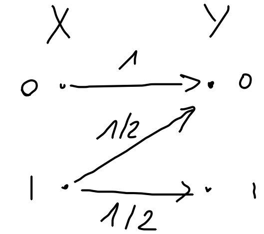
⓪ Canale Z

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \begin{matrix} \leftarrow 1-\alpha \\ \leftarrow \alpha \end{matrix}$$



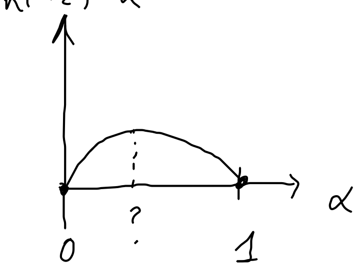
$1-\alpha = P_r[X=0]$

$\alpha = P_r[X=1]$

$$C = \max_{0 \le \alpha \le 1} \Big[\overbrace{h\left(\tfrac{\alpha}{2}\right) - \alpha}^{f(\alpha)} \Big] = \max_{0 \le \alpha \le 1} \Big[-\tfrac{\alpha}{2} \log \tfrac{\alpha}{2} - \left(1-\tfrac{\alpha}{2}\right) \log\left(1-\tfrac{\alpha}{2}\right) - \alpha \Big]$$

$$\boxed{h(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)}$$

$f(\alpha) = h(\alpha/2) - \alpha$



$\alpha = 0 \to h(0) - 0 = 0$

$\alpha = 1 \to h(1/2) - 1 = 0$

$$f'(\alpha) = \underbrace{-\tfrac{1}{2} \log \tfrac{\alpha}{2} - \cancel{\tfrac{\alpha}{2}} \cdot \cancel{\tfrac{2}{\alpha}} \cdot \tfrac{1}{2}}_{\left(-\frac{\alpha}{2} \log \frac{\alpha}{2}\right)'} \underbrace{+ \tfrac{1}{2} \log\left(1-\tfrac{\alpha}{2}\right) + \cancel{\left(1-\tfrac{\alpha}{2}\right)} \frac{+1/2}{\cancel{\left(1-\tfrac{\alpha}{2}\right)}}}_{(\quad)'} - 1$$

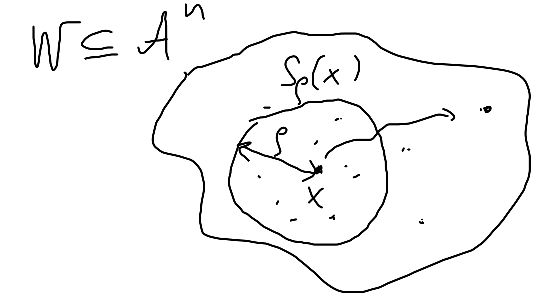
$$= -\tfrac{1}{2} \log \tfrac{\alpha}{2} - \cancel{\tfrac{1}{2}} + \tfrac{1}{2} \log\left(1-\tfrac{\alpha}{2}\right) + \cancel{\tfrac{1}{2}} - 1$$

$$= \tfrac{1}{2} \log \frac{1-\alpha/2}{\alpha/2} - 1 \,.\quad f'(\alpha) = 0 \iff \frac{1-\alpha/2}{\alpha/2} = 4$$

$$/\!/ \; \frac{2}{\alpha} - 1 \iff \frac{2}{\alpha} = 5 \iff \alpha = \tfrac{2}{5}$$

$$C = f\left(\tfrac{2}{5}\right) = h\left(\tfrac{1}{5}\right) - \tfrac{2}{5} \simeq 0.322 \text{ bit}$$

① Distanza di Hamming



n-ple (sequenze di n simboli su un certo alfabeto)

n-pla $\quad x = (x_1, \ldots, x_n) \in A^n$

$\quad\quad\quad y = (y_1, \ldots, y_n) \in A^n$

Distanza di Hamming (tra x e y): $d_H(x,y) = \#\{\text{posizioni } i \ (1 \le i \le n) \text{ tali che } x_i \neq y_i\}$

