Asymptotic integrability

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Abstract: The multiscale expansion is shown to be a convenient tool to define asymptotic integrability up to order N of $1+1$ dispersive nonlinear wave equations. Its connection with complete integrability, an algorithmic test and few examples are discussed. Approximate Lax pair for asymptotically integrable PDEs are also provided.

1 Introduction

Nonlinear wave equations are of great relevance both in many applications, and as dyamical models with infinitely many degrees of freedom. Few of them, such as the Korteweg-de Vries[1] and the Nonlinear Schroedinger [2] equations (PDEs in $1+1$ dimensional space-time), are well-known integrable hamiltonian systems when appropriate boundary conditions are imposed on solutions. In these exeptional cases, the spectral theory provides not only the mapping to action-angle variables but also a tool to analyze the properties of solutions in analogy with Fourier analysis. In general, however, our ability to deal with nonlinear wave equations mainly relies on approximation schemes and numerical experiments.

One of the approaches is the quasi-monochromatic or slowly varying amplitude approximation that was introduced in nonlinear optics and fluid dynamics more than thirty years ago. This method applies whenever the solution, at the first order in a small parameter ϵ, is a monochromatic (carrier) wave whose amplitude slowly varies in space, namely is a function of the slow coordinate $\xi = \epsilon x$. Here the smallness parameter ϵ is the peak amplitude (cfr.[3]). The well-known, and broadly applied, result of this analysis is that the amplitude evolves in the slow time variable $t_2 = \epsilon^2 t$ according to the Nonlinear Schroedinger (NLS) equation, that, since it follows from a generic nonlinear wave equation, has been recognized [4] as a universal model of nonlinear wave propagation in the strongly dispersive regime.

The technique of reducing a nonlinear wave equation to the NLS equation has a long history which starts with the pioneering paper by T.Taniuti and N.Yajima [5] (and see also [6,7,8]). Extended use of the reduction method has proved to be also a way to derive necessary conditions for integrability of PDEs [9,10].

Higher order terms in the ϵ- expansion have been recently [11] considered with the purpose of computing inelastic effects in two-solitary wave collisions, and of capturing the evidence that the original nonlinear wave equation under investigation, being quite generic, is indeed nonintegrable. In fact, this last question stems naturally from the observation that, if the first order reduced equation, namely the NLS equation, is integrable, then the nonintegrability of the original nonlinear wave equation should manifest itself at some higher order of the perturbative expansion. Indeed, the problem we address here is that of detecting such manifestation of nonintegrability; its solution, being based on power expansion in the wave amplitude, provides a precise way to tell the "degree of integrability" of a generic nonlinear wave equation. However, the "genericity" of the PDE to be investigated should be restricted by the condition that its corresponding nonlinear equation in the slow variables, which obtains at the lowest order, be integrable, as the NLS equation, in order to take advantage of the infinitely many independent commuting symmetries it possesses.

In the following we briefly report on the main results we obtain within the formalism introduced in [3] (for a different approach, see [11]), in particular on the definition of asymptotic integrability up to order n (A_n- integrability) which naturally follows from our analysis. As simple examples, we display the conditions for A_1 and A_2 - integrability, while details, and additional results are provided elsewhere [12].

A distinctive ingredient of our method is the introduction of (finitely or infinetely) many slow time variables, $t_n \equiv \epsilon^n t, n = 1, 2, \cdots$. In this respect, we point out that the occurrence of many slow times is as natural as in the well-known Poincaré - Lindstedt perturbation scheme. Indeed, if $q(t)$ is the single anharmonic oscillator coordinate with the initial conditions $q(0) = \epsilon, \;\; \dot{q}(0) = 0$, where ϵ is the (small) perturbation parameter, then, for sufficiently small ϵ, $q(t)$ is periodic,

$$q(t) = f(\theta), \quad \theta = \omega t, \quad f(\theta) = f(\theta + 2\pi). \tag{1.1}$$

