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Reducibility of PDEs with quasi-periodic in time coefficients

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Καὶ ἠγάπησαν οἱ ἄνθρωποι
μᾶλλον τὸ σκότος ἢ τὸ φῶς
E gli uomini preferirono
il buio piuttosto che la luce.

Abstract

The thesis starts from the study of a classical question namely the spectral properties of linear operators on Hilbert spaces. In particular, the thesis focuses on close to diagonal operators with simple spectrum but in which the difference of the eigenvalues accumulates to zero. We are looking for conditions (on the perturbation and on the differences of the eigenvalues) that guarantee that the spectrum is still discrete with simple eigenvalues. Interesting applications are related to the reducibility of linear partial differential equations (PDEs) with quasi-periodic dependence on time. More precisely, we want to reduce the PDE to constant coefficients, by means of a quasi-periodic change of variables. The technical problems that arise are related to the presence of small divisors.

Contents

Introduction

The study of the spectral properties of linear operators on Hilbert spaces is one of the first successes of functional analysis in the end of 1800, starting from the classical result of the spectral decomposition theorem. A great source of motivation came from the development of the quantum mechanics, indeed the study of the spectra of hermitian operators on Hilbert space was central in the explanation of the spectra of atoms.
Here we work on separable Hilbert spaces that we can identify with sequence spaces (e.g. ℓ^2) or weighted sequence spaces (e.g. the Sobolev space h^p) and we would like to understand the properties of the spectrum of an operator on those spaces and in particular the characteristics of the operator that guarantee that it has pure point spectrum. We consider this problem in the perturbative case, namely we consider a family of operators of the form

$$L(\varepsilon) := \Lambda + \varepsilon P$$

where Λ is diagonal and we ask whether this family is diagonalizable and if its spectral properties depend continuously (or more regularly) on ε.
This problem is interesting and it is deeply studied both in finite and in infinite dimension. We will work in infinite dimension and we will focus on the case in which Λ has pure point and simple spectrum.
In finite dimension this property is called "regular semi simple" and it is an open property, namely if $L(\varepsilon)$ has simple spectrum for $\varepsilon = 0$ then there exists a neighbourhood of 0 in which $L(\varepsilon)$ has a simple spectrum. Therefore, if $L(0)$ is diagonalizable then $L(\varepsilon)$ also can be diagonalized and the eigenvalues and eigenfunctions will be analytic functions of ε. In infinite dimension the situation is really more complicated, even if we know that the operator $L(0)$ has simple and pure point spectrum, and spectral properties of L depend strongly on whether the spectrum of Λ has accumulation points.
Let $L(\varepsilon)$ be defined on a sequence space contained in ℓ^2, from the Kato-Rellich Theorem we know that if $L(\varepsilon)$ is an analytic family (in the sense of Kato) of operator-valued functions and $\lambda^{(0)}$ is a nondegenerate pure point eigenvalue of $L(0)$ then for every ε near to 0 there exsists exactly one $\lambda(\varepsilon) \in \sigma(L(\varepsilon))$ near to $\lambda^{(0)}$ and this point is isolated and non degenerate.
Unfortunately, even if every eigenvalue of $L(0)$ is isolated and simple, is not true that we can iterate the Kato-Rellich Theorem and that $L(\varepsilon)$ can be diagonalized. Indeed, the radius of convergence goes to zero with the spectral gap between λ_0 and the other eigenvalues, so some problems occur due to the non uniformity of the distance between

the eigenvalues. Actually, to obtain that result we have to suppose that the difference between the eigenvaulues is *uniformly* bounded from below, that is a really strong property to ask. Under such a strong assumptions one can also prove the diagonalization as an application of the Implicit Function Theorem.

If $L(0)$ has simple and pure point spectrum such that

$$|\lambda_k^{(0)} - \lambda_j^{(0)}| > \alpha > 0 \quad \forall\, j, k \tag{1}$$

and P is small in operator norm then $L(\varepsilon)$ can be diagonalized, namely there exists a change of variables that conjugates it to a diagonal operator, and the spectrum of the diagonalized operator is still simple and pure point.

Finally, the eigenvalues and the eigenvectors are curves in ε in the sense that the eigenvalues forms a denumerable family of smooth functions $\lambda_j^\infty(\varepsilon)$. Same for the eigenvectors. Indeed, one can compute the Taylor series expansion for eigenvalues and eigenvectors and it turns out that the difference of the unperturbed eigenvalues appears at the denominator. So the radius of convergence of the series is inversely proportional to α.

It is interesting and natural to consider operators satisfying assumptions weaker than (1). For instance, we can consider the case when the difference of the eigenvalues accumulates to zero. In this case, if one has some lower bound on the difference of the eigenvalues one can still hope to achieve convergence. However, in general to require the boundedness of the perturbation is not enough, and typically one should impose further conditions on the perturbation.
For instance, in this Thesis we ask to control (in an appropriate operator norm) some "commutators" of P with the "derivative" operator ∂_θ which we shall define below.
Given two linear operators, we will denote $(\mathrm{ad}A)B := [B, A]$ where $[\cdot, \cdot]$ is the usual commutator.
This type of situations, where the unperturbed operator has accumulating eigenvalues, appears in the context of linear operators with quasi-periodic in time coefficients. Indeed, a source of motivation comes from the study of quasi periodic solutions for non linear PDEs on a compact manifold close to an elliptic equilibrium point. KAM (Kolmogorov-Arnold-Moser) theory, born to analyze the dynamics of nearly integrable finite dimensional Hamiltonian system, provides us useful tools to deal with the problem discussed above in the Hamiltonian context.
The main difficulty appearing in KAM theory is to deal with the possible loss of derivatives that arise from the presence of *small divisors*, which prevents the use of a classic Implicit Function Theorem. A possible way to overcome this problem is to apply a fast convergent KAM iterative scheme.
The core of this algorithm is the invertibility of certain linear operators (obtained linearizing the PDE at some approximate quasiperiodic solution). Typically in the applications the spectrum of such linear operator accumulates to zero and this is the so-called small divisor problem.

The idea of the Thesis is to prove diagonalization Theorems with an eye to their application in a PDEs context. We then use such result to study two linear PDEs with quasi periodic in time coefficients and show that they can be reduced to constant coefficients. We work only in the context of linear PDEs, however the estimates that we obtain are exactly the ones needed, in the non linear context, on the linearized equation at an approximate quasiperiodic solution.

In conclusion, we shall be interested in operators acting on function spaces which are perturbation of a diagonal one, namely of an operator that has pure point and that can be written in diagonal form with respect to a countable basis. Therefore it is convenient to work directly in that basis, thus identifying the space with a sequence space. In this thesis we shall confine ourself to Hilbert Spaces.

In our application we shall consider operators acting on the torus $\mathbb{T}^d$ such that the unperturbed part is diagonal with respect to the Fourier basis and hence our sequence spaces are indexed in $\mathbb{Z}^d$.
Consider the scale of Hilbert spaces

$$\mathtt{h}^s \equiv \mathtt{h}^s(\mathbb{Z}^d) := \left\{ \{u_k\}_{k\in\mathbb{Z}^d} \ : \quad |u|_s^2 := \sum \langle k\rangle^{2s}|u_k|^2 < \infty \right\}, \qquad \langle k\rangle := \max\{|k|,1\}\,.$$

In the space $\mathcal{L}(\mathtt{h}^s,\mathtt{h}^{s'})$ of bounded linear operators from $\mathtt{h}^s$ to $\mathtt{h}^{s'}$ we use the standard operator norm

$$\|M\|_{s,s'} := \sup_{|u|_s\le 1} |Mu|_{s'}\,.$$

For $1\le h\le d$ we define the unbounded operator ∂_{θ_h} as $(u_k)_{k\in\mathbb{Z}^d}\mapsto (\mathrm{i}k_h u_k)_{k\in\mathbb{Z}^d}$.

We consider a parameter family of operators

$$L(\xi) = \Lambda_0(\xi) + P_0(\xi) \tag{2}$$

with Λ_0 diagonal with distinct eigenvalues[1]. The parameters ξ are in a compact set $\mathcal{O}_0$ in $\mathbb{R}^n$ with the only condition $n>0$. They will be modulated in order to avoid resonances, so our result will not hold for all ξ but only on a subset defined implicitly. Of course it is very important that this set is not empty. In order to prove this we shall need some Lipschitz dependence both for Λ_0 and P_0. For the precise condition see the hypotheses $(\mathbf{H_1}),(\mathbf{H_2})$ in Section 2.1. To make the Lipschitz dependence quantitative, we shall work with a weighted Lipschitz norm (see definition 1.1.13) denoted by[2] $\|\cdot\|_{s,s}^{\gamma,\mathcal{O}}$.

Theorem 1. *Fix $s>\frac{d}{2},\gamma>0,\tau>d-1$ and $b_1>5d+\tau+1$. Consider an operator as in (2) satisfying the hypotheses above and such that*

$$P_0,\ (\mathrm{ad}\,\partial_{\theta_i})^{b_1}P_0\in\mathcal{L}(\mathtt{h}^s,\mathtt{h}^s)\quad \text{for all } 1<i<d.$$

[1] Clearly this means that it has pure point spectrum. Moreover, returning in the functional space setting we are implicitly requiring that $\Lambda_0(\xi)$ is diagonal respect to the same basis for all ξ.

[2] Here γ is a positive number and $\mathcal{O}$ the set where we want that the operators are Lipschitz.

There exists $\epsilon_\star$ such that, if

$$\|P_0\|_{s,s}^{\gamma,\mathcal{O}_0},\ \|(\mathrm{ad}\,\partial_{\theta_i})^{d+1}P_0\|_{s,s}^{\gamma,\mathcal{O}_0} \le \epsilon_\star\,, \quad \text{for all } 1 < i < d$$

then there exists Lipschitz functions $\lambda_j^{(\infty)}(\xi)$ such that in the set

$$\mathcal{C} := \{\xi \in \mathcal{O}_0 : |\lambda_k^{(\infty)}(\xi) - \lambda_{k'}^{(\infty)}(\xi)| > 2\gamma|k-k'|^{-\tau}\}$$

L is diagonalizable, namely there exists a change of variables close to diagonal that conjugates it to the diagonal operator $\Lambda_\infty = \mathrm{diag}\lambda_k^{(\infty)}$.

This result is proven with a quadratic KAM algorithm. The first step is to define the majorant norm (see (3)) and reformulate the Theorem 1 in terms of that norm (see Theorem 5). Then we introduce an iterative scheme in which at each step we conjugate the operator to a diagonal operator plus a reminder whose norm is quadratically small with respect to the previous step. Then we prove that this algorithm is convergent. At each step the change of variables is of the form e^A where A is the solution of an *Homological Equation* (see Lemma 2.1.2). To solve this equation we have to impose some conditions on the difference of the eigenvalues. In the end we will prove that it is enough to impose the condition only on the final eigenvalues.

The second result that we prove regards the regularity of the change of variable that, by Theorem 1, diagonalizes the operator $L(\xi)$. Namely, we consider an operator as in (2) where P_0 is small in some "low norm" $\mathtt{h}^{s_0}$ and bounded in "higher norms" $\mathtt{h}^s$ with $s_0 < s < s_1$, for a fixed s_1. The theorem above implies that the operator that diagonalizes L is bounded from $\mathtt{h}^{s_0}$ to $\mathtt{h}^{s_0}$. We ask whether this change of variables is more regular, for instance bounded from $\mathtt{h}^s$ to itself.
This type of result is crucial for application to PDEs and typically is achieved by requiring that P_0 is bounded in a stronger norm with respect to the operator one. For example, in [BBM14] the authors work in the decay norm (see definition 1.2.1). Of course, working in a strong norm imposes restriction in the class of applications. To avoid this, a strategy is to use the setting of Modulo-tame operators (see 1.3.1). This idea has been implemented for instance for the Water Waves and Degasperis-Procesi equations ([BBHM18], [FGP19]). In the next Theorem we introduce norms that are weaker then the decay ones and behaves essentially as the Modulo-tame constants.
To do that we need to define a functional structure.

Denoting by $e^{(k)}$ the standard orthonormal basis of $\ell^2(\mathbb{Z}^d) = \mathtt{h}^0(\mathbb{Z}^d)$ (namely $e^{(k)}_{k'} := \delta_{k,k'}$), we may identify an operator with the matrix coefficients $M_k^{k'} := Me^{(k')} \cdot e^{(k)}$, where $\cdot$ denotes the ℓ^2-scalar product.
Given an infinite matrix M we define its *majorant matrix* $\underline{M}$ as

$$(\underline{M})_k^{k'} := |M_k^{k'}|\,.$$

then we define the space of *bounded majorant linear operators* as

$$\mathcal{M}(\mathtt{h}^s, \mathtt{h}^{s'}) := \{M \in \mathcal{L}(\mathtt{h}^s, \mathtt{h}^{s'}) \ \text{ s.t. } \ |M|_{s,s'} < \infty\} \,,$$

where

$$|M|_{s,s'} := \|\underline{M}\|_{s,s'} \,. \tag{3}$$

is called *majorant operator norm.*
In Proposition 1.1.9 we provide an embedding[3] of the bounded linear operators in the bounded majorant linear operators. Hence we could also state the following Theorem with the operator norm instead of the majorant norm. We prefer to use the second one to have a more readable statement.

Fix s_0, s_1 such that $s_1 > s_0 > \frac{d}{2}$. We introduce a new space E_s of couples (M, R) of linear operators such that

$$M \in \bigcap_{s_0 < p < s_1} \mathcal{M}(\mathtt{h}^p, \mathtt{h}^p) \,, \quad R \in \mathcal{M}(\mathtt{h}^{s_0}, \mathtt{h}^s) \quad s_0 < s < s_1 \,.$$

On this space we define a norm $\| \cdot \|_{s,s_0,s_1}$ that it is essentially the natural norm on the product space defined above (for a precise definition see (1.3.2)). Such norm can be used to control the majorant norm (or, since Proposition 1.1.9 hold, the operator) of both the operators of the decomposition. We call "low norm" the norm $\| \cdot \|_{s_0,s_0,s_1}$ (namely when we only require that R is bounded as operator from $\mathtt{h}^{s_0}$ to itself).
Moreover, as in the previous Theorem, we need to control the Lipschitz dependence of the operators. With this purpose, we define a Lipschitz weighted norm $\| \cdot \|^{\gamma,\mathcal{O}}_{s,s_0,s_1}$.

Given an operator $L \in \bigcap_{s_0 < p < s_1} \mathcal{M}(\mathtt{h}^p, \mathtt{h}^p)$ we can always decompose it in a couple (M_L, R_L) that is an element of E_s in such way that $M_L + R_L = L$. One can show that $\| \cdot \|^{\gamma,\mathcal{O}}_{s,s_0,s_1}$ controls the norm of L in a similar way to the Modulo-tame constants (see Remark 1.3.10). This decomposition is not unique, for instance one can trivially associate to L the couples $(L, 0)$ and $(L - \partial_{\theta_1}^{-s_1}, \partial_{\theta_1}^{-s_1})$. The interesting point is to find a "good" representation of L.
After building all this new functional setting we obtain this second result:

Theorem 2. *Fix s_0, s_1 such that $s_1 > s_0 > \frac{d}{2}$. Fix γ, τ and b_1 as in Theorem 1 and $b_2 > b_1$ sufficiently large. Consider an operator L as in (2) satisfying the same hypotheses of Theorem 1 and consider a couple $(M_{P_0}, R_{P_0}) \in E_s$ such that $P_0 = M_{P_0} + R_{P_0}$. Suppose that for some $s_0 < s < s_1$*

$$\| (M_{P_0}, R_{P_0}) \|^{\gamma,\mathcal{O}_0}_{s,s_0,s_1} \,,\ \| ((\mathrm{ad}\partial_{\theta_i})^{b_2} M_{P_0}, (\mathrm{ad}\partial_{\theta_i})^{b_2} R_{P_0}) \|^{\gamma,\mathcal{O}_0}_{s,s_0,s_1} < \infty \quad \text{for all } 1 < i < d. \tag{4}$$

There exists $\epsilon_{\star\star} < \epsilon_\star$ such that if

$$\| (M_{P_0}, R_{P_0}) \|^{\gamma,\mathcal{O}_0}_{s_0,s_0,s_1} \leq \epsilon_{\star\star}$$

[3]One needs to require that the operator maps $\mathtt{h}^s$ in $\mathtt{h}^{s+\beta}$ for some β or control a norm of commutators with the derivative ∂_θ.

then for all $\xi \in \mathcal{C}$ of the Theorem 1 the change of variable U that diagonalize L maps $\mathtt{h}^s$ in itself. Moreover, one can control $|U|_{s,s}$ with the norm $\|(M_{P_0}, R_{P_O})\|_{s,s_0,s_1}$.

Applications

As explained before, a source of interest of the diagonalization theorem is its application in KAM theory. In this context we are interested in reducibility, namely the possibility to find a change of variables quasi periodic in time (with frequency vector $\omega \in \mathbb{R}^{d-1}$) that reduce in constant coefficient a linear PDE. The role of the external parameters ξ is assumed by the frequency ω. We will impose some arithmetic assumption on the frequency ω, for instance we will ask that it is diophantine.

Definition 1. ω is a $(\gamma, \tau)-$diophantine frequency, if for γ, τ positive constant the following bound holds

$$|\omega \cdot l| \geq \frac{\gamma}{|l|^\tau} \qquad \forall\, l \in \mathbb{Z}^{d-1} \setminus \{0\} \tag{5}$$

In order to illustrate the connection between the diagonalization and the reducibity, consider a partial differential equation of the form

$$u_t(x,t) = \mathcal{A}(\omega t) u(x,t) \tag{6}$$

where $\mathcal{A}$ is a linear operator (which in our example is typically unbounded) quasi-periodic in time with diophantine frequency vector $\omega \in \mathbb{R}^{d-1}$ namely $\mathcal{A}(\omega t) = \mathcal{A}(\varphi)$ with $\varphi \in \mathbb{T}^{d-1}$. Recalling that we identify the function space $H^s(\mathbb{T})$ with the corresponding sequence space $\mathtt{h}^s(\mathbb{Z})$, we assume that $\mathcal{A}$ is a sufficiently regular map from $\mathbb{T}^{d-1}$ to $\mathcal{M}(\mathtt{h}^s(\mathbb{Z}), \mathtt{h}^s(\mathbb{Z}))$ $(\varphi \mapsto \mathcal{A}(\varphi))$. We look for quasi-periodic solutions, namely functions $u(x,t) = v(\omega t, x)$ with $v(\varphi, x)$ that solves

$$\mathcal{L} v = (\omega \cdot \partial_\varphi + \mathcal{A}(\varphi)) v = 0\,. \tag{7}$$

We can associate to $\mathcal{A}$ a Töplitz in time (see the definition 1.2.8) operator A acting on $\mathcal{M}(\mathtt{h}^s(\mathbb{Z}^d), \mathtt{h}^s(\mathbb{Z}^d))$.
In this way we associate to $\mathcal{L}$ a linear operator L acting on $\mathtt{h}^s(\mathbb{Z}^d)$ that is not bounded. In Section 3.1, we shall prove that reducing the operator $\mathcal{A}$ is equivalent to diagonalizing the linear operator L via a bounded change of variables with the special property that it is Töplitz in time. This fact is proved in Corollary 2.1.7 under some assumptions on the operator $L = \Lambda + P$. In particular, we ask that Λ is diagonal with eigenvalues $\lambda_k = \lambda_{(l,j)}$ (with $k = (l,j) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$) that are linear in l and that the perturbation P is Töplitz in time.
In conclusion, if we can diagonalize then we can prove the existence of a change of variables quasi periodic in time that reduces to constant coefficient this class of PDEs.
It is important to observe that is not obvious that the change of variables is Töpliz in time and that in general, if the change of variables is not Töpliz in time, is not clear whether reducibility and diagonalization are equivalent.

A concrete application that we will consider in this Thesis is the Airy Equation:

$$u_t + u_{xxx} + V(\omega t, x)u_x = 0, \quad x \in \mathbb{T} \tag{8}$$

The (7) becomes:

$$\mathcal{L}(u) := \Big(\omega \cdot \partial_\varphi + \partial_{xxx} + \varepsilon V(\varphi, x)\partial_x\Big)u = 0, \tag{9}$$

$$V \in H^{s_0} \equiv H^{s_0}(\mathbb{T}^d) := \left\{ u = \sum_{k\in\mathbb{Z}^d} u_k e^{\mathrm{i}k\cdot(\varphi,x)} \ : \ |u|_s^2 := \sum \langle k \rangle^{2s} |u_k|^2 < \infty \right\},$$

where $\langle k \rangle := \max\{|k|, 1\}$.
Observing that in this operator the perturbation is not bounded, we do a well-known change of variables (Iooss-Plotnikov and Toland, [BBM14]) that put it in the form:

$$\omega \cdot \partial_\varphi + \partial_{xxx} + m\partial_x + f(\varphi, x) + g(\varphi, x)\partial_x^{-1} + \text{l.o.t.}$$

After this pseudo-differential transformation, the operator assumes the form $\mathcal{L} = \Lambda + \mathcal{P}$ with $\mathcal{L} = \omega \cdot \partial_\varphi + \partial_{xxx} + m\partial_x$ and $\mathcal{P} = f(\varphi, x) + g(\varphi, x)\partial_x^{-1} + \text{l.o.t.}$. We represent $\mathcal{L}, \mathcal{P}$ as matrices acting on $\mathtt{h}^s(\mathbb{Z}^d)$, denoted respectively as L, P.
The perturbation P is bounded so we can diagonalize it applying Theorem 1 and imposing non-resonance conditions on $\omega \in \mathbb{R}^{d-1}$. We obtain the following result:

Assume that $\omega \in \mathcal{O}$, with $\mathcal{O}$ compact set of $\mathbb{R}^{d-1}$ contained in the set of $(\gamma, \tau)-$diophantine frequencies (see (5)).

Theorem 3. *Fix $b > 4\tau + 2$, $s_0 > \frac{d}{2}$. There exist $\widetilde{s} = \widetilde{s}(s, \tau) > s_0$ and r_0 such that for all $r \le r_0$ the following hold:*
If

$$V \in B_r(H^{\widetilde{s}}) \cap H^{\widetilde{s}+b}$$

then there exists a lipschitz family

$$d_j^\infty(\omega) = \mathrm{i}(j^3 - mj) - r_j^\infty(\omega) \tag{10}$$

with

$$|m| + |r_j^\infty(\omega)| \lesssim_{s_0} |V|_{\widetilde{s}}^{\gamma,\mathcal{O}} \tag{11}$$

and a set $\mathcal{G} \subset \mathcal{O}$ with

$$|\mathcal{O} \setminus \mathcal{G}| \lesssim_{s_0} \gamma \tag{12}$$

such that for $\omega \in \mathcal{G}$ there exists a map $\Phi : H^{s_0} \to H^{s_0}$ invertible, bounded and such that u solves (8) if and only if $z := \Phi^{-1}u$ solves $z_t = \mathrm{diag}(d_j^\infty)z$.

Reducibility results can be used in order to control the time evolution of the Sobolev norm of solutions of (6). To do that we have to require further conditions on the operator.

Consider the space of *real* functions

$$Z := \{u(\varphi, x) = \overline{u(\varphi, x)}\},$$

and of even (in space-time), respectively odd, functions

$$X := \{u(\varphi, x) = u(-\varphi, -x)\} \qquad Y := \{u(\varphi, x) = -u(-\varphi, -x)\} \tag{13}$$

Definition 2. An operator R is said *real* if $R : Z \to Z$ and *reversible* if $R : X \to Y$.

If the operator L is Hamiltonian or Reversible, a Dynamical consequence of Theorem 3 is a stability result:

Lemma 3. For any $s \geq \frac{1}{2}$ and $u_0 \in H^{s_0}(\mathbb{T})$, the unique solution of the equation (6) with initial datum $u(x, 0) = u_0(x)$ satisfies the estimate $\|u(\cdot, t)\|_{H^{s_0}(\mathbb{T})} \leq \|u_0\|_{H^{s_0}(\mathbb{T})}$ uniformly w.r. to $t \in \mathbb{R}$

To prove this consequence, we show that the equation satisfies the hypotheses of the Theorem 1 and therefore the operator L can be conjugated to a diagonal one. Moreover the hypothesis on the Hamiltonian/Reversible perturbation guarantees that the final eigenvalues are purely imaginary and then we have the stability.

Another aim of this thesis is to provide an application of the Theorem 2. In principle we could also apply the second Theorem to the Airy equation; This was done in [BBM14] using the Decay norm. We preferred to use a more interesting example where we do not know how to estimate the perturbation in decay norm.

For $\alpha \in H^s(\mathbb{T}^d)$ denote $\vec{\alpha} := (\alpha, 0, \ldots 0) \in \mathbb{R}^d$ and consider the linear operator

$$\mathtt{C}_\alpha u(\theta) := u(\theta + \vec{\alpha}(\theta)) .$$

whose matrix representation is given by

$$(\mathtt{C}_\alpha)_k^{k'} = \left(\widehat{e^{ik' \cdot \vec{\alpha}(\theta)}}\right)_{k-k'} \tag{14}$$

where $\hat{g}_h$ is the h^{th} Fourier coefficient of the function g.
Let us consider the equation

$$u_t + u_{xxx} + (\mathtt{C}_\alpha \partial_x^{-N})u = 0 \tag{15}$$

As before assume that ω is $(\gamma, \tau)-$diophantine. In the Section 3.3 we prove the following result:

Theorem 4. *Fix $b > 4\tau + 2$, $s_0 > \frac{d}{2}$ and $s_1 > s_0$. There exist $\widetilde{s} = \widetilde{s}(s_0, \tau) > s_0$, r_0 and $N_0 = N_0(b)$ such that for all $r \leq r_0$ and $N \geq N_0$ the following hold:*
If

$$\alpha \in B_r(H^{\widetilde{s}}) \cap H^{s_1+b}$$

then there exist a lipschitz family $d_j^\infty(\omega) = \mathrm{i}j^3 - r_j^\infty(\omega)$ *with*

$$|r_j^\infty(\omega)| \lesssim_{s_0} |\alpha|_{\tilde{s}}^{\gamma,\mathcal{O}} \tag{16}$$

and a set $\mathcal{G}$ *with*

$$|\mathcal{O} \setminus \mathcal{G}| \lesssim_{s_0} \gamma \tag{17}$$

such that for $\omega \in \mathcal{G} \subset \mathcal{O}$ *there exist the map* $\Phi : H^{s_0} \to H^{s_0}$ *invertible, bounded and such that* u *solves* (15) *if and only if* $z := \Phi^{-1}u$ *solves* $z_t = \mathrm{diag}(d_j^\infty)z$.

