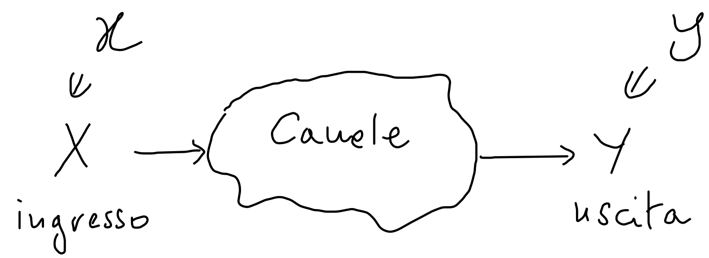
Canali



$P_{Y|X}$

$$\Gamma = \underbrace{\begin{bmatrix} & \vdots & \\ \cdots & p(y_j|x_i) & \cdots \\ & \vdots & \end{bmatrix}}_{|\mathcal{Y}|} \begin{matrix} \leftarrow i \end{matrix} \Big\} |\mathcal{X}|$$

Capacità di un canale: $C = \max_{P_X} I(X;Y)$

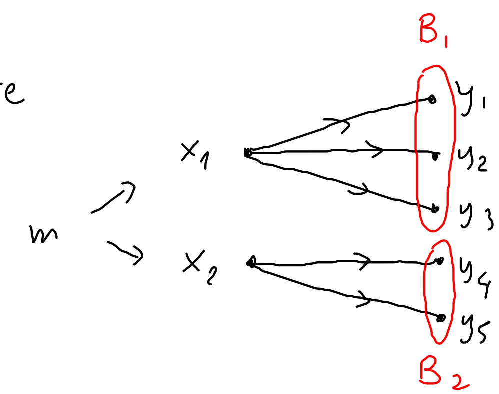
- Canale senza rumore $\quad x_1 \longrightarrow y_1 \quad \vdots$

Capacità: $\log|\mathcal{X}|$

$|\mathcal{X}| = \overset{8 \text{ bit}}{256} \to C = 8 \text{ bit}$

- Canale senza perdite

Ingresso è completamente determinato dall'uscita



Esiste una partizione $B_1, B_2, \dots, B_m$ di $\mathcal{Y}$

tale che $Y \in B_j \Longrightarrow X = x_j$.

$|\mathcal{X}| = m \qquad |\mathcal{Y}| = s > m$

$\forall$ d.p. P_X

$H(X) \leq \log|\mathcal{X}|$

$$C = \max_{P_X} I(X;Y) = \max_{P_X} H(Y) - H(Y|X)$$

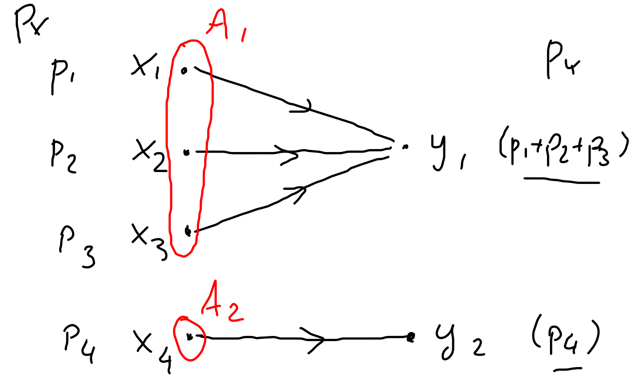
$$= \max_{P_X} H(X) - \underbrace{H(X|Y)}_{\text{(equivocazione)}}^{\to 0} = \max_{P_X} H(X) = \log|\mathcal{X}| = \log m.$$

P_X uniforme su $\mathcal{X}$

Canale deterministico

Uscita è completamente determinata dall'ingresso

$|\mathcal{X}| = s > m \qquad |\mathcal{Y}| = m$



$$p(y_j|x_i) = \begin{cases} 1 & \text{se } x_i \in A_j \\ 0 & \text{se } x_i \notin A_j \end{cases}$$

$p_1 = 1/2$, $p_2 = 0$, $p_3 = 0$, $p_4 = 1/2$

$p_1 = 0$, $p_2 = 1/2$, $p_3 = 0$, $p_4 = 1/2$

$\left.\begin{matrix} 1/4 \\ 1/4 \\ 0 \end{matrix}\right\} 1/2$

$\left.1/2\right\} 1/2$

$\Rightarrow$ Esiste una partizione $A_1, A_2, \dots, A_m$ di $\mathcal{X}$ tale che $X \in A_j \Rightarrow Y = y_j$.

