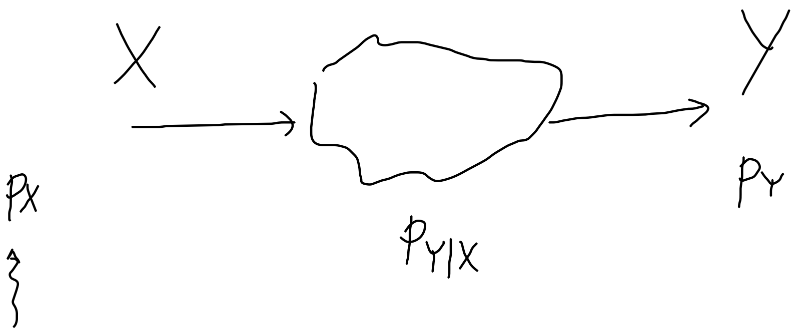
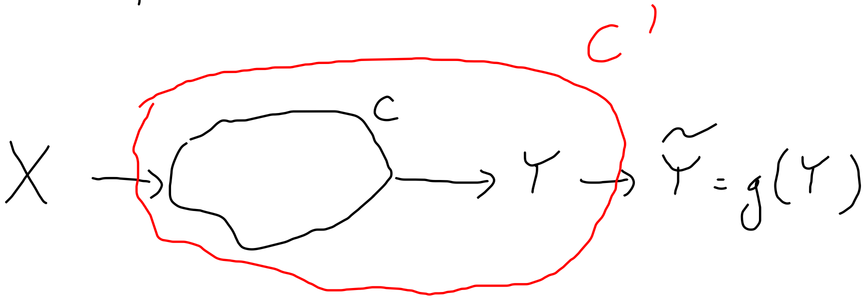
CANALI

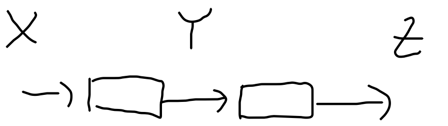
$$C = \max_{p_X} I(X;Y)$$



①



Dimostrare che $C \geq C'$



$I(X;Z) \leq I(X;Y)$

Poiché $\tilde{Y} = g(Y)$, allora $\tilde{Y}$ è indipendente della X data Y

Quindi vale la catena di Markov $X \to Y \to \tilde{Y}$

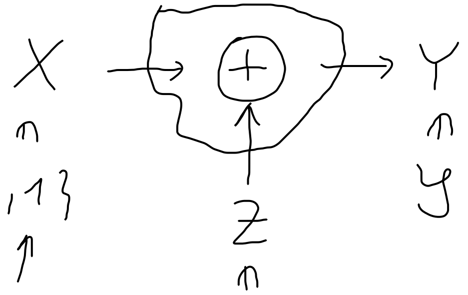
Allora per il 2° teorema di elaborazione dati : $\boxed{I(X;Y) \geq I(X;\tilde{Y})}$ $\forall p_X$

$$\Rightarrow \underset{\substack{\| \\ C}}{\max_{p_X} I(X;Y)} \geq \max_{p_X} I(X;\tilde{Y}) = C' \Rightarrow C \geq C'.$$

Quando $C = C'$? $\Rightarrow I(X;Y) = I(X;\tilde{Y})$

$\Rightarrow$ vale anche $X \to \tilde{Y} \to Y$.

②



$X \in \mathcal{X} = \{0,1\}$, $Y \in \mathcal{Y}$, $Z \in \mathcal{Z} = \{0, \underline{a}\}$

$1-\alpha = Pr[X=0]$, $\alpha = Pr[X=1]$

a è un parametro

$Pr[Z=0] = 1/2$

$Pr[Z=a] = 1/2$

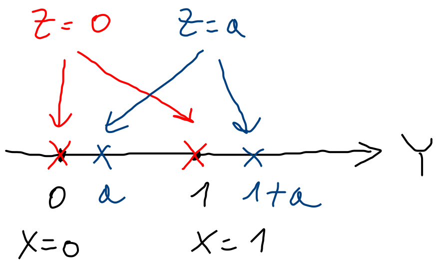
$$\begin{array}{c|cc} Y & Z=0 & Z=a \\ \hline X=0 & 0 & a \\ X=1 & 1 & 1+a \end{array}$$

$a=1$

$$\begin{array}{|cc} \hline 0 & 1 \\ 1 & 2 \end{array}$$

Qual è la capacità del canale (X ingresso, Y uscita)?

$Y = X + Z$



Se $a \neq 0, a \neq \pm 1$:

$Y=0 \Rightarrow X=0$

$Y=a \Rightarrow X=0$

$Y=1 \Rightarrow X=1$

$Y=1+a \Rightarrow X=1$

Sol. $\mathcal{Y} = \{0, a, 1, 1+a\}$

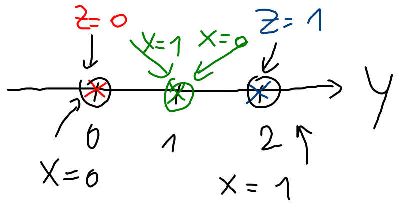
Caso $a=0$: $Y = X+Z = X+0 = X$. Quindi $I(X;Y) = I(X;X) = H(X)$ e $C = \max_{P_X} H(X) = \log|\mathcal{X}| = \log 2 = 1$ bit.

