FOLDING CUBIC ROOTS: MARGHERITA PIAZZOLLA BELOCH'S CONTRIBUTION TO ELEMENTARY GEOMETRIC CONSTRUCTIONS

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Abstract. In this paper we describe the contributions made by Margherita Piazzolla Beloch to the field of elementary geometric construction. In particular we examine her solution of the classical Greek problem of doubling the cube via paper folding. The solution was achieved by creating a new fold which in turn enabled Piazzolla Beloch to construct, also by paper folding, the root of any given cubic polynomial.

Keywords and phrases: Paper folding, projective geometry, roots of cubic polynomial, elementary geometric constructions

Mathematics subject classification: Primary 51M15; Secondary 51N15

1 Introduction

Paper folding, or origami, has been known for centuries as a fine art form, a way to transform a piece of paper into a stunning three dimensional object. In the last 30-40 years it has been understood that paper folding can also be an important scientific and technological tool. One of the pioneers of the application of paper folding to geometrical problems is Margherita Piazzolla Beloch. In her book *Lezioni di matematica complementare, la matematica elementare vista dall'alto* Piazzolla Beloch devotes the last two chapters to elementary solutions of classical geometric problems. It is probable that her deep interest in elementary mathematics, a common thread among Italian algebraic geometers at the beginning of the '900 (cf. [7]), is one of the main reason in Piazzolla Beloch's appreciation and usage of paper folding. Mainly because paper folding is a powerful but simple tool to perform geometric constructions. Following Piazzolla Beloch, see [21, p.353], we call a geometric problem a *problem of third degree* if its resolution requires finding a root of a cubic polynomial irreducible over the rational numbers. Gauss was the first one to state, even though without proof (see [6]), that angle trisection and cube doubling are not solvable by ruler and compass. The first published proof is due to M.L. Wantzel (cf. [30]) who actually proved that every problem of the third degree cannot be solved by ruler and compass. In her book, Piazzolla Beloch after describing and investigating classical solutions via mechanical instruments and higher degree curves turns her attention to a new method: paper folding.

Margherita Piazzolla Beloch's source of information about paper folding was the book written by Sundara Row [23] and an article of Rupp [26]. She published her reflections about paper folding in a series of articles ([18], [19], and [20]), which appeared several years before her lecture notes [21]. Her fundamental contribution was the discovery of what is now often called[1] Beloch's fold: *Given two point* P_1 *and* P_2 *and two lines* r_1 *and* r_2 *then, when it exists, we can fold the line reflecting* P_1 *onto* r_1 *and* P_2 *onto* r_2*.* (Fig. 1)

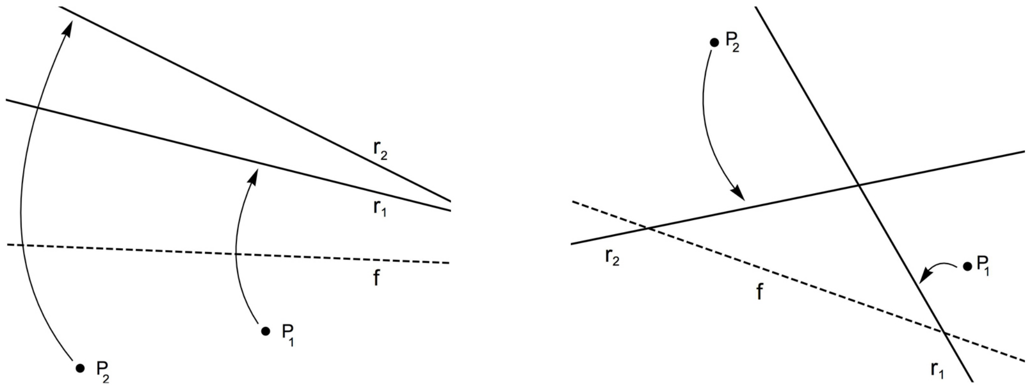


Fig. 1: Two instances of Piazzolla Beloch's fold

It was this fold that enabled Piazzolla Beloch to show that by using paper folding it is possible to double the cube and, with the aid of Lill's graphical method, to solve all cubic equations. As we shall see in section 7, in all the applications that Piazzolla Beloch considered it was evident that one could perform the above fold. Nonetheless it is interesting to establish a sufficient condition for when it is possible to execute Piazzolla Beloch's fold. We provide such a sufficient condition by means of (synthetic) projective geometry, see also [8] where similar conditions are established by means of analytic geometry.