$$C = \max_{P_X} I(X;Y) = \max_{P_X} H(Y) - \underbrace{\overset{0}{\cancel{H(Y|X)}}}_{\text{(ambiguità)}} = \max_{P_X} H(Y) = \log|\mathcal{Y}| = \log m.$$

Scelgo P_X in modo che P_Y sia uniforme

Canale inutile

Canale con matrice di transizione avente righe tutte identiche : $\Gamma = \begin{array}{cc} & \begin{array}{cccc} & & y_j & \end{array} \\ \begin{array}{c} x_1 \\ \\ \\ \\ \end{array} & \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_m \\ \alpha_1 & \alpha_2 & \dots & \alpha_m \\ & & & \\ \alpha_1 & \alpha_2 & \dots & \alpha_m \end{bmatrix} \end{array}$

$$p(y_j) = \sum_i \underbrace{p(x_i)\overbrace{p(y_j|x_i)}^{\alpha_j}}_{p(x_i,y_j)} = \alpha_j \overbrace{\sum_i p(x_i)}^{=1} = \alpha_j$$

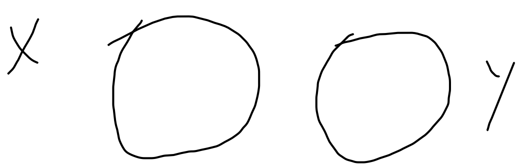
$$= p(y_j|x_i) \quad \forall j \ \forall i$$

$$p(y_j|x_1) = p(y_j|x_2) = p(y_j|x_3) = \dots = p(y_j|x_s) = \alpha_j \qquad \left(\sum_{j=1}^m \alpha_j = 1\right)$$

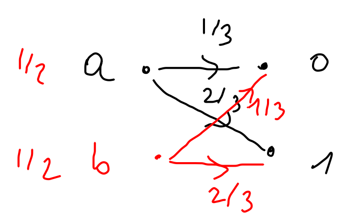
$\Rightarrow$ Y e X sono indipendenti

$\Rightarrow$ $H(Y|X) = H(Y)$ e $I(X;Y) = 0$

$\Rightarrow$ La capacità del canale $C = \max_{p_X} \overset{0}{I(X;Y)} = 0$ $\rightarrow$ non è possibile trasmettere informazione sul canale.



$$\begin{array}{cc} \begin{array}{cc} 0 & 1 \end{array} & \\ \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} & \begin{array}{c} a \\ b \end{array} \end{array}$$



$Y = 0$

$p(x_i | y_j = 0)$

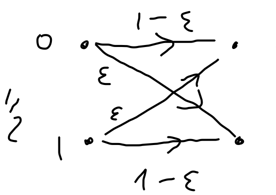
Canale simmetrico

Matrice di canale : $\begin{cases} \text{ogni riga è permutazione di ogni altra riga} \\ \text{ogni colonna è permutazione di ogni altra colonna} \end{cases}$

$|\mathcal{X}| = s \;,\; |\mathcal{Y}| = m$

$$\Gamma = \begin{array}{cc} \begin{bmatrix} 1/6 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 1/2 & 1/6 \end{bmatrix} & \begin{array}{l} \\ \leftarrow x_i \\ \\ \end{array} \end{array}$$

$$\Gamma = \begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix}$$



L'entropia $H(Y \mid X = x_i)$ non dipende da x_i perché le righe contengono gli stessi elementi

$$H\left(\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}\right) = -\tfrac{1}{6}\log\tfrac{1}{6} - \tfrac{1}{3}\log\tfrac{1}{3} - \tfrac{1}{2}\log\tfrac{1}{2}$$

$$H\left(\tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{3}\right) \;//$$

Quindi $H(Y \mid X) = \sum_{i=1}^{s} p(x_i) \underbrace{H(Y \mid X = x_i)}_{\text{non dipende da } i}$

$$= H(Y \mid X = x_1) \underbrace{\sum_{i=1}^{s} p(x_i)}_{1} = H(Y \mid X = x_1).$$

Capacità : $C = \max_{p_X} I(X;Y) = \max_{p_X} \Big[H(Y) - \underbrace{H(Y \mid X = x_1)}_{\text{non dipende da } p_X\,!} \Big] = \Big(\underbrace{\max_{p_X} H(Y)}_{\log m} \Big) - H(Y \mid X = x_1)$