Moreover the two expansions

$$\omega = c_o + \epsilon c_1 + \epsilon^2 c_2 + \cdots \tag{1.2}$$

$$f = \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots \tag{1.3}$$

where the coefficients c_n, are chosen so as to eliminate the secular terms, imply that the solution $q(t)$ is a function not only of the time t but also of the slow times t_n, since combining (1.1) with (1.2) yields

$$\theta = c_o t + c_1 t_1 + c_2 t_2 + c_3 t_3 + \cdots \quad . \tag{1.4}$$

2 Formalism and basic equations

Let

$$Lu = G(u) \tag{2.1}$$

be a $1+1$ wave equation where $u = u(x,t)$ is the dependent (possibly complex) variable, L is a linear differential operator with constant coefficients, and $G(u)$ is an analytic function (with no linear terms) of u and its complex conjugate u^*, and of their x- derivatives at $u = 0$. For instance, we have considered real equations with $L = \partial_t - \partial_x^3$, $G(u) = cu_x^3 + (c_2 u^2 + c_3 u^3 + \cdots)_x$, and $L = \partial_t + \partial_x + a\partial_x^3 + b\partial_x^2\partial_t$, $G(u) = (a_1 u^2 + a_2 u_x^2 + a_3 u_{xx}^2)_x + (a_4 u^2 + a_5 u_x^2)_{xx} + a_6 (u^2)_{xxx}$ this second case being relevant to waves in shallow water. For an arbitrarily given (real) wave number k, it is convenient to introduce the harmonic plane-wave

$$E \equiv \exp[i(kx - \omega t)] \tag{2.2}$$

where $\omega = \omega(k)$ is the dispersion law that is determined by the condition that E be a solution of the linearized equation, $LE = 0$. Let us now set the harmonic expansion

$$u = \sum_{\alpha=-\infty}^{+\infty} u^{(\alpha)} E^\alpha \tag{2.3}$$

where the coefficients $u^{(\alpha)}$ depend only on the slow variables ξ and $t_n, n = 1, 2, \cdots,$ $u^{(\alpha)} = u^{(\alpha)}(\xi, t_1, t_2, \cdots)$, and let us note that the action of differential operators on u can be easily translated into operations on the functions $u^{(\alpha)}$ via the identities

$$\partial_x (E^\alpha u^{(\alpha)}) = E^\alpha (i\alpha k + \epsilon \partial_\xi) u^{(\alpha)} \quad , \tag{2.4a}$$

$$\partial_t (E^\alpha u^{(\alpha)}) = E^\alpha (-i\alpha\omega + \epsilon\partial_1 + \epsilon^2 \partial_2 + \epsilon^3 \partial_3 + \cdots) u^{(\alpha)} \quad , \tag{2.4b}$$

where $\partial_n \equiv \partial/\partial t_n$. It is then clear that the equation

$$L(E^\alpha u^{(\alpha)}) = E^\alpha L^{(\alpha)} u^{(\alpha)} \tag{2.5}$$

defines the linear operator $L^{(\alpha)}$, for the α-th harmonic, that is differential in the slow variables ξ and t_n, and possesses a well-defined formal expansion in powers of ϵ, namely

$$L^{(\alpha)} = L_o^{(\alpha)} + \epsilon L_1^{(\alpha)} + \epsilon^2 L_2^{(\alpha)} + \cdots \tag{2.6}$$

Once the expansion (2.3) of u is inserted in the function $G(u)$, then also $G(u)$ turns out to be expressed in terms of harmonics,

$$G(u) = \sum_{\alpha=-\infty}^{+\infty} G^{(\alpha)} E^\alpha \quad , \tag{2.7}$$

where the coefficients $G^{(\alpha)}$ are formal expressions of the functions $u^{(\beta)}, u^{(\beta)*}$ and their ξ - derivatives. As a result of this setting, the original equation (2.1) is equivalent to the (infinite) set of PDEs

$$L^{(\alpha)}u^{(\alpha)} = G^{(\alpha)}, \quad \alpha = 0, \quad \pm 1, \pm 2, \cdots \quad . \tag{2.8}$$

