Mustafin varieties and moduli of points in the projective plane, II.

(Joint with J. Tevelev)

§ Goals.

1. Examples of n-pointed degen's of $\mathbb{P}^2$ from Mustafin var's.
2. Gerritzen-Piwek's work on comp. moduli of n pts in $\mathbb{P}^2$.
& our contribution.

§ Degen's of $\mathbb{P}^2$ with n pts.

Recall: $R = \mathbb{C}[[t]]$, $K = Q(R)$, $\Delta = \mathrm{Spec}(R)$.

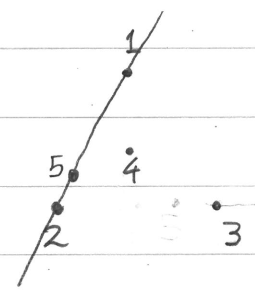
1. $a_1(t), \dots, a_n(t) : \Delta \setminus \{0\} \to \mathbb{P}^2_K = \mathbb{P}^2 \times (\Delta\setminus\{0\})$ in g.l.p.
2. L lattice. $\bar{a}_1(t), \dots, \bar{a}_n(t) : \Delta \to \mathbb{P}(L)$.
3. $\bar{a}_1(0), \dots, \bar{a}_n(0) \in \pi^{-1}(0) = \mathbb{P}^2$.
4. L is stable w.r.t. $a = (a_1, \dots, a_n)$ if $\exists$ 4 pts among $\bar{a}_1(0), \dots, \bar{a}_n(0)$ in g.l.p.
5. $\Sigma_a = \{\text{stable lattice classes}\}$

$$\begin{tikzcd} \mathbb{P}(\Sigma_a) \arrow[d] \\ \Delta \arrow[u, bend left, "\bar{a}_i"] \end{tikzcd}$$

<u>Ex 1</u>: e_1, e_2, e_3 canonical basis of K^3.

$a(t) = ([1:0:0], [0:1:0], [0:0:1], [1:1:1], [1:-1:t])$
(Coordinates are with respect to the dual of e_1, e_2, e_3).
(These are in g.l.p. over K) Compute the 5-pointed central fiber of $\mathbb{P}(\Sigma_a)$.

__Solve:__ For $t \to 0$



$[1:0:0], [0:1:0], [0:0:1], [1:1:1], [1:-1:0]$

$L_0 = e_1R + e_2R + e_3R$ stabilizes 1234, 1345, 2345
We need lattices stabilizing 1235, 1245.

Let $L = t^\alpha e_1 R + t^\beta e_2 R + t^\gamma e_3 R$. How do the 5 pts change in $\mathbb{P}(L)$?

$$\begin{pmatrix} t^\alpha e_1 \\ t^\beta e_2 \\ t^\gamma e_3 \end{pmatrix} = \underbrace{\begin{pmatrix} t^\alpha & 0 & 0 \\ 0 & t^\beta & 0 \\ 0 & 0 & t^\gamma \end{pmatrix}}_{M} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \Rightarrow M^t \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{\substack{\uparrow \\ \text{coordinates} \\ \text{w.r.t. } L}} = (M^t)^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t^{-\alpha} x_1 \\ t^{-\beta} x_2 \\ t^{-\gamma} x_3 \end{pmatrix}$$

Let $L_1 = e_1 R + e_2 R + t e_3 R$. In $\mathbb{P}(L_1)$

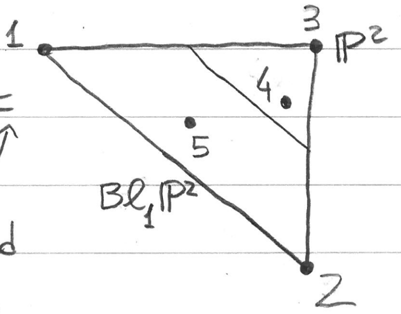
$[1:0:0], [0:1:0], [0:0:t^{-1}], [1:1:t^{-1}], [1:-1:1]$

$[1:0:0], [0:1:0], [0:0:1], [t:t:1], [1:-1:1]$

For $t \to 0$

$[1:0:0], [0:1:0], [0:0:1], [0:0:1], [1:-1:1]$

L_1 stabilizes 1235 & 1245. So, $\Sigma_a = \{[L_0], [L_1]\}$.

$\pi^{-1}(0) =$ 

computed last time

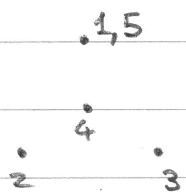
Obtaining the limit points requires more work which we omit.

<u>Ex 2</u>: e_1, e_2, e_3 canonical basis of K^3.

$a(t) = ([1:0:0], [0:1:0], [0:0:1], [1:1:1], [1:t:t^2])$

Compute the 5-pointed central fiber of $\mathbb{P}(\Sigma_a)$.