This paper is organized as follows: section 2 contains a short biography of Margherita Piazzolla Beloch. In section 3 we examine her writings around solutions via mechanical instruments and higher degree curves of classical geometric problems. Section 4 encompasses a brief history of the application of paper folding to geometric problems. In section 5 we introduce Piazzolla Beloch's fold as well as the other six basic folds of paper folding; section 6 contains our deduction of the sufficient condition for executing Piazzolla Beloch's fold, based on synthetic projective geometry and Bezout's theorem. In the last section we exhibit three different approaches (graphical, geometrical and algebraic) to show that Piazzolla Beloch's fold can be used to construct a solution to any given cubic equations. The graphical approach, via Lill's method, was the one used originally by Piazzolla Beloch, and has been revisited several times since then. The other two, although quite straightforward, have not received the same kind of attention.

2 A few biographical notes about Margherita Piazzolla Beloch

The original source of information about Margherita Piazzolla Beloch life is the introduction of the guide [5] describing the "Collection Montesano", a private collection of offprints owned by Domenico Montesano which was acquired by Piazzolla Beloch on behalf of the library of the Mathematics department of Ferrara University.

[1]Most authors who are native English speakers mistakenly think that Piazzolla is her middle name.

Margherita Piazzolla Beloch was born in Frascati, near Rome, in 1879. She graduated in mathematics from Sapienza Università di Roma, with "dignità di stampa"[2], in 1908, under the supervision of Guido Castelnuovo. Her thesis, entitled "On birational transformations of space", was indeed published in the Annali di Matematica Pura ed Applicata, [17], one of the oldest math journal in Italy. Her first official academical appointment was as an assistant to the chair of "Descriptive Geometry" at the University of Pavia, and then in Palermo. In 1927 she became full professor of geometry at the University of Ferrara. In Ferrara she gave courses on many topics, such as descriptive geometry, higher geometry, complementary mathematics, superior mathematics.

In 1955 she retired but continued to be very active publishing widely for several years. In the mid '60 she personally prepared for publication a selection of her works [22], which was finally published in 1967. It consists of over 50 research papers which were divided in the following three topics: algebraic geometry, projective topology, and photogrammetry. The latter field was particularly dear to her. As stated in the preface of [22] it was the Italian Society of Photogrammetry and Topography that in 1961 proposed the publication of her selected works, to show their gratitude and affection to the scientist. As she herself acknowledges this was her "preferred field of study".

3 Classical problems: geometric construction and mechanical devices

Of the many geometrical problems studied by the ancient Greek only three gained enduring fame: doubling a cube, trisecting an angle, and squaring the circle. They owe their celebrity chiefly to the fact that they withstood every attempt to be solved by ruler and compass for centuries, until, in the nineteenth century, it was proven that was impossible to solve them with only ruler and compass, see [11] for a detailed account.

Piazzolla Beloch's interest in these matters was more on the positive side so to say: she was interested on which mechanical devices one needed to solve a particular problem or alternatively which curve of higher degree was needed to obtain the desired construction. We now briefly review the content, relevant to our investigation, of the two chapters of [21] dedicated to geometric construction and mechanical devices. After giving a detailed construction of the cissoid of Diocles, the conchoid of Nicomedes, and the Quadratrix of Hippias, she used them to solve the above mentioned classical problems. Specifically: doubling a cube can be solved by using either the cissoid or the conchoid, trisecting an angle is solved by the conchoid, and squaring the circle require the use of the Quadratrix.

She also remarks that any third degree problem can be reduced, by the use of ruler and compass, to either doubling a cube or trisecting an angle. Since both these problems are solvable by the conchoid, she concludes that every third degree problem is solvable by the combined use of ruler, compass and the conchoid.

Having settled these matters she takes a slightly different point of view. Suppose we fix the set of tools we are allowed to use (where tools mean both mechanical devices and curves). We then call the *frame of problems* of the given set of tools all the geometric problems that set solves. So for example the frame of problems of ruler, compass and

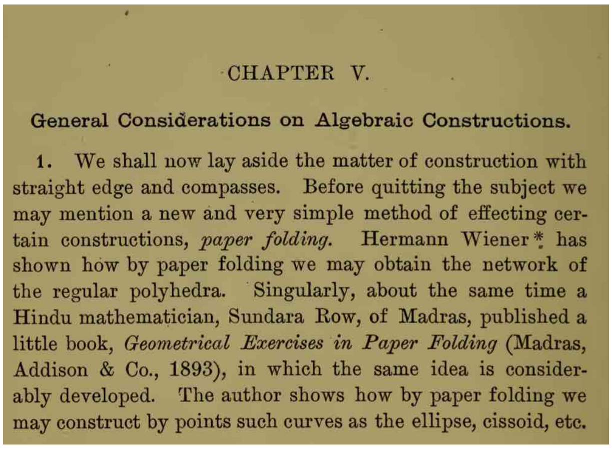
[2]worthy of publication

conchoid of Nicomedes contains all the problem of degree less or equal to three. If two sets of tools have the same frame of problems they are called *equivalent*. For example ruler and compass is equivalent to the ruler and a fixed circle completely drawn. She then determines the frame of problems for several mechanical (hinged) devices, with special regard to those constructed by Alfred B. Kempe in [10].

