FIRST Seminar,
Singular K3 Surfaces

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1 Introduction

In this talk I will discuss the content of [SI].

2 Singular K3 surfaces

Definition 1. A *K3 surface* X is a smooth connected projective 2-dimensional variety over $\mathbb{C}$ such that $K_X \sim 0$ ($\Rightarrow p_g(X) = h^0(X, \omega_X) = 1$) and $q(X) = h^1(X, \mathcal{O}_X) = 0$.

Let us make some observations originating from this definition. Let X be a K3 surface.

(1) $(H^2(X;\mathbb{Z}), \smile)$ is a even, unimodular lattice of signature $(3, 19)$. (Therefore, it is isomorphic to $U \oplus U \oplus U \oplus E_8 \oplus E_8$.)

(2) If we look at $H^2(X;\mathbb{C}) = H^2(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, we have the Hodge decomposition

$$H^2(X;\mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

From which we argue that $\dim H^{1,1}(X) = 20$. We have that $NS(X) \hookrightarrow H^{1,1}(X)$, implying that the Picard number $\rho(X) = \mathrm{rank}(NS(X))$ satisfies $1 \leq \rho(X) \leq 20$.

(3) $NS(X) \subseteq H^2(X;\mathbb{Z})$, and

$$T_X = NS(X)^{\perp H^2(X;\mathbb{Z})},$$

is called the transcendental lattice of X. By the Hodge index theorem, we can argue that T_X has signature $(2, 20 - \rho(X)) = (3, 19) - (1, \rho(X) - 1)$.

Definition 2. A K3 surface X is *singular* if $\rho(X) = 20$.

Observation 1. For a singular K3 surface T_X is an even lattice of signature $(2,0)$. In particular, after choosing a basis $\{y_1, y_2\}$ for T_X, the matrix of the bilinear form looks like

$$\begin{pmatrix} y_1 \cdot y_1 & y_1 \cdot y_2 \\ y_2 \cdot y_1 & y_2 \cdot y_2 \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix},$$

where $a, b, c \in \mathbb{Z}, a > 0, b^2 - 4ac < 0$ (which imply that $c > 0$).

Definition 3. Let $\mathcal{S}_{\text{K3}}$ be the set of singular K3 surfaces.

Definition 4. $\mathcal{Q} = \{\left(\begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix}\right) \mid a, b, c \in \mathbb{Z}, a > 0, c > 0, b^2 - 4ac < 0\}$. Moreover, define the following action

$$\begin{aligned} SL(2,\mathbb{Z}) \times \mathcal{Q} &\to \mathcal{Q}, \\ (\gamma, Q) &\mapsto \gamma^t Q \gamma. \end{aligned}$$

Definition 5. We want to define a map $\mathcal{S}_{\text{K3}} \to \mathcal{Q}/SL(2,\mathbb{Z})$. Let X be a singular K3 surface. Naively, one can do the following. Let $\{y_1, y_2\}$ be a basis for T_X and associate the orbit of $\left(\begin{smallmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{smallmatrix}\right)$. But if we chose a different basis $\{z_1, z_2\}$, then maybe $\left(\begin{smallmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & z_2^2 \end{smallmatrix}\right)$ are not in the same orbit! To solve this problem, we pick bases which are positively oriented: $\{y_1, y_2\}$ is *positively oriented* if

$$\text{Im}(p_X(y_1)/p_X(y_2)) > 0.$$

p_X is the linear functional $H^2(X;\mathbb{Z}) \to \mathbb{C}$ such that $t \mapsto \int_t \omega_X$ (ω_X is defined up to a constant, but then the constant simplifies when we consider the ratio), and it is easy to see that p_X is nonzero on cycles in T_X (consider the long exact sequence in cohomology associated to the exponential short exact sequence). So we have a map $\mathcal{S}_{\text{K3}} \to \mathcal{Q}/SL(2,\mathbb{Z})$ which does not depend on the choice of the basis, and it induces a map

$$\varphi \colon \mathcal{S}_{\text{K3}}/_{\cong} \to \mathcal{Q}/SL(2,\mathbb{Z}).$$