<u>Solve</u>: For $t \to 0$



$[1:0:0], [0:1:0], [0:0:1], [1:1:1], [1:0:0]$.

L_0 stabilizes 1234, 2345.

$L_1 = e_1R + te_2R + t^2e_3R$. Then in $\mathbb{P}(L_1)$, for $t \to 0$

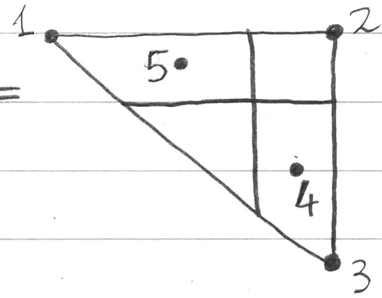
$[1:0:0], [0:1:0], [0:0:1], [0:0:1], [1:1:1]$.

L_1 stabilizes 1235, 1245. We miss 1345!

$L_2 = e_1R + te_1R + te_2R$. Then in $\mathbb{P}(L_2)$, for $t \to 0$

$[1:0:0], [0:1:0], [0:0:1], [0:1:1], [1:1:0]$

L_2 stabilizes 1345. So, $\Sigma_a = \{[L_0], [L_1], [L_2]\}$.

$\pi^{-1}(0) =$ 

§ Gerritzen-Piwek work.

Moduli space of n pts in $\mathbb{P}^2$ in g.l.p. : $B_n = \mathcal{U}/SL_3$, where $\mathcal{U} \subseteq (\mathbb{P}^2)^n$ n-tuples in g.l.p.

GP comp'n of B_n:

$$B_n \hookrightarrow \prod_{\substack{\text{ordered}\\ \text{quintuples}\\ \text{in } \{1,\dots,n\}}} \mathbb{P}^2$$

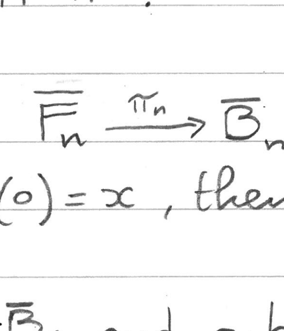
$$[(p_1,\dots,p_n)] \longmapsto (\dots, q_v, \dots)$$

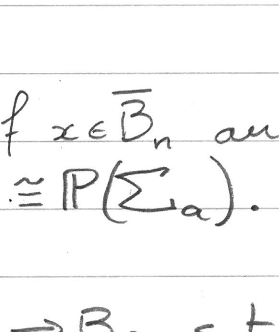
$q_v = f(p_{v_5})$, where $\exists! f \in PGL_3$ sending $p_{v_1},\dots,p_{v_4}$ to $[100], [010], [001], [111]$.

$\overline{B}_n :=$ closure $B_n \subseteq \prod \mathbb{P}^2$.

GP claim: $\exists$ family $\overline{F}_n \xrightarrow{\pi_n} \overline{B}_n$ s.t. if $x \in \overline{B}_n$ and $a: \Delta\setminus\{0\} \to B_n$ with $\overline{a}(0) = x$, then $\overline{a}^* \overline{F}_n \cong \mathbb{P}(\Sigma_a)$.

Thm (S-Tevelev): $\exists\, x \in \overline{B}_6$ and $a, b: \Delta\setminus\{0\} \to B_6$ s.t.
1. $\overline{a}(0) = x = \overline{b}(0)$;
2. $\mathbb{P}(\Sigma_a)_0 \not\cong \mathbb{P}(\Sigma_b)_0$





(Thm (S-Tevelev): $\overline{B}_n \cong \overline{X}(3,n)$, Kapranov's comp'n moduli n lines in $\mathbb{P}^2$.)

§ The correct comp'n of B_n.

Construct embedding $B_n \hookrightarrow \overline{B}_n \times \mathcal{H}$, where $\mathcal{H}$ is the multigraded Hilbert scheme of $(\mathbb{P}^2)^{\binom{n}{4}}$.

$\overline{X}_{GP}(3,n)$ = closure of B_n in $\overline{B}_n \times \mathcal{H}$. $\overline{\mathcal{M}} \to \overline{X}_{GP}(3,6)$ pullback of the family over $\mathcal{H}$.

<u>Thm</u> (S-Tevelev):

1. If $a : \Delta \setminus \{0\} \to B_n$ and $\overline{a} : \Delta \to \overline{X}_{GP}(3,6)$, then $\overline{a}^* \overline{\mathcal{M}} \cong \mathbb{P}(\Sigma_a)$.
2. $\exists\ \overline{X}_{GP}(3,n) \xrightarrow{birat} \overline{B}_n$.
3. We construct n-sections of $\overline{\mathcal{M}} \to \overline{X}_{GP}(3,n)$.
4. $\overline{X}_{GP}(3,5) \cong \overline{M}_{0,5}$.
5. Study $\overline{X}_{GP}(3,6)$ (in progress).