The next set of tools she examines is paper folding. Firstly she remarks that several geometric constructions are more readily done with paper folding than ruler and compass e.g. drawing a line through a point perpendicular to a given line takes only one fold while one has to use the compass three time and the ruler once. She then continues showing how to bisect an angle, find the intersection of a given circle and a given line and lastly find the intersection of two given circles by paper folding. As a consequence paper folding can solve all the problems that ruler and compass can. She then shows that paper folding can actually solve all third degree problems. We examine this latter topic in greater details in sections 5, 6, and 7.

4 A few remarks on the history of geometric constructions via paper folding

As remarked by Piazzolla Beloch in [20], the first author to draw attention to the use of paper folding in the solution of classical geometric problems was Felix Klein in [11]



CHAPTER V.

General Considerations on Algebraic Constructions.

1. We shall now lay aside the matter of construction with straight edge and compasses. Before quitting the subject we may mention a new and very simple method of effecting certain constructions, *paper folding*. Hermann Wiener* has shown how by paper folding we may obtain the network of the regular polyhedra. Singularly, about the same time a Hindu mathematician, Sundara Row, of Madras, published a little book, *Geometrical Exercises in Paper Folding* (Madras, Addison & Co., 1893), in which the same idea is considerably developed. The author shows how by paper folding we may construct by points such curves as the ellipse, cissoid, etc.

Fig. 2: Klein's reference to Wiener's work and Sundara Row's book

In his brief remark Klein mentions the two forefathers of the application of paper folding to geometric problems: Herman Wiener and Sundara Row. While Wiener's contribution to paper folding has been largely forgotten, Sundara Row's book has become a classic and has enjoyed a flurry of re-printings (Fig.3)

The version of *Geometric exercises in paper folding* by Sundara Row that can be found nowadays is the one edited by W.W. Beman and D.E. Smith. They also became interested in the work of Sundara Row thanks to Klein. In fact the first sentence of the preface of their edition of the book reads "*Our attention was first attracted to Sundara Row's book by a reference in Klein's Vorträge über ausgewählte Fragen der Elementargeometrie*", a text which they translated from German into English. Beman and Smith in their edition of Sundara Row's book added several pictures of actual folded paper, see Fig. 4 as an example , in order to replace some of the illustrations present in the original version, which were mere line drawings.

Fig. 3: A selection of covers of reprints of Sundara Row's book

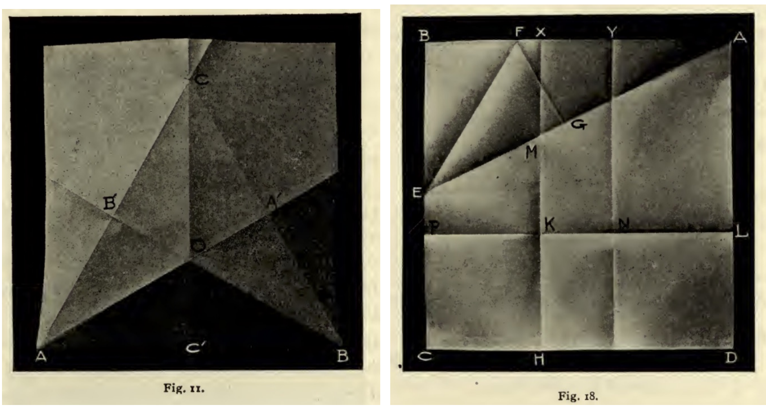


Fig. 4: Two pictures taken from the book by Sundara Row

It is only in the recent article by Michael Friedman [3] that Herman Wiener work on paper folding is examined. As both the book of Sundara Row and the article by Wiener on paper folding appeared in 1893, Friedman asserts that 1893 may be thought of as the first year that saw a modern mathematical treatment of the problem of determining what kind of object can be produced by folding paper. Wiener exhibited some paper objects during an exhibition on mathematical tools and models in 1892 in Munich and wrote a short note [31] for the catalogue of the exhibition, which is admittedly hard to follow being without the illustrations which were present at the exhibition. On the other hand, as Friedman points out, there are some interesting observation in the paper but a general lack of mathematical rigor: for example he gives the instructions to construct a regular pentagon by folding and knotting a strip of paper (Fig. 5), but does not supply a proof of the regularity (a proof can be found in [15]).