3 Main theorem

Theorem 1 ([SI]). *φ is a bijection.*

Proof. φ is injective. Let X, Y be two singular K3 surfaces such that $\varphi([X]) = \varphi([Y])$. Then there exists an isometry $T_X \cong T_Y$ which preserves the orientations. It follows from a result of Mukai that this isometry extends to an isometry $H^2(X;\mathbb{Z}) \cong H^2(Y;\mathbb{Z})$ preserving the Hodge structures. It follows from the Torelli theorem that X and Y are isomorphic.
φ is surjective. Surjectivity of the period map for K3 surfaces was not known yet. The proof in [SI] is very explicit. Consider $Q = \left(\begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix}\right)$. Define

$$\tau_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \ \tau_2 = \frac{b + \sqrt{b^2 - 4ac}}{2}.$$

Define
$$C_i = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau_i,\ \ i = 1, 2.$$
Let $A = C_1 \times C_2$, which is a singular abelian surface (i.e. an abelian surface of maximal Picard number, which is 4) because C_1 and C_2 are isogenous elliptic curves with complex multiplication. Consider the Kummer surface $Km(A)$, which is the minimal desingularization of $A/(pt \sim -pt)$. In [SI] they construct a singular K3 surface X with a symplectic involution ι such that the minimal desingularization of X/ι is $Km(A)$. We have that
$$Q_X = \frac{1}{2} Q_{KM(A)} = Q_A = Q,$$
where the last equality is proved in [SM]. In conclusion, $\varphi([X]) = [Q]$.
Let me add some details about the construction of the K3 surface X. The Kummer surface $Km(A)$ has a configuration of 24 smooth rational curves called the double Kummer pencil. These are the irreducible components of the singular fibers of the two elliptic pencils $Km(A) \to \mathbb{P}^1$ induced by the two projections $A \to C_i$, $i = 1, 2$. We can define a divisor D on X obtained as a $\mathbb{Z}$-combination of some of the lines of the double Kummer pencil in such a way that D is a curve of Kodaira type II^*. Then there exists a unique elliptic pencil $\pi\colon Km(A) \to \mathbb{P}^1$ with D as one of the singular fibers. This elliptic pencil has at least three singular fibers of type

- $II^*, I^*_{b_1}, I^*_{b_2}$ with $b_1 + b_2 \leq 2$, or
- II^*, IV^*, I^*_0.

We base change π with respect to the double cover $\mathbb{P}^1 \to \mathbb{P}^1$ branched at the two points whose corresponding fibers with respect to π are singular of type $I^*_{b_1}, I^*_{b_2}$ or IV^*, I^*_0. This is how we obtain X (which is singular because it dominates $Km(A)$). □

4 Consequences

Corollary 1. *Every singular K3 surface has a model defined over a number field.*

Proof. Clear from the construction in the proof of Theorem 1. □

Corollary 2. *Every singular K3 surface has infinite automorphism group.*

Proof. Given a singular K3 surface X, they find an elliptic pencil $\pi\colon X \to \mathbb{P}^1$ with infinite group of sections (this is called the Mordell-Weil group of an elliptic fibration), and any such section $s\colon \mathbb{P}^1 \to X$ defines an automorphism of X given by $x \mapsto x + s(\mathbb{P}^1) \cap \pi^{-1}(\pi(x))$. □

Later on, Morrison coined the term Shioda-Inose structure in [M]. A K3 surface X admits a Shioda-Inose structure if there is a Nikulin involution ι such that the minimal desingularization Y of X/ι is a Kummer surface such that $T_X = T_Y\left(\frac{1}{2}\right)$. Morrison proved that a K3 surface of Picard number 19 admits such structure, and give precise conditions for K3 surfaces of Picard number $17, 18$ (for Picard number ≤ 16 it is easy to see that there cannot be a Shioda-Inose structure).

References

[M] Morrison, D.R.: *On K3 surfaces with large Picard number*. Invent. Math. 75 (1984), no. 1, 105–121.

[SI] Shioda, T., Inose, H.: *On singular K3 surfaces*. Complex analysis and algebraic geometry, pp. 119–136. Iwanami Shoten, Tokyo, 1977.

[SM] Shioda, T., Mitani, N.: *Singular abelian surfaces and binary quadratic forms*. Classification of algebraic varieties and compact complex manifolds, pp. 259–287. Lecture Notes in Math., Vol. 412, Springer, Berlin, 1974.