Introduction to Algebraic Geometry (MATH 8300) 08/30/2016

Lecture 5

TODAY: • Noetherian topological spaces and decomposition into irreducible components.

Noetherian topological spaces:

Def: A topological space $X$ is noetherian if it satisfies the descending chain condition for closed subsets: for any sequence $Y_1 \supseteq Y_2 \supseteq \dots$ of closed subsets, $\exists r \in \mathbb{Z}_{>0}$ s.t. $Y_r = Y_{r+1} = \dots$.

Observation: As with irreducibility, this is probably a new concept because it is not very interesting when working with the Euclidean topology on $\mathbb{R}^n$ or $\mathbb{C}^n$: a noetherian Hausdorff space must be a finite set with the discrete topology. We will prove it later.

Example: $\mathbb{A}^n$ with the Zariski topology is noetherian.

$$X_1 \supseteq X_2 \supseteq \dots \text{ Zariski closed subsets}$$

$$\Rightarrow I(X_1) \subseteq I(X_2) \subseteq \dots$$

$k[x_1, \dots, x_n]$ is noetherian $\Rightarrow \exists r \in \mathbb{Z}_{>0}$ s.t.

$$I(X_r) = I(X_{r+1}) = \dots$$

$$\Rightarrow Z(I(X_r)) = Z(I(X_{r+1})) = \dots$$

$$\Rightarrow X_r = X_{r+1} = \dots$$

The next proposition produces a lot of noetherian topological spaces.

<u>Prop</u>: Let $X$ be a noetherian topological space and let $S \subseteq X$. Then $S$ is noetherian with its induced topology.

proof: Exercise. $\square$

<u>Corollary</u>: Any Zariski closed subset of $\mathbb{A}^n$ is noetherian.

<u>Recall</u>: A topological space $X$ is <u>quasi-compact</u> if for any open cover $X = \bigcup_i U_i$, $\exists\, i_1, \dots, i_n$ s.t. $X = \bigcup_{j=1}^{n} U_{i_j}$.

<u>Prop</u>: A noetherian topological space $X$ is quasi-compact.

proof: By contradiction. Let $X = \bigcup_i U_i$ and assume that there is no finite subcover. Consider any $U_{i_1}$, and let $U_{i_2}$ such that $U_{i_1} \subsetneq U_{i_1} \cup U_{i_2}$. Let $U_{i_3}$ s.t. $U_{i_1} \cup U_{i_2} \subsetneq U_{i_1} \cup U_{i_2} \cup U_{i_3}$, and keep going. This contradicts the fact that $X$ is noetherian. $\square$

<u>Corollary</u>: Any Zariski closed subset of $\mathbb{A}^n$ is quasi-compact.

Before proving today's main theorem, we need a lemma.

<u>Lemma</u>: Let $X$ be a topological space. Then $X$ is irreducible $\iff$ every time $X = \bigcup_{i=1}^{n} C_i$, $C_i \subseteq X$ closed, then $X = C_i$, $\exists i \in \{1, \dots, n\}$.

proof: $(\Leftarrow)$ ✓

$(\Rightarrow)$ $X = \bigcup_{i=1}^{n} C_i = C_1 \cup \underbrace{\left(\bigcup_{i=2}^{n} C_i\right)}_{\text{closed}} \Rightarrow X = C_1$ or $X = \bigcup_{i=2}^{n} C_i$. If $X = C_1$ we are done. Otherwise, iterate this argument on $X = \bigcup_{i=2}^{n} C_i$. □

<u>Thm</u>: Let $X$ be a noetherian topological space. Then $X = X_1 \cup \dots \cup X_n$, where $X_1, \dots, X_n$ are closed irreducible subspaces of $X$. If $X_i \not\subseteq X_j$ $\forall i \neq j$, then this representation is unique.

proof: Let $S$ be the set of all closed subsets of $X$ which cannot be written as above. Let us assume by contradiction that $S \neq \emptyset$. Then $S$ has a minimal element $Y$ because $X$ is noetherian. Then $Y$ is not irreducible, so $\exists Y_1, Y_2 \subsetneq Y$ closed such that

$Y = Y_1 \cup Y_2$. But now $Y_1, Y_2 \notin \mathcal{S}$ because $Y$ is minimal, so $Y_1 = Y_1^{(1)} \cup \ldots \cup Y_r^{(1)}$ and $Y_2 = Y_1^{(2)} \cup \ldots \cup Y_s^{(2)}$ where $Y_j^{(i)}$ is a closed irreducible subset of $Y_i$ $\forall i, j$. But then $Y = Y_1^{(1)} \cup \ldots \cup Y_r^{(1)} \cup Y_1^{(2)} \cup \ldots \cup Y_s^{(2)}$, which is a contradiction. So $\mathcal{S} = \emptyset$ and $X$ can be written as claimed.

Now assume $X = X_1 \cup \ldots \cup X_n = Y_1 \cup \ldots \cup Y_m$ as in the statement of the theorem with $X_i \not\subseteq X_j$ and $Y_k \not\subseteq Y_\ell$ $\forall i \neq j$ and $\forall k \neq \ell$.

$X_1 = \bigcup_{i=1}^{n} (X_1 \cap Y_i) \Rightarrow X_1 = X_1 \cap Y_{i'}, \exists i' \Rightarrow X_1 \subseteq Y_{i'}$.

Similarly, $Y_{i'} \subseteq X_{i''}, \exists i'' \Rightarrow X_1 \subseteq Y_{i'} \subseteq X_{i''}$
$\Rightarrow 1 = i'' \Rightarrow X_1 = Y_{i'}$. Up to relabeling $Y_1, \ldots, Y_m$, <u>we can</u> assume that $X_1 = Y_1$.

Let $Z := \overline{X \setminus X_1}^{X} = X_2 \cup \ldots \cup X_n = Y_2 \cup \ldots \cup Y_m$, so now we just iterate the same argument. $\square$

<u>Example</u>: $I = \left( (x^2+y^2-1)(x^2y^2-x^2-y^2+1) \right)$, $k = \overline{k}$ (or $|k| = \infty$).

$Z(I)$ has a decomposition into irreducible closed subset:

$$Z(I) = Z(x^2+y^2-1) \cup Z(\overbrace{x^2y^2-x^2-y^2+1}^{(x-1)(x+1)(y-1)(y+1)})$$

$$= Z(x^2+y^2-1) \cup Z(x-1) \cup Z(x+1) \cup Z(y-1) \cup Z(y+1)$$