Caso $a \neq 0, a \neq \pm 1$: $(|\mathcal{Y}|=4)$. Da Y riesco a ricostruire X. $\Rightarrow I(X;Y) = H(X) - \underbrace{H(X|Y)}_{=0} = H(X)$

$\Rightarrow C = \max_{P_X} H(X) = \log|\mathcal{X}| = \log 2 = 1$ bit.

Caso $a = \pm 1$:

$a=1$:

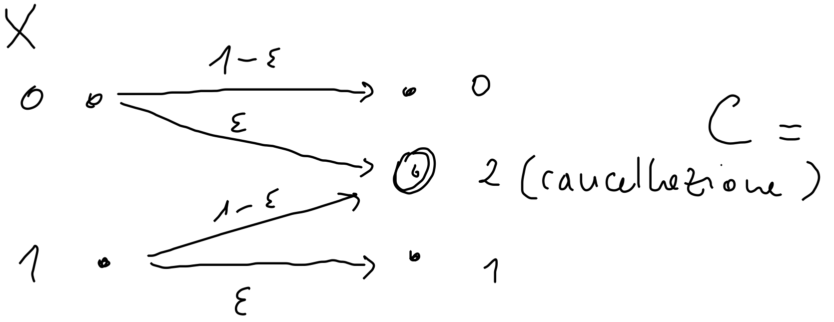


$Pr[Y=0] = Pr[X=0 \wedge Z=0] = 1/2 (1-\alpha)$

$Pr[Y=2] = Pr[X=1 \wedge Z=1] = 1/2 \cdot \alpha$

$Pr[Y=1] = Pr[(X=0 \wedge Z=1) \vee (X=1 \wedge Z=0)] = 1/2 (1-\alpha) + 1/2 \alpha$

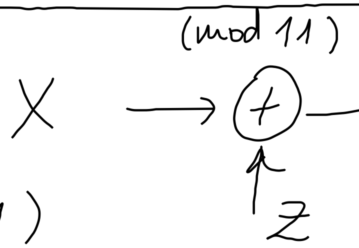
Canale simmetrico con cancellazione



$C = 1 - \varepsilon$

$\Longrightarrow$ nel nostro scenario, $\varepsilon = 1/2$ e $C = 1 - 1/2 = 1/2$ bit.

③ $\mathcal{X} = \{0,1,2,...,10\}$



$\mathcal{Y} = \{0,1,2,...,10\}$

$|\mathcal{Y}| = 11$

$\mathcal{Z} = \{1,2,3\}$ $Y = X + Z \pmod{11}$

$\rightarrow$ $Pr[Z=1] = 1/3$, $Pr[Z=2] = 1/3$, $Pr[Z=3] = 1/3$.

$H(X) - H(X|Y)$

〃 〃 $H(Y) - H(Y|X)$

(a) Trovare la capacità del canale

(b) Determinare la p_X corrispondente (quella che massimizza $I(X;Y)$)

Z è indip. da X

Sol. $H(Y|X) = H(X+Z \pmod{11} \,|\, X) = H(Z|X) = H(Z) = \log 3$

aggiungere X fa traslare (mod 11) di un valore costante

$$C = \max_{p_X} I(X;Y) = \max_{p_X} \left[H(Y) - \underbrace{H(Y|X)}_{\log 3} \right] = \left(\max_{p_X} H(Y) \right) - \log 3$$

$H(Y) \leq \log 11$

$H(Y) = \log 11$ se la p_Y è uniforme

La p_Y può essere uniforme? Sì se scelgo opportunamente p_X : $p_X = \left(1/11, 1/11, \ldots, 1/11 \right)$

$\Rightarrow$ la p_Y è uniforme. $\Rightarrow$ $H(Y) = \log 11$ e p_X $\underline{\text{è uniforme}}$

$\uparrow$

$\Rightarrow$ $C = \log 11 - \log 3 = \log 11/3$.

la p_X che massimizza $H(Y)$

④ "Macchina da scrivere rumorosa"

Consideriamo una macchina da scrivere a 26 tasti $(\overset{x_1}{'A'}, \overset{x_2}{'B'}, 'C', \dots, \overset{x_{26}}{'Z'})$

(a) Se la pressione di ogni tasto produce la corrispondente lettera, quanto vale la capacità?

(b) Se invece:

$$\begin{tikzcd}
'A' \arrow[r, "1/2"] \arrow[dr, "1/2"'] & 'A' & 'B' \arrow[r, "1/2"] \arrow[dr, "1/2"'] & 'B' & 'C' \arrow[r, "1/2"] \arrow[dr, "1/2"'] & 'C' & \dots & 'Z' \arrow[r, "1/2"] \arrow[dr, "1/2"'] & 'Z' \\
& 'B' & & 'C' & & 'D' & & & 'A'
\end{tikzcd}$$

Quanto vale la capacità?