At this point few remarks are in order. Because of the power expansion (2.6), the inverse operator $(L^{(\alpha)})^{-1}$ has itself a well-defined formal expansion in powers of ϵ iff the first term $L_0^{(\alpha)}$, that is just a k-dependent number, $L_0^{(\alpha)}(k)$, is not vanishing, $L_0^{(\alpha)}(k) \neq 0$. Of course, by construction, $L_0^{(1)}(k) = 0$ because of the dispersion law $\omega = \omega(k)$, and therefore the equation (2.8) for the amplitude $u^{(1)}$ of the fundamental harmonic is strictly differential. In this respect, it is convenienct to introduce the following definition: the α-th harmonic, with $\alpha \neq 1$, is *resonant* (or, shortly, (α) is a *resonance*) iff $L_0^{(\alpha)}(k) = 0$. Moreover, a resonance α can be *structural*, if $L_0^{(\alpha)}(k) = 0$ holds for all values of k, or *accidental* if, instead, $L_0^{(\alpha)}(k) = 0$ only for one, or more, particular value of k. Also the following distinction is useful: if two harmonics are at resonance, $L_0^{(\alpha)}(k) = L_0^{(\beta)}(k) = 0$ with $\alpha \neq \beta$, then they are *weakly* resonating if $L_1^{(\alpha)}(k) \neq L_1^{(\beta)}(k)$, and they are *strongly* resonating if, instead, $L_1^{(\alpha)}(k) = L_1^{(\beta)}(k)$. For instance, if the operator L in (2.1) is given by the simple expression

$$L = \partial_t + i\omega(-i\partial_x), \tag{2.9}$$

where the function $\omega(k)$ is analytic on the real axis, then, from the resonance condition

$$L_0^{(\alpha)}(k) = i[\omega(\alpha k) - \alpha\omega(k)] = 0, \tag{2.10}$$

it follows that only $\alpha = 0$ and $\alpha = -1$ can be structural, the former occurring iff $\omega(0) = 0$ and the latter iff $\omega(k)$ is odd, $\omega(-k) = -\omega(k)$. Moreover, since for the operator (2.9) one obtains

$$L_n^{(\alpha)} = \partial_n - (-i)^{n+1}\omega_n(\alpha k)\partial_\xi^n \;\; , \; \omega_n(k) \equiv \frac{1}{n!}\frac{d^n}{dk^n}\omega(k), n \geq 1, \tag{2.11}$$

if the harmonic $\alpha = -1$ is structurally at resonance, then it is strongly resonating with the fundamental harmonic.

The way to go further is based on the ϵ-expansion of the amplitudes $u^{(\alpha)}$ of each harmonic,

$$u^{(\alpha)} = \sum_{n=1} \epsilon^n u^{(\alpha)}(n), \tag{2.12}$$

and on the assumption that the first order coefficient of the fundamental harmonic, $u^{(1)}(1)$, satisfies the NLS equation with respect to the second slow time variable t_2,

$$u^{(1)}_{t_2}(1) = i\omega_2[u^{(1)}_{\xi\xi}(1) - 2c|u^{(1)}(1)|^2 u^{(1)}(1)], \tag{2.13}$$

where c is a costant. That this is the case obviously depends on the particular operator L and nonlinear function $G(u)$, see (2.1). Though a detailed discussion of this matter is outside the scope of this report, we point out few examples to illustrate this point. If there are no resonances, namely $L_0^{(\alpha)}(k) \neq 0$ for all $\alpha \neq 1$, then the NLS equation (2.13) obtains for a generic function $G(u)$. If one resonance occurs, f.i. $\alpha = 0$ with L given by (2.9) and $\omega(0) = 0$, then a generic $G(u)$ does not yield the NLS equation. The same is true when there are two resonances, say $\alpha = 0$ and $\alpha = -1$ for $\omega(k) = -\omega(-k)$; in this case only special choices for $G(u)$ lead to the NLS equation. It is worth noting here that in any case in which the NLS (2.13) obtains, then the first (see below) integrability condition is simply the reality condition

$$\mathrm{Im}(c) = 0 \quad . \tag{2.14}$$

As a simple example, if the operator L has the expression (2.9), if no resonance occurs, and $G(u) = au_x^2 + bu^* u_x$, then this integrability condition reads Re $(ab) = 0$. As for the following presentation of our results, we prefer to choose L of the form (2.9) with $\omega(k) = -\omega(-k)$ and $G(u)$ to be a real odd function, $G(u) = -G(-u)$ so as to deal with the particularly interesting case of real solutions, $u = u^*$. Indeed, any PDE (2.1) in this class satisfies our assumption that (2.13) holds provided all even harmonics have vanishing amplitude,