Some Literature

The behaviour of the spectrum of linear operators on Hilbert spaces under perturbation is of course an interesting problem in itself. Many authors studied what happens to the spectrum of a linear operator after a perturbation. For instance, a theorem by Weyl-von Neumann ensures that any selfadjoint operator (in a separable Hilbert space) can be perturbed by a (arbitrary small) compact (Weyl 1909) or Hilbert-Schmidt (von Neumann 1935) selfadjoint operator so that its spectrum becomes pure point (see [K66]). Also counterexamples are studied, for example see in [TW85].
In this Thesis however our motivation in the choice of hypotheses, different from the theorems that we quote above, is due to the possible application to PDEs.
There are two main contexts that we have in mind:
– The control over time of the Sobolev norms of solutions of linear PDEs whose coefficients depend quasi periodically on time;
– The search of quasi periodic solutions for non linear PDEs on a compact manifold.
In both contexts the first results were on PDEs whose perturbation (respectively the nonlinear part) does not contain derivatives. Recently, thanks to the development of new techniques from pseudo-differential calculus, there have been a series of new results covering also unbounded perturbation; We give an example of these methods in the section 3.2.1.
Let us give a brief description of some of the relevant results.
One of the pioneers in the study of the problem of control of Sobolev norms was Bourgain, that proved an upper bound of the Sobolev norm for Linear Schrödinger operators, both in the case of quasi-periodic bounded potentials with a Diophantine frequency [Bou99a] and for general time dependent potentials [Bou99b]. His result was developed by many autors, among them [D10, BGMR17, BBHM18, M18, M19, M21].
In parallel to the study of the growth of the Sobolev norm there is the study the reducibility of equations with quasi-periodic in time coefficient. This is a key argument in KAM for PDEs. The first results in KAM theory for PDEs are due to Kuksin [K87] , Wayne [W90], Craig-Wayne [CW93], Bourgain [Bou94] and Pöschel [P96], for semi-linear 1-dimensional PDEs with no derivatives in the nonlineariy, later extended for unbounded nonlinearities (see for instance [K98], [Bou99], [LY03], [KP03], [BBP13]. and references therein).
More recently, KAM theory has been developed for quasi-linear equations, namely PDEs

whose nonlinearity contains the same number of derivatives appearing in the linear part. quasi-linear PDEs arise from various physical scenarios such as fluid dynamics and quantum mechanics.
In this case to apply a KAM/Nash Moser scheme one has to deal with the combined problem of small divisor and of the presence of derivative in the nonlinearity. Indeed, the linearized operator is an unbounded perturbation of a diagonal operator and its invertibility is harder to prove since the perturbative effects are stronger.
We mention a series of papers of Ioss-Plotnikov-Toland [IPT05] which constructed *periodic* solution for the 2d Water Waves equation. Later, Baldi-Berti-Haus-Montalto [BBHM18] proved the existence of quasi-periodic solutions for the 2d Water Waves by extending techniques introduced in KdV.
The breakthrough idea developed in the paper mentioned above is to combine tools from microlocal analysis, like pseudo-differential operators, with the classical iterative scheme. In other words, before applying the scheme, one has to perform a pseudo-differential reduction of the operator to an other which is a regularizing perturbation of a diagonal operator.

The difficulties of implementing KAM theory drastically increase when consider equation posed on higher space dimension due to the presence of much stronger resonance phoenomena.
For instance, for KAM thoery most of the existent result regards PDEs with no derivatives in the nonlinearity (Bourgain [Bou98, Bou05], Eliasson-Kuksin [EK09, EK10], Procesi-Procesi [PP15], Berti-Corsi-Procesi [BCP15]). The extension to the unbounded case is really much more recent and involves pseudo-differential calculus and equation with "special structure". We refer for instance to the reducibility results of [BGMR18], [BLM19], [FGMP19], [FG20], [M17], [M19] and the KAM result for Kirkhoff and Euler equations in [CM18] and [BM20].

Ringraziamenti

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Chapter 1

Functional setting

In this chapter we introduce some spaces that we will use along the proofs of chapters 2 and 3. In particular, we will work on scales of Sobolev spaces on the torus $H^s(\mathbb{T}^d)$ and with operators that are linear and bounded between these spaces.
In the first Section we will introduce a first type of operators, the *majortant operators*, that guarantees us a more strong structure. We will also introduce an operator that will behave like the commutator with the derivative.
In the second Section we will introduce the *Decay norm* that we will use in the application on the Airy equation (see the Section 3.2). We will see that this norm satisfies an important property that we will call *tameness*. Then we will generalize this property in the third Section, defining a general class of *"tame" operator*. We will use this space for the second KAM Theorem.

1.1 Space of sequences and majorant operators

We work in the setting of Lipschitz families of linear operators on the scale of Hilbert spaces:

$$\mathtt{h}^s \equiv \mathtt{h}^s(\mathbb{Z}^d) := \left\{ \{u_k\}_{k\in\mathbb{Z}^d} \ : \quad |u|_s^2 := \sum \langle k\rangle^{2s}|u_k|^2 < \infty \right\}, \qquad \langle k\rangle := \max\{|k|, 1\}\,.$$

Remark 1.1.1. *In our notation $\mathtt{h}^s(\mathbb{Z}^d)$ is a sequence space contained in $\ell_2(\mathbb{Z}^d, \mathbb{C})$. Of course working of sequence spaces is equivalent to the usual functions space definitions, provided that one works in the Fourier basis $\left(e^{\mathrm{i}k\cdot\theta}\right)_{k\in\mathbb{Z}^d}$. This gives the identification*

$$(u_k)_{k\in\mathbb{Z}^d} \ \leftrightsquigarrow \ u(\theta) = \sum_{k\in\mathbb{Z}^d} u_k e^{\mathrm{i}\theta\cdot k}$$

With this notation the norm $|\cdot|_s$ is equivalent to the usual Sobolev norm on functions.

Remark 1.1.2. *If $s > \frac{d}{2}$ the norm $|\cdot|_s$ satisfies the algebra and the tameness properties.*

Following the notations on function spaces, given $u, v \in \mathtt{h}^s(\mathbb{Z}^d)$ we denote by uv the convolution of the sequences $(uv)_k := \sum_{h\in\mathbb{Z}^d} u_h v_{k-h}$.

– *for* $u, v \in \mathtt{h}^s$ $\quad |uv|_s \leq |u|_s |v|_s$

– *for* $s_1 > s > \frac{d}{2}$ *and for* $u, v \in \mathtt{h}^s$ $\quad |uv|_{s_1} \leq c(s)|u|_{s_1}|v|_s + c(s_1)|u|_s|v|_{s_1}$, *the constants can be explicitly computed (see [BBM14] and reference therein).*

Denoting by $e^{(k)}$ the standard orthonormal basis of $\ell^2(\mathbb{Z}^d) = \mathtt{h}^0(\mathbb{Z}^d)$ (namely $e^{(k)}_{k'} := \delta_{k,k'}$), we may identify an operator with the matrix coefficients $M_k^{k'} := Me^{(k')} \cdot e^{(k)}$, where $\cdot$ denotes the ℓ^2-scalar product. In the space $\mathcal{L}(\mathtt{h}^s, \mathtt{h}^{s'})$ of bounded linear operators from $\mathtt{h}^s$ to $\mathtt{h}^{s'}$ we use the standard operator norm

$$\|M\|_{s,s'} := \sup_{|u|_s \leq 1} |Mu|_{s'} \,.$$

Lemma 1.1.3. *Let* $A \in \mathcal{L}(\mathtt{h}^{s+\beta}, \mathtt{h}^{s+\beta})$ *such that* $A\partial_x^\beta \in \mathcal{L}(\mathtt{h}^{s+\beta}, \mathtt{h}^{s+\beta})$*, then A is smoothing and* $A \in \mathcal{L}(\mathtt{h}^s, \mathtt{h}^{s+\beta})$*. Moreover*

$$\|A\|_{s,s+\beta} = \|A\partial_x^\beta\|_{s+\beta,s+\beta} + \|A\|_{s+\beta,s+\beta} \tag{1.1.1}$$

Definition 1.1.4. Given an infinite matrix M we define its majorant matrix $\underline{M}$ as[1]

$$(\underline{M})_k^{k'} := |M_k^{k'}| \,.$$

Definition 1.1.5. We define the space of bounded majorant linear operators as

$$\mathcal{M}(\mathtt{h}^s, \mathtt{h}^{s'}) := \{M \in \mathcal{L}(\mathtt{h}^s, \mathtt{h}^{s'}) \ \text{ s.t. } \ |M|_{s,s'} < \infty\} \,,$$

where

$$|M|_{s,s'} := \|\underline{M}\|_{s,s'} \,.$$

is called majorant operator norm.

Remark 1.1.6. *One has that*

$$|M|_{s,s'} \geq \|M\|_{s,s'} \,.$$

Moreover $\mathcal{M}(\mathtt{h}^s, \mathtt{h}^{s'})$ *endowed with the norm* $|\cdot|_{s,s'}$ *is a Banach space.*

We have a partial ordering relation i.e.

$$M \preccurlyeq N \ \Leftrightarrow \ \underline{M} \preccurlyeq \underline{N} \ \Leftrightarrow \ |M_k^{k'}| \leq |N_k^{k'}| \ \ \forall k, k' \,.$$

Note that

$$M \preccurlyeq N \implies |M|_{s,s'} \leq |N|_{s,s'} \,.$$

and that

$$\underline{MN} \preccurlyeq \underline{M}\ \underline{N} = \underline{\underline{M}\ \underline{N}} \,.$$

This implies that $\mathcal{M}(\mathtt{h}^s, \mathtt{h}^s)$ is a Banach algebra, namely

$$|MN|_s \leq |M|_s |N|_s \,.$$

[1] This is defined in terms of a preferred basis, say the standard ℓ^2-basis above, on the other hand the other definitions are intrinsic.

Definition 1.1.7. For every $1 \le h \le d$, we define the operator $\mathtt{d}_h$:

$$(M)_k^{k'} \to (\mathtt{d}_h M)_k^{k'} := \mathrm{i}(k_h - k'_h) M_k^{k'}$$

and the operator $\langle \mathtt{d}_h \rangle$

$$(M)_k^{k'} \to (\langle \mathtt{d}_h \rangle M)_k^{k'} := \langle k_h - k'_h \rangle M_k^{k'}$$

Remark 1.1.8. *i.* $\mathtt{d}_h M = -\mathrm{ad}(\partial_h) M = [\partial_h, M]$[2], $\qquad \partial_h := \mathrm{diag}_{\mathrm{k} \in \mathbb{Z}^{\mathrm{d}}}\ \mathrm{i}\, k_h$.

ii. Note that it is in general not true that $\mathtt{d}_h$ *maps bounded matrices into bounded matrices.*

iii. Note that $\mathtt{d}_h$ *satisfies the Leibniz rule*

$$\mathtt{d}_h(MN) = \mathtt{d}_h(M)N + M\mathtt{d}_h(N).$$

iv. Note that $\underline{M} \preceq \langle \mathtt{d} \rangle \underline{M} = \underline{\langle \mathtt{d} \rangle M}$

v. Note that $\underline{\langle d \rangle A} \preceq \underline{A} + \sum_h \underline{\mathtt{d}_h A}$

Let us finally denote by $\mathtt{d}_h^b$ the composition of $\mathtt{d}_h$ with itself b times

$$(\mathtt{d}_h^b M)_k^{k'} = \mathrm{i}^b (k_h - k'_h)^b M_k^{k'}.$$

and in the same way

$$(\langle \mathtt{d}_h \rangle^b M)_k^{k'} = \langle k_h - k'_h \rangle^b M_k^{k'}$$

iv and *v* of remark 1.1.8 hold also for $\mathtt{d}_h^b$.
We now state a crucial result that allows to move from the majorant norm to the operator norm. The proof of that result is the Appendix.

Proposition 1.1.9. *(L. Biasco) Let* $A, \mathtt{d}_h^\beta A \in \mathcal{L}(\mathtt{h}^s, \mathtt{h}^{s'})$ *for every* $1 \le h \le d$*, and*

$$\beta := [d/2] + 1,$$

then $A \in \mathcal{M}(\mathtt{h}^s, \mathtt{h}^{s'})$ *(i.e.* $\underline{A} \in \mathcal{L}(\mathtt{h}^s, \mathtt{h}^{s'})$*) and*

$$|A|_{s,s'} \le \|A\|_{s,s'} + \ c_d \sum_{1 \le h \le d} \|\mathtt{d}_h^\beta A\|_{s,s'} \le C_d(\|A\|_{s,s'+\beta} + \|A\|_{s-\beta,s'}), \tag{1.1.2}$$

for a suitable $c_d, C_d > 1$.

Lemma 1.1.10. *For* $b, n \ge 1$ *and* $1 \le h \le d$*, we have the following estimates*

$$|\mathrm{ad}(A)^n B|_{s,s} \le 2^n |A|_{s,s}^n |B|_{s,s}, \tag{1.1.3}$$

$$|\langle \mathtt{d} \rangle^b [A,B]|_{s,s} \le 2^{b+1}(|\langle \mathtt{d} \rangle^b A|_{s,s} |B|_{s,s} + |A|_{s,s} |\langle \mathtt{d} \rangle^b B|_{s,s}), \tag{1.1.4}$$

$$|\langle \mathtt{d} \rangle^b \mathrm{ad}(A)^n B|_{s,s} \le 2^{n(b+1)}(n|\langle \mathtt{d} \rangle^b A|_{s,s} |A|_{s,s}^{n-1} |B|_{s,s} + |A|_{s,s}^n |\langle \mathtt{d} \rangle^b B|_{s,s}) \tag{1.1.5}$$

[2] We denote $\mathrm{ad}(A)B := [B, A]$.

Proof. We have that

$$|[A,B]|_{s,s} \le 2|A|_{s,s}|B|_{s,s}\,,$$

then (1.1.3) follows by induction over n. (1.1.4) follows by

$$|k_h - k'_h|^b |\sum_j A^j_k B^{k'}_j| \le \sum_{j:|k_h-j_h|\le|j_h-k'_h|} (2|j_h-k'_h|)^b |A^j_k||B^{k'}_j| + \sum_{j:\,|k_h-j_h|\ge|j_h-k'_h|} (2|k_h-j_h|)^b |A^j_k||B^{k'}_j|.$$

Finally (1.1.5) follows by induction over n, (1.1.3) and (1.1.4). □

Definition 1.1.11. We define the projections[3]

$$(\Pi_K M)^{k'}_k := \begin{cases} M^{k'}_k & \text{if} \quad |k-k'|\le K \\ 0 & \text{otherwise} \end{cases} \qquad \Pi^\perp_K = \mathbb{I} - \Pi_K\,.$$

Note that

$$\Pi_K M,\ \Pi^\perp_K M \ \preccurlyeq\ M\,. \tag{1.1.6}$$

Lemma 1.1.12. *Let $K \in \mathbb{N}$, then*

$$|\Pi^\perp_K M|_{s,s'} \le K^{-1} |\mathtt{d} M|_{s,s'}\,.$$

Proof. We have that

$$\Pi^\perp_K M \preccurlyeq K^{-1} \sum_{h=1}^d \underline{\mathtt{d}_h M}\,.$$

□

Lipschitz families of matrices Given $\xi \in \mathcal{O}$ compact set of $\mathbb{R}^n$, we consider a Lipschitz family $\mathcal{O} \ni \xi \mapsto M(\xi)$ of bounded operators and for $\gamma \ge 0$,.

Definition 1.1.13. We define the Lipschitz norm

$$|M|^{\mathrm{lip},\mathcal{O}}_{s,s'} := \sup_{\xi\neq\eta\in\mathcal{O}} \frac{|M(\xi)-M(\eta)|_{s,s'}}{|\xi-\eta|}$$

and the weighted Lipschitz norms

$$|M|^{\gamma,\mathcal{O}}_{s,s'} := \sup_{\xi\in\mathcal{O}} |M(\xi)|_{s,s'} + \gamma \sup_{\xi\neq\eta\in\mathcal{O}} \frac{|M(\xi)-M(\eta)|_{s,s'}}{|\xi-\eta|}\,.$$

By $\|\cdot\|^{\gamma,\mathcal{O}}_{s,s'}$ we shall define the weighted Lipschitz norm corresponding to the usual operator norm $\|\cdot\|_{s,s'}$.

Remark 1.1.14. *By definition we immediately have that the weighted Lipschitz norm satisfies the same bounds of Lemma 1.1.10.*

[3] This is an *intrinsic* definition.

Lemma 1.1.15. *(Monotonicity of the norm $|\cdot|_{s,s'}^{\frac{\gamma}{M},\mathcal{O}}$).*

$$\begin{aligned} M_1 > M_2 \quad &\Longrightarrow \quad |A|_{s,s'}^{\frac{\gamma}{M_1},\mathcal{O}} < |A|_{s,s'}^{\frac{\gamma}{M_2},\mathcal{O}} \\ \gamma_1 > \gamma_2 \quad &\Longrightarrow \quad |A|_{s,s'}^{\frac{\gamma_1}{M},\mathcal{O}} > |A|_{s,s'}^{\frac{\gamma_1}{M},\mathcal{O}} \\ \mathcal{O}_1 \subset \mathcal{O}_2 \quad &\Longrightarrow \quad |A|_{s,s'}^{\frac{\gamma}{M},\mathcal{O}_1} > |A|_{s,s'}^{\frac{\gamma}{M},\mathcal{O}_2} \end{aligned}$$

As the same way we define the weighted lipschitz norms for lipiscitz functions $f(\xi) \in \mathtt{h}^s$

$$|f|_s^{\gamma,\mathcal{O}} := \sup_{\xi\in\mathcal{O}} |f(\xi)|_{s,s'} + \gamma \sup_{\xi\neq\eta\in\mathcal{O}} \frac{|f(\xi)-f(\eta)|_{s,s'}}{|\xi-\eta|}\,.$$

1.2 Decay Norm

We now introduce a special class of linear operators with "off-diagonal" decay. We shall prove that this property guarantees the boundness in majorant norm and some "tame estimates".

This space of operators is defined by the **s-*decay norm*** defined as follow:

$$|A|_s^{\mathrm{dec}} := \left(\sum_{h\in\mathbb{Z}^d} \langle h\rangle^{2s} \sup_{i-j=h} |A_i^j|^2 \right)^{1/2}\,. \tag{1.2.1}$$

Note that

$$|\underline{A}|_s^{\mathrm{dec}} = |A|_s^{\mathrm{dec}}\,.$$

In the following Lemma we prove that the decay norm control the majorant norm introduced in the previous Section.

Lemma 1.2.1. *For $A \in \mathcal{M}(\mathtt{h}^s,\mathtt{h}^s)$ we have that*

$$|A|_{s,s} \leq C(s)|A|_s^{\mathrm{dec}}\,. \tag{1.2.2}$$

Proof. Let $\mathtt{a} \in \mathtt{h}^s$ with $\mathtt{a}_h := \sup_{i-j=h}|A_i^j|$. Note that $|\mathtt{a}|_s = |A|_s^{\mathrm{dec}}$. Given $u \in \mathtt{h}^s$, define $\underline{u} \in \mathtt{h}^s$ through $\underline{u}_h := |u_h|$. Note that $|\underline{u}|_s = |u|_s$. Then

$$|Au|_s^2 = \sum_i \langle i\rangle^{2s} \left| \sum_j A_i^j u_j \right|^2 \leq \sum_i \langle i\rangle^{2s} \left| \sum_j \mathtt{a}_{i-j}\underline{u}_j \right|^2 = |\mathtt{a}\underline{u}|_s^2 \leq (C(s)|\mathtt{a}|_s|\underline{u}|_s)^2$$

since $\mathtt{h}^s$ is a Banach algebra when $s > d/2$, see Remark 1.1.2. $\square$

Informally a matrix with bounded decay norm is a linear operator which can be "well approxiamted" by the multiplication operator $M_a : u \mapsto au$ according to the definitions of Remark 1.1.2. Using the Fourier series to identify $\mathtt{h}^s$ with H^s, the operator is $u(\theta) \mapsto a(\theta)u(\theta)$, where $a(\theta) \in H^s$ is the function associated to the sequence $a \in \mathtt{h}^s$.

Remark 1.2.2. *Note that by the reasoning above, given $v \in \mathtt{h}^s$ with $s > d/2$, the multiplication operator $M_v :\ u \mapsto vu$ is bounded in decay norm with*

$$|M_v|_s^{\mathrm{dec}} = |v|_s\,.$$

Let us define the Lipschitz decay norm

$$|M|_s^{\mathrm{dec,lip},\mathcal{O}} := \sup_{\xi\neq\eta\in\mathcal{O}} \frac{|M(\xi)-M(\eta)|_s^{\mathrm{dec}}}{|\xi-\eta|}$$

and the weighted Lipschitz decay norms

$$|M|_s^{\mathrm{dec},\gamma,\mathcal{O}} := \sup_{\xi\in\mathcal{O}} |M(\xi)|_s^{\mathrm{dec}} + \gamma \sup_{\xi\neq\eta\in\mathcal{O}} \frac{|M(\xi)-M(\eta)|_s^{\mathrm{dec}}}{|\xi-\eta|}\,.$$

The following is a technical Lemma with the properties of the s-decay norm. The proof can be found in the Appendix of [BBM14].

Lemma 1.2.3. *Let $s_0 > \frac{d}{2}$*

i. ***Alegebra*** $|MN|_{s_0}^{\mathrm{dec},\gamma,\mathcal{O}} \leq C(s_0)|M|_{s_0}^{\mathrm{dec},\gamma,\mathcal{O}}|N|_{s_0}^{\mathrm{dec},\gamma,\mathcal{O}}$ *for all $M, N \in \mathcal{M}(\mathtt{h}^{s_0}, \mathtt{h}^{s_0})$*

ii. ***Interpolation*** *For $s > s_0$ one has for all $M, N \in \mathcal{M}(\mathtt{h}^{s_0}, \mathtt{h}^{s_0})$*

$$|MN|_s^{\mathrm{dec},\gamma,\mathcal{O}} \leq C(s)|M|_{s_0}^{\mathrm{dec},\gamma,\mathcal{O}}|N|_s^{\mathrm{dec},\gamma,\mathcal{O}} + C(s_0)|M|_s^{\mathrm{dec},\gamma,\mathcal{O}}|N|_{s_0}^{\mathrm{dec},\gamma,\mathcal{O}}$$

iii. ***Smoothing*** *For $b > 0$ one has* $|\Pi_K^{\perp} M|_s^{\mathrm{dec},\gamma,\mathcal{O}} \leq K^{-b}|M|_{s+b}^{\mathrm{dec},\gamma,\mathcal{O}}$

iv. ***Tameness*** $|\langle \mathtt{d}\rangle^b A|_s^{\mathrm{dec}} \leq |A|_{s+b}^{\mathrm{dec}}$

Definition 1.2.4. We say that a scale of Banach algebras $(\mathcal{B}_s, |\cdot|_s)$ satisfies the (asymmetric) tameness product property, if for $s > s_0$

$$|AB|_s \leq C(s_0)|A|_s|B|_{s_0} + C(s)|A|_{s_0}|B|_s \quad \forall\, A, B \in \mathcal{B}.$$

Lemma 1.2.5. *Let's consider a Banach algebra $\mathcal{B}_s$ of operators with the (asymmetric) tameness product property as above.Then for all $k \geq 2$ one has*

$$|A^k|_{s_0} \leq (2C(s_0))^{k-1}|A|_{s_0}^k$$

$$|A^k|_s \leq kC(s)(2C(s_0))^{k-2}|A|_{s_0}^{k-1}|A|_s$$

Proof. Both follow by induction on k and from the properties of the decay norm. □

The tame product property has the following very important property w.r.t. totally convergent power series .

Lemma 1.2.6. *Under the hypotheses of the previous Lemma, given a complex sequence* $\{m_k\}_{k\in\mathbb{N}}$ *and* $A \in \mathcal{B}_s$ *with*

$$|A|_{s_0} < \frac{1}{2C(s_0)} \liminf_k |m_k|^{-\frac{1}{k}},$$

we have that there exists[4]

$$B := \sum_{k=0}^{\infty} m_k A^k .$$

Moreover

$$|B|_s \le |m_0| + C(s)|A|_s \sum_{k=1}^{\infty} |m_k| \; k (2C(s_0)|A|_{s_0})^{k-1} < \infty .$$

Lemma 1.2.7. *The spaces of linear operators* $\mathtt{h}^s \to \mathtt{h}^s$ *with bounded s-decay norm form a scale of Banach algebras with the tame product property, with* $C(s)$ *equal to the constants* $c(s)$ *in Remark 1.1.2.*

1.2.1 Töplitz in time Operator

In view of the application of the Theorem 3, we introduce a special class of operators which behave as multiplication operators w.r.t. the "time variables" $\varphi \in \mathbb{T}^{d-1}$.

Definition 1.2.8. We say that an operator P is Töplitz in time if for all $h \in \mathbb{Z}^{d-1}$ one has

$$P_k^{k'} = P_{(l,j)}^{(l',j')} = P_{(l+h,j)}^{(l'+h,j')} \qquad k = (l,j) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$$

or equivalently if it can be written in the form

$$P_k^{k'} = \mathcal{P}_j^{j'}(l - l'), \qquad k = (l,j) \in \mathbb{Z}^{d-1} \times \mathbb{Z} .$$

The two definitions are equivalent by just setting $\mathcal{P}_j^{j'}(l) := P_{(l,j)}^{(0,j')}$ for all $l \in \mathbb{Z}^{d-1}$, indeed $\mathcal{P}_j^{j'}(l - l') = P_{(l-l',j)}^{(0,j')} = P_{(l,j)}^{(l',j')}$ (by using the first definition with $h = l'$). In the following Lemma we provide an example of an operator töplitz both in space and time.