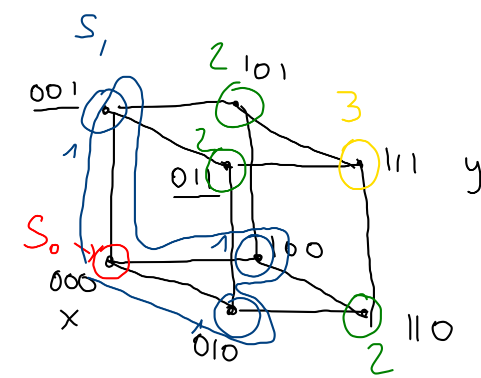
Sfera di Hamming (centrata in x e di raggio ρ):

$$S_\rho(x) \stackrel{\Delta}{=} \{n\text{-ple } y : d_H(x,y) \le \rho\}$$

Esempio: $A = \{0,1\}$

$n = 3$

$A^n = \{000, 001, 010, 011, 100, 101, 110, 111\}$



(Iper) Cubo di Hamming

$d_H(\cdot,\cdot)$ è una metrica:

c.1 – $d_H(x,y) \geq 0$ e $= 0$ se e solo se $x = y$ ✓

c.2 – $d_H(x,y) = d_H(y,x)$ ✓

c.3 – $d_H(x,z) \le d_H(x,y) + d_H(y,z)$ ✓

se $x_i \neq z_i$

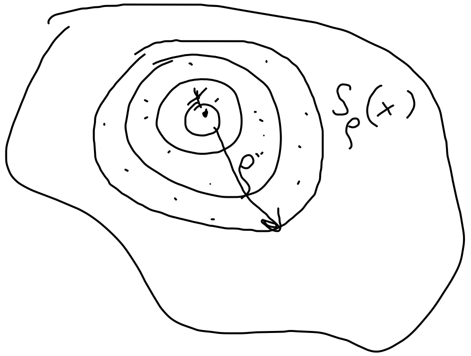
$\#$posizioni i in cui $x_i \neq z_i$ $\le$ $\#$pos i in cui $x_i \neq y_i$ $+$ $\#$pos i in cui $y_i \neq z_i$

$+1 \qquad +1 \text{ o } x_i \neq y_i \qquad +1 \text{ o } y_i \neq z_i$

Proposizione. Per ogni $\underbrace{x}_{n\text{-pla}} \in \{0,1\}^n$, per ogni $0 \le p \le 1/2$, si ha

$$|\underbrace{S_{pn}}_{\rho = p \cdot n}(x)| \le 2^{nh(p)}$$

$$p \le 1/2 \Rightarrow \frac{\overset{\le 1/2}{p}}{\underbrace{1-p}_{\ge 1/2}} \le 1$$



Dim. $$1 = (p + (1-p))^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \ge \sum_{i=0}^{pn} \binom{n}{i} p^i (1-p)^{n-i} =$$

$$= \sum_{i=0}^{pn} \binom{n}{i} p^i (1-p)^{-i} (1-p)^n = \sum_{i=0}^{pn} \binom{n}{i} (1-p)^n \left(\frac{p}{1-p}\right)^i \ge$$

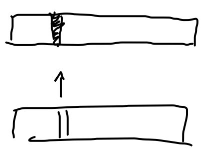
$$\ge \underbrace{\sum_{i=0}^{pn} \binom{n}{i}}_{|S_{pn}(x)|} \underbrace{(1-p)^n \left(\frac{p}{1-p}\right)^{pn}}_{\text{non dipende da } i} = |S_\rho(x)|\, p^{pn} (1-p)^{n(1-p)}$$

$$= |S_{pn}(x)|\, 2^{pn \log p + (1-p) n \log(1-p)} = |S_{pn}(x)| \cdot 2^{-nh(p)}$$

$$z = 2^{\log z}$$

$$1 \ge |S_{pn}(x)| \cdot 2^{-nh(p)}$$

$$2^{nh(p)} \ge |S_{pn}(x)|.$$



$\binom{n}{0} = 1$ (x stesse)

