Es.

① v.a. X, Y

Y

	p_{XY}	1	2	3	4	← y
X	1	$1/8$	$1/16$	$1/16$	$1/4$	$1/2$
	2	$1/16$	$1/8$	$1/16$	0	$1/4$
	3	$1/32$	$1/32$	$1/16$	0	$1/8$
	4	$1/32$	$1/32$	$1/16$	0	$1/8$
	↑					↑
	$\mathcal{X}$	$1/4$	$1/4$	$1/4$	$1/4$	$p(X)$

↖ $p(Y)$

y

	$p(X\|Y=y)$	1	2	3	4
X	1	$1/2$	$1/4$	$1/4$	1
	2	$1/4$	$1/2$	$1/4$	0
	3	$1/8$	$1/8$	$1/4$	0
	4	$1/8$	$1/8$	$1/4$	0

Calcolare $H(X)$, $H(X|Y)$, e $I(X;Y)$

$$H(X) = +\tfrac{1}{2}\log 2 + \tfrac{1}{4}\log 4 + \tfrac{1}{8}\log 8 + \tfrac{1}{8}\log 8 = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{3}{8} + \tfrac{3}{8} = \tfrac{7}{4}.$$

$\Rightarrow$ $H(X|Y) = \sum_{y\in\mathcal{Y}} p(y)\, H(X|Y=y)$ → devo calcolare prima $H(X|Y=1), \dots, H(X|Y=4)$

$$H(X|Y=y) = -\sum_{x\in\mathcal{X}} p(x|y)\log p(x|y) \qquad p(x|y) = \frac{p(x,y)}{p(y)}$$

$$H(X|Y=1) = +\tfrac{1}{2}\log 2 + \tfrac{1}{4}\log 4 + \tfrac{1}{8}\log 8 + \tfrac{1}{8}\log 8 = \tfrac{7}{4}.$$

$H(X|Y=2) = 7/4$ $\qquad$ $H(X|Y=3) = \log 4 = 2$ $\qquad$ $H(X|Y=4) = 0$

$$7/16 + 7/16$$

$$\Rightarrow H(X|Y) = \tfrac{1}{4}\cdot\tfrac{7}{4} + \tfrac{1}{4}\cdot\tfrac{7}{4} + \tfrac{1}{4}\cdot 2 + \cancel{\tfrac{1}{4}\cdot 0}$$
$$= \tfrac{7}{8} + \tfrac{1}{2} = \tfrac{11}{8}.$$

$$I(X;Y) = H(X) - H(X|Y) =$$
$$= \tfrac{7}{4} - \tfrac{11}{8} = \tfrac{3}{8}.$$

Es.

② sistema SIGSALY, one-time ped

$X \oplus 1 = 1 - X$

$p_X = (\overset{0}{\alpha}, \overset{1}{1-\alpha})$

$$\begin{tikzcd}
X \arrow[r, Rightarrow] & \oplus \arrow[r, Rightarrow] & Z \arrow[r, Rightarrow] & \oplus \arrow[r, Rightarrow] & X \\
& Y \arrow[u, Rightarrow] & {} \arrow[u, squiggly] & Y \arrow[u, Rightarrow] &
\end{tikzcd}$$

$\rightarrow \underline{p_Y = (1/2, 1/2)}$

$Z = X \oplus Y$

$\boxed{Z = (X+Y) \bmod 2}$

(a) Supponiamo che $p_X = (\alpha, 1-\alpha)$, $p_Y = (1/2, 1/2)$ e che X e Y siano v.a. indipendenti.

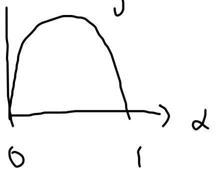
Dimostrare che $H(X|Z) = H(X)$ (Z non fornisce alcuna informazione su X).

(b) Supponiamo che $p_Y = (1/4, 3/4)$. Cambia qualcosa?

(a) $H(X) = h(\alpha)$ $\qquad I(X;Z) \overset{?}{=} 0$

$$\overset{\shortparallel}{H(X) - H(X|Z)}$$

$h(\alpha) = -\alpha \log \alpha - (1-\alpha) \log(1-\alpha)$



$$I(X;Z) = D(p_{XZ} \| p_X \cdot p_Z) \underset{p_{XZ} = p_X \cdot p_Z}{=} 0 \Rightarrow H(X) - H(X|Z) = 0 \quad \text{QED}$$

$$\left[\sum_{x \in \mathcal{X}, z \in \mathcal{Z}} p(x,z) \log \frac{p(x,z)}{p(x) \cdot p(z)}\right]$$