Lemma 1.2.9 (Multiplication operator)**.** *Let* $V = \sum_k V_k e_k$. *The multiplication operator* $h \mapsto Vh$ *is represented by the Töplitz matrix* $T_j^{j'} = V_{j-j'}$ *and*

$$|T|_s^{\mathrm{dec}} = |V|_s .$$

Moreover if, given $\xi \in \mathcal{O}$, $V(\xi)$ *is a Lipschitz family of functions, then*

$$|T|_s^{\mathrm{dec},\gamma,\mathcal{O}} = |V|_s^{\gamma,\mathcal{O}} .$$

[4]The series converges in the norm of $\mathcal{B}_s$ in the sense that there exists $\lim_{n\to\infty} \sum_{k=0}^{n} m_k A^k$.

An example of a Töplitz in time operator is then for instance an operator of the form TA where T is defined in the Lemma above, while A is "time independent" in the sense that $A_k^{k'}$ does not depend on the indexes l, l'.
Note that if $\mathcal{P}$ is a Töplitz in time operator with finite decay norm then:

$$|\mathcal{P}|_s^{\rm dec} = \left(\sum_{l\in\mathbb{Z}^{d-1}, j\in\mathbb{Z}} \langle l, j\rangle^{2s} \sup_{j_1 - j_2' = j} |\mathcal{P}_{j_1}^{j_2'}(l)|^2 \right)^{1/2} .$$

Remark 1.2.10. *We can identify the matrix P with a one-parameter family of operators acting on $\mathtt{h}^s(\mathbb{Z})$ depending on the angle φ, namely*

$$\mathcal{P}_j^{j'}(\varphi) = \sum_{h\in\mathbb{Z}^{d-1}} \mathcal{P}_j^{j'}(h) e^{ih\cdot\varphi}$$

This means that $\mathcal{P}$ is a map from $\mathbb{T}^{d-1}$ to $\mathcal{M}(\mathtt{h}^s(\mathbb{Z}), \mathtt{h}^s(\mathbb{Z}))$ where

$$\mathtt{h}^s(\mathbb{Z}) := \{(u_j)_{j\in\mathbb{Z}} : |u|_s^2 := \sum_{j\in\mathbb{Z}} \langle j\rangle^{2s} |u_j|^2 < \infty\} .$$

Conversely, if we consider a sufficiently regular map $\mathcal{A}$ from $\mathbb{T}^{d-1}$ to $\mathcal{M}(\mathtt{h}^s(\mathbb{Z}), \mathtt{h}^s(\mathbb{Z}))$, we can always associate to it a Töplitz in time operator A acting on $\mathcal{M}(\mathtt{h}^s(\mathbb{Z}^d), \mathtt{h}^s(\mathbb{Z}^d))$.

We have hence shown that the space of the (sufficiently regular) maps from $\mathbb{T}^{d-1}$ to $\mathcal{M}(\mathtt{h}^s(\mathbb{Z}), \mathtt{h}^s(\mathbb{Z}))$ is identified with the subspace of $\mathcal{M}(\mathtt{h}^s(\mathbb{Z}^d), \mathtt{h}^s(\mathbb{Z}^d))$. We denote this by $\mathcal{P} \leftrightsquigarrow P$

Moreover the following Lemma holds:

Lemma 1.2.11. *Consider P as above and define $N_s(\varphi) := |\mathcal{P}(\varphi)|_{s,s}$ to be the majorant operator norm of $\mathcal{P}(\varphi)$ on $\mathtt{h}^s(\mathbb{Z})$. One has*

$$\sup_{\varphi\in\mathbb{T}^{d-1}} |\mathcal{P}(\varphi)|_{s,s} \le |P|_s^{\rm dec}, \quad |N_s(\varphi)|_{H^{s_1}(\mathbb{T}^{d-1})} \le |P|_{s+s_1}^{\rm dec}$$

Note that if for every φ the operator $\mathcal{A}(\varphi)$ is invertible, then the associated operator A is invertible and $\mathcal{A}^{-1} \leftrightsquigarrow A^{-1}$.

Remark 1.2.12. *A notable property of this identification is behaviour with respect to commutator with time derivatives. For simplicity let us consider the angle φ_1 starting with $\mathcal{L} \leftrightsquigarrow L$. Define $N := [\partial_{\varphi_1}, L]$. An explicit computation give*

$$N_{(j,l)}^{(j',l')} = \mathrm{i}(l_1 - l_1') L_{(j,l)}^{(j',l')} = \mathrm{i}(l_1 - l_1') \mathcal{L}_j^{j'}(l - l')$$

Hence N is Töplitz in time and $N \leftrightsquigarrow \mathcal{N} = \partial_{\varphi_1}\mathcal{L}$. By linearity this holds for every linear combination of ∂_{φ_i}.

Lemma 1.2.13. *The family $\mathcal{T}$ of the Töplitz in time operator is an Algebra.*

Let us consider an operator of the form $\Lambda_0 + P_0$ with $\Lambda_0 = \mathrm{diag}\lambda_k^{(0)}$. With the Theorem 5 we prove that there exist a change of variables that diagonalize that operator, namely that there exists L such that in a certain set of parameters

$$U^{-1}(\Lambda_0 + P_0)U = \Lambda_\infty \qquad \Lambda_\infty := \mathrm{diag}\lambda_k^{(\infty)} .$$

In view of applications to PDE's we to prove that if the initial eigenvalues $\lambda_k^{(0)}$ are linear respect l[5] as in (10) and the perturbation is Töpliz in time then also the change of variables U is Töplitz in time. To this purpose we need the following technical Lemmata.

Lemma 1.2.14. *Let P be a Töplitz in time operator and let λ_k be linear in l. Then A, defined as*

$$A_k^{k'} := \frac{P_k^{k'}}{\lambda_k - \lambda_{k'}}$$

is a Töplitz in time operator.

Lemma 1.2.15. *Let P be a Töplitz in time operator. Then $\Pi_K P$ is Töplitz in time operator.*

Lemma 1.2.16. *Let P and Q be a Töplitz in time operators. Then $[P, Q]$ is Töplitz in time operator.*

Lemma 1.2.17. *Let P be a Töplitz in time operator. Then e^P is Töplitz in time operator.*

We omit the proof of this Lemmata since they easly follow by definition.

1.3 A general class of "tame" operators

Now we introduce a further class of operators that satisfies the tame property of Definition 1.2.4, just like the decay norm defined in (1.2.1) but containing a wider class of operators.
We will use this class to prove a diagonalization algorithm in "high norm", namely start from an operator $\Lambda_0 + P_0$ with Λ_0 diagonal and with simple eigenvalues and P_0 with some smallness conditions and assume that it is conjugated to a diagonal form by L, we study the relationship between the regularity of P_0 and the one of L. Note that if we know that P_0 is small in some norm $|\cdot|_s$ then the change of variable U is clearly also bounded in the same norm and the point of the property 1.2.4 is to prove the convergence of the algorithm in high norm by requiring only the smallness condition on the low norm $|\cdot|_{s_0}$. This kind of phenomena can be seen in a simple context, for instance in Lemma 1.2.6. In KAM algorithm to prove such results in high norm one either needs to control P_0 in decay norm or one introduce the class of modulo tame operators.

[5]Recall that we are splitting $k \in \mathbb{Z}^d$ as (l, j) with $l \in \mathbb{Z}^{d-1}, j \in \mathbb{Z}$

Definition 1.3.1. We say that a linear operator A is *tame* w.r.t. a non-decreasing sequence $\{\mathfrak{M}_A(s)\}_{s=s_0}^{s_1}$ if

$$\|Au\|_s \leq \mathfrak{M}_A(s)\|u\|_{s_0} + \mathfrak{M}_A(s_0)\|u\|_s \qquad u \in \mathtt{h}^s\,, \tag{1.3.1}$$

for any $s_0 \leq s \leq s_1$. We call $\mathfrak{M}_A(s)$ a TAME CONSTANT for the operator A. We say that a linear operator A is *modulo-tame* if $\underline{A}$ is tame.

Fix $s_0 > d/2$. For $s \geq s_0$ let us consider the Banach space

$$\mathcal{M}_{s_0,s} := \bigcap_{s_0 \leq p \leq s} \mathcal{M}(\mathtt{h}^p, \mathtt{h}^p) \quad \text{endowed with the norm} \quad |\cdot|_{\mathcal{M}_{s_0,s}} := \sup_{s_0 \leq p \leq s} |\cdot|_{p,p}\,.$$

Remark 1.3.2. *Note that in the definition 1.3.1 we can take as* $\mathfrak{M}_{\underline{A}}(s) = \sup_{s_0 \leq s \leq s_1} |A|_{s,s}$ *This means that all the operators in* $\mathcal{M}_{s_0,s_1}$ *are modulo tame.*

Since in KAM algorithm we need smallness condition on $\mathfrak{M}_{\underline{A}}(s_0)$, the idea is to look for the "best possible" constants.
The purpose of this Section is to define a class of operator that behave as the modulo tame operators but that form a Banach space.

Note that for $s' \geq s$ we have

$$\mathcal{M}_{s_0,s'} \subseteq \mathcal{M}_{s_0,s}\,, \qquad \text{with} \quad |\cdot|_{\mathcal{M}_{s_0,s}} \leq |\cdot|_{\mathcal{M}_{s_0,s'}}\,,$$

namely a scale of Banach spaces. Moreover every $\mathcal{M}_{s_0,s}$ is a Banach algebra

$$|M_1 M_2|_{\mathcal{M}_{s_0,s}} \leq |M_1|_{\mathcal{M}_{s_0,s}} |M_2|_{\mathcal{M}_{s_0,s}}\,.$$

Definition 1.3.3. Fix $s_1 > s_0$, for every $s \geq s_0$ we define the vector space[6] $E_s = E_{s,s_0,s_1}$ as the space whose elements are the couples

$$A = (M, R) \;:\quad M \in \mathcal{M}_{s_0,s_1}\,, \quad R \in \mathcal{M}(\mathtt{h}^{s_0}, \mathtt{h}^s)$$

with finite norm

$$|\!|\!|(M,R)|\!|\!|_s = |\!|\!|(M,R)|\!|\!|_{s,s_0,s_1} := \sup_{s_0 \leq p \leq s_1} |M|_{p,p} + |R|_{s_0,s}\,. \tag{1.3.2}$$

Remark 1.3.4. *Note that* E_{s,s_0,s_1}*, endowed with the above norm, are scales of Banach spaces:*

$$\begin{aligned}
s_0 \leq s_0' < s_1' \leq s_1\,,\; s_0' \leq s' \leq s \quad &\Longrightarrow \quad E_{s,s_0,s_1} \subseteq E_{s',s_0',s_1'}\,, \quad |\!|\!|\cdot|\!|\!|_{s',s_0',s_1'} \leq |\!|\!|\cdot|\!|\!|_{s,s_0,s_1}\,. \\
s < s' \quad &\Longrightarrow \quad |\!|\!|\cdot|\!|\!|_{s,s_0,s_1} \leq |\!|\!|\cdot|\!|\!|_{s',s_0,s_1}\,. \\
s_0 < s_0' \quad &\Longrightarrow \quad |\!|\!|\cdot|\!|\!|_{s,s_0,s_1} \geq |\!|\!|\cdot|\!|\!|_{s,s_0',s_1}\,. \\
s_1 < s_1' \quad &\Longrightarrow \quad |\!|\!|\cdot|\!|\!|_{s,s_0,s_1} \leq |\!|\!|\cdot|\!|\!|_{s,s_0,s_1'}\,.
\end{aligned}$$

[6] With sum $(M_1, R_1) + (M_2, R_2) := (M_1 + M_2, R_1 + R_2)$ and multiplication by a scalar $k(M,R) := (kM, kR)$.

Definition 1.3.5. We define the (associative but non commutative) product

$$(M_1, R_1) \star (M_2, R_2) := (M_1 M_2, M_1 R_2 + R_1 M_2 + R_1 R_2)\,.$$

We denote $A^k := A \star \cdots \star A$, k times, and $\mathbb{I} = (\mathbb{I}, 0)$ as the identity.

Lemma 1.3.6. *For $A_1 := (M_1, R_1)$, $A_2 := (M_2, R_2) \in E_s$, we get the tame product property*[7] *as in Definition 1.2.4 with constants $C(s) = 1$:*

$$s_0 \le s \le s_1 \qquad \Longrightarrow \qquad \| A_1 \star A_2 \|_s \le \| A_1 \|_{s_0} \| A_2 \|_s + \| A_1 \|_s \| A_2 \|_{s_0}\,.$$

Proof.

$$\| (M_1, R_1) \star (M_2, R_2) \|_s = \sup_{s_0 \le p \le s_1} |M_1 M_2|_{p,p} + |M_1 R_2 + R_1 M_2 + R_1 R_2|_{s_0,s}$$

$$|R_1|_{s_0,s} |R_2|_{s_0,s_0} + |R_1|_{s_0,s} |M_2|_{s_0,s_0} + |M_1|_{s,s} |R_2|_{s_0,s} + \sup_{s_0 \le p \le s_1} |M_1|_{p,p} |M_2|_{p,p}$$

$$\le \| (R_1, M_1) \|_{s_0} \| (R_2, M_2) \|_s + \| (R_1, M_1) \|_s \| (R_2, M_2) \|_{s_0}$$

□

In E_s we have a partial ordering

$$(M_1, R_1) \preccurlyeq (M_2, R_2) \;\Leftrightarrow\; M_1 \preccurlyeq M_2\,, \quad R_1 \preccurlyeq R_2\,. \tag{1.3.3}$$

Lemma 1.3.7. *For $k \ge 1$ and $s \ge s_0$, we have*

$$\| A^k \|_s \le 2^{k-1} \| A \|_{s_0}^{k-1} \| A \|_s\,. \tag{1.3.4}$$

Lemma 1.3.8. *Given a complex sequence $\{m_k\}_{k \in \mathbb{N}}$ and $A \in E_s$, $s \ge s_0$ with*

$$\| A \|_{s_0} < \frac{1}{2} \liminf_k |m_k|^{-\frac{1}{k}}\,,$$

we have that there exists[8]

$$B := \sum_{k=0}^{\infty} m_k A^k\,.$$

Moreover

$$\| B \|_s \le |m_0| + \| A \|_s \sum_{k=1}^{\infty} |m_k|\, 2^{k-1} \| A \|_{s_0}^{k-1} < \infty\,.$$

[7] In particular this implies that E_{s,s_0,s_1} is a Banach algebra for $s_0 \le s \le s_1$.

[8] The series converges in the norm of E_s in the sense that there exists $\lim_{n\to\infty} \sum_{k=0}^{n} m_k A^k$.

Definition 1.3.9. For $s_0 \le s \le s_1$, we define the bounded linear operator $\mathcal{S} : E_s = E_{s,s_0,s_1} \to \mathcal{M}_{s_0,s}$ defined as

$$\mathcal{S}(M,R) := M + R\,.$$

In particular

$$\|\mathcal{S}\|_{\mathcal{L}(E_s,\mathcal{M}_{s_0,s})} = 1\,, \qquad \text{for} \quad s_0 \le s \le s_1\,, \tag{1.3.5}$$

since

$$|\mathcal{S}(M,R)|_{\mathcal{M}_{s_0,s}} = |M+R|_{\mathcal{M}_{s_0,s}} = \sup_{s_0\le p\le s} |M+R|_{p,p} \le \|(M,R)\|_{s,s_0,s_1}\,, \qquad \text{for} \quad s_0 \le s \le s_1\,.$$

Remark 1.3.10. *Note that if $A = \mathcal{S}(\mathtt{A})$ then A is tame modulus, according to Definition 1.3.1, with constants $\mathfrak{M}_A(s) = \|\mathtt{A}\|_s$. Indeed setting $\mathtt{A} = (M,R)$ we have*

$$|Au|_s \le |M|_{s,s}|u|_s + |R|_{s_0,s}|u|_{s_0} \le \sup_{s_0\le p\le s_1} |M|_{p,p}|u|_s + |R|_{s_0,s}|u|_{s_0} \le \|\mathtt{A}\|_{s_0}|u|_s + \|\mathtt{A}\|_s|u|_{s_0}$$

Note that, beside the sum, $\mathcal{S}$ also preserves the product:

$$\mathcal{S}(A \star B) = (\mathcal{S}A)\cdot(\mathcal{S}B)\,. \tag{1.3.6}$$

Remark 1.3.11. *By continuity of $\mathcal{S}$,*

$$\mathcal{S}B = \sum_{k=0}^{\infty} m_k(\mathcal{S}A)^k. \tag{1.3.7}$$

Notice that $\mathcal{S}$ is not injective.
We remark that since $\mathcal{M}s_0, s_1 \cap \mathcal{M}(\mathtt{h}^{s_0},\mathtt{h}^s) \neq \emptyset$, given any operator A in $\mathcal{M}_{s_0,s_1}$ and any $R \in \mathcal{M}_{s_0,s_1} \cap \mathcal{M}(\mathtt{h}^{s_0},\mathtt{h}^s)$ then the operator $(A-R,R) \in E_s$ and $\mathcal{S}(A-R,R) = A$.

1.3.1 Properties of E_s

Recalling the definition of $\mathtt{d}_h$ and of $\mathtt{d}$ in 1.1.7 and the definition of Π_K in 1.1.11, with abuse of notation we define for $A = (M,R) \in E_s$ the operators

$$\mathtt{d}_h A := (\mathtt{d}_h M, \mathtt{d}_h R)\,, \quad \Pi_K A := (\Pi_K M, \Pi_K R)\,, \quad \langle\mathtt{d}\rangle A := (\langle\mathtt{d}\rangle M, \langle\mathtt{d}\rangle R)$$

Lemma 1.3.12. *For $A, B \in E_s$ we have the bounds*

$$\|\Pi_K^\perp A\|_s \le K^{-1}\sum_{h=1}^{d} \|\mathtt{d}_h A\|_s := K^{-1}\|\mathtt{d}A\|_s$$

$$\|\Pi_K^\perp A\|_s \le K^{-b}\|\langle\mathtt{d}\rangle^b A\|_s$$

Lemma 1.3.13. *For any $A = (M, R)$ we define the linear operator $E_s \to E_s$*

$$\mathrm{ad}A : B \to A \star B - B \star A$$

Then

$$\| A \star B \|_s \leq \| A \|_{s_0} \| B \|_s + \| A \|_s \| B \|_{s_0}$$

$$\| (\mathrm{adA})\mathrm{B} \|_s \leq 2\Big(\| A \|_{s_0} \| B \|_s + \| A \|_s \| B \|_{s_0} \Big)$$

$$\| (\mathrm{adA})^{\mathrm{k}}\mathrm{B} \|_s \leq 2^k \Big(\| A \|_{s_0}^k \| B \|_s + k \| A \|_{s_0}^{k-1} \| A \|_s \| A \|_{s_0} \Big)$$

Set

$$U = \langle \mathtt{d} \rangle^b (\mathrm{ad}A)^k B$$

then U satisfies the bounds

$$\begin{aligned} \| U \|_s \leq\ & 2^{k(b+1)} k \Big(\| \langle \mathtt{d} \rangle^b A \|_s \| A \|_{s_0}^{k-1} \| B \|_{s_0} + \| A \|_s \| A \|_{s_0}^{k-1} \| \langle \mathtt{d} \rangle^b B \|_{s_0} + \| \langle \mathtt{d} \rangle^b A \|_{s_0} \| A \|_{s_0}^{k-1} \| B \|_s \Big) \\ & + 2^{k(b+1)} \Big(k(k-1) \| A \|_s \| A \|_{s_0}^{k-2} \| \langle \mathtt{d} \rangle^b A \|_{s_0} \| B \|_{s_0} + \| A \|_{s_0}^k \| \langle \mathtt{d} \rangle^b B \|_s \Big) \end{aligned}$$

The proof of this Lemma is in the Appendix.

Given a Lipschitz family of couples (M, R) we define as usual the weighted Lipschitz norms

$$\| A \|_s^{\gamma, \mathcal{O}} := \sup_{s_0 \leq p \leq s_1} |M|_{p,p}^{\gamma, \mathcal{O}} + |R|_{s_0, s}^{\gamma, \mathcal{O}}$$

where we recall that

$$|A|_{s,s'}^{\gamma, \mathcal{O}} := \sup_{\xi \in \mathcal{O}} |A(\xi)|_{s,s'} + \gamma \sup_{\xi \neq \eta \in \mathcal{O}} \frac{|A(\xi) - A(\eta)|_{s,s'}}{|\xi - \eta|}$$

Remark 1.3.14. *By definition we immediately have that the weighted Lipschitz norm satisfies the same bounds of Lemma 1.3.13*

1.3.2 A "good" decomposition

In this Section we show an example of non trivial operators in $\mathfrak{S}E_s$. We indeed show that an element with finite decay norm is in $\mathfrak{S}E_s$ with a nice control of the norm.
Given $L \in \mathcal{M}(\mathtt{h}^s, \mathtt{h}^s)$ we set

$$(L^B)_k^{k'} := \begin{cases} L_k^{k'} & \text{if } |k - k'| < |k|/2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad (L^U)_k^{k'} := \begin{cases} L_k^{k'} & \text{if } |k - k'| > |k|/2 \\ 0 & \text{otherwise} \end{cases}$$

The first term is known as the *Bony* part and we will call the second one the *Ultraviolet* part.

For $s \geq s_0$ we have

$$|L^B|_{s,s} \leq 2^{s-s_0}|L|_{s_0,s_0}\,, \qquad\qquad |L^U|_{s_0,s} \leq 2^s C(s_0)|L|_s^{\rm dec}\,. \tag{1.3.8}$$

Indeed, given $u \in \mathtt{h}^s$ we have

$$\begin{aligned}
&\sum_k \langle k\rangle^{2s}\Big(\sum_{k':|k-k'|\leq|k|/2}|L_k^{k'}||u_{k'}|\Big)^2 = \sum_k \langle k\rangle^{2s_0}\Big(\sum_{k':|k-k'|\leq|k|/2}|L_k^{k'}|\langle k\rangle^{s-s_0}|u_{k'}|\Big)^2\\
&\leq 2^{2(s-s_0)}\sum_k\langle k\rangle^{2s_0}\Big(\sum_{k'}|L_k^{k'}|\langle k'\rangle^{s-s_0}|u_{k'}|\Big)^2 = 2^{2(s-s_0)}|L\tilde u|_{s_0}^2 \leq 2^{2(s-s_0)}|L|_{s_0,s_0}^2|\tilde u|_{s_0}^2
\end{aligned}$$

since $|k| \leq 2|k'|$ and defining $\tilde u$ as $\tilde u_k := \langle k\rangle^{s-s_0}|u_k|$, so that $|\tilde u|_{s_0} = |u|_s$. This prove the estimate on L^B.
Given $u \in \mathtt{h}^{s_0}$, by the Cauchy-Schwarz inequality

$$\begin{aligned}
&\sum_k\langle k\rangle^{2s}\Big(\sum_{k':|k-k'|>|k|/2}|L_k^{k'}||u_{k'}|\Big)^2 = \sum_k\Big(\sum_{k':|k-k'|>|k|/2}\frac{\langle k\rangle^s|k-k'|^s}{|k-k'|^s\langle k'\rangle^{s_0}}|L_k^{k'}|\langle k'\rangle^{s_0}|u_{k'}|\Big)^2\\
&\leq 2^{2s}\sum_k\Big(\sum_{k'}\frac{|k-k'|^s}{\langle k'\rangle^{s_0}}|L_k^{k'}|\langle k'\rangle^{s_0}|u_{k'}|\Big)^2 \leq 2^{2s}|u|_{s_0}^2\sum_k\sum_{k'}\frac{|k-k'|^{2s}}{\langle k'\rangle^{2s_0}}|L_k^{k'}|^2\\
&= 2^{2s}|u|_{s_0}^2\sum_{k'}\frac{1}{\langle k'\rangle^{2s_0}}\sum_k|k-k'|^{2s}|L_k^{k'}|^2 \leq 2^{2s}|u|_{s_0}^2\sum_{k'}\frac{1}{\langle k'\rangle^{2s_0}}\sum_h\langle h\rangle^{2s}\sup_{k-k'=h}|L_k^{k'}|^2\,.
\end{aligned}$$

Lemma 1.3.15. *Let $L \in \mathcal{M}(\mathtt{h}^s,\mathtt{h}^s)$ be an operator with finite s-decay norm, then there exists an operator $A^L \in E_s$ such that $\mathfrak{S}A^L = L$ and*

$$\|A^L\|_{s,s_0,s_1} \leq 2^{s_1-s_0}|L|_{s_0,s_0} + 2^sC(s_0)|L|_s^{\rm dec} \leq c(s,s_0,s_1)|L|_s^{\rm dec}\,, \tag{1.3.9}$$

with $c(s,s_0,s_1) := 2^{s_1-s_0}+2^sC(s_0)$.