$\binom{n}{1} = n$

$\binom{n}{2}$

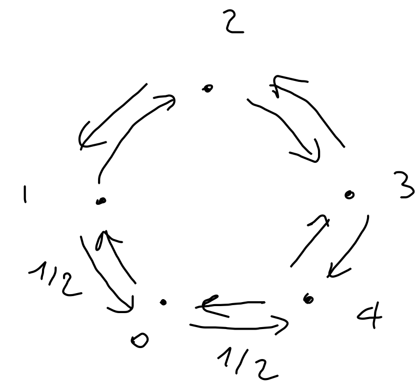
Anche per un alfabeto <u>non binario</u>,

$$|S_\rho(x)| \le \frac{n}{2}\, 2^{n h(\rho/n)}$$

$\rho = pn$

$\rho/n = p$

② Canale con alfabeto (ingresso e uscita) $\mathcal{X}=\mathcal{Y}=\{0,1,2,3,4\}$



$$p(y|x) = \begin{cases} 1/2 & \text{se } y = x+1 \pmod 5 \text{ o } y = x-1 \pmod 5 \\ 0 & \text{altrimenti} \end{cases}$$

✓ Il canale è simmetrico

Calcolare la capacità del canale.

Sol.

Matrice di transizione del canale:

$$\begin{array}{c|c|c}
 & y & p_x \\
\text{ingressi} & \begin{array}{c} 0\\1\\2\\3\\4 \end{array} \left[\begin{array}{ccccc} 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{array}\right] & \begin{array}{c} 1/5\\1/5\\1/5 \leftarrow x\\1/5\\1/5 \end{array} \\
 & \underbrace{\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}}_{\text{uscite}} &
\end{array}$$

$$C = \max_{p_X} I(X;Y) = \max_{p_X} \left[H(Y) - \underbrace{H(Y|X)} \right]$$

$$= \left(\max_{p_X} H(Y) \right) - \underbrace{H(Y|X=x_1)}_{\text{non dipende da } p_X}$$

Quando il canale è simmetrico,

$$H(Y|X) = \underbrace{H(Y|X=x_1)}$$

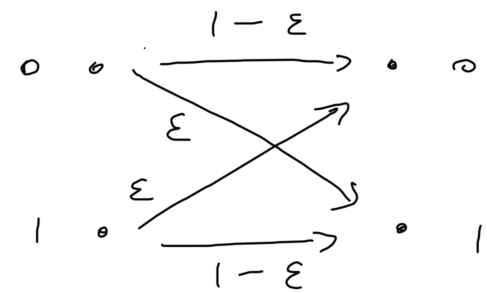
Nel nostro caso, $H(Y|X=x_1) = 1 = 1/2 \log 2 + 1/2 \log 2$

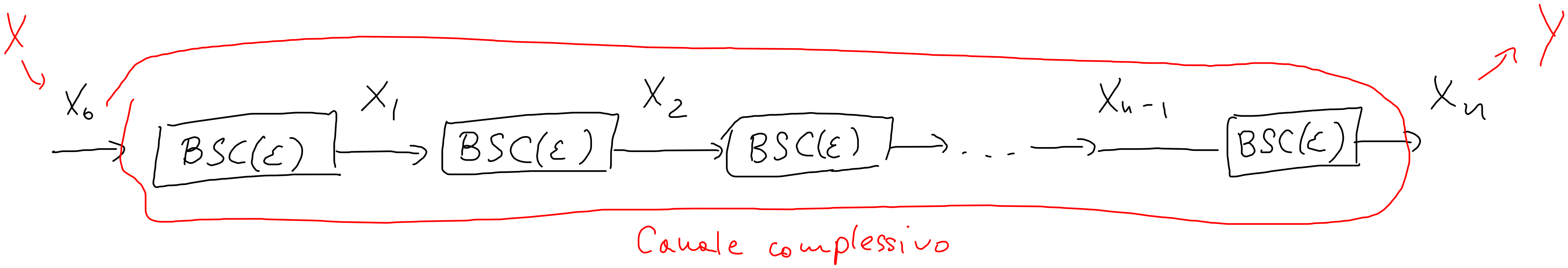
$$C = \left(\max_{p_X} H(Y) \right) - 1 = \log|\mathcal{Y}| - 1 = \log 5 - 1$$