$$u^{(2\alpha)} = 0 \quad , \tag{2.15}$$

this being consistent with the odd parity property of $G(u)$. Note that the condition (2.15) eliminates the resonance $\alpha = 0$, and thus only the strong resonance $\alpha = -1$ is left; moreover, because of the reality of u, the condition

$$u^{(-\alpha)} = u^{(\alpha)*} \tag{2.16}$$

holds with the consequence that we need to focus our attention only on the differential equation $L^{(1)}u^{(1)} = G^{(1)}$, while all other equations yield, by recursion, the expression of $u^{(\alpha)}(n)$, for $|\alpha| \neq 1$, in terms of $u^{(1)}(m), u^{(1)*}(l)$ and their ξ- derivatives. As a by-product of the ϵ- expansion, one can show that only the harmonics $u^{(\pm 1)}$ give contributions of order ϵ since $u^{(\alpha)}(n) = 0$ for $n < |\alpha|$. Moreover, because of the special rôle played by the functions $u^{(1)}(n)$, we introduce the following simpler notation $u^{(1)}(n) \equiv u(n), L_n^{(1)} \equiv L_n$ and $G^{(1)}(n) \equiv G(n)$, where, of course,

$$G^{(\alpha)} = \sum_{n=2} \varepsilon^n G^{(\alpha)}(n) \ , \tag{2.17}$$

with the additional assumption that the functions $u(n)$, $(n = 1, 2, 3 \ldots)$ be infinitely differentiable with respect to the variable ξ, with the notation

$$u_\ell(n) \equiv \partial_\xi^\ell \, u(n) \ , \qquad u_o(n) = u(n) \ . \tag{2.18}$$

By inserting the expansions (2.6), (2.12) and (2.7) into the equation (2.8) for $\alpha = 1$, it is easily seen that the basic equation is the triangular set of PDEs

$$L_1\ u(n-1) + L_2\ u(n-2) + \ldots + L_{n-1}\ u(1) = G(n)\ , \quad n \geq 2\ , \tag{2.19}$$

where $G(n)$ is a differential polynomial of the functions $\{u(m),\ u^*(m)\}$ with unit gauge index, this meaning that $G(n) \longrightarrow \exp(i\theta)G(n)$ if $u(n) \longrightarrow \exp(i\theta)u(n)$. In this context it is convenient to introduce the finite-dimensional vector space $\mathcal{P}(n)$ of differential polynomials that are i) nonlinear in $\{u(m),\ u^*(m)\}$, ii) with unit gauge index, and iii) of order n in ϵ. The vector spaces $\mathcal{P}(n)$ can be easily identified by specifying the basis of monomials; thus, for instance, $\mathcal{P}(2)$ is empty, $\mathcal{P}(3)$ is 1-dimensional, $\mathcal{P}(3) = \{u^2(1)\ u^*(1)\}$, $\mathcal{P}(4)$ is 4-dimensional, $\mathcal{P}(4) = \{u(2)\ u(1)\ u^*(1),\ u^2(1)\ u^*(2),\ u_1(1)\ u(1)u^*(1),\ u^2(1)\ u_1^*(1)\}$, and so forth. Therefore the right-hand side of equation (2.19), $G(n)$, is a vector of $\mathcal{P}(n)$, with the implication that $G(2) = 0$, $G(3)$ is proportional to $u^2(1)\,u^*(1)$, and so on. As for the left-hand side of the basic equation (2.19), we point out that the number of operators L_n equals that of the slow time variables t_n, see the expression (2.11) for $\alpha = 1$. On the other hand, one can easily show that the number of slow time variables to be introduced is finite iff the dispersion function $\omega(k)$ is a polynomial, the number M of slow time variables $t_1, t_2, ..., t_M$, being the degree of this polynomial.