Proof. We set

$$A^L := (L^B, L^U)\,, \qquad \text{so that} \quad \mathfrak{S}A^L = L\,.$$

By (1.3.8) and (1.2.2)

$$\|A^L\|_{s,s_0,s_1} \leq 2^{s_1-s_0}|L|_{s_0,s_0} + 2^sC(s_0)|L|_s^{\rm dec} \leq c(s,s_0,s_1)|L|_s^{\rm dec}\,, \tag{1.3.10}$$

with $c(s,s_0,s_1) := 2^{s_1-s_0}+2^sC(s_0)\,.$ ☐

Remark 1.3.16. *We expect that the class of operator that are in $\mathfrak{S}E_s$ is larger than the class of operators with finite decay norm. Indeed, if we consider the operator $\mathtt{c}_\alpha\partial_x^{-N}$ as in 4 we are not able to say that it has finite decay norm. On the other hand, we can calculate the modulo tame constant and also show in Lemma 3.3.1 that it is in $\mathfrak{S}E_s$. The estimates on these norms are sufficiently sharp to apply the KAM algorithm in "high norm".*

Chapter 2

Diagonalization

2.1 Abstract KAM algorithm

Let $n, d \in \mathbb{N}$, consider $\mathcal{O}_0$ an open set of $\mathbb{R}^n$, fix $M_0, \gamma > 0$, $s_0 > \frac{d}{2}$ and a lipschitz family of operators of the form

$$\mathcal{O}_0 \ni \xi \mapsto \Lambda_0(\xi) + P_0(\xi) \tag{2.1.1}$$

with the following properties:

- (**H1**) the operator Λ_0 has the form

$$\Lambda_0(\xi) = \operatorname{diag}_{k \in \mathbb{Z}^d}(\lambda_k^{(0)}(\xi))$$

  where $\lambda_k^{(0)}(\xi) \in \mathbb{C}$ for all $k \in \mathbb{Z}^d$ and

  1. $\lambda_k^{(0)} \neq \lambda_h^{(0)}$ for all $k \neq h$
  2. $|\lambda_h - \lambda_k|^{\text{lip},\mathcal{O}_0} \leq M_0|h-k|$ where we recall that, given f lipschitz function, $|f|^{\text{lip},\mathcal{O}} := \sup_{\xi \neq \eta \in \mathcal{O}} \frac{|f(\xi)-f(\eta)|}{|\xi-\eta|}$.

- (**H2**) the operator $\langle \mathtt{d} \rangle^b P_0 \in \mathcal{M}(\mathtt{h}^{s_0}, \mathtt{h}^{s_0})$ for some $b \geq 1$.

Recalling Definition 1.1.7, we define

$$\delta_0 := \gamma^{-1} |\langle \mathtt{d} \rangle^b P_0|_{s_0, s_0}^{\frac{\gamma}{M}, \mathcal{O}_0} \tag{2.1.2}$$

which is finite by (**H2**) and

$$\varepsilon_0 := \gamma^{-1} |P_0|_{s_0, s_0}^{\frac{\gamma}{M}, \mathcal{O}_0} \tag{2.1.3}$$

which is finite and bounded by δ_0 by Remark 1.1.8 item *iv*.

Theorem 5. *Fix $\tau > d-1$ and $b > 4\tau + 2$. Consider an operator of the form (2.1.1) such that (**H1**) and (**H2**) hold. Recalling the definitions (2.1.2), (2.1.3), there exists $\varepsilon_\star = \varepsilon_\star(d, \tau, s_0, M_0, \delta_0)$ such that if*

$$\varepsilon_0 \leq \varepsilon_\star$$

then there exist Lipschitz functions $\lambda_j^{(\infty)}(\xi)$ defined for $\xi \in \mathcal{O}_0$ such that for any ξ in the set

$$\mathcal{C} := \{\xi \in \mathcal{O}_0 : |\lambda_k^{(\infty)} - \lambda_{k'}^{(\infty)}| > 2\gamma|k-k'|^{-\tau} \quad 0 < |k-k'|\} \tag{2.1.4}$$

$\Lambda^{(0)} + P_0$ is diagonalizable in $\mathtt{h}^{s_0}$, namely there exists a constant $\mathfrak{C} = \mathfrak{C}(\tau, b)$ and a linear invertible change of variables $U(\xi)$ such that

$$|U - \mathbb{I}|_{s_0,s_0}^{\gamma,\mathcal{C}} \leq \mathfrak{C}\varepsilon_0 \,,$$

and

$$U^{-1}(\Lambda_0 + P_0)U = \Lambda^{(\infty)} := \mathrm{diag}(\lambda_k^{(\infty)})$$

Proof of the Theorem 1. By the first inequality of Lemma 1.1.9, the smallness condition of Theorem 1 implies the one of the Theorem 5, of course provided that $\epsilon_\star$ is small enough. The lower bound on b_1 comes from the corresponding one on b. □

Remark 2.1.1. *Observe that A priori the Cantor set $\mathcal{C}$ in (2.1.4) could be empty. For example, if the operator (2.1.1) does not satisfies the hypothesis **(H1.1)** then $\mathcal{C}$ is empty.*

The proof of this Theorem is based on the following iteration
KAM reduction procedure Fix any $K \gg 1$ and consider any operator of the form

$$\Lambda(\xi) + P(\xi)\,, \qquad \Lambda(\xi) = \mathrm{diag}_{k\in\mathbb{Z}^d}\lambda_k(\xi) \tag{2.1.5}$$

with Λ defined for $\xi \in \mathcal{O}_0$ and P defined in some compact set $\mathcal{O} \subseteq \mathcal{O}_0$. Fix M so that

$$|\lambda_h - \lambda_k|^{\mathrm{lip},\mathcal{O}_0} \leq M|h-k| \tag{2.1.6}$$

and assume that P satisfies

$$\gamma^{-1}K^{2\tau+1}|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} < 2^{-4b}. \tag{2.1.7}$$

Let

$$\mathcal{O}_\Lambda^{(K)} := \{\xi \in \mathcal{O}_0 : \quad |\lambda_k(\xi) - \lambda_{k'}(\xi)| > \gamma\left\langle k-k'\right\rangle^{-\tau} \,, \quad 0 < |k-k'| \leq K\}$$

and fix $\mathcal{O}^+ = \mathcal{O}_\Lambda^{(K)} \cap \mathcal{O}$.

We look for a solution $\mathcal{A}$ of the *homological* equation

$$[\Lambda, A] + \Pi_K P = [P]\,, \qquad [P] := \mathrm{diag}_{k\in\mathbb{Z}^d}P_k^k \tag{2.1.8}$$

Starting from the operator in (2.1.5), we want to consider the change of variables generated by A, solution of the homological equation, and we want to determinate the conjugated operator. To do this we need to prove that A is well defined and bounded, that it generates a change of variables $Q = e^A$ and look what we obtain when we conjugate. We will obtain an operator of the form

$$Q^{-1}(\Lambda + P)Q =: \Lambda^+(\xi) + P^+(\xi)$$

where Λ^+ will by define for $\xi \in \mathcal{O}^+ \subseteq \mathcal{O}_0$ while P^+ will be of smaller norm. Since we want that the new diagonal part is again defined on all $\mathcal{O}_0$, we will do an extension using the Kiertzbraun Theorem[1].

Lemma 2.1.2 (Homological Equation)**.** *For all $\xi \in \mathcal{O}^+$ there exists a unique solution A of the* (2.1.8)*. Moreover, A is majorant analytic with the bounds:*

$$|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+} \le \gamma^{-1}K^{2\tau+1}|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}, \qquad |\langle \mathtt{d}\rangle^b A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+} \le \gamma^{-1}K^{2\tau+1}(|\langle \mathtt{d}\rangle^b \mathcal{P}|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}) \tag{2.1.9}$$

Proof. Since $\Lambda = \mathrm{diag}_{k\in\mathbb{Z}^d}\lambda_k$ we have $[\Lambda, A]_k^{k'} = (\lambda_k - \lambda_{k'})A_k^{k'}$ and (2.1.8) amounts to

$$(\lambda_k - \lambda_{k'})A_k^{k'} = P_k^{k'} - \Pi_K P_k^{k'}$$

whose solution is

$$A_k^{k'} = \begin{cases} \frac{\mathcal{P}_k^{k'}}{\lambda_k - \lambda_{k'}} & 0 < |k-k'| \le K \\ 0 & \text{otherwise} \end{cases}$$

The estimates are obvious from the definitions, indeed for $a = 0, b$

$$\langle \mathtt{d}\rangle^a A \preccurlyeq \gamma^{-1}K^\tau \langle \mathtt{d}\rangle^a P\,, \qquad \langle \mathtt{d}\rangle^a \Delta_\xi A \preccurlyeq \gamma^{-1}K^\tau \langle \mathtt{d}\rangle^a \Delta_\xi P + \gamma^{-2}MK^{2\tau+1}\langle \mathtt{d}\rangle^a P\,. \tag{2.1.10}$$

where by $\Delta_\xi f$ we denote the variation in $\xi \in \mathcal{O}$, i.e

$$\Delta_\xi f = \frac{|f(\xi) - f(\eta)|}{|\xi - \eta|}\,, \quad \eta \in \mathcal{O} \tag{2.1.11}$$

□

Having proved that $A \in \mathcal{M}(\mathtt{h}^{s_0}, \mathtt{h}^{s_0})$ for all $\xi \in \mathcal{O}^+$, we consider $Q := e^A$, which is invertible and in $\mathcal{M}(\mathtt{h}^{s_0}, \mathtt{h}^{s_0})$. The following Lemma gives the properties of the conjugated operator.

Lemma 2.1.3 (KAM step)**.** *Under the hypothesis and the notations of the KAM reduction procedure described above, the following holds:*

- *The change of variables $Q := e^A$ is well defined and invertible as a majorant bounded operator from H^{s_0} to itself, with the bounds*

$$|Q - \mathbb{I}|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+} \le 2|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+} \tag{2.1.12}$$

[1] See in the Appendix 7

– *There exists a diagonal operator* $\Lambda^+(\xi)$ *defined for* $\xi \in \mathcal{O}_0$ *such that for* $\xi \in \mathcal{O}$,

$$\sup_k |\lambda_k - \lambda_k^+|^{\frac{\gamma}{M},\mathcal{O}_0} \leq |\mathcal{P}|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} \tag{2.1.13}$$

where $\Lambda^+(\xi) = \operatorname{diag} \lambda_k^+(\xi)$, *such that setting for* $\xi \in \mathcal{O}^+$

$$Q^{-1}\left(\Lambda(\xi) + P(\xi)\right) Q =: \Lambda^+(\xi) + P^+(\xi)$$

– *The following bounds hold*

$$|\lambda_h^+ - \lambda_k^+|^{\mathrm{lip},\mathcal{O}_0} \leq M_+|h-k|\,, \quad M_+ := M(1 + 2\gamma^{-1}|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}) \tag{2.1.14}$$

$$|P^+|_{s_0,s_0}^{\frac{\gamma}{M^+},\mathcal{O}^+} \leq 4\gamma^{-1}K^{2\tau+1}(|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}})^2 + K^{-b}|\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} \tag{2.1.15}$$

$$|\langle \mathtt{d}\rangle^b P^+|_{s_0,s_0}^{\frac{\gamma}{M^+},\mathcal{O}^+} \leq |\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} + 2^{4b}\gamma^{-1}K^{2\tau+1}|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}|\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} \tag{2.1.16}$$

Proof. The first statement comes from the definition of Q and the bound (2.1.7). By definition, recalling the Lemma A.4.1, for all $\xi \in \mathcal{O}^+$

$$\begin{aligned}
\Lambda^+ + P^+ = e^{[\cdot,A]}\left(\Lambda + P\right) &= \Lambda + P + [\Lambda + P, A] + \sum_{k\geq 2} \frac{\operatorname{ad}(A)^k}{k!}(\Lambda + P) \\
&= \Lambda + P + [\Lambda, A] + [P, A] + \sum_{k\geq 2} \frac{\operatorname{ad}(A)^k}{k!}\Lambda + \sum_{k\geq 2} \frac{\operatorname{ad}(A)^k}{k!}P \\
&= \Lambda + P + [\Lambda, A] + \sum_{k\geq 1} \frac{\operatorname{ad}(A)^k}{(k+1)!}[\Lambda, A] + \sum_{k\geq 1} \frac{\operatorname{ad}(A)^k}{k!}P\,.
\end{aligned}$$

Recalling that A solves the homological equation (2.1.8)

$$[\Lambda, A] = \Pi_K P - [\Pi_K P]\,, \qquad [M]_k^{k'} := \delta_{k,k'} M_k^{k'}\,,$$

for all $\xi \in \mathcal{O}^+$, we obtain

$$e^{[\cdot,A]}\left(\Lambda + P\right) = \Lambda + P + [P] - \Pi_K P + \sum_{k\geq 1} \frac{\operatorname{ad}(A)^k}{(k+1)!}(\Pi_K P - [\Pi_K P]) + \sum_{k\geq 1} \frac{\operatorname{ad}(A)^k}{k!}P$$

Now we set

$$P^+ := \Pi_K^\perp P + \sum_{k\geq 1} \frac{\operatorname{ad}(A)^k}{k!}P - \sum_{k\geq 2} \frac{\operatorname{ad}(A)^{k-1}}{k!}(\Pi_K P - [\Pi_K P])\,. \tag{2.1.17}$$

It remains to define Λ_+. For $\xi \in \mathcal{O}^+$ we get directly $\lambda_k^+ = \lambda_k + (P_k^k)$. By Kirtzbraun's Theorem we extend P_k^k (which is defined only in $\mathcal{O}$) to $\mathcal{O}_0$ preserving the norm $\|\cdot\|^{\frac{\gamma}{M},\mathcal{O}}$. We set $\lambda_k^+ := \lambda_k + (P_k^k)^{\mathrm{ext}}$. This proves (2.1.13), indeed by the definition of λ^+

$$\sup_k |\lambda - \lambda^+|^{\frac{\gamma}{M},\mathcal{O}_0} = \sup_k |(P_k^k)^{\mathrm{est}}|^{\frac{\gamma}{M},\mathcal{O}_0} = \sup_k |P_k^k|^{\frac{\gamma}{M},\mathcal{O}} \leq |P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}\,.$$

and, with abuse of notation, calling Λ^+ also the operator after the extension

$$\Lambda^+ = \Lambda + [P]^{\rm ext}.$$

The bounds on the Lipschitz variation of λ_k^+ follow directly.
From the definition on P and from the Lemma 1.1.10 we obtain the following estimates

$$\begin{aligned}
|P^+|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+} &\le K^{-b}|\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} + \sum_{k\ge 1}\frac{(2|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^k}{k!}|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} + \sum_{k\ge 2}\frac{(2|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^{k-1}}{k!}|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}\\
&\le K^{-b}|\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} + 2|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+}\Big(\sum_{k\ge 1}(2|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^{k-1}(\frac{1}{k!}+\frac{1}{(k+1)!})\Big)|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}
\end{aligned}$$

$$\begin{aligned}
&|\langle \mathtt{d}\rangle^b P^+|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+} \le |\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}\\
&+\sum_{k\ge 1}\frac{2^{k(b+1)}}{k!}\Big(k|\langle \mathtt{d}\rangle^b A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+}(|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^{k-1}|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} + (|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^k|\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}\Big)\\
&+\sum_{k\ge 1}\frac{2^{k(b+1)}}{(k+1)!}\Big(k|\langle \mathtt{d}\rangle^b A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+}(|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^{k-1}|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}} + (|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^k|\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}\Big)\\
&\le 2^{b+1}|\langle \mathtt{d}\rangle^b A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+}\Big(\sum_{k\ge 1}(2^{b+1}A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^{k-1}(\frac{1}{(k-1)!}+\frac{k}{(k+1)!})\Big)|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}\\
&+2^{b+1}|A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+}\Big(\sum_{k\ge 1}(2^{b+1}A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^{k-1}(\frac{1}{k!}+\frac{1}{(k+1)!})\Big)|\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}\\
&\le 2^{b+1}\gamma^{-1}K^{\tau}|\langle \mathtt{d}\rangle^b P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}|P|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}}\sum_{k\ge 1}(2^{b+1}A|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}^+})^{k-1}\mathfrak{m}(k)\\
&\mathfrak{m}(k) := \Big(\frac{1}{(k-1)!}+\frac{k}{(k+1)!}+\frac{1}{k!}+\frac{1}{(k+1)!}\Big)
\end{aligned}$$

Of course the same bounds hold with M replaced by M^+ (since M^+ is larger). $\square$

Proposition 2.1.4 (KAM iteration). *Fix any $\gamma > 0$, $1 < \chi < 2$ and $\tau > 0$. Choose $c > 0$ so that $\frac{\chi(2\tau+1)}{b} < (2\tau+1)c < 2-\chi$ (recall that $b > (2\tau+1)\chi$ by Theorem 5) For any operator*

$$L_0 = \Lambda_0 + P_0 \tag{2.1.18}$$

satisfying the same assumption of the Theorem 5 with

$$(\varepsilon_\star) := \min\left\{\frac{1}{4\sup_{n\ge 0}\{2^{n+1}e^{-\chi^n}\}}, \Big(\frac{1}{2^{4b+1}C^*}\Big)^{\frac{b}{b-(2\tau+1)}}\Big(\frac{1}{4\delta_0}\Big)^{\frac{2\tau+1}{b-(2\tau+1)}}\right\},$$

where $C^* := \sup_{n\ge 0}\{2^{n+1}e^{[1-c(2\tau+1)]\chi^n}\}$

Fix K_0 such that

$$\left(\frac{4\delta_0}{\varepsilon_0}\right)^{\frac{1}{b}} \leq K_0 \leq \left(\frac{1}{2^{4b+1}C^*\varepsilon_0}\right)^{\frac{1}{2\tau+1}}.$$

Define

$$\varepsilon_n = \varepsilon_0 e^{-\chi^n} \quad \text{and} \quad \delta_n = \delta_0 \sum_{j=0}^{n} 2^{-j} \quad K_n := K_0 e^{c\chi^n} \quad M_n = M_0 \sum_{j=0}^{n} 2^{-j}.$$

Then one has recursively for $n \geq 1$:
$(\mathbf{S1})_n$ *Given*

$$L_{n-1} = \Lambda_{n-1} + P_{n-1}$$

with $\Lambda_{n-1} = \mathrm{diag}\lambda_k^{(n-1)}$ defined for all $\xi \in \mathcal{O}_0$ satisfying

$$|\lambda_h^{(n-1)} - \lambda_k^{(n-1)}|^{\mathrm{lip},\mathcal{O}_0} \leq M_{n-1}|h-k|$$

and P_{n-1} defined for $\xi \in \mathcal{O}_{n-1}$. We define[2]

$$\mathcal{O}_n := \mathcal{O}_{\Lambda_{n-1}}^{K_{n-1}} \cap \mathcal{O}_{n-1}$$

and for all $\xi \in \mathcal{O}_n$ we define A_{n-1} as

$$(A_{n-1})_k^{k'} = \begin{cases} \frac{(P_{n-1})_k^{k'}}{(\lambda_k^{(n-1)} - \lambda_{k'}^{(n-1)})} & 0 < |k-k'| \leq K_{n-1} \\ 0 & \textit{otherwise} \end{cases}. \tag{2.1.19}$$

Then exist a diagonal operator

$$\Lambda_n(\xi) = \mathrm{diag}(\lambda_k^{(n)}(\xi)) \tag{2.1.20}$$

defined for $\xi \in \mathcal{O}_0$, and an operator P_n defined for $\xi \in \mathcal{O}_n$ such that setting $Q_n = e^{A_{n-1}}$ and $L_n := Q_n^{-1}L_{n-1}Q_n$ Then one has

$$L_n = \Lambda_n + P_n.$$

Moreover $Q_n : H^{s_0} \to H^{s_0}$ and the following properties hold:

$$|A_{n-1}|_{s_0,s_0}^{\frac{\gamma}{M_{n-1}},\mathcal{O}_n} \leq \varepsilon_0 K_{n-1}^{2\tau+1}, \tag{2.1.21}$$

$$\gamma^{-1}|P_n|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n} \leq \varepsilon_n, \qquad \gamma^{-1}|\langle \mathtt{d}\rangle^b P_n|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n} \leq \delta_n \tag{2.1.22}$$

[2] where we recall that

$$\mathcal{O}_\Lambda^{(K)} := \{\xi \in \mathcal{O}_0 : \quad |\lambda_k(\xi) - \lambda_{k'}(\xi)| > \gamma \langle k-k' \rangle^{-\tau}, \quad 0 < |k-k'| \leq K\}$$

$(\mathbf{S2})_n$ *For* $n \geq 1$

$$|\lambda_k^{(n)} - \lambda_k^{(n-1)}|^{\frac{\gamma}{M_{n-1}},\mathcal{O}_0} \leq \gamma\varepsilon_0 e^{-\chi^n}, \qquad \forall\, k \in \mathbb{Z}. \tag{2.1.23}$$

Thus for all $\xi \in \mathcal{O}_0$, $\lambda_k^{(n)}$ *is a Cauchy sequence in* n *uniformly in* k*. Moreover*

$$|\lambda_h^{(n)} - \lambda_k^{(n)}|^{\mathrm{lip},\mathcal{O}_0} \leq M_n|h-k|\,.$$

$(\mathbf{S3})_n$ *For all* $\xi \in \mathcal{O}_n$*, the sequence of changes of variables*

$$U_n := Q_1 Q_2 \dots Q_n \tag{2.1.24}$$

satisfy the following bound

$$|U_n - U_{n-1}|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n} \leq \mathrm{Const.}2^{-n}$$

Proof. Item $(\mathbf{S1})_1$ follows directly from Lemmata 2.1.2 and 2.1.3, from the choose of K_0 that implies $K_0^{2\tau+1}\varepsilon_0 < 2^{-4b}$ so that formula (2.1.7) holds (recall that $(2\tau+1)c < 1$). $(\mathbf{S2})_1$, follows from Kirtzbraun and $(\mathbf{S3})_1$ from formula (2.1.12).
We proceed by induction:
$(\mathbf{S1})_{n+1}$. Since $(2\tau+1)c < 1$

$$\gamma^{-1}K_n^{2\tau+1}|P_n|_{s_0,s_0}^{\gamma,\mathcal{O}_n} \leq K_0^{2\tau+1}\varepsilon_0 < 2^{-4b}\,,$$

we can apply the Lemma 2.1.3 with $\Lambda = \Lambda_n$ and $P = P_n$. So, with A_n defined in (2.1.19) with $n-1 \rightsquigarrow n$, there exists a diagonal operator $\Lambda_{n+1}(\xi) = \mathrm{diag}(\lambda_k^{(n+1)}(\xi))$ with the eigenvalues $\lambda_k^{(n+1)}(\xi)$ defined on $\mathcal{O}_0$ such that for $\xi \in \mathcal{O}_{n+1}$

$$Q_{n+1}^{-1}(\Lambda_n(\xi) + P_n(\xi))Q_{n+1} := \Lambda_{n+1} + P_{n+1} \qquad Q_{n+1} := e^{A_n}$$

The bounds (2.1.21) for A_n follow from (2.1.9) and from the bounds (2.1.22) on P_n. In order to prove the bounds (2.1.22) for P_{n+1} we use (2.1.15) and (2.1.16) and the bounds (2.1.22) on P_n.

$$\begin{aligned}
|P_{n+1}|_{s_0,s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} &\leq 4\gamma^{-1}K_n^{2\tau+1}(|P_n|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n})^2 + K_n^{-b}|\langle \mathtt{d}\rangle^b P_n|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n} &(2.1.25)\\
&\leq 8\gamma K_0^{2\tau+1}\varepsilon_0^2 e^{-\chi^n[2-(2\tau+1)c]} + 4\gamma K_0^{-b}\delta_0 e^{-bc\chi^n} &(2.1.26)\\
|\langle \mathtt{d}\rangle^b P_{n+1}|_{s_0,s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} &\leq |\langle \mathtt{d}\rangle^b P_n|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n} + 2^{4b}\gamma^{-1}K_n^{2\tau+1}|P_n|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n}|\langle \mathtt{d}\rangle^b P_n|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n} &(2.1.27)\\
&\leq \gamma\delta_n + \gamma 2^{4b}K_n^{2\tau+1}\varepsilon_n &(2.1.28)\\
&\leq \gamma\delta_0\Big(\sum_{j=0}^n 2^{-j} + 2^{4b+1}K_0^{2\tau+1}\varepsilon_0 e^{-[1-c(2\tau+1)]\chi^n}\Big)
\end{aligned}$$

Let us verify that $\gamma^{-1}|P_{n+1}|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_{n+1}} \leq \varepsilon_{n+1}$, indeed

$$8K_0^{2\tau+1}\varepsilon_0^2 e^{-\chi^n[2-(2\tau+1)c]} + 4K_0^{-b}\delta_0 e^{-\chi^n bc} \leq \varepsilon_0 e^{-\chi^{n+1}}$$

is equivalent to

$$8K_0^{2\tau+1}\varepsilon_0^2 e^{-\chi^n[2-(2\tau+1)c-\chi]} + 4K_0^{-b}\delta_0 e^{-\chi^n(-\chi+bc)} \leq \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2}$$

Now the first bound follows by the definition of c, which reads $2-(2\tau+1)c-\chi > 0$ and by the definition of K_0 which implies $\varepsilon_0 K_0^{2\tau+1}C^* < 1$. The second bound follows by the definition of c which gives $bc-\chi \geq 0$ and again by the definition of K_0 which implies $8K_0^{-b}\delta_0 < \varepsilon_0$. In the same way

$$\sum_{j=0}^{n} 2^{-j} + 2^{4b+1}K_0^{2\tau+1}\varepsilon_0 e^{-[1-c(2\tau+1)]\chi^n} \leq \sum_{j=0}^{n+1} 2^{-j}\,.$$

Indeed, since $(2\tau+1)c < 2-\chi$ implies $c(2\tau+1) < 1$ and by the definition of K_0 we have

$$\sum_{j=0}^{n} 2^{-j} + 2^{4b+1}K_0^{2\tau+1}\varepsilon_0 e^{-[1-c(2\tau+1)]\chi^n} \leq \sum_{j=0}^{n} 2^{-j} + 2^{4b+1}\varepsilon_0 \frac{C^*}{\varepsilon_0} e^{-[1-c(2\tau+1)]\chi^n} \leq \sum_{j=0}^{n+1} 2^{-j}$$

Moreover, if $\varepsilon_0 \leq \frac{1}{4\sup_{n\geq 0}\{2^{n+1}e^{-\chi^n}\}}$ then we have that

$$M_n(1+2\varepsilon_n) = M_0 \sum_{i=0}^{n} 2^{-i}(1+2\varepsilon_0 e^{-\chi^n} \leq M_{n+1}$$

Finally, to be sure that

$$\left(\frac{4\delta_0}{\varepsilon_0}\right)^{\frac{1}{b}} \leq K_0 \leq \left(\frac{1}{2^{4b+1}C^*\varepsilon_0}\right)^{\frac{1}{2\tau+1}}.$$

we need the hypothesis

$$\varepsilon_0 \leq \left(\frac{1}{2^{4b+1}C^*}\right)^{\frac{b}{b-(2\tau+1)}} \left(\frac{1}{4\delta_0}\right)^{\frac{2\tau+1}{b-(2\tau+1)}}, \quad C^* := \sup_{n\geq 0}\{2^{n+1}e^{[1-c(2\tau+1)]\chi^n}\}$$

$(\mathbf{S2})_{n+1}$. By (2.1.13),

$$\sup_k |\lambda_k^{(n+1)} - \lambda_k^{(n)}|^{\frac{\gamma}{M_n},\mathcal{O}_0} \leq |P_{n+1}|_{s_0,s_0}^{\frac{\gamma}{M_n},\mathcal{O}_0}$$

The Lipschitz bound follows by 2.1.14 and the definition of ε_0.
$(\mathbf{S3})_{n+1}$

$$\begin{aligned}
|U_{n+1}-U_n|_{s_0,s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} &\leq |U_n|_{s_0,s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} |Q_{n+1}-\mathbb{I}|_{s_0,s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} \\
&\leq 2|A_n|_{s_0,s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}\left(1+|P_0|_{s_0,s_0}^{\frac{\gamma}{M_0},\mathcal{O}_0}\right) \\
&\leq 2\gamma^{-1}K_{n+1}^{2\tau+1}|P_{n+1}|_{s_0,s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}\left(1+|P_0|_{s_0,s_0}^{\frac{\gamma}{M_0},\mathcal{O}_0}\right) \\
&\leq 2K_0^{2\tau+1}e^{-[1-c(2\tau+1)]\chi^{n+1}}\varepsilon_0(1+\varepsilon_0) \\
&\leq 2K_0^{2\tau+1}2^{-[1-c(2\tau+1)]\chi^{n+1}}\varepsilon_0(1+\varepsilon_0) \\
&\leq 2^{-(n+1)}K_0^{2\tau+1}\varepsilon_0 \sup_{k\geq 0}\left\{2^{-[1-c(2\tau+1)]\chi^{k+1}+k+3}\right\}
\end{aligned}$$

$\square$

We define $\mathcal{O}_\infty = \cap_n U_n$. The sequence U_n is Cauchy, in the sense

$$|U_n - U_{n-1}|_{s_0,s_0}^{\frac{\gamma}{M_0},\mathcal{O}_\infty} < \text{Const}.2^{-n} \tag{2.1.29}$$

Let us now discuss the limit of the Cauchy sequences in (2.1.23) and (2.1.24).
We define for all $k \in \mathbb{Z}^d$

$$U := \lim_{n\to\infty} U_n\,, \qquad \lambda_k^{(\infty)} := \lim_{n\to\infty} \lambda_k^{(n)}\,.$$

Note that U is defined for $\xi \in \mathcal{O}_\infty$, while $\lambda_k^{(\infty)}$ is defined for $\xi \in \mathcal{O}_0$.

