

Lecture 6

TODAY: • Dimension and coordinate rings, a first exposure;
 • Some projective geometry.

From now on, I will assume $k = \mathbb{C}$, really.
Dimension and coordinate rings, a first exposure.

Def: Let X be a noetherian topological space. Then we define the dimension of X to be the maximum number of strict inclusions that we can obtain considering chains of irreducible closed subsets of X :

$$\underbrace{X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_K}_{\text{length } K}$$

We denote it by $\dim(X)$.

Example:

(a) $\dim(\{\text{pt}\}) = 0$

(b) $\dim(A^1) = 1$

Proof: $\{\text{pt}\} \subsetneq A^1$.

↑
 nothing else
 can be inserted
 here

Use linear
 subspaces

(c) $\dim(A^n) = n$ (trickier). Easy to see that $n \leq \dim(A^n)$

Def: Let R be a ring (commutative with unit). Define the Krull dimension of R to be the maximum number of strict inclusion that we can obtain considering chains of prime ideals in R . We denote it by $\dim(R)$

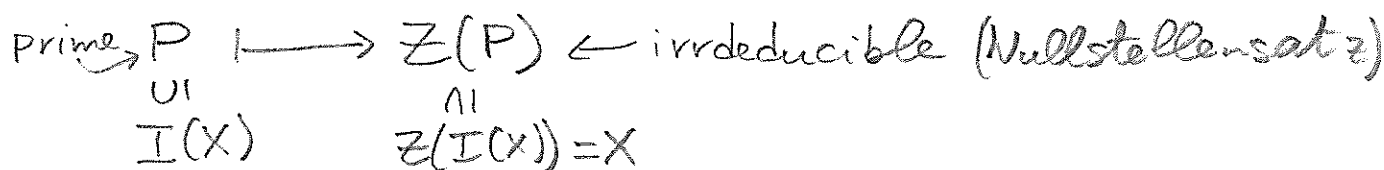
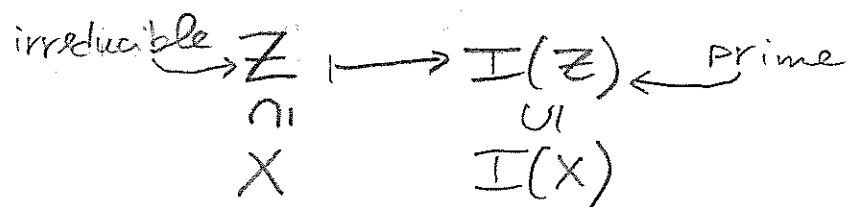
Example:

(a) $\dim(k) = 0$;

(b) $\dim(k[x]) = 1$, because $k[x]$ is a PID ;

(c) $\dim(k[x_1, \dots, x_n]) = n$ (Commutative algebra fact).

Observation: Let $X \subseteq A^n$ Zariski closed. Then it follows from the Nullstellensatz that there is a bijection between irreducible closed subsets of X and prime ideals in $k[x_1, \dots, x_n]$ containing $I(X)$.



Therefore:

$$\dim(X) = \dim\left(\frac{k[x_1, \dots, x_n]}{I(X)}\right)$$

Def: Let $X \subseteq A^n$ Zariski closed. Then $k[x_1, \dots, x_n]/I(X)$ is the coordinate ring of X and is denoted by $k[X]$.

Observation: $k[X]$ is a finitely generated k -algebra ^{with no nilpotent}. Conversely any finitely generated k -algebra ^{with no nilpotents} B is isomorphic to $k[x_1, \dots, x_n]/I$ for some n and some

radical ideal I .

$$k[x_1, \dots, x_e] \xrightarrow{\varphi} B = k[\gamma_1, \dots, \gamma_r]$$

$x_i \longmapsto \gamma_i$
and extended

$$\Rightarrow B \cong k[x_1, \dots, x_e] / \text{Ker}(\varphi).$$

Also, X is irreducible $\Leftrightarrow k[X]$ is a domain.

Thm: $X \subseteq A^m$ Zariski closed (recall that we assumed $k = \bar{k}$). Then $\dim(X) = \dim(k[X])$.

Corollary: $\dim(A^n) = \dim(k[x_1, \dots, x_n] / (0)) = \dim(k[x_1, \dots, x_n]) = n$.

