Introduction to Algebraic Geometry (MATH 8300) 08/27/2016

Lecture 6

TODAY: • Dimension and coordinate rings, a first exposure;
• Some projective geometry.

From now on, I will assume $k=\bar{k}$, really.

Dimension and coordinate rings, a first exposure.

Def: Let X be a noetherian topological space. Then we define the dimension of X to be the maximum number of strict inclusions that we can obtain considering chains of irreducible closed subsets of X:

$$\underbrace{X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_K}_{\text{length } K},$$

We denote it by $\dim(X)$.

Example:

(a) $\dim(\{pt\}) = 0$

(b) $\dim(\mathbb{A}^1) = 1$

proof: $\{pt\} \subsetneq \mathbb{A}^1$.

nothing else can be inserted here

(c) $\dim(\mathbb{A}^n) = n$ (trickier). Easy to see that $n \leq \dim(\mathbb{A}^n)$ Use linear subspaces

Def: Let R be a ring (commutative with unit). Define the Krull dimension of R to be the maximum number of strict inclusion that we can obtain considering chains of prime ideals in R. We denote it by $\dim(R)$

Example:

(a) $\dim(k) = 0$;

(b) $\dim(k[x]) = 1$, because $k[x]$ is a PID;

(c) $\dim(k[x_1, .., x_n]) = n$ (Commutative algebra fact).

Observation: Let $X \subseteq \mathbb{A}^n$ Zariski closed. Then it follows from the Nullstellensatz that there is a bijection between irreducible closed subsets of X and prime ideals in $k[x_1, \dots, x_n]$ containing $I(X)$.

$$\begin{array}{ccc} \text{irreducible } Z & \longmapsto & I(Z) \text{ prime} \\ \cap\!\!| & & \cup\!\!| \\ X & & I(X) \end{array}$$

$$\begin{array}{ccc} \text{prime } P & \longmapsto & Z(P) \leftarrow \text{irrdeducible (Nullstellensatz)} \\ \cup\!\!| & & \cap\!\!| \\ I(X) & & Z(I(X)) = X \end{array}$$

Therefore:

$$\dim(X) = \dim\left(\frac{k[x_1, \dots, x_n]}{I(X)}\right)$$

Def: Let $X \subseteq \mathbb{A}^n$ Zariski closed. Then $k[x_1, \dots, x_n]/I(X)$ is the coordinate ring of X and is denoted by $k[X]$.

Observation: $k[X]$ is a finitely generated k-algebra with no nilpotent. Conversely any finitely generated k-algebra with no nilpotents B is isomorphic to $k[x_1, \dots, x_n]/I$ for some n and some radical ideal I.

$$k[x_1, \dots, x_e] \xrightarrow{\varphi} B = k[\gamma_1, \dots, \gamma_e)$$
$$x_i \longmapsto \gamma_i$$
and extended

$$\Rightarrow \quad B \cong k[x_1, \dots, x_e] / \mathrm{Ker}(\varphi).$$

Also, X is irreducible $\Leftrightarrow$ $k[X]$ is a domain.

__Thm__: $X \subseteq \mathbb{A}^n$ Zariski closed (recall that we assumed $k = \bar{k}$). Then $\dim(X) = \dim(k[X])$.

__Corollary__: $\dim(\mathbb{A}^n) = \dim\left(k[x_1, \dots, x_n]/(0)\right) = \dim(k[x_1, \dots, x_n]) = n$.

__Some projective geometry.__

__Def__: Let $n \geq 0$. The projective n-space is the set

$$\mathbb{P}^n = \left(\mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\}\right) / \sim \ ,$$

where $(a_0, \dots, a_n) \sim (b_0, \dots, b_n) :\Leftrightarrow \exists\, c \in k \setminus \{0\}$ s.t. $(a_0, \dots, a_n) = c(b_0, \dots, b_n)$.

An element in $\mathbb{P}^n$ is denoted by $[a_0 : \dots : a_n]$.

__Goal__: Define Zariski topology on $\mathbb{P}^n$.

__Def__: A polynomial $F \in k[X_0, \dots, X_n]$ is homogeneous if $F(tX_0, \dots, tX_n) = t^d F(X_0, \dots, X_n)$. d is called the degree of the homogeneous polynomial F.

Equivalently, a polynomial $F \in k[X_0, \ldots, X_n]$ is homogeneous of degree d if it can be written as the sum of degree d monomials in $k[X_0, \ldots, X_n]$.

Example: $F(X,Y,Z) = X^3Y + 7X^2Z^2 - Y^4$

Observation: The reason why we consider homogeneous polynomials F is because it makes sense for a point $[a_0 : \ldots : a_n] \in \mathbb{P}^n$ to be a zero of F.

Say $F(a_0, \ldots, a_n) = 0$ But, given $c \in k \setminus \{0\}$, $[ca_0 : \ldots : ca_n] = [a_0 : \ldots : a_n]$ we also want that $F(ca_0, \ldots, ca_n) = 0$. But

$$F(ca_0, \ldots, ca_n) = c^d F(a_0, \ldots, a_n) = c^d \cdot 0 = 0.$$

Def: An ideal $I \subseteq k[X_0, \ldots, X_n]$ is homogeneous if it is generated by homogeneous polynomials.
The homogeneous ideal $(X_0, \ldots, X_n)$ is called the irrelevant ideal.

Def: For a homogeneous ideal $I \subseteq k[X_0, \ldots, X_n]$, define $V(I) = \{ p \in \mathbb{P}^n \mid F(p) = 0 \ \forall F \in I \}$.

Examples:

(a) $V((0)) = \mathbb{P}^n$

(b) $V(k[X_0, \ldots, X_n]) = \emptyset$

(c) $V((X_0, \ldots, X_n)) = \emptyset$.

Def: $X \subseteq \mathbb{P}^n$ is Zariski closed if $X = V(I)$ for some homogeneous ideal $I \subseteq k[X_0, \dots, X_n]$.

Prop: Zariski closed subsets of $\mathbb{P}^n$ give a topology on $\mathbb{P}^n$.

proof: $\mathbb{P}^n = V((0))$, $\emptyset = V(k[X_0, \dots, X_n])$

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right)$$

$$V(I_1) \cup V(I_2) = V(I_1 I_2)$$

$(\subseteq)$ ✓ $(\supseteq)$ By contradiction, $p \in V(I_1 I_2)$,
$P \notin V(I_1) \cup V(I_2)$.
$\Rightarrow \exists\, F_1 \in I_1, F_2 \in I_2$ s.t. $F_1(P), F_2(P) \neq 0$.
But $F_1(P) F_2(P) = 0$, which cannot be.
$\square$

Def: Let $X \subseteq \mathbb{P}^n$ Zariski closed. Define $I(X)$ to be the ideal generated by homogeneous polynomials in $k[X_0, \dots, X_n]$ vanishing on X.

Thm: Let $I \subseteq k[X_0, \dots, X_n]$ be a homogeneous ideal different from the irrelevant ideal. Then

(a) $V(I) = \emptyset \iff (X_0, \dots, X_n) \subseteq \sqrt{I}$;

(b) If $V(I) \neq \emptyset$, then $I(V(I)) = \sqrt{I}$.