Corollary 2.1.5. *The operator U conjugates the operator L_0 defined in (2.1.18) to a diagonal operator.*

Proof. By construction

$$U_n^{-1} L_0 U_n = \Lambda_n + P_n$$

with U_n, Λ_n, P_n are defined respectively in (2.1.24), (2.1.20) and in $(\mathbf{S1})_\mathbf{n}$ and with the bound

$$|P_n|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}_n} \le \varepsilon_0 e^{-\chi^n} \to 0$$

Since U is the limit of U_n,

$$U_n^{-1} L_0 U_n \to U^{-1} L_0 U = \Lambda_\infty$$

$\square$

In the previous corollary we have constructed a diagonalizing change of variables for all ξ in the intersection of $\mathcal{O}_n$. In the following we show that such set contains the set $\mathcal{C}$ defined in 2.1.4.

Corollary 2.1.6 (Final Eigenvalues)**.** *For all $n \in \mathbb{N}$, $k \in \mathbb{Z}^d$*

$$|\lambda_k^{(\infty)} - \lambda_k^{(n)}|^{\frac{\gamma}{2M_0},\mathcal{O}_0} \le C\gamma\varepsilon_0 e^{\chi^n}$$

and consequently

$$\mathcal{C} \subset \cap_{n\ge1} \mathcal{O}_n$$

Proof.

$$\begin{aligned}
|\lambda_k^{(\infty)} - \lambda_k^{(n)}|^{\frac{\gamma}{2M_0},\mathcal{O}_0} &= |\sum_{h\ge n} \lambda_k^{(h+1)} - \lambda_k^{(h)}|^{\frac{\gamma}{2M_0},\mathcal{O}_0} \\
&\le \sum_{h\ge n} |\lambda_k^{(h+1)} - \lambda_k^{(h)}|^{\frac{\gamma}{2M_0},\mathcal{O}_0} \overset{(2.1.23)}{\le} \gamma\varepsilon_0 \sum_{h\ge n} e^{-\chi^h} \\
&= \gamma\varepsilon_0 e^{-\chi^n}\Big(1 + \sum_{h\ge n+1} e^{\chi^n-\chi^h}\Big) \le \gamma\varepsilon_0 e^{-\chi^n} \sum_{h\ge n+1} \frac{1}{h^2} h^2 e^{\chi^n-\chi^h} \\
&\le \gamma\varepsilon_0 e^{-\chi^n} (\sup_{h\ge n+1} h^2 e^{\chi^n-\chi^h}) \sum_{h\ge n+1} \frac{1}{h^2} \le C\gamma\varepsilon_0 e^{-\chi^n}
\end{aligned}$$

Let $\xi \in \mathcal{C}$. For every $n \in \mathbb{N}$

$$|\lambda_k^{(n)} - \lambda_{k'}^{(n)}| \geq |\lambda_k^{(\infty)} - \lambda_{k'}^{(\infty)}| - \left(|\lambda_k^{(\infty)} - \lambda_k^{(n)}|^{\frac{\gamma}{2M_0},\mathcal{O}_0} + |\lambda_{k'}^{(\infty)} - \lambda_{k'}^{(n)}|^{\frac{\gamma}{2M_0},\mathcal{O}_0}\right)$$
$$\geq 2\gamma|k-k'|^{-\tau} - 2C\gamma\varepsilon_0 e^{\chi^n} \geq \gamma|k-k'|^{-\tau} + \gamma K_n^{-\tau} - 2C\gamma\varepsilon_0 e^{\chi^n} \geq \gamma|k-k'|^{-\tau}$$

□

Proof of Theorem 5. We apply the iteration Lemma 2.1.4 to Λ_0 and P_0. For the previous corollaries we obtain the thesis. □

Let us now consider a special class of operators, where we have a separation in "space" and "time" variables. Set $k = (l, j)$, $l \in \mathbb{Z}^{d-1}, j \in \mathbb{Z}$ and consider an operator L_0 of the form (2.1.1) satisfying al the hypotheses of Theorem 5, such that one has

$$\lambda_{(l,j)} = \mathrm{i}\omega \cdot l + \Delta_j\,, \quad P_{(l,j)}^{(l',j')} = \mathcal{P}_j^{j'}(l-l')\,,$$

that is P is Töplitz in time, following Definition 1.2.8.

Corollary 2.1.7. *The change of variables U, which diagonalizes L_0 is Töplitz in time namely is of the form $U_{(l,j)}^{(l',j')} = \mathcal{U}_j^{j'}(l-l')$ and the eigenvalues of L_0 have the form*

$$\lambda_{l,j}^{(\infty)} = \mathrm{i}\omega \cdot l + \Delta_j^{(\infty)}\,.$$

Proof. We proceed by induction. Assume that U_{n-1}, P_{n-1} are Töplitz in time and that

$$\lambda_k^{(n-1)} = \mathrm{i}\omega \cdot \ell + (\Delta_{n-1})_j.$$

Since at the step n the change of variables is $U_n = U_{n-1}\mathcal{Q}_n$ where $\mathcal{Q}_n = e^{A_n}$ with A_n defined in (2.1.19), by the Lemmata 1.2.13, 1.2.14, 1.2.15, 1.2.17 U_n is Töplitz in time. Similarly

$$\lambda_k^{(n)} := \lambda_k^{(n-1)} + (\mathcal{P}_n)_k^k(0) = \mathrm{i}\omega \cdot \ell + (\Delta_n)_j, \qquad \Delta_j^{(n)} := \Delta_j^{(n-1)} + (\mathcal{P}_n)_j^j(0).$$

So, we have that $\lim_{n\to\infty} U_n$ is Töplitz in time and

$$\Delta_j^{(\infty)} = \sum_{n=0}^{\infty} (\mathcal{P}_n)_j^j(0)\,.$$

□

2.2 Diagonalization algorithm in high norm

Let $n, d \in \mathbb{N}$, consider $\mathcal{O}_0$ a compact set of $\mathbb{R}^n$, fix $M_0, \gamma > 0$ and $1 < \chi < 2$ and fix s_0, s_1 so that

$$s_1 > s_0 > \frac{d}{2}\,. \tag{2.2.1}$$

Consider a lipschitz family of operators of the form

$$\mathcal{O}_0 \ni \xi \mapsto L_0 := \Lambda_0(\xi) + P_0(\xi) \tag{2.2.2}$$

and assume that it satisfies

- (**H1**) (see page 16)
- (**H2′**) There exists the decomposition

$$\mathcal{P}_0 = \mathcal{S}(\mathtt{P}_0)^3 , \quad \mathtt{P}_0 = (M_{\mathtt{P}_0}, R_{\mathtt{P}_0}).$$

  such the operator $\langle \mathtt{d}\rangle^b \mathtt{P}_0 \in E_{s,s_0,s_1}$ (see the definition 1.3.3), for $s_0 \le s \le s_1$ and some $b \ge 1$

We define

$$\hat{\varepsilon}_0 := \gamma^{-1} |\mathtt{P}_0|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}_0} \qquad \hat{\delta}_0 := \gamma^{-1} |\langle \mathtt{d}\rangle^b \mathtt{P}_0|_{s_0,s_0}^{\frac{\gamma}{M},\mathcal{O}_0} \tag{2.2.3}$$

$$\eta_0 := \gamma^{-1} |\mathtt{P}_0|_{s,s}^{\frac{\gamma}{M},\mathcal{O}_0} \qquad \mu_0 := \gamma^{-1} |\langle \mathtt{d}\rangle^b \mathtt{P}_0|_{s,s}^{\frac{\gamma}{M},\mathcal{O}_0} \tag{2.2.4}$$

All the quantities above are finite by Remarks 1.1.8, 1.3.4.
Note that ε_0 and δ_0 defined respectively in (2.1.3) and (2.1.2) are smaller than the corresponding $\hat{\varepsilon}_0$ and $\hat{\delta}_0$.Therefore the operator L_0 defined in (2.1.1) satisfies the hypotheses of the Theorem 5. With an abuse of notation we shall not distinguish between $\hat{\varepsilon}_0, \hat{\delta}_0$ and ε_0, δ.
We shall use the same notation in the following Theorem where we discuss the regularity of U (that is the change of variables that diagonalizes L_0). We recall that the Theorem 5, that we can apply since we have the bound in 1.3.10, ensures that $U \in \mathcal{M}(\mathtt{h}^{s_0}, \mathtt{h}^{s_0})$.

Theorem 6. *Given s_0, s_1 as in (2.2.1),τ as in the Theorem 5 and*

$$b > \max\{4\tau + 1 , \frac{(2\tau + 1)\chi}{2 - \chi}\} + 1 .$$

, consider an operator of the form (2.1.1) such that (**H1**) *and* (**H2′**) *hold. There exists $\varepsilon_{\star\star} = \varepsilon_{\star\star}(d, \tau, s_0, M_0, \delta_0, \eta_0, \mu_0) \le \varepsilon_\star$ such that if*

$$\varepsilon_0 < \varepsilon_{\star\star}$$

then for all $\xi \in \mathcal{C}$, defined in (2.1.4) the change of variables $U(\xi) \in \mathcal{M}(\mathtt{h}^s, \mathtt{h}^s)$. Moreover there exists $\mathtt{U} \in E_s$ with $U = \mathcal{S}\mathtt{U}$ such that

$$|\mathtt{U} - \mathbb{I}|_{s_0}^{\gamma,\mathcal{C}} \le \mathfrak{C}_2 \varepsilon_0 \tag{2.2.5}$$

$$|\mathtt{U} - \mathbb{I}|_{s}^{\gamma,\mathcal{C}} \le \mathfrak{C}_2 \eta_0 \tag{2.2.6}$$

[3] Recall that $\mathcal{S}$(M,R):= M+R .

Note that by the remark 1.3.10, (2.2.6) implies that U is modulo-tame with tameness constants $\mathfrak{M}_U(s)$ bounded $\|\mathtt{U}\|_s$

KAM reduction procedure We use the same KAM step as in 2.1.3, but we assume stronger conditions on $\mathcal{P}$.
Recall that in KAM reduction procedure of Section 2.1, given an operator $L = \Lambda + P$ as in (2.1.5) with Λ satisfying (2.1.6), we produce A as in Lemma 2.1.2. Denoting by $Q = e^A$ we consider the conjugated operator $L^+ := \Lambda^+ + P^+ = Q^{-1}LQ$. In the following Lemma we shall assume that $P = \mathfrak{S}(\mathtt{P})$ and show that all the resulting operators are in $\mathfrak{S}(E_s)$ with appropriate bounds.

Lemma 2.2.1 (KAM step)**.** *In the setting of Lemma 2.1.3, we assume that* $P = \mathfrak{S}(\mathtt{P})$ *with* $\mathtt{P} = (M_\mathtt{P}, R_\mathtt{P})$ *and*

$$2^{4b}\gamma^{-1}K^{2\tau+1}\|\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}} < 1\,, \quad \|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}}, \|\mathtt{P}\|_{s}^{\frac{\gamma}{M},\mathcal{O}}, \|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s}^{\frac{\gamma}{M},\mathcal{O}} < \infty\,. \tag{2.2.7}$$

We use the notation of the Lemmata 2.1.2 and 2.1.3.
We have the following

- $A = \mathfrak{S}(\mathtt{A})$ *with the bounds:*

$$\|\langle\mathtt{d}\rangle^b\mathtt{A}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}^+} \le \gamma^{-1}K^{2\tau+1}(\|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}})\,, \qquad \|\mathtt{A}\|_{s}^{\frac{\gamma}{M},\mathcal{O}^+} \le \gamma^{-1}K^{2\tau+1}\|\mathtt{P}\|_{s}^{\frac{\gamma}{M},\mathcal{O}}\,, \quad \forall s_0 \le s \le s_1 \tag{2.2.8}$$

- *The change of variables* $Q = e^A = \mathfrak{S}(\mathtt{Q})$ *with* $\mathtt{Q} = e^{\mathtt{A}}$*:*

$$\|\mathtt{Q} - \mathbb{I}\|_{s}^{\frac{\gamma}{M},\mathcal{O}^+} \le 2\|\mathtt{A}\|_{s}^{\frac{\gamma}{M},\mathcal{O}^+}\,, \quad \mathbb{I} = (\mathbb{I}, 0) \tag{2.2.9}$$

  This implies that Q *is well defined and invertible as a tame majorant bounded operator from* $\mathtt{h}^s$ *to itself.*

- *For* $\xi \in \mathcal{O}^+$*, we have* $P^+ = \mathfrak{S}(\mathtt{P}^+)$ *with the bounds*

$$\|\mathtt{P}^+\|_{s_0}^{\frac{\gamma}{M^+},\mathcal{O}^+} \le 4\gamma^{-1}K^{2\tau+1}(\|\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}})^2 + K^{-b}\|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}} \tag{2.2.10}$$
$$\|\langle\mathtt{d}\rangle^b\mathtt{P}^+\|_{s_0}^{\frac{\gamma}{M^+},\mathcal{O}^+} \le \|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}} + 2^{4b}\gamma^{-1}K^{2\tau+1}\|\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}}\|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}}$$
$$\|\mathtt{P}^+\|_{s}^{\frac{\gamma}{M^+},\mathcal{O}^+} \le K^{-b}\|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s}^{\frac{\gamma}{M},\mathcal{O}} + 2^{4b}\gamma^{-1}K^{2\tau+1}\|\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}}\|\mathtt{P}\|_{s}^{\frac{\gamma}{M},\mathcal{O}}$$
$$\|\langle\mathtt{d}\rangle^b\mathtt{P}^+\|_{s}^{\frac{\gamma}{M^+},\mathcal{O}^+} \le \|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s}^{\frac{\gamma}{M},\mathcal{O}} \tag{2.2.11}$$
$$+ 2^{4b}\gamma^{-1}K^{2\tau+1}\Big(2\|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s}^{\frac{\gamma}{M},\mathcal{O}}\|\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}} + 2\|\mathtt{P}\|_{s}^{\frac{\gamma}{M},\mathcal{O}}\|\langle\mathtt{d}\rangle^b\mathtt{P}\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}} + \|\mathtt{P}\|_{s}^{\frac{\gamma}{M},\mathcal{O}}\|\langle\mathtt{d}\rangle^b\mathtt{P}^+\|_{s_0}^{\frac{\gamma}{M},\mathcal{O}}\Big)$$

Proof. Recalling that $\mathtt{P} = (M_\mathtt{P}, R_\mathtt{P})$ we construct $\mathtt{A} = (M_\mathtt{A}, R_\mathtt{A})$ explicitly by setting for all $\xi \in \mathcal{O}^+$

$$(M_{\mathtt{A}})_k^{k'} = \begin{cases} \frac{(M_{\mathtt{P}})_k^{k'}}{(\lambda_k - \lambda_{k'})} & |k-k'| \le K \\ 0 & \text{otherwise} \end{cases}, \quad (R_{\mathtt{A}})_k^{k'} = \begin{cases} \frac{(R_{\mathtt{P}})_k^{k'}}{(\lambda_k - \lambda_{k'})} & |k-k'| \le K \\ 0 & \text{otherwise} \end{cases},$$

so that the bounds follow trivially from (2.1.10)

$$\langle \mathtt{d} \rangle^a \mathtt{A} \preccurlyeq \gamma^{-1} K^\tau \langle \mathtt{d} \rangle^a \mathtt{P}\,, \qquad \langle \mathtt{d} \rangle^a \Delta_\xi \mathtt{A} \preccurlyeq \gamma^{-1} K^\tau \langle \mathtt{d} \rangle^a \Delta_\xi \mathtt{P} + \gamma^{-2} M K^{2\tau+1} \langle \mathtt{d} \rangle^a \mathtt{P}\,, \quad \text{for } a = 0, b. \tag{2.2.12}$$

The second statement follows from formula (1.3.7) and Lemma (1.3.8), which implies (2.1.12), since

$$\| e^{\mathtt{A}} - \mathbb{I} \|_s^{\frac{\gamma}{M}, \mathcal{O}^+} \le \| \mathtt{A} \|_s^{\frac{\gamma}{M}, \mathcal{O}^+} \sum_{k=1}^\infty \frac{(\| \mathtt{A} \|_{s_0}^{\frac{\gamma}{M}, \mathcal{O}^+})^{k-1}}{k!}\,, \quad \| \langle \mathtt{d} \rangle^b e^{\mathtt{A}} \|_{s_0}^{\frac{\gamma}{M}, \mathcal{O}^+} \le \| \langle \mathtt{d} \rangle^b \mathtt{A} \|_{s_0}^{\frac{\gamma}{M}, \mathcal{O}^+} \sum_{k=1}^\infty \frac{(2^b \| \mathtt{A} \|_{s_0}^{\frac{\gamma}{M}, \mathcal{O}^+})^{k-1}}{k!}$$

The third statement comes from 2.1.17 by setting

$$\mathtt{P}^+ = \Pi_K^\perp \mathtt{P} + \sum_{k \ge 1} \frac{\mathrm{ad}(\mathtt{A})^k}{k!} \mathtt{P} - \sum_{k \ge 2} \frac{\mathrm{ad}(\mathtt{A})^{k-1}}{k!} (\Pi_K \mathtt{P} - [\mathtt{P}])\,, \qquad [\mathtt{P}] = ([M_{\mathtt{P}}], [R_{\mathtt{P}}])$$

in the first bound we apply Lemma1.3.12 in order to bound the first summand and Lemma 1.3.8 with $s = s_0$ for the other two. In the second bound we use Lemma 1.3.13 with $s = s_0$ and $a = b$. In the third and fourth bounds we use Lemma 1.3.13 with $s_0 \le s \le s_1$ and respectively $a = 0, b$.

□

We now restate the proposition 2.1.4 under the stronger hypotheses of the KAM step above. Recall that in proposition 2.1.4 we constructed a list of operators $A_n, Q_n, U_n, L_n = \Lambda_n + P_n$ such that A_n solves the Homological equation (2.1.8) so it is defined in (2.1.19), $Q_n = e^{A_{n-1}}$, $U_n = Q_1 \dots Q_n$ and $L_n = Q_n^{-1} L_{n-1} Q_n$. Recall that A_{n-1} and P_n are defined in $\mathcal{O}_n$ and Λ_n in $\mathcal{O}_0$. Finally the sets $\mathcal{O}_n$ are defined recursively by

$$\mathcal{O}_n := \mathcal{O}_{\Lambda_{n-1}}^{K_{n-1}} \cap \mathcal{O}_{n-1} \quad K_n = K_0 e^{c\chi^n}$$

Proposition 2.2.2 (KAM iteration 2)**.** *Fix any $\tau > 0$ $\gamma > 0$, $1 < \chi < 2$ and $0 < \alpha < 1$. Fix b as in Theorem 6. Choose c so that*

$$\frac{\chi}{b - \alpha} < c < \frac{2 - \chi}{2\tau + 1}\,.$$

For any operator

$$L_0 = \Lambda_0 + P_0$$

as in (2.2.2) satisfying $(\mathbf{H1}), (\mathbf{H_2'})$ *there exist $\varepsilon_{\star\star}$ and $\mathfrak{K}_1 < \mathfrak{K}_2$ such that if $\varepsilon_0 \le \varepsilon_{\star\star}$, defining*

$$\varepsilon_n = \varepsilon_0 e^{-\chi^n} \quad , \quad \delta_n = \delta_0 \sum_{j=0}^n 2^{-j} \qquad \eta_n = \eta_0 e^{-\chi^n} \quad \text{and} \quad \mu_n = \eta_0 K_n^\alpha.$$

and fixing $\mathfrak{K}_1 < K_0 < \mathfrak{K}_2$ then one has recursively for $n \geq 1$:

$(\mathbf{S1})_n$ *Given*

$$L_{n-1} = \Lambda_{n-1} + P_{n-1}$$

as in $(\mathbf{S1})_n$ *of the proposition 2.1.4 such that $P_{n-1} = \mathfrak{S}(\mathtt{P}_{n-1})$ for some $\mathtt{P}_{n-1} \in E_s$, then the operator A_{n-1}, defined in* (2.1.19) *for all $\xi \in \mathcal{O}_n$, satisfies $A_{n-1} = \mathfrak{S}(\mathtt{A}_{n-1})$ with $\mathtt{A}_{n-1}$ defined as*

$$(\mathtt{A}_{n-1})_k^{k'} = \begin{cases} \frac{(\mathtt{P}_{n-1})_k^{k'}}{(\lambda_k^{(n-1)} - \lambda_{k'}^{(n-1)})} & 0 < |k-k'| \leq K_{n-1} \\ 0 & \textit{otherwise} \end{cases} . \tag{2.2.13}$$

The operator $L_n = Q_n^{-1} L_{n-1} Q_n$ is of the form $\Lambda_n + P_n$ with Λ_n as in (2.1.20)*. Moreover there exists $\mathtt{P}_n \in E_s$ such that $P_n = \mathfrak{S}(\mathtt{P}_n)$ and the following bounds hold:*

$$|\mathtt{A}_{n-1}|_{s_0,s_0}^{\frac{\gamma}{M_{n-1}},\mathcal{O}_n} \leq \varepsilon_0 K_{n-1}^{2\tau+1}, \tag{2.2.14}$$

$$\begin{aligned} \gamma^{-1} \| \mathtt{P}_n \|_{s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n} &\leq \varepsilon_n\,, \qquad \gamma^{-1} \| \langle \mathtt{d} \rangle^b \mathtt{P}_n \|_{s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n} \leq \delta_n \\ \gamma^{-1} \| \mathtt{P}_n \|_{s}^{\frac{\gamma}{M_n},\mathcal{O}_n} &\leq \eta_n\,, \quad \gamma^{-1} \| \langle \mathtt{d} \rangle^b \mathtt{P}_n \|_{s}^{\frac{\gamma}{M_n},\mathcal{O}_n} \leq \mu_n \end{aligned} \tag{2.2.15}$$

$(\mathbf{S2})_n$ *we have $U_n = \mathfrak{S}(\mathtt{U}_n)$ where $\mathtt{U}_n$ is a Cauchy sequence, in the same sense of 2.1.29, since the following bound holds*

$$\| \mathtt{U}_n - \mathtt{U}_{n-1} \|_s^{\frac{\gamma}{M_n},\mathcal{O}_n} \leq 2^{-n} \mathfrak{C}_2 \| \mathtt{P}_0 \|_s^{\frac{\gamma}{M_n},\mathcal{O}_n} ,$$

Proof. For $n = 0$ there is nothing to prove (we use the condition $\mu_0 < \eta_0 K_0^\alpha$ provided by the choice of $\mathfrak{K}_1$); by induction assume that $(\mathbf{Si})_n$ holds then (since $c(2\tau+1) < 1$ and $K_0^{2\tau+1}\varepsilon_0 < 2^{-4b}$)

$$\gamma^{-1} K_n^{2\tau+1} \| \mathtt{P}_n \|_{s_0}^{\frac{\gamma}{M_n},\mathcal{O}_n} = \gamma^{-1} K_0^{2\tau+1} \varepsilon_0 e^{-[1-c(2\tau+1)]\chi^n} \leq 2^{-4b}$$

and $(\mathbf{S1})_{n+1}$ follows by the KAM step. Indeed (2.2.14) follows from (2.2.8), as for (2.2.15), the first two bounds follow from the first two bounds in (2.2.10) by reasoning as in (2.1.25).
Finally, for the last two bounds of (2.2.15) we want that

$$K_n^{-b} \mu_n + 2^{4b} K_n^{2\tau+1} \varepsilon_n \eta_n \leq \eta_{n+1} \tag{2.2.16}$$

$$\mu_n + 2^{4b+1} K_n^{2\tau+1} \varepsilon_n \mu_n + 3 \cdot 2^{4b} K_n^{2\tau+1} \eta_n \delta_n \leq \mu_{n+1} \tag{2.2.17}$$