It must be noted that also Piazzolla Beloch contributions went unnoticed for several year as mathematicians generally lost interest in the subject. It was only in the late '80 that, thanks to the efforts of Huzita Humizaki with the aid of Benedetto Scimemi, there was a revival of interest in paper folding. The first International meeting on Origami Science and Technology, organized by Humizaki, was held in Ferrara in December 1988 [1]. It is rather peculiar that even though the first pages of the proceeding of this

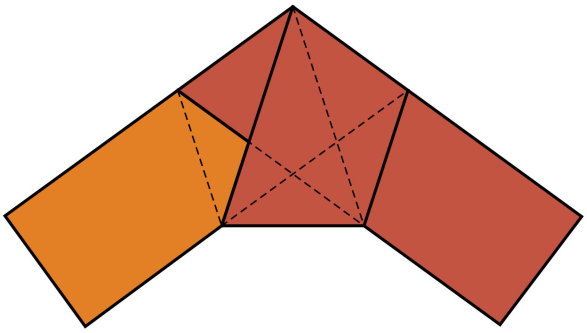
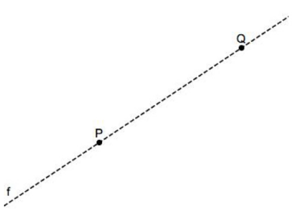


Fig. 5: The construction of a regular pentagon by Wiener

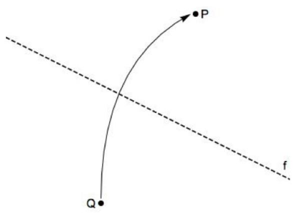
meeting host a reproduction of Piazzolla Beloch's article on paper folding many of the subsequent authors were not aware of her contributions. The Origami meeting have now become a regular event, the last of which was held at Tokyo university in 2014 and its proceedings are published by the American Mathematical Society [16].

5 Margherita Piazzolla Beloch innovative folding

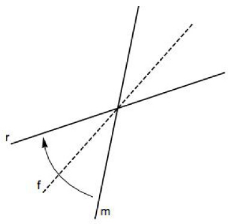
Margherita Piazzolla Beloch seems to have been the first one to discover that problems of the third degree can be solved by means of paper folding. At that time the applications of paper folding to geometry were still in their infancy and a list of basic folds was not available. We now have such a complete list of basic foldings (see 6), sometimes improperly called axioms. It has to be noted that some foldings were discovered several times over, see the introduction of [9] and [13], for more details.



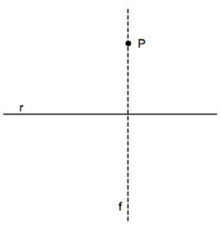
(O1) we can fold the line through two points.



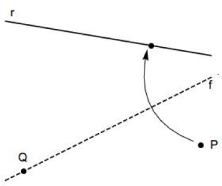
(O2) we can fold the median of the segment joining two points.



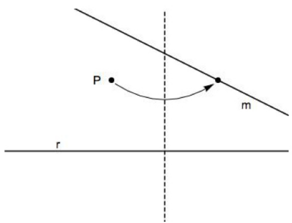
(O3) we can fold the bisector of an angle.



(O4) we can fold the line perpendicular to a given line passing through a given point.



(O5) Given two point P and Q and a line r, we can fold the line passing through Q and reflecting P onto r.



(O7) Given a point P and two lines r and m we can fold the line perpendicular to r reflecting P ont m

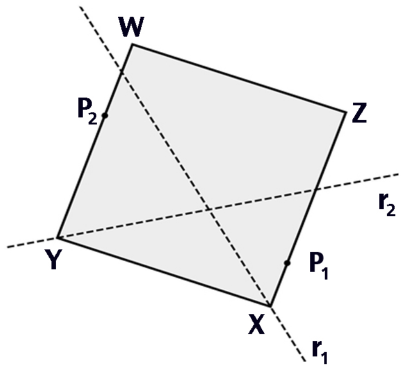
Fig. 6: Six of the seven basic paper foldings

There is still one basic fold: Piazzolla Beloch's fold. This fold appeared first in print in the short note [18], an extract from the lecture notes for her course *Matematiche complementari* containing the new result about paper folding. Let us recall its definition: *Given two point P_1 and P_2 and two lines r_1 and r_2 then, when it exists, we can fold the line reflecting P_1 onto r_1 and P_2 onto r_2.* (cf. Fig. 1). To see that, when possible, one can actually perform this fold we quote B. Scimemi (see [28]) "taking advantage of translucency, first superimpose P_1 on r_1, then let P_1 run along r_1 and meanwhile pay attention to the motion of the point P_2, waiting until it comes to lie on the line r_2".

The folds O1-O5 were already extensively used by Sundara Row in [23]. Fold O7 became widely known only after Koshiro Hatori announced its discovery in 2002, but Justin (see [12]) already listed it as one of his seven basic fold, 13 years earlier.

