, b_k$  are related to  $\frac{1}{n}(1, r)$  by the following continued fraction:

$$\frac{n}{r} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}}$$

## 3 $\mathbb{Q}$ -Gorenstein smoothing of surface quotient singularities

**Definition 3.1.** A surface quotient singularity which admits a  $\mathbb{Q}$ -Gorenstein smoothing is called a singularity of *class T*.

**Theorem 3.2** ([KSB88]). *A singularity of class  $T$  is either an ADE singularity or a cyclic quotient singularity  $\frac{1}{dn^2}(1, dna - 1)$  with  $\text{g.c.d.}(a, n) = 1$ .*

*Proof.* The fact that ADE singularities are of class  $T$  is simple to check. We show that  $\frac{1}{dn^2}(1, dna - 1)$  is of class  $T$ . For the converse implication, we refer to [KSB88].

Assume that  $p \in X$  is such singularity. Before starting, we give the big picture of the proof:

$$\begin{array}{ccc} Y & \hookrightarrow & \mathcal{Y} \\ \mathbb{Z}_n \downarrow & & \downarrow \mathbb{Z}_n \\ X & \hookrightarrow & \mathfrak{X}. \end{array}$$

Write  $X = Y/\langle \zeta \rangle$  where  $\zeta$  is a primitive  $n$ -th root of unity. Therefore  $Y$  has a singularity of type

$$\frac{1}{dn}(1, dna - 1) = \frac{1}{dn}(1, -1) = \frac{1}{dn}(1, dn - 1) = A_{dn-1},$$

which is given by  $xy - z^{dn} = 0$  for suitable local coordinates on  $Y$ . We have that  $\zeta$  acts on  $\mathbb{C}^3$  by

$$\zeta \cdot (x, y, z) = (\zeta x, \zeta^{-1}y, \zeta^a z).$$

The only fixed point of the  $\langle \zeta \rangle$ -action is  $(0, 0, 0) \in Y$ .

Now let  $\Delta = \mathbb{C}[[t]]$ ,  $\mathcal{Y} = \{xy - z^{dn} + t = 0\} \rightarrow \Delta$ , and  $\mathfrak{X} = \mathcal{Y}/\langle \zeta \rangle$ . We can extend the  $\langle \zeta \rangle$ -action on  $\mathcal{Y}$  by setting  $\zeta \cdot t = t$ . Then  $\mathfrak{X} \rightarrow \Delta$  is a smoothing of  $X$ , because the  $\langle \zeta \rangle$ -action does not have fixed points on the fibers over  $t \in \Delta \setminus \{0\}$ . In addition,  $\mathfrak{X} \rightarrow \Delta$  is a  $\mathbb{Q}$ -Gorenstein smoothing because  $K_{\mathfrak{X}/\Delta} = K_{\mathfrak{X}}$  is  $\mathbb{Q}$ -Cartier. The reason for this is that  $\mathcal{Y}$  is smooth and  $\mathfrak{X}$  is the quotient of  $\mathcal{Y}$  by a finite group, so  $\mathfrak{X}$  is  $\mathbb{Q}$ -factorial.  $\square$

**Example 3.3.**

- (1)  $\frac{1}{3}(1, 1)$  is not of class  $T$ ;
- (2)  $\frac{1}{4}(1, 1) = \frac{1}{1 \cdot 2^2}(1, 1 \cdot 2 \cdot 1 - 1)$  is of class  $T$ . The exceptional divisor on the minimal resolution is [4];
- (3)  $\frac{1}{16}(1, 7) = \frac{1}{4 \cdot 2^2}(1, 4 \cdot 2 \cdot 1 - 1)$  is of class  $T$ . The exceptional divisor on the minimal resolution is [3, 2, 2, 3] because

$$\begin{aligned} \frac{16}{7} &= 3 - \frac{1}{7} \\ &= 3 - \frac{1}{2 - \frac{1}{\frac{5}{3}}} \\ &= 3 - \frac{1}{2 - \frac{1}{2 - \frac{1}{3}}}. \end{aligned}$$

The next proposition shows that a singularity of class  $T$  can be recognized by the shape of the exceptional divisor of its minimal resolution.

**Proposition 3.4** ([Wah81, KSB88]).

- (i)  $[4]$  and  $[3, 2, \dots, 2, 3]$  are of class  $T$ ;
- (ii) If  $[b_1, \dots, b_k]$  is of class  $T$ , then  $[2, b_1, \dots, b_{k-1}, b_k + 1]$  and  $[b_1 + 1, b_2, \dots, b_k, 2]$  are of class  $T$ ;
- (iii) A singularity of class  $T$  which is not ADE is obtained from a singularity in (i) to which we apply the steps in (ii).

**Remark 3.5.** Say we have a projective variety  $X$  with only one singularity of class  $T$ . Can we  $\mathbb{Q}$ -Gorenstein smooth  $X$ ? The answer in general is no! We can locally  $\mathbb{Q}$ -Gorenstein smooth the singularity, however we may not be able to do it globally. The same kind of problem arises if  $X$  has several singularities of class  $T$ . How do we know that we can  $\mathbb{Q}$ -Gorenstein-smooth all of them simultaneously?

