① Lancio una moneta finché non esce testa

X = numero di lanci $\qquad \mathcal{X}$

Quanto vale $H(X)$? $\qquad \left(\sum_{n=0}^{\infty} n r^n = r/(1-r)^2 \quad \forall r \in [0,1) \right).$

$$H(X) = - \sum_{x \in \mathcal{X}} p(x) \log p(x)$$

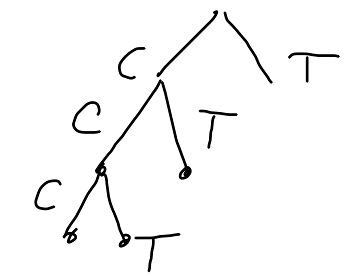
$$= + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \cdot (k)$$

$r = 1/2$

$$= \frac{r}{(1-r)^2} = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = 2$$

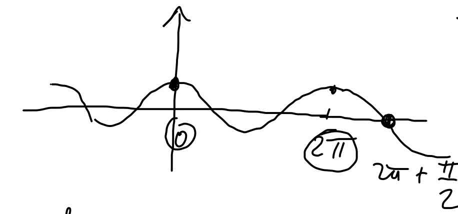
2 bit.

X	p(X)
1	1/2
2	1/4
3	1/8
⋮	
k	$1/2^k$



$$\log \frac{1}{2^k} = -k$$

② Sia X una v.a. con valori su un sottoinsieme finito $\mathcal{X}$ di $\mathbb{R}$



$\mathcal{X} = \{\dots, 0, \dots, 2\pi, \dots\}$

$\downarrow$

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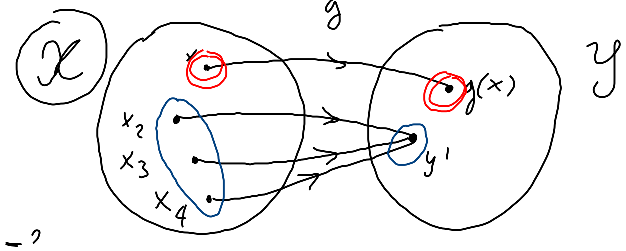
Che relazione c'è tra $H(X)$ e $H(Y)$ quando:

g non è biettiva su $\mathcal{X}$

$\boxed{p(y) = p(x_1)}$

(a) $Y = 2^X$? $\mathcal{X} \Rightarrow H(Y) = H(X)$



$\mathcal{X} = \{0, 2\pi + \frac{\pi}{2}\}$

g è biettiva su $\mathcal{X}$

(b) $Y = \cos X$? $\mathcal{X}$ $H(Y) \le H(X)$ $\quad x \quad p(x)$

$p(y') = p(x_2) + p(x_3) + p(x_4)$, con $p(x_2) > 0$, $p(x_3) > 0$, $p(x_4) > 0$

$\Downarrow$

$p(y') > p(x_2)$

$p(y') > p(x_3)$

$p(y') > p(x_4)$

$y = g(x) \qquad g: \mathcal{X} \to \mathcal{Y}$

$$\boxed{p(y) = \sum_{x: g(x) = y} p(x)} \Rightarrow \text{Se } y = g(x), \quad \sum_{x:g(x)=y} p(x) \log p(x) \overset{=?}{\le} \sum_{x:g(x)=y} p(x) \log p(y) = (\log p(y)) \cdot p(y) \quad \forall y \in \mathcal{Y}$$

$H(X) = H(g(X))$

$\boxed{H(X) \ge H(g(X))} = H(Y).$

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x) = -\sum_{y \in \mathcal{Y}} \sum_{x: g(x)=y} p(x) \log p(x) \overset{=?}{\ge} -\sum_{y \in \mathcal{Y}} p(y) \log p(y) = H(Y).$$

vale con $=$ sse g è biettiva su $\mathcal{X}$

$\mathcal{X} = \{x_1, x_2, \dots, x_k\}$

$\{\{x_{j_1}, \dots, x_{j_a}\}, \{x_{k_1}, \dots, x_{k_r}\}, \dots, \{\ \}\}$

$\to y_1 \longrightarrow y_2$ —

③ Qual è il minimo di $H(p_1, \dots, p_n) = H(\vec{p})$ quando $\vec{p}$ varia sull'insieme delle d.p. n-dimensionali? Determinare tutti i vettori $\vec{p}$ che realizzano il minimo.

$H(X)$
$\uparrow$
v.a.

p_i

$$H(\vec{p}) = -\sum_{i=1}^{n} \underbrace{p_i}_{\geq 0} \underbrace{\log p_i}_{\leq 0 \ (p_i \leq 1)} \geq 0$$

$= 0$? sse $p_i \log p_i = 0 \quad \forall i = 1, \dots, n$

$\iff p_i \in \{0, 1\} \quad \forall i = 1, \dots, n$

$\iff \vec{p}$ è della forma: $(0, 0, \dots, 0, 1, 0, \dots, 0)$

Ci sono n d.p. $\vec{p}$ tali che $H(\vec{p}) = 0$

$$\begin{cases} (1, 0, \dots, 0) & \longrightarrow X = x_1 \text{ sempre} \\ (0, 1, \dots, 0) & \longrightarrow X = x_2 \text{ sempre} \\ \vdots & \\ (0, 0, \dots, 1) & \end{cases}$$

ENTROPIA E VARIANZA

ENTROPIA $\quad X \in \mathcal{X} \quad \mathcal{X}$ alfabeto finito di simboli

VARIANZA $\quad X \in \mathbb{R}$ numeri reali $\quad E[(X - EX)^2] \geq 0$

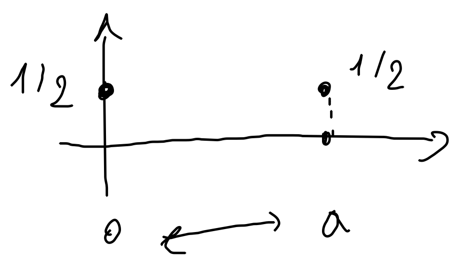
Se $\mathcal{X} \subseteq \mathbb{R}$? Possiamo dire qualcosa?