Piazzolla Beloch introduced O6 to solve the following problem: *Given two points P_1 and P_2 and two lines r_1 and r_2 construct a square with vertices X, Y, W, and Z so that X and Y lie on r_1 and r_2 and P_1 (respectively P_2) lies on the line joining X and Z (respectively Y and W)* (Fig. 7):

Costruire un quadrato di cui due lati opposti passino rispettivamente per due punti dati, e i due rimanenti vertici situati sui rimanenti lati stiano rispettivamente su due rette date



Given two points, P_1 and P_2, and two lines, r_1 and r_2, construct a square with two opposite edges passing through the two given points, and the remaining vertices lying on the two given lines

Fig. 7: Piazzolla Beloch's square

This square construction is the key to construct solutions of third degree problems by paper folding. To show why fold O6 enables us to construct the square in Fig. 7, we have to give a geometrical interpretation of fold O5 and O6. As was already known to Sundara Row the folding line in O5 is the tangent line, in Q, to the parabola, $\mathcal{P}_P^r$, having r has directrix and P has focus. In fact as Piazzolla Beloch remarks in [18] one can reconstruct a parabola by folding its tangents in the following way [3]: use the bottom edge of a sheet of paper as the directrix of the parabola, and mark the focus at a given distance from it. Then fold the paper on itself, without moving the focus making sure that the bottom edge of the sheet passes through the focus.

The folding line produced by fold O6 is then a common tangent to $\mathcal{P}_{P_1}^{r_1}$ and $\mathcal{P}_{P_2}^{r_2}$. To show way fold O6 enable us to construct the square of Fig. 7, we argue along the lines of [18]: consider the parabola $\mathcal{P}_1$ (respectively $\mathcal{P}_2$) having focus in P_1 (resp. P_2) and whose tangent line in the vertex is r_1 (resp. r_2). Then we perform fold O6 and the

3 "*Prendendo cioè l'orlo (rettilineo) di un foglio di carta come direttrice della parabola, e segnando il fuoco alla data distanza da questa, basta tener fermo il fuoco e ripiegare la carta su se stessa in modo che l'orlo ripiegato venga a passare per il fuoco.*" [18, p.187]

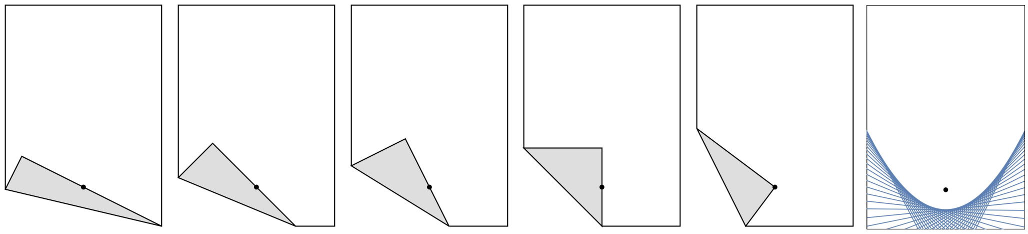


Fig. 8: Folding the tangents of a parabola

folding line, call it f, is tangent to both $\mathcal{P}_1$ and $\mathcal{P}_1$. Let X (respectively Y) be the intersection of f and r_1 (resp. r_2), see Fig. 9

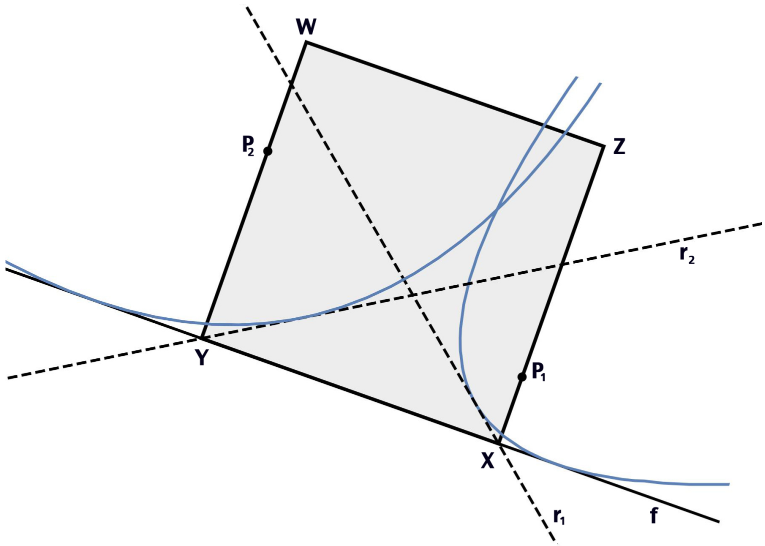


Fig. 9: The underlying parabolae in Piazzolla Beloch's square

The line passing through X (resp. Y) and perpendicular to f passes through the focus of $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) which is P_1, (resp. P_2). Therefore the square having the segment $\overline{XY}$ as an edge is the desired square.