This can be done if  $H^2(T_X) = 0$ , where recall  $T_X = \mathcal{H}om(\Omega_X, \mathcal{O}_X)$ . This is actually a standard argument: consider an affine cover  $X = \cup_\alpha U_\alpha$  where each  $U_\alpha$  contains at most one  $T$  singularity. Then we can  $\mathbb{Q}$ -Gorenstein-smooth each  $U_\alpha$  individually, and the condition  $H^2(T_X) = 0$  guarantees that we can glue all the smoothings together.

In practice, the condition  $H^2(T_X) = 0$  can be tricky to check, but in this case other methods are employed: the vanishing  $H^2(T_X) = 0$  is implied by the vanishing of the cohomology group  $H^2(T_{\tilde{X}}(-\log E))$ , where  $\tilde{X}$  is the minimal resolution of  $X$  and  $E$  is the reduced exceptional divisor (see [LP06, Section 2]).

## 4 Back to general deformation theory: the functor $\text{Def}' X$

Let  $p \in X$  be the germ of a singularity of class  $T$ . Then  $X$  is  $\mathbb{Q}$ -Gorenstein (i.e.,  $NK_X$  is Cartier for some nonzero integer  $N$ ). The *index* of  $X$  is defined to be the smallest positive integer  $N$  such that  $NK_X$  is Cartier. With this setting we have the following lemma.

**Lemma 4.1.** [Hac04, Lemma 3.3] *Let  $\mathfrak{X} \rightarrow \Delta$  be a  $\mathbb{Q}$ -Gorenstein smoothing of  $X$ . Then  $N$  is the least positive integer such that  $\omega_{\mathfrak{X}/\Delta}^{[N]}$  is invertible ( $\omega_{\mathfrak{X}/\Delta}^{[N]} = i_* \omega_{\mathfrak{X}^\circ/\Delta}^{\otimes N}$  where  $i: \mathfrak{X}^\circ \hookrightarrow \mathfrak{X}$  is the Gorenstein locus, or equivalently,  $\omega_{\mathfrak{X}/\Delta}^{[N]} = (\omega_{\mathfrak{X}/\Delta}^{\otimes N})^{**}$ ).*

**Definition 4.2.** Define the functor

$$\text{Def}' X(S) = \{\text{flat families } \pi: \mathfrak{X} \rightarrow S \mid \pi^{-1}(s) = X, \exists s \in S, \text{ and } \omega_{\mathfrak{X}/S}^{[N]} \text{ is invertible}\} / \cong.$$

$S$  is taken to be the spectrum of a finite local  $\mathbb{C}$ -algebra. Then  $\text{Def}' X$  is pro-represented (i.e. it is a small filtered colimit of representable functors) by a subspace of  $\text{Def} X$ .

**Theorem 4.3.** *The points of  $\text{Def}' X$  are in 1-to-1 correspondence with the points of  $\text{Def} Y$  corresponding to  $\mathbb{Z}_N$ -equivariant smoothings of  $Y$ , where*

$$Y = \underline{\text{Spec}}_X(\mathcal{O}_X \oplus \omega_X^{[1]} \oplus \dots \oplus \omega_X^{[N-1]}).$$

(Recall, this is the relative  $\text{Spec}$  of a coherent sheaf of algebras  $\mathcal{A}$ .  $\underline{\text{Spec}}_X(\mathcal{A}) \rightarrow X$  is an affine morphism and the preimage of  $\text{Spec}(R)$  is  $\text{Spec}(\Gamma(\text{Spec}(R), \mathcal{A}))$ .) Observe that  $Y$  has an  $A_{dn-1}$  singularity, which is easier to study.

The previous theorem is important because it gives us a more tangible notion of deformation. Arie will talk more about the importance of  $\text{Def}' X$  to study  $\text{Def} X$ .

**Example 4.4.** We already know from Theorem 3.2 that  $\frac{1}{3}(1, 1)$  is not of class  $T$ . But now Theorem 4.3 gives us another way to see this.  $\mathbb{Q}$ -Gorenstein smoothings of  $\frac{1}{3}(1, 1)$  are in bijection with  $\mathbb{Z}_3$ -equivariant smoothings  $\mathcal{Y} \rightarrow \Delta$  of  $\mathbb{C}^2$ . But the only possibility is  $\mathcal{Y} = \mathbb{C}^2 \times \Delta$ , and after quotienting by  $\mathbb{Z}_3$  to obtain  $\mathfrak{X}$  we get that all fibers of  $\mathfrak{X} \rightarrow \Delta$  are singular (this actually requires some checking). So there is no  $\mathbb{Q}$ -Gorenstein smoothing of  $\frac{1}{3}(1, 1)$ .

**Remark 4.5.** One could require that all the powers  $\omega_{\mathfrak{X}/S}^{[n]}$  to commute with base change (this condition implies that  $\omega_{\mathfrak{X}/S}^{[N]}$  is invertible, where  $N$  is the index). This moduli functor was considered in [Hac04], and it gives a better behaved moduli space.

## References

- [Hac04] P. Hacking, *Compact moduli of plane curves*, 2004.
- [LP06] Y. Lee and J. Park, *A simply connected surface of general type with  $p_g = 0$  and  $K^2 = 2$* , 2006.
- [KSB88] J. Kollár and N. Shepherd-Barron, *Threefolds and deformations of surface singularities*, 1988.
- [Wah81] J. Wahl, *Smoothing of normal surface singularities*, 1981.