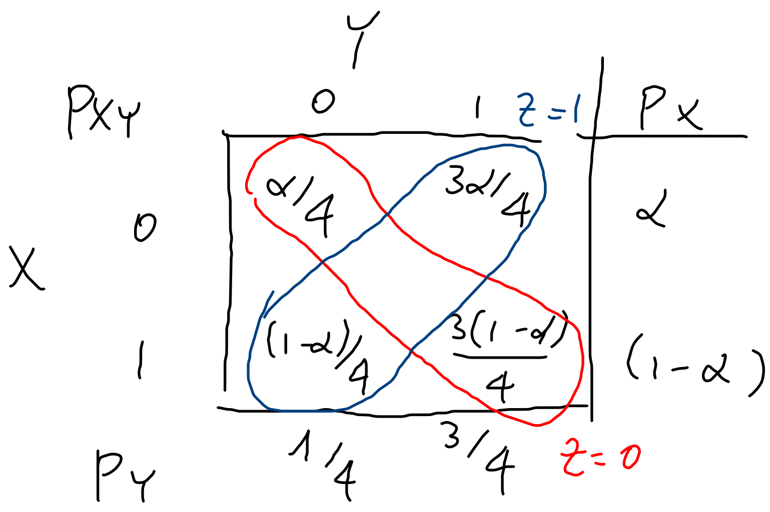
$$\begin{array}{c|c|cc}
 & & Y & \\
 & P_{XY} & 0 & 1 \\
\hline
X & 0 & \alpha/2 & \alpha/2 \\
 & 1 & (1-\alpha)/2 & (1-\alpha)/2
\end{array}$$

$Z=1$: $\alpha/2$ ($X=0,Y=1$), $(1-\alpha)/2$ ($X=1,Y=0$)

$z=0$: $\alpha/2$ ($X=0,Y=0$), $(1-\alpha)/2$ ($X=1,Y=1$)

$\rightarrow$

$$\begin{array}{c|c|cc|c}
 & & Z & & \\
 & P_{XZ} & 0 & 1 & p(X) \\
 & & z=0 & z=1 & \downarrow \\
\hline
X & 0 & \alpha/2 & \alpha/2 & \alpha \\
 & 1 & \frac{(1-\alpha)}{2} & (1-\alpha)/2 & 1-\alpha \\
\hline
 & p(z) \rightarrow & 1/2 & 1/2 &
\end{array}$$



p_{XY}	$Y=0$	$Y=1$	p_X
$X=0$	$\alpha/4$	$3\alpha/4$	α
$X=1$	$(1-\alpha)/4$	$\frac{3(1-\alpha)}{4}$	$(1-\alpha)$
p_Y	$1/4$	$3/4$	

$z=1$

$z=0$

p_{XZ}	$Z=0$	$Z=1$	p_X
$X=0$	$\alpha/4$	$3\alpha/4$	α
$X=1$	$\frac{3(1-\alpha)}{4}$	$\frac{(1-\alpha)}{4}$	$(1-\alpha)$
p_Z	$3/4-\alpha/2$	$1/4+\alpha/2$	

Non è più vero che $p_{XZ} = p_X \cdot p_Z$! Per esempio

$$\Pr[X=0, Z=0] = \alpha/4 \neq \Pr[X=0] \cdot \Pr[Z=0] = \alpha \cdot (3/4 - \alpha/2)$$

(a meno che $1/4 = 3/4 - \alpha/2$, ovvero $\alpha = 1/2$)

e quindi, in generale $H(X|Z) < H(X)$

$I(X;Z) > 0$ ⇒ Osservare Z mi permette di inferire informazione su X.

ENTROPIA CONGIUNTA E ENTROPIA CONDIZIONATA

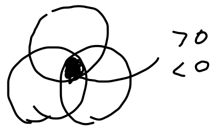
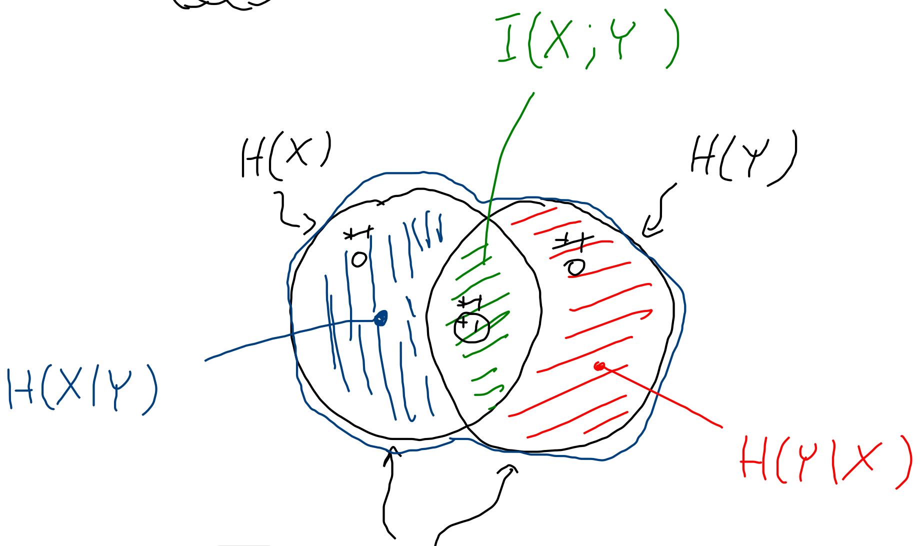