6 Piazzolla Beloch's fold and projective duality

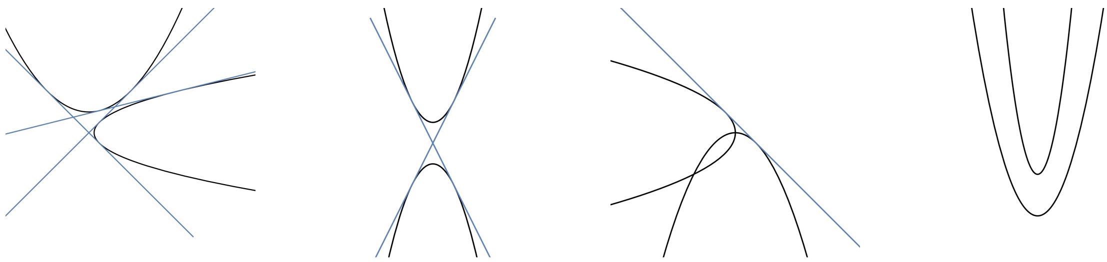
As we already remarked is not always possible to perform fold O6 and most probably Piazzolla Beloch was well aware of this. Moreover, as we shall see in the last section, all the instance of fold O6 that she needed to perform in her investigation were clearly possible. Even though several authors seem unaware that fold O6 can not be performed in all cases, already in [8] there is a discussion, in terms of analytic geometry of when it is possible, see also [2] for an approach closer to the one presented here. We would like now to give a simple criterion for when two parabolae do have a common tangent in terms of projective geometry and projective duality in the plane. Projective duality was well known to Margherita Piazzolla Beloch, who extensively worked on projective algebraic curves. We briefly recall the basic facts about projective duality, referring the interested reader to [24].

So let $\mathbb{A}^2(\mathbb{R})$ denote the (affine, real) plane and $\mathbb{P}_2(\mathbb{R}) = \mathbb{A}^2(\mathbb{R}) \cup \ell^\infty$ the projective real plane obtained from $\mathbb{A}^2(\mathbb{R})$ adjoining ℓ^∞ the "line at infinity" whose points represent the directions of lines in $\mathbb{A}^2(\mathbb{R})$. Formally points in $\mathbb{P}_2(\mathbb{R})$ are represented by three homogeneous coordinates $P[X : Y : Z]$ and $\mathbb{A}^2(\mathbb{R})$ correspond to the open subset given by $Z \neq 0$. It is straightforward to check that all parabolae in the affine plane are actually tangent to the line at infinity, the tangency point being the direction of the symmetry axis of the parabola.

Now we come to projective duality in the plane. The dual projective plane $\mathbb{P}_2^*(\mathbb{R})$ is the set of all lines in $\mathbb{P}_2(\mathbb{R})$, and points in $\mathbb{P}_2(\mathbb{R})$ correspond to lines in $\mathbb{P}_2^*(\mathbb{R})$. Given a conic $\mathcal{C}\in\mathbb{P}_2(\mathbb{R})$ its dual conic $\mathcal{C}^*$ is the set of all tangent lines to $\mathcal{C}$. It is readily seen that $\mathcal{C}^*$ is a conic as well. Therefore the common tangents to two parabolae $\mathcal{C}_1$ and $\mathcal{C}_2$ are the point of intersections of the dual conics $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$.

By Bézout's theorem [4, section 5.3], there are exactly 4 points of intersection if counted with the appropriate multiplicity. In this simple case the multiplicity is one unless $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ have the same tangent line at the point of intersection, in which case the multiplicity is 2. This means that there is a line which not only is tangent to both $\mathcal{C}_1$ and $\mathcal{C}_2$, but also that the point of tangency is the same.

Now let us go back to the problem of finding the common tangent to two given parabolae $\mathcal{C}_1$ and $\mathcal{C}_2$. As we already remarked the line at infinity is tangent to both $\mathcal{C}_1$ and $\mathcal{C}_2$, thus $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ always intersect in the point Q_0 of $\mathbb{P}_2^*(\mathbb{R})$ corresponding to the line at infinity of $\mathbb{P}_2(\mathbb{R})$. It follows that the remaining points of intersection can be found by solving the cubic equation obtained by eliminating the appropriate variable, unless the multiplicity of Q_0 is two, which happens if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ have parallel symmetry axes. If $\mathcal{C}_1$ and $\mathcal{C}_2$ have parallel symmetry axes we are left with a quadratic equation which may or may not have real solutions, and both cases do happen. If $\mathcal{C}_1$ and $\mathcal{C}_2$ do not have parallel symmetry axis then the common tangents in $\mathbb{A}^2(\mathbb{R})$, different from the line at infinity, are either 3 or 1 and both cases do happen (10).



a Three common tangents   b Two common tangents   c One common tangent   d No common tangent