$a/2$

$$X = \begin{cases} 0 & \text{con prob. } 1/2 \\ a \neq 0 & \text{con prob. } 1/2 \end{cases}$$

$a \in \mathbb{R}$

$\mathcal{X} = \{0, a\}$



$$H(X) = \log |\mathcal{X}| = \log K = \log 2 = 1 \text{ bit.}$$

$$\underline{Var(X)} = E[X^2 - 2X \cdot EX + (EX)^2]$$

$$= E[X^2] - 2E[X]E[X] + (EX)^2$$

$$= E[X^2] - (EX)^2 = \frac{1}{2}\cancel{0^2} + \frac{1}{2}a^2 - \frac{1}{4}a^2 = \frac{a^2}{4}.$$

dipende da a.

$(\mathcal{X},\mathcal{Y}) \ni X,Y$ v.a. indipendenti $\Longleftrightarrow p_{XY} = p_X \cdot p_Y$

$\Downarrow \quad \not\Uparrow$

$\mathbb{R} \ni X,Y$ v.a. scorrelate $\Longleftrightarrow Cov(X,Y) = 0 \Longleftrightarrow E[(X-\overset{0}{EX})(Y-\overset{0}{EY})] = 0$

$$I(X;Y) = D(p_{XY} \,\|\, p_X \cdot p_Y)$$

Esempio

p_{XY}	$Y=-1$	0	$+1$	p_X
$X=-1$	0	1/4	0	1/4
0	1/4	0	1/4	1/2
$+1$	0	1/4	0	1/4
p_Y	1/4	1/2	1/4	

$Pr[X=-1]$

$Pr[X=0]$

$$= \sum_{x\in\mathcal{X},\, y\in\mathcal{Y}} p(x,y) \log \frac{p(x,y)}{p(x)\cdot p(y)}$$

$$= \frac{1}{4} \log \frac{1/4}{1/4 \cdot 1/2} + \frac{1}{4} \log \frac{1/4}{1/2 \cdot 1/4} +$$

$$+ \frac{1}{4} \log \frac{1/4}{1/4 \cdot 1/2} + \frac{1}{4} \log \frac{1/4}{1/2 \cdot 1/4} = \log 2 = 1 \text{ bit} > 0 \Rightarrow X \text{ e } Y$$

<u>non</u> sono indipendenti

$EX = 0 \quad EY = 0 \qquad Cov(X,Y) = E[XY] = \frac{1}{4}\cancel{(-1)\cdot 0} + \cancel{\frac{1}{4}\, 0\, (-1)} + \cancel{\frac{1}{4}(0)1} + \cancel{\frac{1}{4}}1 \cdot 0$

$= 0 \Rightarrow X$ e Y sono scorrelate

④ X, Y v.a.

		c	d	p_X
X	a	0	$1/8$	$1/8$
	b	$3/4$	$1/8$	$7/8$
	p_Y	$3/4$	$1/4$	

$Pr[X = a \wedge Y = d] = 1/8$

p_{XY} $\quad p_X \quad p_Y$

$Pr[X = \dots]$

Calcolare $H(X)$, $H(X|Y)$ e $I(X;Y)$.

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log(x) = -\frac{1}{8}\log\frac{1}{8} - \frac{7}{8}\log\frac{7}{8} \approx 0.544 \text{ bit}$$

$$H(X|Y) = \sum_{y \in \mathcal{Y}} p(y)\, H(X|Y=y)$$

$$H(X|Y=c) = -\sum_{x \in \mathcal{X}} p(x|Y=c) \log p(x|Y=c) = -p(X=a|Y=c)\log p(X=a|Y=c) + -p(X=b|Y=c)\log p(X=b|Y=c) = 0$$

$$\underbrace{\frac{0}{3/4}\log\frac{0}{3/4}}_{0} + \underbrace{\frac{3/4}{3/4}\log\frac{3/4}{3/4}}_{0}$$

$$H(X|Y=d) = 1 \text{ bit}$$

$$H(X|Y) = \frac{3}{4}\cdot 0 + \frac{1}{4}\cdot 1$$

$H(X)$: misure di incertezza nella v.a. X

Quanta informazione mi manca per conoscere X?

ESEMPIO

X = risultato di un dado a 6 facce

onesto → $p_X = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) \rightarrow H(X) = \log_2 6 \approx 2.58$ bit

truccato → $p_X = (0.001, 0.001, 0.001, 0.001, 0.001, 0.995) \rightarrow H(X) \approx 0.057$ bit

$I(Y;X) = I(X;Y)$: misure dell'incertezza "comune" a X e Y

Quanta informazione ottengo su X osservando Y? (o vice versa)

$$I(X;Y) = H(X) - H(X|Y)$$

$H(X)$: Informazione mancante su X

$H(X|Y)$: Informazione mancante su X se conosco già Y