Some projective geometry.

Def: Let $n \geq 0$. The projective n -space is the set

$$\mathbb{P}^n = (A^{n+1} \setminus \{(0, \dots, 0)\}) / \sim,$$

where $(a_0, \dots, a_n) \sim (b_0, \dots, b_n) : \Leftrightarrow \exists c \in k \setminus \{0\}$ s.t.
 $(a_0, \dots, a_n) = c(b_0, \dots, b_n)$.

An element in \mathbb{P}^n is denoted by $[a_0 : \dots : a_n]$.

Goal: Define Zariski topology on \mathbb{P}^n .

Def: A polynomial $F \in k[x_0, \dots, x_n]$ is homogeneous if $F(tx_0, \dots, tx_n) = t^d F(x_0, \dots, x_n)$. d is called the degree of the homogeneous polynomial F .

Equivalently, a polynomial $F \in k[X_0, \dots, X_n]$ is homogeneous of degree d if it can be written as the sum of degree d monomials in $k[X_0, \dots, X_n]$.

Example: $F(X, Y, Z) = X^3Y + 7X^2Z^2 - Y^4$

Observation: The reason why we consider homogeneous polynomials F is because it makes sense for a point $[a_0 : \dots : a_n] \in \mathbb{P}^n$ to be a zero of F .

Say $F(a_0, \dots, a_n) = 0$. But, given $c \in k \setminus \{0\}$,

$[ca_0 : \dots : ca_n] = [a_0 : \dots : a_n]$ we also want that

$F(ca_0, \dots, ca_n) = 0$. But

$$F(ca_0, \dots, ca_n) = c^d F(a_0, \dots, a_n) = c^d \cdot 0 = 0.$$

Def: An ideal $I \subseteq k[X_0, \dots, X_n]$ is homogeneous if it is generated by homogeneous polynomials.

The homogeneous ideal (X_0, \dots, X_n) is called the irrelevant ideal.

Def: For a homogeneous ideal $I \subseteq k[X_0, \dots, X_n]$, define $V(I) = \{P \in \mathbb{P}^n \mid F(P) = 0 \forall F \in I\}$.

Examples:

(a) $V((0)) = \mathbb{P}^n$

(b) $V(k[X_0, \dots, X_n]) = \emptyset$

(c) $V((X_0, \dots, X_n)) = \emptyset$.

Def: $X \subseteq \mathbb{P}^n$ is Zariski closed if $X = V(\mathcal{I})$ for some homogeneous ideal $\mathcal{I} \subseteq k[X_0, \dots, X_n]$.

Prop: Zariski closed subsets of \mathbb{P}^n give a topology on \mathbb{P}^n .

Proof: $\mathbb{P}^n = V(0)$, $\emptyset = V(k[X_0, \dots, X_n])$

$$\bigcap_{\alpha} V(\mathcal{I}_{\alpha}) = V\left(\sum_{\alpha} \mathcal{I}_{\alpha}\right)$$

$$V(\mathcal{I}_1) \cup V(\mathcal{I}_2) = V(\mathcal{I}_1 \mathcal{I}_2)$$

(\subseteq)^v (\supseteq) By contradiction, $p \in V(\mathcal{I}_1 \mathcal{I}_2)$,
 $p \notin V(\mathcal{I}_1) \cup V(\mathcal{I}_2)$.

$\Rightarrow \exists F_1 \in \mathcal{I}_1, F_2 \in \mathcal{I}_2$ s.t. $F_1(p), F_2(p) \neq 0$.
But $F_1(p)F_2(p) = 0$, which cannot be.

Def: Let $X \subseteq \mathbb{P}^n$ Zariski closed. Define $\mathcal{I}(X)$ to be the ideal generated by homogeneous polynomials in $k[X_0, \dots, X_n]$ vanishing on X . \square

Thm: Let $\mathcal{I} \subseteq k[X_0, \dots, X_n]$ be a homogeneous ideal different from the irrelevant ideal. Then

(a) $V(\mathcal{I}) = \emptyset \Leftrightarrow (X_0, \dots, X_n) \subseteq \sqrt{\mathcal{I}}$;

(b) If $V(\mathcal{I}) \neq \emptyset$, then $\mathcal{I}(V(\mathcal{I})) = \sqrt{\mathcal{I}}$.