Sol. (a) $C = \max_{p_X}\big(\underbrace{H(X) - \overset{0}{\cancel{H(X|Y)}}}_{I(X;Y)}\big) = \max_{p_X} H(X) \overset{p_X \text{ uniforme}}{=} \log|\mathcal{X}| = \log 26$

(b) $C = \max_{p_X}\big(H(Y) - \underline{H(Y|X)}\big)$

$$H(Y|X) = \sum_{i=1}^{26} p(x_i) \underbrace{H(Y|X=x_i)}_{\Downarrow} = \sum_{i=1}^{26} p(x_i) \cdot 1 = 1$$

$$-\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2}\log\tfrac{1}{2} = h_2(\tfrac{1}{2}) = 1 \text{ bit} = H((0,0,0,\dots,\underset{i\text{-esimo}}{\tfrac{1}{2}},\underset{(i+1)\text{-esimo}}{\tfrac{1}{2}},0,\dots,0))$$

$$C = \max_{p_X} \left(H(Y) - 1 \right) = \left(\max_{p_X} H(Y) \right) - 1$$

$$H(Y) \leq \log |\mathcal{Y}| = \log 26$$

$H(Y) = \log|\mathcal{Y}|$ sse la p_Y è uniforme

La p_Y può essere uniforme?

Sì: prendo p_X uniforme

$(p_X = (1/26, 1/26, \ldots, 1/26))$

$\Rightarrow p_Y = (1/26, 1/26, \ldots, 1/26)$

$$\Rightarrow C = \log|\mathcal{Y}| - 1 = (\log 26) - 1 = \log 13 .$$

p_Y dipende da $\begin{cases} p_X \\ p_{Y|X} \end{cases}$

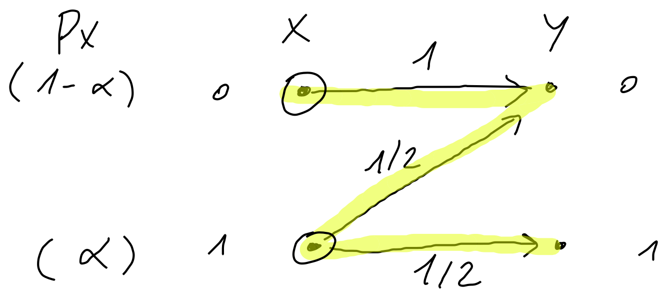
$$p(y_j) = \sum_{i=1}^{26} p(x_i) \cdot \overbrace{p(y_j|x_i)}^{\in\{0, 1/2\}} =$$

$$= 1/2 \, \underset{1/26}{p(x_j)} + 1/2 \, \underset{1/26}{p(x_{j-1})}$$

$$= 1/26 \quad \forall j$$

⑤ "Canale Z"

$$\Gamma = \begin{array}{cc} \left[\begin{array}{c|c} 1 & 0 \\ \hline 1/2 & 1/2 \end{array}\right] & \begin{array}{l} \overset{x}{0} \\ 1 \end{array} \\ \begin{array}{cc} 0 & \quad 1 \end{array} & \end{array}$$

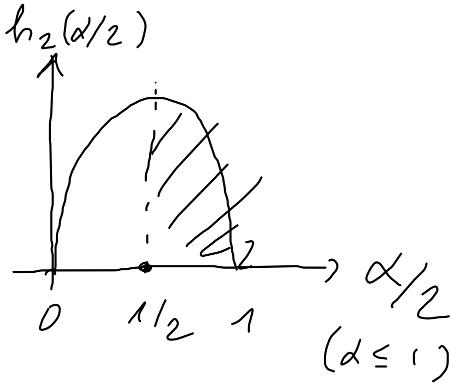


Py

$(1-\alpha)\cdot 1 + \alpha \cdot 1/2 = 1 - \alpha/2$

$\alpha \cdot 1/2 = \alpha/2$

$\alpha = \Pr[X=1]$, $1-\alpha = \Pr[X=0]$



Trovare la capacità del canale e la corrispondente px.

Sol. $H(Y|X) = \Pr[X=0]\, H(Y|X=0) + \Pr[X=1]\, H(Y|X=1)$

$$(1-\alpha)\, H(Y|X=0) + \alpha\, H(Y|X=1)$$

$$= (1-\alpha)\cdot 0 + \alpha \cdot \log 2 = \alpha$$

$$H(Y) = H((1-\alpha/2, \alpha/2)) = h_2(\alpha/2)$$

$$C = \max_{px}[H(Y) - H(Y|X)] = \max_{px}\left[h_2(\alpha/2) - \alpha\right] = \max_{0\le\alpha\le 1}\left(h_2(\alpha/2) - \alpha\right)$$