(2.2.16) is equivalent to ask

$$K_0^{\alpha-b} e^{-(b-\alpha)c\chi^n} + 2^{4b} K_0^{2\tau+1} \varepsilon_0 e^{-[2-c(2\tau+1)]\chi^n} \leq e^{-\chi^{n+1}}$$

To have that we need that both the addends are smaller than $\frac{e^{-\chi^{n+1}}}{2}$
So we have to ask:

$$2K_0^{\alpha-b}e^{-(b-\alpha)c\chi^n}\le e^{-\chi^{n+1}}\Rightarrow 2K_0^{\alpha-b}e^{-[(b-\alpha)c-\chi]\chi^n}\le 1$$
$$2^{4b+1}K_0^{2\tau+1}\varepsilon_0 e^{-[2-c(2\tau+1)]\chi^n}\le e^{-\chi^{n+1}}\Rightarrow 2^{4b+1}K_0^{2\tau+1}\varepsilon_0 e^{-[2-c(2\tau+1)-\chi]\chi^n}<1$$

This inequalities hold true provided that

- $bc-\alpha c-\chi>0$
- $K_0>2^{\frac{1}{b-a}}$
- $c(2\tau+1)+\chi<2$
- $K_0<\left(\frac{1}{2^{4b+1}\varepsilon_0}\right)^{\frac{1}{2\tau+1}}$

To obtain (2.2.17) we can ask

$$K_0^{\alpha}e^{\alpha c\chi^n}+2^{4b+1}K_0^{2\tau+1+\alpha}\varepsilon_0 e^{-[1-c(2\tau+1+\alpha)]\chi^n}+3{\cdot}2^{4b+1}K_0^{2\tau+1}e^{-[1-c(2\tau+1)]\chi^n}\delta_0\le K_0^{\alpha}e^{\alpha c\chi^{n+1}}$$

Also this time we have to ask that every addend is smaller than a fraction of $e^{\alpha c\chi^{n+1}}$ but this time we don't want to do it uniformly in 3 parts but we ask that the first addend is smaller than

$$\frac{e^{\alpha c\chi^{n+1}}}{C_1}\,,\qquad C_1:=e^{\alpha c(\chi-1)}$$

and the other two addends smaller than

$$\frac{e^{\alpha c\chi^{n+1}}}{C_2}\,,\qquad C_2:=\frac{1}{2}\Big(1-\frac{1}{C_1}\Big)$$

With this choose we obtain the following conditions to ask

- $c(2\tau+1+\alpha-\alpha\chi)<1$
- $K_0<\left(\frac{1}{2^{4b+1}C_1\varepsilon_0}\right)^{\frac{1}{2\tau+1}}$
- $c(2\tau+1-\alpha\chi)<1$
- $K_0>\Big(3\cdot 2^{4b+1}C_2\delta_0\Big)^{\frac{1}{\alpha-2\tau-1}}$

So we can take

$$\mathfrak{K}_1=\max\left\{\Big(\frac{4\delta_0}{\varepsilon_0}\Big)^{\frac{1}{b}},\Big(\frac{\mu_0}{\eta_0}\Big)^{\frac{1}{\alpha}},\Big(3\cdot 2^{4b+1}C_2\delta_0\Big)^{\frac{1}{\alpha-2\tau-1}}\right\}$$
$$\mathfrak{K}_2=\min\left\{\Big(\frac{1}{2^{4b+1}\varepsilon_0C^*}\Big)^{\frac{1}{2\tau+1}},\Big(\frac{1}{2^{4b+1}\varepsilon_0}\Big)^{\frac{1}{2\tau+1}}\right\}$$

To be sure that $\mathfrak{K}_1 < \mathfrak{K}_2$ we need $\varepsilon_0 \leq \varepsilon_\star$ where $\varepsilon_{\star\star}$ is defined as

$$\min\Big\{\varepsilon_\star, \Big(\frac{1}{(4\delta_0)^{2\tau+1}(2^{4b+1}C_3)^b}\Big)^{\frac{1}{b-\tau-1}}, \Big(\frac{1}{C_3 2^{4b+1}}\Big)\Big(\frac{\eta_0}{\mu}\Big)^{\frac{2\tau+1}{\alpha}}, \Big(\frac{1}{C_3 2^{4b+1}}\Big)\Big(\frac{1}{3\cdot 2^{4b+1}C_2\delta_0}\Big)^{\frac{2\tau+1}{\alpha-(2\tau+1)}}\Big\}$$

where $C_3 := \max\{1, C^*\}$
$(\mathbf{S2})_{n+1}$ Recalling definition 1.3.5 we set $\mathtt{Q}_{n+1} = e^{\mathtt{A}_n}$ and $\mathbb{I} = (\mathbb{I}, 0)$.

$$\begin{aligned}
\|\mathtt{U}_{n+1} - \mathtt{U}_n\|_s^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} &\leq \|\mathtt{U}_n\|_s^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}\|\mathtt{Q}_{n+1} - \mathbb{I}\|_{s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} + \|\mathtt{U}_n\|_{s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}\|\mathtt{Q}_{n+1} - \mathbb{I}\|_s^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} \\
&\leq 2\|\mathtt{U}_n\|_s^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}\|\mathtt{A}_{n+1}\|_{s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} + 2\|\mathtt{U}_n\|_{s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}\|\mathtt{A}_{n+1}\|_s^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} \\
&\leq 2\gamma^{-1}K_{n+1}^{2\tau+1}(\|\mathtt{U}_n\|_s^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}\|\mathtt{P}_{n+1}\|_{s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} + \|\mathtt{U}_n\|_{s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}\|\mathtt{P}_{n+1}\|_s^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}) \\
&\leq 2\gamma^{-1}K_{n+1}^{2\tau+1}((1 + \|\mathtt{P}_0\|_s^{\frac{\gamma}{M_{n+1}},\mathcal{O}_0})\|\mathtt{P}_{n+1}\|_{s_0}^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}} + \|\mathtt{P}_{n+1}\|_s^{\frac{\gamma}{M_{n+1}},\mathcal{O}_{n+1}}) \\
&\leq 2K_{n+1}^{2\tau+1}((1+\eta_0)\varepsilon_{n+1} + \eta_{n+1}) \leq 2^{(c(2\tau+1)-1)\chi^{n+1}+1}((1+\eta_0)\varepsilon_0 + \eta_0) \\
&\leq \eta_0 2^{-(n+1)}\sup_{k\geq 0} 2^{(c(2\tau+1)-1)\chi^{k+1}+3+k}
\end{aligned}$$

Note that $(1+\eta_0)\varepsilon_0 + \eta_0 \leq 3\eta_0$ since $\varepsilon_0 < \min(1, \eta_0)$.

□

Proof of Theorem 6. We apply the iteration Lemma 2.2.2 to Λ_0 and P_0 hence we obtain the thesis. □

Chapter 3

Applications

In this chapter we prove the Theorems 3-4. Our purpose is to prove such Theorems as an application of our diagonalization algorithms, so before turning to the proofs let us discuss the connection between reducibility and diagonalization.

3.1 Reducibility and diagonalization

Consider the quasi-periodic linear dynamical system (6)

$$\dot{u} = \mathcal{A}(\omega t)u \tag{3.1.1}$$

where for $t \in \mathbb{R}$, $u(t) \in \mathtt{h}^s(\mathbb{Z})$ while $\mathcal{A}$ is a map $\varphi \mapsto \mathcal{A}(\varphi)$ from $\mathtt{h}^s(\mathbb{Z})$ to $\mathtt{h}^{s'}(\mathbb{Z})$.
We want to describe how this system changes under the action of a transformation of the phase space that depend quasi-periodically on time.

To this purpose we consider a map $\varphi \mapsto \mathcal{U}(\varphi)$ as in the remark 1.2.10 that for every φ is invertible.
Setting $v(t) = (\mathcal{U}(\omega t))^{-1}u(t)$, and substituting in (3.1.1) we obtain that v solves the new dynamical system:

$$\dot{v}(t) = \mathcal{B}(\omega t)v(t)\,, \qquad \mathcal{B}(\varphi) := (\mathcal{U}(\varphi))^{-1}(\mathcal{A}(\varphi)\mathcal{U}(\varphi) - \omega \cdot \partial_\varphi \mathcal{U}(\varphi))\,.$$

Definition 3.1.1. We say that $\mathcal{U}$ reduces (3.1.1) if $\mathcal{B}$ does not depend on angles. We say that the dynamical system is reducible if there exists a change of variables that reduces it.

We now want to assiciate to the time-dependent dynamical system (3.1.1) a linear operator acting on $\mathtt{h}^{s_0}(\mathbb{Z}^d)$. Since for every family of operators $\mathcal{A}(\varphi)$ we can associate a Töplitz in time operator A, recalling the notation of the Section 1.2.1, we define

$$L := \omega \cdot \partial_\varphi - A\,.$$

Note that L is not Töplitz in time.

Lemma 3.1.2. *The reducibility of the dynamical system corresponds to the fact that L can be diagonalized by a Töplitz in time change of variables.*

Proof. Let U be the Töplitz in time matrix that diagonalizes L. By definition $U^{-1}LU$ is diagonal and moreover

$$U^{-1}LU = \omega \cdot \partial_\varphi - U^{-1}(AU - [\omega \cdot \partial_\varphi, U]) = \omega \cdot \partial_\varphi - A^\infty \,.$$

By the Lemmata 1.2.13, 1.2.14, 1.2.15, 1.2.17 and 1.2.12 A_∞ is Töplitz in time and it is diagonal hence $A^\infty = \mathrm{diag} a_j^\infty$.
Moreover,

$$U^{-1}(AU - [\omega \cdot \partial_\varphi, U]) \leftrightsquigarrow (\mathcal{U}(\varphi))^{-1}(\mathcal{A}(\varphi)\mathcal{U}(\varphi) - \omega \cdot \partial_\varphi \mathcal{U}(\varphi)) \,.$$

So we proved that diagonalizable implies reducible. The other implication is clear if $\mathcal{B}$ is diagonal and angle-independent. □

3.2 Proof of the Theorem 3

Consider the equation (8) and its associated operator

$$\mathcal{L} := \omega \cdot \partial_\varphi + \partial_{xxx} + \varepsilon V(\varphi, x)\partial_x \,, \tag{3.2.1}$$

Remark 3.2.1. *Note that this operator maps the space of functions whose mean value in x is zero in to itself. So we could also work in this subspace. Note also that if V is even then $\mathcal{L}$ maps even (see definition 13) functions into odd functions.*

In the subsection 3.2.1 we perform a regularization procedure which conjugates the operator $\mathcal{L}$ to the operator $\mathcal{L}_3$ defined in 3.2.9. This is well-known change of variables (Iooss-Plotnikov and Toland, [BBM14]), usually called *Descent method*.
In this context the change of variables are families of bounded linear operators on $H^s(\mathbb{T}) \equiv \mathtt{h}^s(\mathbb{Z})$. Following remark 1.2.10, we shall then envision this operator as acting on $\mathtt{h}^s(\mathbb{Z}^d)$.
We do two steps of regularization. The first one is a change of space variable (translation) and the second one a conjugation by pseudo-differential operators.
Then we will arrive to an operator of the form

$$\mathcal{L}_3 = \omega \cdot \partial_\varphi + \partial_{xxx} + m\partial_x + \mathcal{R}$$

where m is a constant and $\mathcal{R}$ is a $\varphi-$dependent family of bounded operators on $H^s(\mathbb{T}) \equiv \mathtt{h}^s(\mathbb{Z})$.
Using the notation of the remark 1.2.10, we can associate to $\mathcal{R}$ an operator R acting on $\mathtt{h}^s(\mathbb{Z}^d)$ and Töplitz in time.
By reference to the notation of the Theorem 5 now we have that the role of parameters $\xi \in \mathbb{R}^n$ is assumed by[1] $\omega \in \mathcal{O} \subset \mathbb{R}^{d-1}$ and

$$\Lambda_0(\xi) \rightsquigarrow \Lambda_0(\omega) = \omega \cdot \partial_\varphi + \partial_{xxx} + m\partial_x \,, \quad P_0 \rightsquigarrow R \,, \quad \mathcal{O}_0 \rightsquigarrow \mathcal{O}.$$

[1]So observe that in this application $n = d - 1$

Λ_0 satisfies the hypothesis (**H1**) of page 16, indeed $\Lambda_0(\omega) = \mathrm{diag}_{k=(l,j)\in\mathbb{Z}^d}\lambda_k^{(0)}$ with

$$\lambda_k^{(0)} = \mathrm{i}(\omega \cdot l - j^3 + mj) \tag{3.2.2}$$

where it is clear that $\lambda_k^{(0)} \neq \lambda_{k'}^{(0)}$ if $k \neq k'$ and

$$|\lambda_k - \lambda_{k'}|^{\mathrm{lip},\mathcal{O}} \leq |l - l'| < |k - k'|.$$

Then we will estimate the reminder $\mathcal{R}$ in view to apply the Theorem 5 and prove Theorem 3.

Remark 3.2.2. *After the steps of regularization, we will obtain the operator $\mathcal{L}_3$ defined above and if V is even, it is odd.*
Indeed, if V is even the function e (that we will define in (3.2.3)) is odd, a ((3.2.8)) is even in φ and odd in x and then p_0 (3.2.6) has mean value zero.
Note also that, since $\Lambda_0 + P_0$ maps even functions in odd ones and since the change of variable $Q = e^A$ is a parity-preserving and reality preserving operator then als(o the operator $\Lambda_1 + P_1 := Q^{-1}\Lambda_0 + P_0 Q$ maps even functions in odd functions. Iterating this we obtain that the final operator Λ_∞ maps even functions in odd functions, is reality preserving and it is diagonal and hence it has purely imaginary eigenvalues.
Therefore, the final operator is odd and we can apply the Lemma 3 and obtain the stability result.

Notation We shall systematically use the notation

$$A \lesssim_h B$$

where h is a parameter or a list of parameters, to denote that there exists a constant $C(k)$ depending on k, such that $A < C(k)B$.

3.2.1 Regularization

Step 1 We do a first change of variables defining $y = x + p(\varphi)$. So we define

$$\mathcal{T}h(\varphi, x) := h(\varphi, x + p(\varphi)).$$

Its inverse is

$$\mathcal{T}^{-1}v(\varphi, y) := v(\varphi, y - p(\varphi)).$$

$$\mathcal{T}^{-1}\omega \cdot \partial_\varphi \mathcal{T} = \omega \cdot \partial_\varphi + [\omega \cdot \partial_\varphi p(\varphi)]\partial_y \qquad \mathcal{T}^{-1}\partial_x \mathcal{T} = \partial_y$$

So if we conjugate $\mathcal{L}$ we obtain

$$\mathcal{T}^{-1}\mathcal{L}\mathcal{T} = \omega \cdot \partial_\varphi + \partial_{yyy} + e(\varphi, y)\partial_y$$

with

$$e(\varphi, y) = \omega \cdot \partial_\varphi p(\varphi) + \mathcal{T}^{-1}V(\varphi, y). \tag{3.2.3}$$

We are looking for $p(\varphi)$ such that $e(\varphi, y)$ has the mean value in y constant in φ, namely

$$\frac{1}{2\pi}\int_{\mathbb{T}} e(\varphi, y)\, dy = m \in \mathbb{R} \quad \forall\, \varphi \in \mathbb{T}^d.$$

$$m = \frac{1}{2\pi}\int_{\mathbb{T}} e(\varphi, y)\, dy = \frac{1}{2\pi}\int_{\mathbb{T}} \omega\cdot\partial_\varphi p(\varphi)+\mathcal{T}^{-1}V(\varphi, y)\, dy = \omega\cdot\partial_\varphi p(\varphi)+\frac{1}{2\pi}\int_{\mathbb{T}} \mathcal{T}^{-1}V(\varphi, y)\, dy$$

$$\omega\cdot\partial_\varphi p(\varphi) = m - \frac{1}{2\pi}\int_{\mathbb{T}} \mathcal{T}^{-1}V(\varphi, y)\, dy$$

This equation has periodic solution $p(\varphi)$ if and only if

$$\int_{\mathbb{T}^d} m - \frac{1}{2\pi}\int_{\mathbb{T}} \mathcal{T}^{-1}V(\varphi, y)\, dy\, d\varphi = 0$$

$$m = \int_{\mathbb{T}^d} \frac{1}{2\pi}\int_{\mathbb{T}} V(\varphi, y)\, dy\, d\varphi$$

So we need to have:

$$m := \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{T}^{d+1}} V(\varphi, y)\, dy\, d\varphi \qquad p(\varphi) := (\omega\cdot\partial_\varphi)^{-1}\Big(m - \int_{\mathbb{T}} V(\varphi, y)\, dy\Big) \tag{3.2.4}$$

With this choice of $p(\varphi)$, after renaming the variable $y = x$, we obtain:

$$\mathcal{L}_2 = \mathcal{T}^{-1}\mathcal{L}\mathcal{T} = \omega\cdot\partial_\varphi + \partial_{xxx} + e(\varphi, x)\partial_x$$

with $e(\varphi, x)$ (recall (3.2.3)) such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} e(\varphi, y)\, dy = m \in \mathbb{R} \quad \forall\, \varphi \in \mathbb{T}^d.$$

Step 2 Now we do a second change of variables. Consider an operator of the form:

$$\mathcal{S} = \mathbb{I} + a(\varphi, x)\partial_x^{-1} \quad a(\varphi, x) : \mathbb{T}^{d+1} \to \mathbb{R}$$

Note that $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = \pi_0$, where π_0 is the L^2-projector on the subspace $H_0 := \{u(\varphi, x) \in L^2(\mathbb{T}^{d+1}) : \int_{\mathbb{T}} u(\varphi, x)\, dx = 0\}$.

$$\mathcal{L}_2\mathcal{S} - \mathcal{S}(\omega\cdot\partial_\varphi + \partial_{xxx} + m\partial_x) = p_1\partial_x + p_0 + p_{-1}\partial_x^{-1}$$

with

$$p_1 := e(\varphi, x) + 3\partial_x a(\varphi, x) - m \tag{3.2.5}$$

$$p_0 := \Big(3\partial_{xx}a(\varphi, x) + e(\varphi, x)a(\varphi, x) - a(\varphi, x)\Big)\pi_0 \tag{3.2.6}$$

$$p_{-1} := \omega\cdot(\partial_\varphi a(\varphi, x)) + \partial_{xxx}a(\varphi, x) + e(\varphi, x)\partial_x a(\varphi, x) \tag{3.2.7}$$

We are looking for a space-periodic function $a(\varphi, x)$ such that $p_1 = 0$ so it has to be defined as

$$a(\varphi, x) := \frac{1}{3}\partial_x^{-1}(m - e(\varphi, x)) \tag{3.2.8}$$

So if we conjugate $\mathcal{L}_2$ through $\mathcal{S}$ we obtain

$$\mathcal{L}_3 = \mathcal{S}^{-1}\mathcal{L}_2\mathcal{S} = \omega \cdot \partial_\varphi + \partial_{xxx} + m\partial_x + \mathcal{R}, \quad \mathcal{R} := \mathcal{S}^{-1}(p_0 + p_{-1}\partial_x^{-1}) \,. \tag{3.2.9}$$

3.2.2 Estimates on $\mathcal{R}(\varphi)$ and R

Recall the assumption of the Theorem 3 so consider $\omega \in \mathcal{O}$ a subset of the $(\gamma, \tau)-$diophantine vectors, and fix $b > 4\tau + 2$, $s_0 > \frac{d}{2}$.

Lemma 3.2.3. *There exist $\sigma = \sigma(\tau)$, such that if $V \in H^{s+\sigma}$ then $p(\varphi)$ defined in (3.2.4) belongs to H^{s_0} and the the following bound holds*

$$|p(\varphi)|_{s_0}^{\gamma,\mathcal{O}} \leq \gamma^{-1}|V|_{s+\sigma}^{\gamma,\mathcal{O}}$$

Proof. It is clear that if $\mathcal{O}$ is contained in the set of $(\gamma, \tau)-$diophantine vectors and $f \in H^{s+2\tau+1}$, one has

$$|(\omega \cdot \partial_\varphi)^{-1} f|_s \leq \gamma^{-1}|f|_{s+\tau}$$
$$|(\omega \cdot \partial_\varphi)^{-1} f|_s^{\gamma,\mathcal{O}} \leq \gamma^{-1}|f|_{s+2\tau+1}^{\gamma,\mathcal{O}}$$

indeed

$$\begin{aligned}|(\omega \cdot \partial_\varphi)^{-1} f|_s^2 &= \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2s}\Big|\Big((\omega\cdot\partial_\varphi)^{-1}f\Big)_k\Big|^2 = \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}\Big|\frac{f_k}{\omega\cdot l}\Big|^2 \leq \\ &\leq \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}\Big|\gamma^-1 f_k\langle l\rangle^\tau\Big|^2 \leq \gamma^{-2}\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2(s+\tau)}\big|f_k\big|^2\end{aligned}$$

$$\begin{aligned}|(\omega \cdot \partial_\varphi)^{-1} f|_s^{\mathrm{lip},\mathcal{O}} = &\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}\Big|\frac{\Delta_\omega f_k}{\omega\cdot l}\Big|^2 \\ &+ \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}\Big|f_k\Delta_\omega\frac{1}{\omega\cdot l}\Big|^2 \leq \gamma^{-1}|\Delta_\omega f|_{s+\tau} + \gamma^{-2}|f|_{s+2\tau+1}\end{aligned}$$

and

$$|(\omega \cdot \partial_\varphi)^{-1} f|_s^{\gamma,\mathcal{O}} \leq \gamma^{-1}|f|_{s+\tau} + \gamma^{-1}|f|_{s+2\tau+1} + \gamma(\gamma^{-1}|\Delta_\omega f|_{s_0+\tau})$$

so the Lemma is proved with $\sigma = 2\tau + 1$. □

Lemma 3.2.4. *For m defined in (3.2.4), $e(\varphi, x)$ defined in (3.2.3) and $a(\varphi, x)$ defined (3.2.8), the following bounds hold:*

i. $|\mathcal{T}^{-1}V|_s^{\gamma,\mathcal{O}} \lesssim_{s,d} \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+1}^{\gamma,\mathcal{O}}$

ii. $|m| \le |V|_s^{\gamma,\mathcal{O}}$

iii. $|e|_s^{\gamma,\mathcal{O}}$, $|m-e|_s^{\gamma,\mathcal{O}}$, $|a|_s^{\gamma,\mathcal{O}} \lesssim_{s,d} \gamma^{-1}|V|^{\gamma,\mathcal{O}}_{s+\frac{d}{2}+2\tau+2}$

Proof. i. By the composition Lemma A.5, the Sobolev embedding and the Lemma 3.2.3 we have

$$
\begin{aligned}
|\mathcal{T}^{-1}V|_s^{\gamma,\mathcal{O}} &\lesssim_{s,d} \left(|V|_{s+1}^{\gamma,\mathcal{O}} + |p(\varphi)|_{s,\infty}^{\gamma,\mathcal{O}}|V|_2^{\gamma,\mathcal{O}}\right) \\
&\lesssim_{s,d} \left(|V|_{s+1}^{\gamma,\mathcal{O}} + |p(\varphi)|_{s+\frac{d}{2}}^{\gamma,\mathcal{O}}|V|_2^{\gamma,\mathcal{O}}\right) \lesssim_{s,d} \\
&\lesssim_{s,d} \left(|V|_{s+1}^{\gamma,\mathcal{O}} + \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+1}^{\gamma,\mathcal{O}}|V|_2^{\gamma,\mathcal{O}}\right) \lesssim_{s,d} \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+1}^{\gamma,\mathcal{O}}
\end{aligned}
$$

ii. The bound holds since m is the mean value of V.

iii. By definition of e and a and by the Lemma 3.2.3 and the first item of this Lemma we have:

$$
\begin{aligned}
|e|_s^{\gamma,\mathcal{O}} &= \left|\omega\cdot\partial_\varphi p(\varphi) + \mathcal{T}^{-1}V\right|_s^{\gamma,\mathcal{O}} \le |p|_{s+1}^{\gamma,\mathcal{O}} + |\mathcal{T}^{-1}V|_s^{\gamma,\mathcal{O}} \lesssim_{s,d} \\
&\lesssim_{s,d} \gamma^{-1}|V|_{s+2\tau+2}^{\gamma,\mathcal{O}} + \gamma^{1}|V|_{s+\frac{d}{2}+2\tau+2}^{\gamma,\mathcal{O}} \lesssim_{s,d} \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+2}^{\gamma,\mathcal{O}}
\end{aligned}
$$

$$
\begin{aligned}
|m-e|_s^{\gamma,\mathcal{O}} &\lesssim_{s,d} |V|_s^{\gamma,\mathcal{O}} + \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+2}^{\gamma,\mathcal{O}} \lesssim_{s,d} \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+2}^{\gamma,\mathcal{O}} \\
|a|_s^{\gamma,\mathcal{O}} &= \left|\frac{1}{3}\partial_x^{-1}(m-e(\varphi,x))\right|_s^{\gamma,\mathcal{O}} \le \frac{1}{3}|m-e|_s^{\gamma,\mathcal{O}} \lesssim_{s,d} \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+2}^{\gamma,\mathcal{O}}
\end{aligned}
$$

□

Recalling the definition of the operators p_0, p_1 defined respectively in (3.2.6) and (3.2.7), we have the following Lemma.