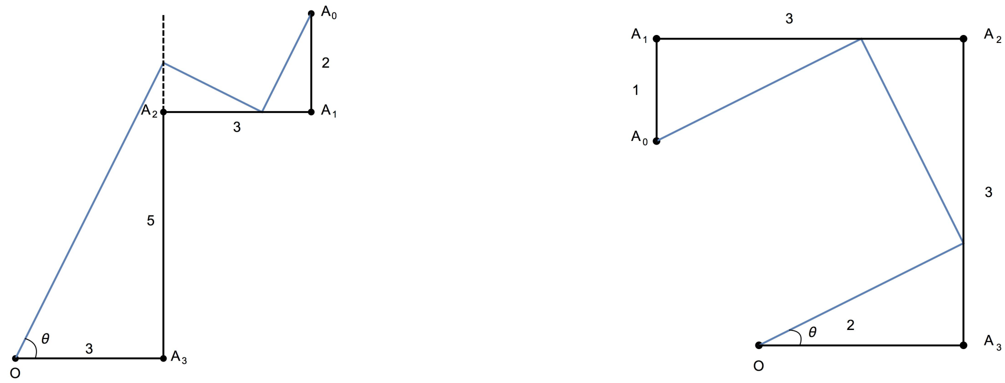
Fig. 10: Common tangents to two parabolae

7 Folding solutions of cubic equations

In this section we will show in three different ways, graphic, geometric, and algebraic, how Margherita Piazzolla Beloch's folding enables us to construct solution of cubic equations by paper folding. As is clear from her writings she was aware of all of them but only wrote down the details for the graphical one. The graphical method is based on Lill's method (see [14], [11]) to find real roots of polynomials (with real coefficients) of every degree.

Lill's method was used by Margherita Piazzolla Beloch in her original solution of cubic equations by paper foldings in [20]. Since then, a few authors dwell on the subject, notably Benedetto Scimemi [27] and Thomas C. Hull [9]. Therefore we will be rather brief on Lill's method and concentrate on the application of Piazzolla Beloch's fold. Being a graphical method we will relay heavily on illustrations.

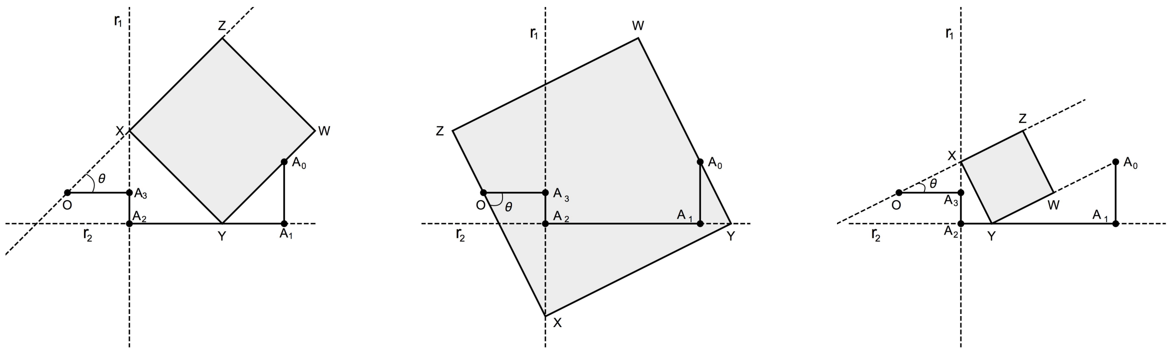
Lill's method, which we describe here only for cubic polynomials, goes as follows (we use the convention of [27]): given a cubic polynomial $P(X) = a_3x^3 + a_2x^2 + a_1x + a_0$ we construct a 4 edge polygonal chain $OA_3A_2A_1A_0$, with only right angles, where the starting point is the origin O, the length of the segment ending in A_i is $|a_i|$ and in A_i the turn is made clockwise if $a_ia_{i-1} \geq 0$ and counterclockwise if $a_ia_{i-1} \leq 0$. As usual a picture is worth a thousand words so we refer the reader to Fig. 11. To find a root Lill proceeds as follows: draw a line from the origin, forming an angle of θ radiants with the x-axis and whenever it hits a edge of the Lill's polygon, or its continuation, it bounces off on the line orthogonal to r, choosing the direction that ensures that it will hit the next edge of the Lill's polygon (or its continuation). Since we are in degree 3, the Lill's polygon has at most 4 edges and we have to perform only two reflections. If after the second reflection we end up with a line which passes through the point P_0 then $\tan(\theta)$ is a root of the cubic polynomial associated to the polygon. Also in this case it is much more illuminating to look at a picture so we refer the reader to Fig. 11.



a $\tan(\theta) = 2$ is a root of $3x^3 - 5x^2 - 3x + 2$ b $\tan(\theta) = \frac{1}{2}$ is a root of $2x^3 - 3x^2 + 3x - 1$