3 Secularities and reduced equations

For arbitrarily given initial data, it turns out that the solutions $u(n)$ of the triangular system (2.19) are not bounded as $t_2 \to \infty$ as a consequence of secularities. However, as shown in [3], bounded solutions exist if appropriate conditions are satisfied. In particular, necessary conditions are

$$L_1 u(n) = 0\ , \qquad \partial_n u(1) = K_n[u(1)]\ , \qquad n = 1, 2, \ldots, \tag{3.1}$$

where the vector fields K_n are the commuting flows of the NLS hierarchy, $[K_n, K_m] = 0$, say (see (2.13) and (2.18))

$$K_2 = i\omega_2(u_2(1) - 2cu^2(1)u^*(1)) \tag{3.2}$$

is the NLS flow, K_3 is the complex modified Korteweg-de Vries flow and so on (here c is assumed to be real so as to deal with the integrable NLS, see (2.14)). This finding clearly implies that $L_n u(1)$ belongs to the vector space $\mathcal{P}(n+1)$, so that the triangular system (2.19) can be recast in the simpler form (note that $L_2 u(1) = G(3)$ is the NLS equation)

$$L_2 u(n-2) + L_3 u(n-3) + \ldots + L_{n-2} u(2) = \tilde{G}(n)\ , \quad n \geq 4\ , \tag{3.3}$$

where the differential polynomial $\tilde{G}(n) \epsilon\, \mathcal{P}(n)$ may obviously depend on $u(\ell)$ only for ℓ up to $n-2$, through the monomials $u(n-2)u(1)u^*(1)$ and $u^*(n-2)u^2(1)$.

For future reference, it is convenient to introduce also the subspace $\mathcal{P}_m(n)$ of $\mathcal{P}(n)$, for $1 \leq m \leq n-2$, as the space of differential polynomials which depend on $u(\ell)$ and $u^*(\ell)$ only for $1 \leq \ell \leq m$ (of course, $\mathcal{P}_{n-2}(n) = \mathcal{P}(n)$).

The multiple-scale equations (3.3) can be rewritten in a more convenient form by introducing the linear operators

$$M_n \equiv \partial_n - K'_n[u(1)] \ , \qquad n \geq 2 \ , \tag{3.4}$$

where $K'_n[u(1)]$ is the Frechét derivative (with respect to $u(1)$) of $K_n[u(1)]$, and by observing that

$$(M_n - L_n)\, u(m) \epsilon \mathcal{P}_m(n+m) \ . \tag{3.5}$$

Indeed, we can prove that the equations (3.3) now read

$$M_2 u(n-2) + M_3 u(n-3) + \ldots + M_{n-2} u(2) = F(n) \ , \qquad n \geq 4 \ , \tag{3.6}$$

with the twofold virtue that, as for the linear operators L_n, also the operators (3.4) commute with each other,

$$[M_n, M_m] = 0 \ , \tag{3.7}$$

and that $F(n) \epsilon \mathcal{P}_{n-3}(n)$.

4 Integrability

At this point we note that the reduced multiple-scale equations (3.6) hold for any equation in the class (2.1) we are considering, and do not bring, therefore, any information on the integrability properties of the original equation (2.1) (except for the reality condition (2.14)). However, a relation to integrability stems from the following facts. First, in the linear (and trivially integrable) case, $M_n = L_n$ and $F(n) = 0$, and the triangular system (3.6) splits into the set of evolution (with respect to each slow time) equations $M_\ell u(n) = 0$, $\ell \geq 2$, $n \geq 2$, that are obviously compatible with each other. Second, we show in [12] that, if the original equation (2.1) is C-integrable (i.e. linearizable by a change of the dependent variable u, see [4]) or S-integrable (i.e. integrable by the Spectral Transform, see f.i. [13]), then, also in this case, the system (3.6) has the splitting property

$$M_\ell u(n) = f_\ell(n) \ \epsilon \mathcal{P}_{n-1}(\ell + n) \ , \tag{4.1}$$

where the differential polynomials $f_\ell(n)$ satisfy the compatibility conditions

$$M_m f_\ell(n) = M_\ell f_m(n) \quad . \tag{4.2}$$

Because of the splitting property (4.1) of integrable equations, and of the fact that the multiple-scale equations obtain in a neighborhood of $\varepsilon = 0$, it is natural to define the equation (2.1), $Lu = G(u)$, asymptotically integrable up to order n, or A_n-integrable, if the forcing terms $F(m)$ for $m \leq n+2$, are precisely such that all the reduced multiple-scale equations $M_2 u(m) + \ldots + M_m\, u(2) = F(m+2)$ for $m = 2, 3, \ldots, n+1$ split into (compatible) equations of the form (4.1), namely $M_\ell u(m-\ell+2) = f_\ell(m-\ell+2)$ for $\ell = 2, \ldots, m$. The case $n = 1$ can be included in this definition by considering that the equation $M_2 u(1) = F(3)$ is the NLS equation for which the integrability condition is $\mathrm{Im}(c) = 0$, see (2.14) where c is the parameter that appears in (3.2).