Lemma 3.2.5. *The following bound hold:*

i. $|p_0|_s^{\gamma,\mathcal{O}} \lesssim_{s,d} \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+4}^{\gamma,\mathcal{O}}$

ii. $|p_{-1}|_s^{\gamma,\mathcal{O}} \lesssim_{s,d} \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+5}^{\gamma,\mathcal{O}}$

Proof. These bound follow easily from the definition of the operators and the algebra property of the decay norm. □

Let us now define

$$
A := \frac{1}{3}\partial_x^{-1}(m-e(\varphi,x))\partial_x^{-1}\,,
$$

This is a bounded operator on the space $\mathtt{h}(\mathbb{Z}^d)$ with finite decay norm, since it is a composition of a multiplication operator, see Remark 1.2.2 with diagonal operators. We have

$$
|A|_s^{\mathrm{dec},\gamma,\mathcal{O}} \lesssim_s |V|_{s+\frac{d}{2}+2\tau+2}\,.
$$

In particular this implies that A belongs to a Banach algebra with the tame product property.
Also to the operators $\mathcal{S}$ and $\mathcal{T}$ we can associate the operators S, T acting on $\mathtt{h}^s(\mathbb{Z}^d)$ and Töplitz in time.

Lemma 3.2.6. *With the same hypothesis of the Lemma 1.2.5, if* $C(s)|A|_s < \frac{5-\sqrt{13}}{6}$ *then*

$$|S^{-1} - \mathbb{I}|_s \leq C(s)|A|_s(1 + 3C(s)|A|_s).$$

Proof. As a consequence of the Lemma 1.2.5 we deduce that

$$\begin{aligned}|S^{-1} - \mathbb{I}|_s &= \left|\sum_{k=1}^{\infty} A^k\right|_s \leq \sum_{k=1}^{\infty} |A^k|_s \leq C(s)|A|_s \sum_{k=1}^{\infty} k\Big(C(s)|A|_s\Big)^{k-1} \\ &= C(s)|A|_s \frac{1}{(1 - C(s)|A|_s)^2} \leq C(s)|A|_s(1 + 3C(s)|A|_s)\end{aligned}$$

In the last inequality we use the smallness condition of the hypothesis. □

Remark 3.2.7. *In view of the Lemma 1.2.3 the decay norm satisfies the hypothesis of the Lemma 1.2.5. Hence this Lemma holds also for the decay norm so:*

$$|S^{-1} - \mathbb{I}|_s^{\mathrm{dec},\gamma,\mathcal{O}} \leq C(s)|A|_s^{\mathrm{dec},\gamma,\mathcal{O}}(1 + 3C(s)|A|_s^{\mathrm{dec},\gamma,\mathcal{O}}).$$

By remark 3.2.7 and Lemma 3.2.6 we deduce

$$\begin{aligned}|S^{-1}|_s^{\mathrm{dec},\gamma,\mathcal{O}} &= |\mathbb{I} + (S^{-1} - \mathbb{I})|_s^{\mathrm{dec},\gamma,\mathcal{O}} \leq |\mathbb{I}|_s^{\mathrm{dec},\gamma,\mathcal{O}} + |S^{-1} - \mathbb{I}|_s^{\mathrm{dec},\gamma,\mathcal{O}} \\ &\overset{3.2.6}{\leq} 1 + C(s)|A|_s^{\mathrm{dec},\gamma,\mathcal{O}}(1 + 3C(s)|A|_s^{\mathrm{dec},\gamma,\mathcal{O}}) \leq 1 + 2C(s)|A|_s^{\mathrm{dec},\gamma,\mathcal{O}} \\ &\lesssim_{s,d} 1 + \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+2}^{\gamma,\mathcal{O}}\end{aligned}$$

And finally, for the algebra property we obtain the estimate of the operator $\mathcal{R}$:

$$\begin{aligned}|R|_s^{\mathrm{dec},\gamma,\mathcal{O}} &\leq |S^{-1}|_s^{\mathrm{dec},\gamma,\mathcal{O}}|p_0 + p_{-1}\partial_x^{-1}|_s^{\mathrm{dec},\gamma,\mathcal{O}} \lesssim_{s,d} (1 + \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+2}^{\gamma,\mathcal{O}})\gamma^{-1}|V|_{2s+2\tau+5}^{\gamma,\mathcal{O}} \\ &\lesssim_{s,d} \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+5}^{\gamma,\mathcal{O}}\end{aligned}$$

Recalling the tameness property of the decay norm (see Lemma 1.2.3), we have also

$$|\langle \mathtt{d}\rangle^b \mathcal{R}|_s^{\mathrm{dec},\gamma,\mathcal{O}} \lesssim_{s,d} \gamma^{-1}|V|_{s+\frac{d}{2}+2\tau+5+b}^{\gamma,\mathcal{O}}$$

Proof of Theorem 3. First of all, observe that $\mathcal{R} \rightsquigarrow R$ is a Töplitz in time operator. Indeed, by

- p_0, p_{-1}, a are moltiplication operator so they are Töplitz in both time and space,
- ∂_x^{-1} in Töplitz in time since it acts only on the space.

– $\mathbb{I}$ is Töplitz

follows that $p_0 + p_{-1}\partial_x^{-1}$ and S^{-1} is Töplitz in time and so R. Therefore, for Lemma 3.1.2 reducing (8) is equivalent to diagonalizing the operator

$$L_3 = \omega \cdot \partial_\varphi + \partial_{xxx} + m\partial_x + R\,.$$

By the discussion in Section 3.2.2 we can apply Theorem 5 to $\mathcal{L}_3$ Then there exists a list $\{\lambda_k^\infty\}_{k\in\mathbb{Z}^d} \in \mathcal{C}$. It remains to prove that the set $\mathcal{C}$ defined in (2.1.4) satisfies the measure estimate in 12. We will do that in the next section. In conclusion $\Phi := T \circ S \circ U$ is the operator searched in the Theorem 3. $\square$

3.2.3 Measure estimates and conclusion of the proof

We proved that the operator R satisfies the hypothesis of the Theorem 5 so we there exists a list $\{\lambda_k^{(\infty)}\}_{k\in\mathbb{Z}^d} \in \mathcal{C}$ defined in (2.1.4).
We want to give a precise formulation of the set $\mathcal{C}$ and show that it is not empty for γ sufficiently small. By Corollary 2.1.7, we have that the $\lambda_k^{(\infty)}$ have the form

$$\lambda_k^{(\infty)} = \mathrm{i}(\omega \cdot l - j^3 + mj) + r_j^\infty(\omega)\,.$$

It remains to prove is that the set of ω such that

$$\mathcal{G} := \left\{ |\lambda_k(\omega) - \lambda_{k'}(\omega)| > \frac{2\gamma}{\langle l - l'\rangle^\tau}, \quad k = (l,j), l \in \mathbb{Z}^d, j \in \mathbb{Z} \right\} \subset \mathcal{C} \tag{3.2.10}$$

satisfies (12). We are dropping the superscript (∞) in λ_k, r_j for easier notation.
First of all we consider two trivial cases:
Case 1 : $j = j'$ and $\nu := l - l' \neq 0$
For the corollary 2.1.7 we know that eigenvalues are of the form

$$\lambda_k = \mathrm{i}\omega \cdot l + \Delta_j = \mathrm{i}\omega \cdot l - \mathrm{i}j^3 + \mathrm{i}mj + r_j(\omega)$$

so the set of the ω that satisfies 3.2.10 is the set of the diophantine numbers.
Indeed $\lambda_k(\omega) - \lambda_{k'}(\omega) = \omega \cdot (l - l') = \omega \cdot \nu$.
Case 2 $\nu = 0 \to j \neq j'$

$$\begin{aligned}
|\lambda_k(\omega) - \lambda_{k'}(\omega)| &= |-\mathrm{i}(j^3 - j'^3) + m\mathrm{i}(j - j') + r_j - r_{j'}| \\
&\geq |j^3 - j'^3| - |m||j - j'| - |r_j| - |r_{j'}| = \\
&= |j - j'|\Big(|j^2 + j'^2 + jj'| - m\Big) - 2C\gamma\varepsilon_0 \\
&\geq (1 - \varepsilon_0) - 2C\gamma\varepsilon_0 \geq \frac{1}{2}
\end{aligned}$$

where we use that $|r_j| \le C\gamma\varepsilon_0$ for the corollary 2.1.6 with $n = 0$.
Now let's consider the general case. Define the set

$$\mathcal{R}^{\nu}_{j,j'} := \left\{ \omega \in \mathcal{O}_0 : |\lambda_k(\omega) - \lambda_{k'}(\omega)| \le \frac{\gamma}{\langle \nu \rangle^{\tau}} \right\}$$

Ask the measure of the set of the intersection of all the set that satisfies the condition 3.2.10 is equivalent to measure the complementary set of

$$\bigcup_{\nu \in \mathbb{Z}^d j,j' \in \mathbb{Z}} \mathcal{R}^{\nu}_{j,j'}.$$

For the additivity of the measure we need to calculate the measure of $\mathcal{R}^{\nu}_{j,j'}$. Before the general case, let's see an easy one: $r_j = r_{j'} = 0$
Intuitively we are measuring the volume of the intersection of the space between the 2 planes

$$\mathrm{i}\omega \cdot \nu - \mathrm{i}(j^3 - j'^3) + \mathrm{i}m(j - j') = \pm \frac{\gamma}{\langle l - l' \rangle^{\tau}}$$

and the hypercube $\omega \in [1,2]^d$ so we can imagine that at most it is $\zeta^{d-1} 2 \frac{\gamma}{\langle \nu \rangle^{\tau}}$ where ζ is the diagonal of the hypercube. Let's prove it in a more rigorous way.
Since are constant in ω, for convenience, we call $L_{j,j'} = j^3 - j'^3 - m(j - j')$ and $\alpha = \frac{\gamma}{\langle \nu \rangle^{\tau}}$. We want to measure $\mathrm{i}\omega \cdot \nu - \mathrm{i}L_{j,j'} = t$ with $|t| \le \alpha$. We do a first orthogonal change of variables such that the new variables ξ_i satisfy $\xi_1 \parallel \nu$ and $\xi_i \perp \nu$, $i \ne 1$ so the planes become $\xi_1|\nu| + L_{j,j'} = t$.
Now we do an other change of variable to explicit ξ_1

$$F : (\xi_1, x_2, \dots, \xi_d) \to (t, x_2, \dots, \xi_d)$$

$$\xi_1 = \frac{t - L_{j,j'}}{|\nu|} \qquad d\xi_1 = \frac{dt}{|\nu|}$$

Now the measure is the product the integral

$$\int_{-\alpha}^{\alpha} \frac{dt}{|\nu|} dt = \frac{2\alpha}{|\nu|} = 2\frac{\gamma}{\langle \nu \rangle^{\tau+1}}$$

and the integral of the other ξ_i in the hypercube that at most are ζ^{d-1}. So we obtain

$$\mu(\mathcal{R}^{\nu}_{j,j'}) \sim \frac{\gamma}{\langle \nu \rangle^{\tau+1}}$$

For the general case the strategy is the same, the new difficulty is that$L_{j,j'}$ depends on ω and after the first change of variables depends on ξ_i so we need the Impicit Function Theorem in Lipschitz class to explicit ξ_1 and conclude in the same way of the case 1.

Lemma 3.2.8. *Fix ν. If $\mathcal{R}^{\nu}_{j,j'} \ne \emptyset$ then $j, j' \le D\sqrt{|\nu|}$.*

Proof.

$$|j^3 - j'^3| = |j - j'||j^2 + jj' + j'^2| \geq (j^2 + j'^2 - |jj'|)|j - j'| \geq \frac{1}{2}(j^2 + j'^2)|j - j'|$$

Suppose that $\min j, j' > \sqrt{|\nu|}$

$$\begin{aligned}
|\lambda_k(\omega) - \lambda_{k'}(\omega)| &\geq | - \mathrm{i}(j^3 - j'^3) + \mathrm{i}m(j - j') + r_j - r_{j'}| \\
&\geq |j^3 - j'^3| - |m||j - j'| - |r_j| - |r_{j'}| \\
&\geq \frac{1}{2}(j^2 + j'^2)|j - j'| - |m||j - j'| - |r_j| - |r_{j'}| \\
&= |j - j'|[\frac{1}{2}(j^2 + j'^2) - m] - |r_j| - |r_{j'}| \\
&\geq (|\nu| - \varepsilon_0) - 2C\gamma\varepsilon_0) \geq \frac{1}{4}
\end{aligned}$$

Then the corresponding resonant set $\mathcal{R}^{\nu}_{j,j'}$ is empty for γ sufficiently small.

□

$$\begin{aligned}
\mu\Big(\bigcup_{\nu\in\mathbb{Z}^d\, j,j'\in\mathbb{Z}} \mathcal{R}^{\nu}_{j,j'}\Big) &= \sum_{\nu\in\mathbb{Z}^d\, j,j'\in\mathbb{Z}} \mu(\mathcal{R}^{\nu}_{j,j'}) \leq \sum_{\nu\in\mathbb{Z}^d\, j,j'\in\mathbb{Z}} \frac{\gamma}{\langle\nu\rangle^{\tau+1}} \\
&\overset{3.2.8}{\leq} \text{Const.} \sum_{r:\, |r|^2\in\mathbb{N}} \langle r\rangle \frac{\gamma}{\langle r\rangle^{\tau+1}} \langle r\rangle^{d-1} = \sum_{r:\, |r|^2\in\mathbb{N}} \frac{\gamma}{\langle r\rangle^{\tau-d+1}}
\end{aligned}$$

that is summable if and only if $\tau - d + 1 > 1$, i.e. $\tau > d$.

3.3 Proof of the Theorem 4

Consider $\alpha \in H^S(\mathbb{T}^d)$ with S sufficiently large be a function such that $|\alpha|_{s_0+\sigma}$ sufficiently small. Denote $\vec{\alpha} := (\alpha, 0, \ldots 0) \in \mathbb{R}^d$ and consider the linear operator

$$\mathtt{C}_\alpha u(\theta) := u(\theta + \vec{\alpha}(\theta))\,.$$

whose matrix representation is given by

$$(\mathtt{C}_\alpha)^{k'}_k = \big(\widehat{e^{ik'\cdot\vec{\alpha}(\theta)}}\big)_{k-k'} \tag{3.3.1}$$

where $\hat{g}_h$ is the h^{th} Fourier coefficient of the function g.
Indeed

$$\mathtt{C}_\alpha e^{ik'\cdot\theta} = e^{ik'\cdot(\theta+\vec{\alpha}(\theta))} = \sum_k \big(\widehat{e^{ik'\cdot(\theta+\vec{\alpha}(\theta))}}\big)_k e^{ik'\cdot\theta} = \sum_k \big(\widehat{e^{ik'\cdot\vec{\alpha}(\theta)}}\big)_{k-k'} e^{i(k-k')\cdot\theta}$$

Writing $\theta = (x, \varphi)$ we have $\mathtt{C}_\alpha u(\theta) = u(x + \alpha(\theta), \varphi)$ and consider the operator[2]

$$P := \Big(\mathtt{C}_\alpha - \mathbb{I}\Big)\partial_x^{-N} \qquad \text{and } A^P = (P^B, P^U)\,.$$

We want to apply the Theorem 6 so we need to prove the smallness of A^L in low norm and the finiteness of $\langle d\rangle^b A^L$ in high norm.
Consider the equation (15) and its associated operator

$$\mathcal{L} := \omega \cdot \partial_\varphi + \partial_{xxx} + (\mathtt{C}_\alpha - \mathbb{I})\partial_x^{-N}\,, \tag{3.3.2}$$

By reference to the notation of Theorem 6 now we have that again the role of $\xi \in \mathbb{R}^n$ is assumed by $\omega \in \mathcal{O} \subset \mathbb{R}^{d-1}$ and

$$\Lambda_0(\xi) \rightsquigarrow \Lambda_0(\omega) = \omega \cdot \partial_\varphi + \partial_{xxx}\,, \quad P_0 \rightsquigarrow (\mathtt{C}_\alpha - \mathbb{I})\partial_x^{-N}\,, \quad \mathcal{O}_0 = \mathcal{O}\,.$$

Λ_0 satisfy the hypothesis (**H1**) of page 16, indeed $\Lambda_0 = \mathrm{diag}_{k=(l,j)\in\mathbb{Z}^d}\lambda_k^{(0)}$ with

$$\lambda_k^{(0)} = \omega \cdot l + j^3$$

where it is clear that $\lambda_k^{(0)} \neq \lambda_{k'}^{(0)}$ if $k \neq k'$ and

$$|\lambda_k - \lambda_{k'}|^{\mathrm{lip},\mathcal{O}} \le |l - l'| < |k - k'|.$$

Now estimate the perturbation $(\mathtt{C}_\alpha - \mathbb{I})\partial_x^{-N}$ in view to apply the Theorem 6 and prove 4.
Since L does not depend on ω, in the following estimates we will not use the weighted norm.

Lemma 3.3.1. *We have that*

$$\| A^P \|_{s_0} \le 2^{s_1+s_0+2}|\alpha|_{s_0+2\beta+1} \tag{3.3.3}$$
$$\| \langle \mathtt{d}\rangle^b A^P \|_s \lesssim_{s,d} 2^{s_1-s_0}|\alpha|_{s+\beta+\frac{d}{2}+b+1} + \mathtt{K}_s(1 + |\alpha|^2_{s+d+1}) \tag{3.3.4}$$

Proof. First of all, note that for the Mean value Theorem

$$\Big(\mathtt{C}_\alpha - \mathbb{I}\Big)\partial_x^{-N} = \alpha(x,\varphi)\int_0^1 \mathtt{C}_{\tau\alpha}\partial_x^{-N+1}\,d\tau \tag{3.3.5}$$

[2]Note that $P = \mathfrak{S}A^P$.

Using that, for the first inequality we have:

$$
\begin{aligned}
\| A^P \|_{s_0} &= \sup_{s_0 \le p \le s_1} |P^B|_{p,p} + |P^U|_{s_0,s_0} \overset{(1.3.8)}{\le} \sup_{s_0 \le p \le s_1} 2^{p-s_0} |L|_{s_0,s_0} + |L^U|_{s_0,s_0} \\
&\le 2^{s_1-s_0+1} |L|_{s_0,s_0} \overset{(3.3.5)}{=} 2^{s_1-s_0+1} |\alpha \int_0^1 \mathtt{C}_{\tau\alpha} \partial_x^{-N+1} \, d\tau|_{s_0,s_0} \\
&\le 2^{s_1-s_0+1} |\alpha|_{s_0} | \int_0^1 \mathtt{C}_{\tau\alpha} \partial_x^{-N+1} \, d\tau|_{s_0,s_0} \\
&\le 2^{s_1-s_0+1} |\alpha|_{s_0} | \sup_{0<\tau<1} |\mathtt{C}_{\tau\alpha} \partial_x^{-N+1}|_{s_0,s_0} \\
&\overset{(1.1.9)}{\lesssim_d} \; 2^{s_1-s_0+1} |\alpha|_{s_0} \sup_{0<\tau<1} \Big(\|\mathtt{C}_{\tau\alpha} \partial_x^{-N+1}\|_{s_0,s_0+\beta} + \|\mathtt{C}_{\tau\alpha} \partial_x^{-N+1}\|_{s_0-\beta,s_0} \Big) \\
&\overset{(1.1.1)}{\lesssim_d} \; 2^{s_1-s_0+1} |\alpha|_{s_0} \sup_{0<\tau<1} \Big(\|\mathtt{C}_{\tau\alpha} \partial_x^{-N+\beta+1}\|_{s_0+\beta,s_0+\beta} + \|\mathtt{C}_{\tau\alpha}\|_{s_0+\beta,s_0+\beta} \\
&+ \|\mathtt{C}_{\tau\alpha} \partial_x^{-N+\beta+1}\|_{s_0,s_0} + \|\mathtt{C}_{\tau\alpha}\|_{s_0,s_0} \Big) \\
&\lesssim_d 2^{s_1-s_0+2} |\alpha|_{s_0} \sup_{0<\tau<1} \Big(\|\mathtt{C}_{\tau\alpha}\|_{s_0+\beta,s_0+\beta} + \|\mathtt{C}_{\tau\alpha}\|_{s_0,s_0} \Big) \\
&\lesssim_d 2^{s_1-s_0+2} |\alpha|_{s_0} \sup_{0<\tau<1} \Big(\sup_{|u|_{s_0+\beta}<1} (|\mathtt{C}_{\tau\alpha} u|_{s_0+\beta}) + \sup_{|u|_{s_0}<1} (|\mathtt{C}_{\tau\alpha} u|_{s_0}) \Big) \\
&\overset{(A.5)}{\lesssim_{d,s_0}} \; 2^{s_1-s_0+2} |\alpha|_{s_0} \sup_{0<\tau<1} \Big(\sup_{|u|_{s_0+\beta}<1} (|u|_{s_0+\beta} + |D\tau\alpha|_{s_0+\beta-1} |u|_1) \\
&+ \sup_{|u|_{s_0}<1} (|u|_{s_0} + |D\tau\alpha|_{s_0-1} |u|_1) \Big) \\
&\lesssim_{d,s_0} 2^{s_1-s_0+2} |\alpha|_{s_0} \sup_{0<\tau<1} \Big(2 + |\tau\alpha|_{s_0+2\beta-1} + |\tau\alpha|_{s_0-1} \Big) \\
&\lesssim_{d,s_0} 2^{s_1-s_0+2} |\alpha|_{s_0} \Big(2 + |\alpha|_{s_0+2\beta-1} + |\alpha|_{s_0-1} \Big) \\
&\lesssim_{d,s_0} 2^{s_1-s_0+2} |\alpha|_{s_0+2\beta-1}
\end{aligned}
$$

Remark 3.3.2. *Note that here we used that $N > \beta + 1$*

Observing that $(\langle \mathtt{d} \rangle^b P)^B = \langle \mathtt{d} \rangle^b P^B$ and the same for L^U, we estimate

$$
\| \langle \mathtt{d} \rangle^b A^P \|_s = \sup_{s_0 < p < s_1} |\langle \mathtt{d} \rangle^b P^U|_{p,p} + |\langle \mathtt{d} \rangle^b P^B|_{s_0,s} \overset{(1.3.8)}{\le} \sup_{s_0 < p < s_1} 2^{p-s_0} |\langle \mathtt{d} \rangle^b P|_{s_0,s_0} + |\langle \mathtt{d} \rangle^b P^U|_{s_0,s}
$$

Let us start with the estimate on the Bony part and let us do it with $b = 1$.
First of all observe that $[\partial_{\theta_i}, (\mathtt{c}_\alpha - \mathbb{I})\partial_x^{-N}] = [\partial_{\theta_i}, \mathtt{c}_\alpha \partial_x^{-N}]$, $\quad \theta = (\theta_1, \theta_2, \dots) = (x, \varphi)$.
Then the action of commutator between ∂_{θ_i} and $\mathtt{c}_\alpha \partial_x^{-N}$ on a function u assume the

form:

$$
\begin{aligned}
[\partial_{\theta_i}, \mathtt{c}_\alpha \partial_x^{-N}]u &= \partial_{\theta_i}[\mathtt{c}_\alpha \partial_x^{-N} u] - \mathtt{c}_\alpha \partial_x^{-N}(\partial_{\theta_i} u) \\
&= \partial_{\theta_i}(\partial_x^{-N} u(x+\alpha(\theta),\varphi)) - (\partial_x^{-N}\partial_{\theta_i} u)(x+\alpha(\theta),\varphi) \\
&= \partial_x^{-N} u_x(x+\alpha(\theta),\varphi))\partial_{\theta_i}\alpha(\theta) + (\partial_x^{-N}\partial_{\theta_i} u)(x+\alpha(\theta),\varphi) - (\partial_x^{-N}\partial_{\theta_i} u)(x+\alpha(\theta),\varphi) \\
&= \partial_x^{-N} u_x(x+\alpha(\theta),\varphi))\partial_{\theta_i}\alpha(\theta)
\end{aligned}
$$

In view of the estimate of the of the Bony part of the operator $\langle \mathtt{d}\rangle^b L$, using the previous formula, the following estimate hold:

$$
\begin{aligned}
|\partial_x^{-N+\beta} u_x(x+\alpha(\theta),\varphi))\partial_{\theta_i}\alpha(\theta)|_s &\leq |\partial_x^{-N+\beta} u_x(x+\alpha(\theta),\varphi))|_s |\partial_{\theta_i}\alpha(\theta)|_s \\
&\leq |u(x+\alpha(\theta),\varphi))|_{s+1}|\alpha(\theta)|_{s+1} \\
&\lesssim_{d,s} |\alpha(\theta)|_{s+1}(|u|_{s+1} + |D\alpha|_{s,\infty}|u|_1) \\
&\lesssim_{d,s} |\alpha(\theta)|_{s+1}(|u|_{s+1} + |\alpha|_{s+\frac{d}{2}}|u|_1)
\end{aligned}
$$

In the end, using the Lemma 1.1.9, the previous estimate, the composition Lemma A.5 and the Sobolev embedding and recalling that $N > \beta + 1$, we obtain:

$$
\begin{aligned}
|[\partial_{\theta_i}, \mathtt{c}_\alpha \partial_x^{-N}]|_{s,s} &\overset{1.1.9}{\lesssim_d} \|[\partial_{\theta_i}, \mathtt{c}_\alpha \partial_x^{-N}]\|_{s,s+\beta} + \|[\partial_{\theta_i}, \mathtt{c}_\alpha \partial_x^{-N}]\|_{s-\beta,s} \\
&= \|[\partial_{\theta_i}, \mathtt{c}_\alpha \partial_x^{-N+\beta}]\|_{s+\beta,s+\beta} + \|[\partial_{\theta_i}, \mathtt{c}_\alpha \partial_x^{-N+\beta}]\|_{s,s} \\
&= \sup_{|u|_{s+\beta}\leq 1} |\partial_x^{-N} u_x(x+\alpha(\theta),\varphi))\partial_{\theta_i}\alpha(\theta)|_{s+\beta} \\
&\quad + \sup_{|u|_s\leq 1} |\partial_x^{-N} u_x(x+\alpha(\theta),\varphi))\partial_{\theta_i}\alpha(\theta)|_s \\
&\lesssim_{d,s} \sup_{|u|_{s+\beta}\leq 1} |\alpha(\theta)|_{s+\beta+1}(|u|_{s+\beta+1} + |\alpha|_{s+\beta+\frac{d}{2}}|u|_1) \\
&\quad + \sup_{|u|_s\leq 1} |\alpha(\theta)|_{s+1}(|u|_{s+1} + |\alpha|_{s+\frac{d}{2}}|u|_1) \\
&\lesssim_{d,s} |\alpha(\theta)|_{s+\beta+1}(1 + |\alpha|_{s+\beta+\frac{d}{2}}) + |\alpha(\theta)|_{s+1}(1 + |\alpha|_{s+\frac{d}{2}}) \\
&\lesssim_{d,s} |\alpha|_{s+\beta+\frac{d}{2}+1}
\end{aligned}
$$

Hence we have that

$$
|\langle \mathtt{d}\rangle^b P|_{s,s} \lesssim_{d,s} |\alpha|_{s+\beta+\frac{d}{2}+b+1}\,.
$$

Consider now ultraviolet part $\langle \mathtt{d}\rangle^b L^U$. Since

$$
\Big((\mathtt{C}_\alpha - \mathbb{I})_k^{k'}(\partial_x^{-N})^{k'}\Big)^U = \Big((\mathtt{C}_\alpha)_k^{k'}\Big)^U (ik')^{-N}\,.
$$

then we have

$$
|\langle \mathtt{d}\rangle^b P^U|^2_{s_0,s} \overset{(1.3.8)}{\leq} C_s\Big(\sum_h |h|^{2s} \sup_{\substack{k-k'=h \\ |h|>|k|/2}} |(\mathtt{C}_\alpha)_k^{k'}|^2\Big) \overset{3.3.3}{\leq} (1+|\alpha|^2_{s+d+1}C_s)\mathtt{K}_s\Big(\sum_h |h|^{2s}\frac{1}{|h|^{2s+d+1}}\Big)\,,
$$

where in the second inequality we have used the Lemma 3.3.3 that we state and prove below. $\square$

Lemma 3.3.3. *Consider $\alpha \in H^{s+d+1}(\mathbb{T})$ and such that $|\alpha|_{1,\infty} := \sum_{|\beta|\le 1} |D^\beta u|_{L^\infty} \le \frac{1}{2}$. Then for $|j|/|\xi| \ge 2/3$ one has*

$$|\big(\widehat{e^{i\xi\alpha(x)}}\big)_j| = |\int_0^{2\pi} e^{i(\xi\alpha(x)-jx)}dx| \le \frac{\mathtt{K}_s}{|j|^{s+\frac{d}{2}+1}}(1+|\alpha|_{s+d+1}),$$

for a suitable constant $\mathtt{K}_s > 1$.