Diagramma informazionale



$$H(X,Y) = -\sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p(x,y)\log p(x,y)$$

(*) $$H(Y|X) = -\sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p(x,y)\log p(y|x)$$

Regola della catena:

Prop. Si ha $H(X,Y) = H(X) + H(Y|X)$.

$$H(X,Y) = -\sum_x \sum_y p(x,y)\log p(x,y) \overset{\text{Bayes}}{=}$$

$$= -\sum_x \sum_y p(x,y)\log\left[p(y|x)\cdot p(x)\right] =$$

$$= -\sum_x \underbrace{\sum_y p(x,y)}_{p(x)}\log p(x) \underbrace{- \sum_x \sum_y p(x,y)\log p(y|x)}_{(*)}$$

$$= \underbrace{-\sum_x p(x)\log p(x)}_{H(X)} + H(Y|X) \quad . \qquad \text{QED}$$

$$\boxed{I(X;Y) = H(X) + H(Y) - H(X,Y)}$$

$$\boxed{I(X;Y) = H(Y) - H(Y|X)}$$

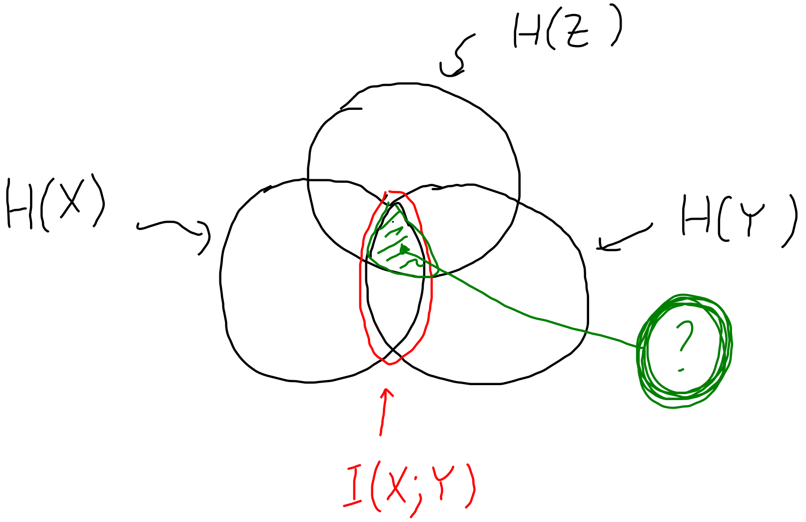
$X \cap Y = Y \setminus (Y \setminus X)$

$H(X,Y) = H(Y) + H(X|Y)$

, ⟷ $\cup$ (unione)

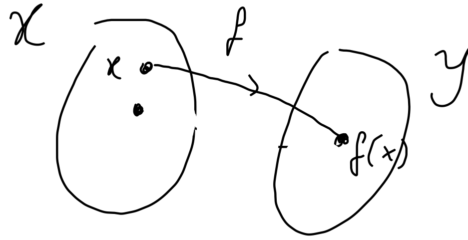
; ⟷ $\cap$ (intersez.)

| ⟷ $\setminus$ (diff tra insiemi)



$$X \cap Y \cap Z \overset{?}{\longleftrightarrow} I(X;Y;Z) \begin{matrix} \geq 0 \\ < 0 \end{matrix} \quad \overset{\neq}{\curvearrowright}$$

$$I(X;Y;Z) \neq \underbrace{D(p_{XYZ} \| p_X p_Y p_Z)} \geq 0$$



ENTROPIA DI FUNZIONI DI V.A.

X una v.a. e $f(X)$ una sua funzione ($f : \mathcal{X} \to \mathcal{Y}$)

Avevamo visto che $H(f(X)) \leq H(X)$, con $=$ sse f è biettiva su $\mathcal{X}$

Dim.

$$H(X, \overset{Y}{\widetilde{f(X)}}) \overset{\text{catena}}{=} \overset{H(X) + H(Y|X)}{H(X) + \underbrace{H(f(X)|X)}_{0}} \overset{\checkmark}{=} H(X)$$

0 perché condizionato a $X=x$, $f(X)$ è costante

catena $\parallel$

$$\overset{H(Y) + H(X|Y)}{\underline{H(f(X))} + H(\underline{X|f(X)})}$$

$$H(X) = H(f(X)) + \overbrace{H(X|f(X))}^{\geq 0} \Rightarrow H(X) \geq H(f(X)).$$

con $=$ sse f è biettiva su $\mathcal{X}$.