$H(Y)$ è massima quando la p_Y è uniforme (in quel caso $H(Y) = \log|\mathcal{Y}|$)

La p_Y è uniforme (in questo caso) quando la p_X è uniforme

3 $BSC(\varepsilon)$





Caratterizzare il canale ottenuto mettendo in serie n canali $BSC(\varepsilon)$.

$$\Gamma_{BSC(\varepsilon)} = \begin{array}{cc} \begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix} & \begin{array}{l} X_0=0 \\ X_0=1 \end{array} \\ \begin{array}{cc} X_1=0 & \quad X_1=1 \end{array} & \end{array}$$

$$P_{X_0} = \left(Pr[X_0=0] \, , \, Pr[X_0=1] \right)$$

$$P_{X_0} \cdot \Gamma = \left(Pr[X_0=0]Pr[X_1=0|X_0=0] + Pr[X_0=1]Pr[X_1=0|X_0=1] \, , \, Pr[X_0=0] \cdot Pr[X_1=1|X_0=0] + Pr[X_0=1]Pr[X_1=1|X_0=1] \right)$$

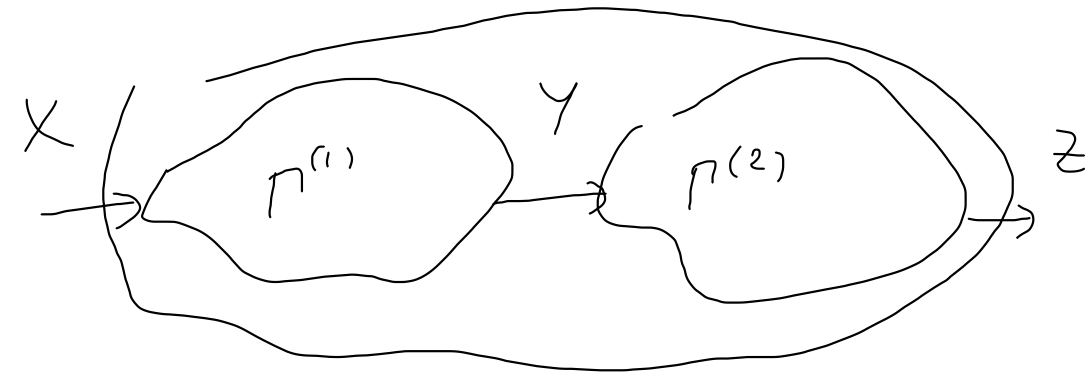
$1\times 2 \quad 2\times 2 \quad 1\times 2$

$$= \left(Pr[X_0=0 \wedge X_1=0] + Pr[X_0=1 \wedge X_1=0] \, , \, Pr[X_0=0 \wedge X_1=1] + Pr[X_0=1 \wedge X_1=1] \right)$$

$$= \left(Pr[X_1=0] \, , \, Pr[X_1=1] \right) = P_{X_1}$$

$$P_{X_1} = P_{X_0} \cdot \Gamma$$

$$P_{X_2} = P_{X_1} \cdot \Gamma = P_{X_0} \cdot \Gamma^2$$



$$\Gamma = \Gamma^{(1)} \cdot \Gamma^{(2)}$$

$$\dots$$

$$P_{X_n} = P_{X_0} \cdot \Gamma^n_{BSC(\varepsilon)}$$

$$(1-2\varepsilon) < 1$$

Si può mostrare per induzione su n che:

Canale inutile

$$\Gamma^n_{BSC(\varepsilon)} = \begin{bmatrix} \frac{1}{2}\left(1+(1-2\varepsilon)^n\right) & \frac{1}{2}\left(1-(1-2\varepsilon)^n\right) \\ \frac{1}{2}\left(1-(1-2\varepsilon)^n\right) & \frac{1}{2}\left(1+(1-2\varepsilon)^n\right) \end{bmatrix} \xrightarrow{n \to \infty} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Capacità zero