Esempio: canale simmetrico binario

$$\Gamma = \underbrace{\begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}}_{2}$$

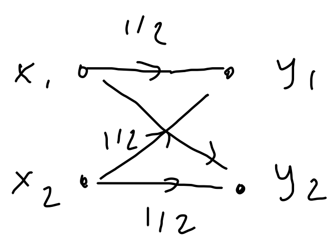
$$C = \log 2 - H((1-\varepsilon, \varepsilon)) = 1 - \overbrace{\left[-(1-\varepsilon)\log(1-\varepsilon) - \varepsilon \log \varepsilon\right]}^{H(1-\varepsilon,\, \varepsilon)}$$

$h_2(\varepsilon)$ funzione entropia binaria

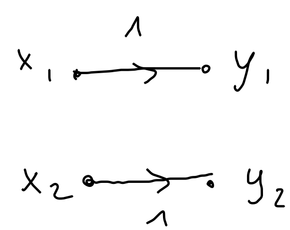
$$= 1 - h_2(\varepsilon)$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

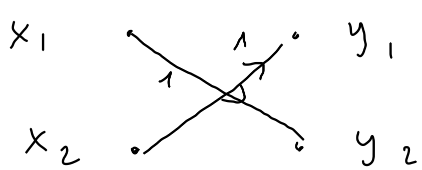
$(\varepsilon = 1/2)$

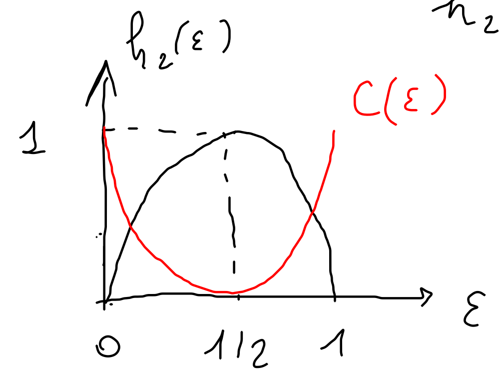


$(\varepsilon = 0)$



$(\varepsilon = 1)$



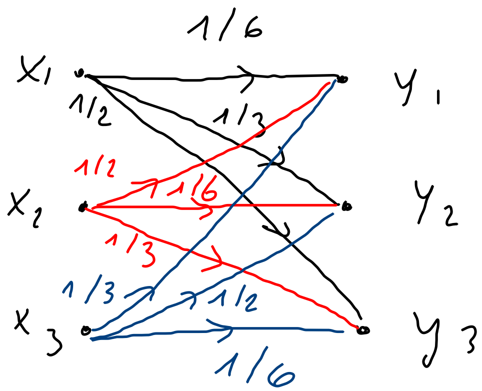


Se $\varepsilon = 0$ o $\varepsilon = 1$, $C = 1$

Se $\varepsilon = 1/2$, $C = 0$

Esempio.

$$\Gamma = \underbrace{\begin{bmatrix} 1/6 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}}_{3} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} 3$$



$$C = \log |Y| - H(Y \mid X = x_1) = \log 3 - H\left(1/2, 1/6, 1/3\right) =$$

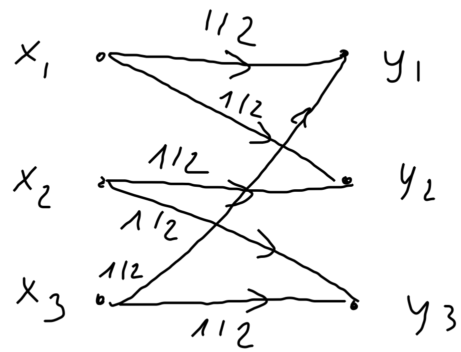
$$= \log 3 - 1/2\, \cancel{\log 2} - 1/3 \log 3 - 1/6 \log(2 \cdot 3)$$

$$= \log 3 - 1/2 \cdot 1 - 1/3 \log 3 - 1/6\, \cancel{\log 2} - 1/6 \log 3$$

$$= (1 - 1/3 - 1/6) \log 3 - 1/2 - 1/6$$

$$= 3/6 \log 3 - 4/6 = 1/2 \log 3 - 2/3 .$$

$$\Gamma = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$



$$-\tfrac{1}{2}\overset{(-1)}{\log \tfrac{1}{2}} - \tfrac{1}{2}\overset{(-1)}{\log \tfrac{1}{2}} = 1$$