Proof. We set $\eta := \xi/k$, with $|\eta| \le 3/2$, and note that $y = x - \eta\alpha(x)$ is a well defined change of variables on the circle with inverse $x = y + \eta\zeta(y,\eta)$, by the composition Lemma A.5. Note that $\zeta \in H^{s+d+1}$ again by Lemma A.5 and $|\beta|_{1,\infty} \le 2|\alpha|_{1,\infty}$. Now with this in mind, we make the change of variable $x = y + \eta\zeta(y,\eta)$ in the integral and we get

$$\big(\widehat{e^{i\xi\alpha(x)}}\big)_j = \int_0^{2\pi} (1+\eta\zeta_y(y,\eta))e^{-ijy}dy = \hat{g}_j$$

where

$$g(y,\eta) := 1 + \eta\zeta_y(y,\eta).$$

Since, if $g \in H^s$

$$|\widehat{g}_j| \le \frac{|g|_{s-1}}{|j|^{s-1}},$$

by the standard composition rules in Sobolev spaces, we have

$$|\big(\widehat{e^{i\xi\alpha(x)}}\big)_j| = \frac{|g|_{s-1}}{|j|^{s-1}} \lesssim_{d,s} (1+|D\zeta|_{s-2}) \lesssim_{d,s} (1+|D\alpha|_{s-2}) \lesssim_{d,s} (1+|\alpha|_{s-1})$$

$\square$

Note that the same property holds also if α depends on φ.

Proof of Theorem 4. The perturbation $(\mathtt{C}_\alpha - \mathbb{I})\partial_x^{-N}$ is Töplitz since ∂_x^{-N} acts only on the space, $\mathbb{I}$ is Töplitz and

$$(\mathtt{C}_\alpha)_k^{k'} = \big(\widehat{e^{ik'\cdot\vec{\alpha}(\theta)}}\big)_{k-k'} = \big(\widehat{e^{ij'\alpha(\theta)+(j'-j)x}}\big)_{l-l'}.$$

Therefore, for Lemma 3.1.2 reduce (15) is equivalent to diagonalize the operator

$$\omega\cdot\partial_\varphi + \partial_{xxx} + (\mathtt{C}_\alpha - \mathbb{I})\partial_x^{-N}.$$

From Lemma 3.3.1 we have

$$\| A^P \|_{s_0} \le 2^{s_1+s_0+2}|\alpha|_{s_0+2\beta+1}, \quad \| \langle \mathtt{d}\rangle^b A^P \| \lesssim_{s,d} 2^{s_1-s_0}|\alpha|_{s+\beta+\frac{d}{2}+b+1} + \mathtt{K}_s(1+|\alpha|^2_{s+d+1})$$

Then the operator P satisfies the hypothesis of the Theorems 5 and 6 so we there exists a list $\{\lambda_k^{(\infty)}\}_{k\in\mathbb{Z}^d} \in \mathcal{C}$ defined in (2.1.4) such that in this set $\mathcal{C}$ the Töplitz in time operator

P associated to the operator (3.3.2) is conjugated to $\Lambda_\infty = \mathrm{diag_k} \lambda_k^{(\infty)}$.
By the Corollary 2.1.7 we have that

$$\lambda_k^{(\infty)} = \mathrm{i}\omega \cdot l - \mathrm{i}j^3 + r_j^{(\infty)}$$

Thus we are in the setting of the Section 3.2.3 with $m = 0$. Therefore, also in this case the set $\mathcal{C}$ is not empty. $\square$

Appendix A

Technical results and useful tools

In this Chapter we collect the proofs of technical results and some useful tools.

A.1 Proof of Proposition 1.1.9

Let $s \geq 0$. Since H^s is a subspace of $H^0 = \ell^2$, denoting by $e^{(k)}$, $k \in \mathbb{Z}$ the standard orthonormal basis of ℓ^2, namely $e^{(k)} = (\ldots, 0, 1, 0, \ldots)$, we have that the standard basis of H^s, is

$$e^{(k,s)} = (\ldots, 0, \langle k \rangle^{-s}, 0, \ldots) = D^{-s} e^{(k)} ,$$

where D^s is the infinite matrix

$$D^s := \mathrm{diag}_k \langle k \rangle^s .$$

If $x \in H^s$ has ℓ^2-coordinates $x = (x_k)$, namely $x = \sum_k x_k e^{(k)}$, its H^s-coordinates are $\hat{x}^{(s)} = (\hat{x}_k^{(s)})_{k \in \mathbb{Z}}$, namely

$$x = D^{-s} \hat{x}^{(s)} = \sum_k \hat{x}_k^{(s)} e^{(k,s)} .$$

Note that $\hat{x}^{(s)} \in \ell^2$ and[1]

$$|x|_s = |\hat{x}^{(s)}|_0 . \tag{A.1.1}$$

Let $s' \geq 0$ and consider a bounded linear operator $A \in \mathcal{L}(H^s, H^{s'})$. Since H^s and $H^{s'}$ are subspace of $\ell^2 = H^0$, we can represent A as an infinite matrix[2] $A = (A_k^{k'})_{k,k' \in \mathbb{Z}}$ in the ℓ^2-coordinates, namely, if $x \in H^s$ and $y \in H^{s'}$ with ℓ^2-coordinates $x = (x_{k'})$, $y = (y_k)$, and $y = Ax$, then $y_k = \sum_{k'} A_k^{k'} x_{k'}$. On the other hand, using the coordinates of H^s and $H^{s'}$, namely writing $x = D^{-s} \hat{x}^{(s)}$ and $y = D^{-s'} \hat{y}^{(s')}$, we get, by $y = Ax$, that

$$\hat{y}^{(s)} = \hat{A} \hat{x}^{(s)} , \qquad \text{where} \quad \hat{A} := D^{s'} A D^{-s} \in \mathcal{L}(H^0, H^0) .$$

[1] This is the usual isometry between the separable Hilbert space H^s and $H^0 := \ell^2$.

[2] With abuse of notation we denote by A both the operator and its ℓ^2-representation by the infinite matrix $A = (A_k^{k'})$.

In particular

$$\|A\|_{s,s'} = \|\hat{A}\|_{0,0}\,, \tag{A.1.2}$$

since by (A.1.1) (used with s and s')

$$\|A\|_{s,s'} = \sup_{|x|_s=1} |Ax|_{s'} = \sup_{|\hat{x}^{(s)}|_0=1} |AD^{-s}\hat{x}^{(s)}|_{s'} = \sup_{|\hat{x}^{(s)}|_0=1} |\widehat{AD^{-s}\hat{x}^{(s)}}|_0 = \sup_{|\hat{x}^{(s)}|_0=1} |\hat{A}\hat{x}^{(s)}|_0 = \|\hat{A}\|_{0,0}\,.$$

We immediately have that, for every A,

$$\widehat{(\partial A)} = \partial \hat{A}\,, \qquad \widehat{(\underline{A})} = \underline{(\hat{A})}\,. \tag{A.1.3}$$

Lemma A.1.1. *If $M, \partial_m^\beta M \in \mathcal{L}(H^0, H^0)$ for every $1 \le m \le d$, and*

$$\beta := \lfloor d/2 \rfloor + 1\,,$$

then $M \in \mathcal{M}(H^0, H^0)$ (i.e. $\underline{M} \in \mathcal{L}(H^0, H^0)$) and

$$|M|_{0,0} \le \|M\|_{0,0} + \; c_d \sum_{1 \le m \le d} \|\partial_m^\beta M\|_{0,0}\,.$$

Then Proposition 1.1.9 is a direct consequence of (A.1.2), (A.1.3) and Lemma A.1.1 (applied with $M := \hat{A}$). Indeed we have

$$\begin{aligned} |A|_{s,s'} = \|\underline{A}\|_{s,s'} = \|\widehat{\underline{A}}\|_{0,0} = |\hat{A}|_{0,0} \le \|\hat{A}\|_{0,0} + \; c_d \sum_{1\le m\le d} \|\partial_m^\beta \hat{A}\|_{0,0} \\ = \|A\|_{s,s'} + \; c_d \sum_{1\le m\le d} \|\widehat{(\partial_m^\beta A)}\|_{0,0} = \|A\|_{s,s'} + \; c_d \sum_{1\le m\le d} \|\partial_m^\beta A\|_{s,s'}\,. \end{aligned}$$

It remains to prove Lemma A.1.1. We first note that

$$\sum_{k'} |M_k^{k'}|\,, \; \sum_k |M_k^{k'}| \;\le\; \|M\|_{0,0} + \; c_d \sum_{1\le m\le d} \|\partial_m^\beta M\|_{0,0} =: \mu\,, \tag{A.1.4}$$

for a suitable $c_d > 1$. Fix $k' \in \mathbb{Z}^d$; let us split

$$\sum_k |M_k^{k'}| = |M_{k'}^{k'}| + S_1 + S_2 + \ldots + S_d\,, \tag{A.1.5}$$

where S_n, $1 \le n \le d$, is the sum over $k_1, \ldots, k_d \in \mathbb{Z}$ such that $n - d$ indexes are equal to the respective k'_j and the other d are different; for example one of the addenda of S_n is

$$\sum_{k_1 \ne k'_1} \cdots \sum_{k_n \ne k'_n} |M^{k'}_{(\tilde{k}, \hat{k}')}|$$

where, for brevity, $\tilde{k} := (k_1, \ldots, k_n)$ and $\hat{k}' := (k'_{n+1}, \ldots, k'_d)$. Denoting

$$\beta := \lfloor d/2 \rfloor + 1\,,$$

we estimate this the term[3] as, by Cauchy-Schwarz inequality,

$$\begin{aligned}
\sum_{k_1\neq k_1'}\cdots\sum_{k_n\neq k_n'}|M^{k'}_{(\tilde{k},\hat{k}')}| &= \sum_{k_1\neq k_1'}|M^{k'}_{(\tilde{k},\hat{k}')}|^{1/n}\cdots\sum_{k_n\neq k_n'}|M^{k'}_{(\tilde{k},\hat{k}')}|^{1/n}\\
&= \sum_{k_1\neq k_1'}\frac{|\partial_1^\beta M^{k'}_{(\tilde{k},\hat{k}')}|^{1/n}}{|k_1-k_1'|^{\beta/n}}\cdots\sum_{k_n\neq k_n'}\frac{|\partial_n^\beta M^{k'}_{(\tilde{k},\hat{k}')}|^{1/n}}{|k_n-k_n'|^{\beta/n}}\\
&\leq \left(\sum_{k_1\neq k_1'}\frac{1}{|k_1-k_1'|^{2\beta/n}}\right)^{1/2}\left(\sum_{k_1\neq k_1'}|\partial_1^\beta M^{k'}_{(\tilde{k},\hat{k}')}|^{2/n}\right)^{1/2}\\
&\cdots\left(\sum_{k_n\neq k_n'}\frac{1}{|k_n-k_n'|^{2\beta/n}}\right)^{1/2}\left(\sum_{k_n\neq k_n'}|\partial_n^\beta M^{k'}_{(\tilde{k},\hat{k}')}|^{2/n}\right)^{1/2}\\
&\leq c_d\left(\sum_{k_1\neq k_1'}\cdots\sum_{k_n\neq k_n'}|\partial_1^\beta M^{k'}_{(\tilde{k},\hat{k}')}|^{2/n}\cdots|\partial_n^\beta M^{k'}_{(\tilde{k},\hat{k}')}|^{2/n}\right)^{1/2}\\
&\leq c_d\left(\sum_{k_1\neq k_1'}\cdots\sum_{k_n\neq k_n'}\left(|\partial_1^\beta M^{k'}_{(\tilde{k},\hat{k}')}|^{2}+\cdots+|\partial_n^\beta M^{k'}_{(\tilde{k},\hat{k}')}|^{2}\right)\right)^{1/2}\\
&\leq c_d\left(\sum_{k\in\mathbb{Z}^d}|\partial_1^\beta M^{k'}_k|^2+\cdots+\sum_{k\in\mathbb{Z}^d}|\partial_n^\beta M^{k'}_k|^2\right)^{1/2}\\
&= c_d\left(|(\partial_1^\beta M)e^{(k')}|_0^2+\cdots+|(\partial_n^\beta M)e^{(k')}|_0^2\right)^{1/2}\\
&\leq c_d\left(\|\partial_1^\beta M\|_{0,0}^2+\cdots+\|\partial_n^\beta M\|_{0,0}^2\right)^{1/2}\\
&\leq c_d\left(\|\partial_1^\beta M\|_{0,0}+\cdots+\|\partial_n^\beta M\|_{0,0}\right)\\
&\leq c_d\sum_{1\leq m\leq d}\|\partial_m^\beta M\|_{0,0}\,,
\end{aligned}$$

[3]The other ones being analogous.

for[4] a suitable different constants $c_d > 1$. The same estimate holds for the other addenda in S_n and for all the term S_n's. Noting that $|M_{k'}^{k'}| \le |Me^{(k')}|_0 \le |M|_{0,0}$, we have proved (A.1.4) for $\sum_k |M_{k'}^k|$. The inequality for

$$\sum_{k'} |M_{k'}^k| = \sum_{k'} |(M^T)_k^{k'}|$$

where M^T is the transpose matrix of M, follows similarly noting that $\|M^T\|_{0,0} = \|M\|_{0,0}$ and that $\partial(A^T) = -(\partial A)^T$. This completes the proof of (A.1.4). Let us finish the proof of Lemma A.1.1. By Cauchy-Schwarz we get

$$\begin{aligned} |M|_{0,0}^2 &= \|\underline{M}\|_{0,0}^2 = \sup_{|x|_0=1} |\underline{M}x|_0^2 \le \sum_k \Big(\sum_{k'} |M_k^{k'}||x_{k'}|\Big)^2 \le \sum_k \Big(\sum_{k'} |M_k^{k'}|\Big)\Big(\sum_{k'} |M_k^{k'}||x_{k'}|^2\Big) \\ &\overset{(A.1.4)}{\le} \mu \sum_k \sum_{k'} |M_k^{k'}||x_{k'}|^2 = \mu \sum_{k'} \sum_k |M_k^{k'}||x_{k'}|^2 \le \mu^2 \sum_{k'} |x_{k'}|^2 = \mu^2 , \end{aligned}$$

proving Lemma A.1.1.

A.2 Proof of the Lemma 1.3.13

Proof. The estimates on U follows by induction on k: recalling that

$$\|\langle \mathtt{d}\rangle^b(\mathrm{ad}A)B\|_s \le 2^{(b+1)}\Big(\|\langle \mathtt{d}\rangle^b A\|_s\|B\|_{s_0} + \|A\|_s\|\langle \mathtt{d}\rangle^b B\|_{s_0} + \|\langle \mathtt{d}\rangle^b A\|_{s_0}\|B\|_s + \|A\|_{s_0}\|\langle \mathtt{d}\rangle^b B\|_s\Big)$$

$$\|\langle \mathtt{d}\rangle^b(\mathrm{ad}A)B\|_{s_0} \le 2^{(b+1)}\Big(\|\langle \mathtt{d}\rangle^b A\|_{s_0}\|B\|_{s_0} + \|A\|_{s_0}\|\langle \mathtt{d}\rangle^b B\|_{s_0}\Big), \quad \|(\mathrm{adA})\mathrm{B}\|_{s_0} \le 2\|A\|_{s_0}\|B\|_{s_0}$$

[4] In order to get the third inequality we have used that for $a_1, \dots a_n \ge 0$ one has that the product $a_1, \cdots a_n \le \max_{1\le m\le n} a_m^n \le \sum_{1\le m\le n} a_m^n$.

$$
\begin{aligned}
\|\langle \mathtt{d}\rangle^b(\mathrm{ad}A)^{k+1}B\|_s &= \|\langle \mathtt{d}\rangle^b(\mathrm{ad}A)^k(\mathrm{ad}AB)\|_s \le 2^{k(b+1)}k\Big(\|\langle \mathtt{d}\rangle^b A\|_s\|A\|_{s_0}^{k-1}\|\mathrm{ad}AB\|_{s_0}\\
&\quad + \|A\|_s\|A\|_{s_0}^{k-1}\|\langle \mathtt{d}\rangle^b\mathrm{ad}AB\|_{s_0} + \|\langle \mathtt{d}\rangle^b A\|_{s_0}\|A\|_{s_0}^{k-1}\|\mathrm{ad}AB\|_s\Big)\\
&\quad + 2^{k(b+1)}\Big(k(k-1)\|A\|_s\|A\|_{s_0}^{k-2}\|\langle \mathtt{d}\rangle^b A\|_{s_0}\|\mathrm{ad}AB\|_{s_0} + \|A\|_{s_0}^k\|\langle \mathtt{d}\rangle^b\mathrm{ad}AB\|_s\Big)\\
&\le 2^{k(b+1)}k\Big(\|\langle \mathtt{d}\rangle^b A\|_s\|A\|_{s_0}^{k-1}2\|A\|_{s_0}\|B\|_{s_0} + \|A\|_s\|A\|_{s_0}^{k-1}2^{b+1}(\|\langle \mathtt{d}\rangle^b A\|_{s_0}\|B\|_{s_0}\\
&\quad + \|A\|_{s_0}\|\langle \mathtt{d}\rangle^b B\|_{s_0}) + 2\|\langle \mathtt{d}\rangle^b A\|_{s_0}\|A\|_{s_0}^{k-1}(\|A\|_{s_0}\|B\|_s + \|A\|_s\|B\|_{s_0})\Big)\\
&\quad + 2^{k(b+1)}k(k-1)\|A\|_s\|A\|_{s_0}^{k-2}\|\langle \mathtt{d}\rangle^b A\|_{s_0}2\|A\|_{s_0}\|B\|_{s_0}\\
&\quad + 2^{k(b+1)}\Big(\|A\|_{s_0}^k 2^{(b+1)}\Big(\|\langle \mathtt{d}\rangle^b A\|_s\|B\|_{s_0}\\
&\quad + \|A\|_s\|\langle \mathtt{d}\rangle^b B\|_{s_0} + \|\langle \mathtt{d}\rangle^b A\|_{s_0}\|B\|_s + \|A\|_{s_0}\|\langle \mathtt{d}\rangle^b B\|_s\Big)\Big)\\
&\le 2^{k(b+1)}k\Big(2\|\langle \mathtt{d}\rangle^b A\|_s\|A\|_{s_0}^k\|B\|_{s_0} + \|A\|_s\|A\|_{s_0}^{k-1}2^{b+1}(\|\langle \mathtt{d}\rangle^b A\|_{s_0}\|B\|_{s_0} + \|A\|_{s_0}\|\langle \mathtt{d}\rangle^b B\|_{s_0})\\
&\quad + 2\|\langle \mathtt{d}\rangle^b A\|_{s_0}\|A\|_{s_0}^{k-1}(\|A\|_{s_0}\|B\|_s + \|A\|_s\|B\|_{s_0})\Big)\\
&\quad + 2^{k(b+1)}\Big(2k(k-1)\|A\|_s\|A\|_{s_0}^{k-1}\|\langle \mathtt{d}\rangle^b A\|_{s_0}\|B\|_{s_0}\\
&\quad + 2^{(b+1)}\|A\|_{s_0}^k\|\langle \mathtt{d}\rangle^b A\|_s\|B\|_{s_0} + 2^{(b+1)}\|A\|_{s_0}^k\|A\|_s\|\langle \mathtt{d}\rangle^b B\|_{s_0}\\
&\quad + 2^{(b+1)}\|A\|_{s_0}^k\|\langle \mathtt{d}\rangle^b A\|_{s_0}\|B\|_s + 2^{(b+1)}\|A\|_{s_0}\|\langle \mathtt{d}\rangle^b B\|_s)\Big)\\
&\le 2^{(k+1)(b+1)}(k+1)\Big(\|\langle \mathtt{d}\rangle^b A\|_s\|A\|_{s_0}^k\|B\|_{s_0} + \|A\|_s\|A\|_{s_0}^k\|\langle \mathtt{d}\rangle^b B\|_{s_0}\\
&\quad + \|\langle \mathtt{d}\rangle^b A\|_{s_0}\|A\|_{s_0}^k\|B\|_s\Big)\\
&\quad + 2^{(k+1)(b+1)}\Big(k(k+1)\|A\|_s\|A\|_{s_0}^{k-1}\|\langle \mathtt{d}\rangle^b A\|_{s_0}\|B\|_{s_0} + \|A\|_{s_0}^{k+1}\|\langle \mathtt{d}\rangle^b B\|_s\Big)
\end{aligned}
$$

A.3 Lipschitz extention Theorem

Theorem 7 (Kirszbraun Theorem). *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ a Lipschitz function. Then f can be extended to $\mathbb{R}^n$ keeping the Lipschitz constant of the original function.*

Remark A.3.1. *In the case $m = 1$ one such extension is given by*

$$\tilde{f}(x) := \inf_{y\in\mathbb{E}}(f(y) + Lip(f)|x-y|),$$

where $Lip(f)$ is the Lipschitz constant of f.

$\square$

A.4 Lie exponentiation Formula

Lemma A.4.1. *Let's consider the system*

$$\begin{cases} \dot{B}(t) = \mathrm{ad(A)B(t)} \\ B(0) = B \end{cases} .$$

The solution of this system is $e^{t\mathrm{ad(A)}}B$. *It holds that* $e^{-A}Be^{A} = e^{ad(A)}B$.

Proof. Setting

$$e^{t\mathrm{ad(A)}}B := \sum_{k\geq 0} \frac{t^k \mathrm{ad(A)^k}}{k!} B$$

we see that it totally converges so

$$\frac{d}{dt} e^{t\mathrm{ad(A)}}B = \frac{d}{dt} \sum_{k\geq 0} \frac{t^k \mathrm{ad(A)^k}}{k!} B = \sum_{k\geq 0} \frac{t^{k-1} \mathrm{ad(A)^k}}{(k-1)!} B = \mathrm{ad(A)e^{tad(A)}B}$$

thus by the existence and uniqueness Theorem this is the solution.

□

A.5 Composition lemma

Let $p : \mathbb{R}^d \to \mathbb{R}^d$ be a $2\pi-$periodic function in $W^{s,\infty}$, $s \geq 1$, with $|p|_{1,\infty} \leq \frac{1}{2}$[5]. Let $f(x) = x + p(x)$. Then:

i. f is invertible, its inverse is $f^{-1}(y) = g(y) = y + q(y)$ where q is $2\pi-$periodic, $q \in W^{s,\infty}(\mathbb{T}^d, \mathbb{R}^d)$, and $|q|_{s,\infty} \leq C|p|_{s,\infty}$. More precisely,

$$|q|_{L^\infty} = |p|_{L^\infty}, \quad |Dq|_{L^\infty} \leq 2|Dp|_{L^\infty}, \quad |Dq|_{s-1,\infty} \leq 2|Dp|_{s-1,\infty}.$$

where the constant C depends on d, s.

ii. If $u \in H^s(\mathbb{T}^d, \mathbb{C})$, then $u \circ f(x) = u(x + p(x))$ is also in $H^s(\mathbb{T}^d, \mathbb{C})$, and, with the same C as in *i*,

$$|u \circ f(x)|_s \leq C(|u|_s + |Dp|_{s-1,\infty}|u|_1) \tag{A.5.1}$$
$$|u \circ f(x) - u|_s \leq C(|p|_{L^\infty}|u|_{s+1} + |p|_{s,\infty}|u|_2) \tag{A.5.2}$$
$$|u \circ f(x)|_s^{\gamma,\mathcal{O}} \leq C(|u|_{s+1}^{\gamma,\mathcal{O}} + |p|_{s,\infty}^{\gamma,\mathcal{O}}|u|_2^{\gamma,\mathcal{O}}). \tag{A.5.3}$$

(A.5.1), (A.5.2), (A.5.3) also hold for $u \circ g$

The proof can be found in [BBM14] in the Appendix.

[5] $|u|_{s,\infty} := \sum_{|\beta|\leq s} |D^\beta u|_{L^\infty}$

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