With this definition of A_n-integrability, it remains to set up a computational method to determine the degree n of asymptotic integrability of a given nonlinear wave equation (2.1). This is based on the following facts [12]. Of the system (4.2) of compatibility equations, only the set of equations

$$M_2 f_3(n) = M_3 f_2(n) \quad , \qquad n = 2, 3, \ldots \tag{4.3}$$

has to be tested, all other equations being consequently satisfied as they do not add further conditions on the nonlinear part $G(u)$ of (2.1); moreover one can prove that the kernel of the operator M_ℓ in the vector space $\mathcal{P}(n)$ is empty. We note also that the differential operator M_2 is rapresented by a rectangular matrix taking a vector of $\mathcal{P}_{n-1}(n+3)$ into a vector of $\mathcal{P}_{n-1}(n+5)$. As a consequence, the solution $f_3(n)$ of equation (4.3) exists iff the vector $f_2(n)$ satisfies appropriate conditions, that are, therefore, those which entail asymptotic integrability. The actual computations have been performed by computer since the algebraic complexity rapidly increases with n. In order to show this, we give below the dimensionality of the first few vector spaces $\mathcal{P}_\ell(n)$ in the notation $\mathcal{P}_\ell(n) \to \dim(\mathcal{P}_\ell(n))$: $\mathcal{P}_1(3) \to 1$, $\mathcal{P}_1(4) \to 2$, $\mathcal{P}_1(5) \to 5$, $\mathcal{P}_1(6) \to 8$, $\mathcal{P}_2(4) \to 4$, $\mathcal{P}_2(5) \to 12$, $\mathcal{P}_2(6) \to 26$, $\mathcal{P}_3(5) \to 14$, $\mathcal{P}_3(6) \to 34$. Thus, the A_2-integrability condition on

$$f_2(2) = (\alpha_1 + i\beta_1) u(1) u_1(1) u^*(1) + (\alpha_2 + i\beta_2) u^2(1) u_1^*(1) \tag{4.4}$$

reads $\beta_1 = \beta_2 = 0$ if $c \neq 0$, while no condition on α_j and β_j is required if $c = 0$. The A_3-integrability condition on the 12-dimensional vector $f_2(3)$ is given by 15 real equations so that the general vector $f_2(3)$ depends on 9 real arbitrary parameters. The explicit expression of these, conditions are given in [12]. Once these conditions have been found, they can be used to test the order of the asymptotic integrability of a given wave equation of the form (2.1) (with appropriate specifications, see Section 2). Examples of particular wave equations that have been tested by this method are also reported in [12]. Among them we report here the equation (α and β being arbitrary real parameters)

$$\alpha(u_t + u_x + u_{xxx} + 6uu_x) + \beta(u_t + \frac{2}{3} uu_x)_{xx} = 0 \tag{4.4}$$

which has found to be A_3-integrable. In fact, since its nonlinearity is only quadratic, we expect it to be completely integrable. This equation may also be of interest in the motion of shallow water (se also [14]). Finally, since integrable equations are A_∞-integrable, we conjecture that, conversely, A_∞-integrable equations are indeed (either C or S-) integrable.