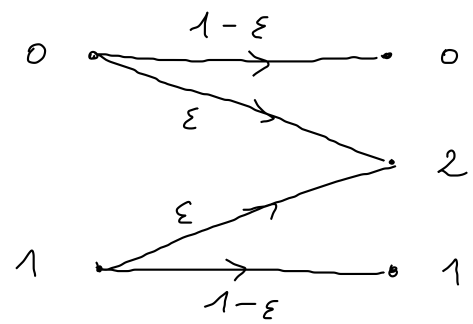
$$C = \log 3 - H\left(\left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right)\right) = \log 3 - 1$$

$$H\left(\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right)$$

(simmetrico)

Canale con cancellazione

$$\Gamma = \underbrace{\begin{pmatrix} 1-\varepsilon & 0 & \varepsilon \\ 0 & 1-\varepsilon & \varepsilon \end{pmatrix}}_{\substack{\text{3 colonne} \\ (0,\ 1,\ 2)}} \Big\}\ \text{2 righe} \quad \begin{array}{l} 0 \ \ Pr[X=0] \\ 1 \ \ Pr[X=1] \end{array}$$



in questo caso
(ho cancellazione)

$$P_X = (\underbrace{Pr[X=0]}_{1-\alpha},\ \underbrace{Pr[X=1]}_{\alpha}) = (1-\alpha,\ \alpha)$$

Poiché le righe sono una permutazione dell'altra,

$$H(Y|X) = \sum_i p(x_i)\, \underbrace{H(Y|X=x_i)}_{\text{non dip. da } i} = H(Y|X=x_1) \cdot 1 = H(1-\varepsilon, 0, \varepsilon) = -(1-\varepsilon)\log(1-\varepsilon) - \varepsilon \log \varepsilon$$
$$= h_2(\varepsilon)$$

Capacità: $C = \max_{P_X} I(X;Y) = \max_{P_X} H(Y) - H(Y|X) = \max_{P_X} [H(Y) - h_2(\varepsilon)] = \left(\max_{P_X} H(Y)\right) - h_2(\varepsilon)$.

Quanto vale $H(Y)$?

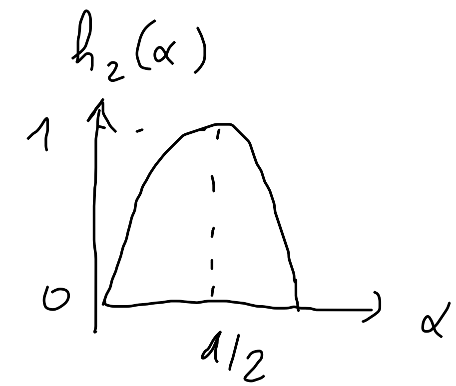
$$P_Y \begin{cases} p(Y=0) = \sum_x p(x, Y=0) = \underset{X=0}{(1-\alpha)} \cdot \underset{Y=0|X=0}{(1-\varepsilon)} + \underset{X=1}{\alpha} \cdot \underset{Y=0|X=1}{0} = (1-\alpha)(1-\varepsilon) \\ p(Y=1) = \sum_x p(x, Y=1) = (1-\alpha) \cdot 0 + \alpha(1-\varepsilon) = \alpha(1-\varepsilon) \\ p(Y=2) = \sum_x p(x, Y=2) = (1-\alpha) \cdot \varepsilon + \alpha \cdot \varepsilon = \varepsilon. \end{cases}$$

$\varepsilon \qquad \alpha$

$$H(Y) = -(1-\alpha)(1-\varepsilon)\log[(1-\alpha)(1-\varepsilon)] - \alpha(1-\varepsilon)\log[\alpha(1-\varepsilon)] - \varepsilon\log\varepsilon$$

$$= \dots$$

$$= (1-\varepsilon)\,h_2(\alpha) + h_2(\varepsilon)$$



$$C = \max_{\alpha}\left[(1-\varepsilon)\,h_2(\alpha) + \cancel{h_2(\varepsilon)}\right] \cancel{- h_2(\varepsilon)}$$

$$= 1-\varepsilon$$

ottengo il massimo quando $\alpha = 1/2$ ovvero $p_X = (1/2, 1/2)$.