Fig. 11: Finding roots of cubic polynomials via Lill's polygons

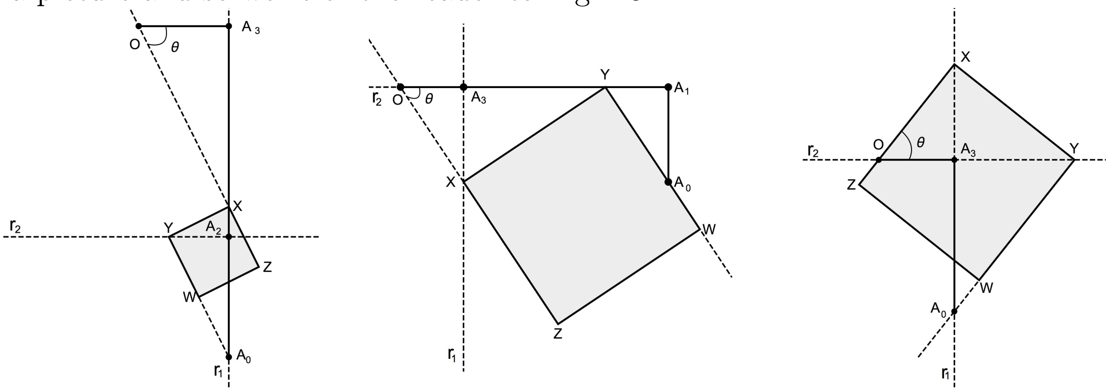
Piazzola Beloch's square construction shows that we can always find, by paper folding, such an angle θ. To see this suppose first that all the coefficients of $P(x)$ are not zero. Then we can apply Piazzola Beloch square construction to the points O and A_0 and the lines r_1 joining A_3 and A_2, and r_2 joining A_2 and A_1. Since r_1 and r_2 are perpendicular, hence not parallel, we are sure that we can perform Piazzolla Beloch's fold and construct a square with O on one edge or on its continuation, A_0 on the parallel edge and two vertices on r_1 and r_2. Therefore the line passing trough the origin on which the edge of the square lies is the line required to find the root of the cubic polynomial by Lill's method.



a $\alpha = 1$ b $\alpha = -2$ c $\alpha = \frac{1}{2}$

Fig. 12: Piazzolla Beoch' s squares correspondig to the three real roots of $x^3 - \frac{x^2}{2} + \frac{5}{2}x + 1$

It is possible to apply Piazzolla Beloch's construction also if some of the coefficients vanishes, one has just to be careful in the choice of the lines r_1 and r_2. Namely one line will be the one connecting the two vertices of the polygon and the other one has to be choose perpendicular to the previous one passing to the vertex corresponding to the coefficient preceding the missing one. One argues similarly if there are two missing coefficients. Note that we only need to deal with polynomials of the following form $x^3 + a_2x^2 + a_0$, $x^3 + a_1x + a_0$ and $x^3 + a_0$. As before, everything is clearer if one looks at a picture and so we refer the reader to Fig. 13.



a $P(x) = x^3 + \frac{7}{3}x^2 - \frac{4}{3}$ b $P(x) = x^3 - \frac{13}{4}x - \frac{3}{2}$ c $P(x) = x^3 - 2$

Fig. 13: Piazzolla Beloch's square construction for polynomials with missing terms

It has to be noticed that in order to construct the cubic root of a given number, one does not need Lill's method but it is enough to use Piazzolla Beloch's square construction directly as shown in [18].

Next we turn to geometry. Let $P(x) = a_0 + a_1x + a_2x^2 + x^3$ be a generic monic cubic polynomial. It suffices to show that we can find its real roots via paper folding. To do so we have to exhibit two parabolae whose common tangents have slopes that are precisely the real roots of $P(x)$. Piazzolla Beloch's square construction suggests that should be enough to search among parabolae that have symmetry axis either horizontal or vertical. So we consider the following two families of parabolae

$$\mathcal{P}_c : y = x^2 + c \qquad \mathcal{P}_{a,b} : x = ay^2 + b.$$

A straightforward computation shows that the slopes of the lines that are common tangents to $\mathcal{P}_c$ and $\mathcal{P}_{a,b}$ are the roots of $t^3 - 4bt^2 - 4ct + \frac{1}{a}$. The self-evident choice of a, b, c shows that we can recover the polynomial $P(x)$.

We come at last to the algebraic approach. Our construction is based on Cardano's formula for solving the cubic equations. First of all a general cubic polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + x^3$ can be reduced, by a linear change of variable, to a polynomial of the form $x^3 + px + q$. Next Cardano's formula gives us a real root for this type of polynomials. Namely:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Since it is possible by paper folding to construct, without the aid of Lill's polygon, both the square root and the cubic root of a given quantity, it follows, again, that by using paper folding we can construct the root of cubic polynomial.

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