5 Spectral theory

If a PDE in the class (2.1) (with appropriate conditions, see section 2.) is S-integrable (f.i. the mKdV equation [15], $u_t - u_{xxx} = au^2 u_x$, or the equation [16] $u_t - u_{xxx} = -\frac{1}{8}u_x^3 + a(\cosh u - 1)u_x$), then a spectral method of solution (for appropriate boundary conditions) is available via the corresponding Lax pair of linear differential operators (see, f.i., [17]). Here we shortly sketch the way to build up a (necessarily approximate) spectral approach to those PDEs (2.1) which are not integrable, but only asymptotically integrable up to order N. According to the previous sections, the distinctive feature of a A_n-integrable PDE is that the slowly varying amplitudes $u(n)$, for $1 \leq n \leq N$, evolve with respect to the slow times t_m according to the hierarchies of commuting flows (see (3.1), (4.1) and (3.4))

$$\partial_m u(1) = K_m[u(1)] \quad , \quad m = 1, 2, ..., \tag{5.1}$$

$$\partial_m u(n) = K'_m[u(1)]u(n) + f_m(n), \quad m = 1, 2, .., \ 2 \leq n \leq N, \tag{5.2}$$

(with $f_1(n) = 0$, see (3.1)). On the oher hand, it is well-known [18] that the evolution equations (5.1) result as compatibility conditions for a pair of linear differential equations, one of them being the Zakharov-Shabat spectral equation

$$\psi_\xi(0) + iq\sigma_3\psi(0) = U(1)\psi(0), \tag{5.3}$$

where q is the spectral variable, σ_3 is the Pauli matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $U(1) = \begin{pmatrix} 0 & u(1) \\ cu^*(1) & 0 \end{pmatrix}$, and $\psi(0)$ (for this notation, see below) is a 2×2 matrix solution. Our aim here is to show that also the linear (in $u(n)$) nonhomogeneous equations (5.2) obtain as compatibility conditions for a pair of linear operators. To this purpose, we first consider a S-integrable PDE in the class (2.1), and we derive a Lax pair for the corresponding equations (5.1) and (5.2) by means of the multiscale expansion of the spectral problem. Then we conversely construct a Lax pair for a given set of compatible flows (5.1) and (5.2) by analogy with the completely integrable case.

The first multiple-scale analysis of a spectral problem has been introduced in [19], where the Z-S equation (5.3) is derived from the scalar stationary Schoedinger equation $\psi_{xx} + k^2\psi = u(x,t)\psi$. We have extended [12] that analysis in two ways, first to a quite general class of spectral problems (say, to $N \times N$ matrix linear equations), and, second, to all higher orders in ϵ. Without going into any detail, it is sufficient to point out here that the multiple-scale technique generally leads to the following triangular linear system

$$\psi_\xi(n) = \sum_{m=0}^{n} X(n-m+1)\psi(m) \quad , n = 0, 1, 2, ..., \tag{5.4}$$

where $\psi(n)$ is a 2×2 matrix of order n in ϵ, and $X(n)$ is a 2×2 matrix, that depends on ξ and on the spectral variable q, and is order n in ϵ. The first equation (5.4), i.e. for $n=0$, is just the Z-S equation (5.3) since $X(1)=-iq\sigma_3+U(1)$ (it should be kept in mind that ∂_ξ and q are order 1 in ϵ). Moreover, the entries of $X(n)$ are polynomials in q and differential polynomials in $\{u(m),u^*(m)\}$ where $X_{11}(n)$ and $X_{22}(n)$ are gauge invariant while $X_{12}(n)(X_{21}(n))$ have gauge index $1(-1)$ with respect to the gauge transformation $u(m)\to e^{i\theta}u(m)$. For instance

$$X(2)=\begin{pmatrix} iac|u(1)|^2 & u(2)+ibu_\xi(1) \\ cu^*(2)-icbu^*_\xi(1) & -iac|u(1)|^2 \end{pmatrix}, \tag{5.5}$$

where a and b are real constants.

Next we derive the hierarchy of evolution equations associated with the spectral system (5.4). This derivation can be conveniently achieved in three steps. First, we transform the spectral equations (5.4) into new equations,

$$\tilde\psi_\xi(n)=\sum_{m=0}^{n}\tilde X(n-m+1)\tilde\psi(m), \tag{5.6}$$

where the new matrices $\tilde X(n)$ have the following properties:
i) $\tilde X(1)=X(1)$, ii) $\tilde X(n)$ are q-independent and off-diagonal for $n\geq 2$. The appropriate gauge transformation which leads to (5.6) is found to be

$$\tilde\psi(n)=\psi(n)+\sum_{m=1}^{n}G(m)\psi(n-m), \tag{5.7a}$$

$$\tilde X(n)=X(n)+G_\xi(n-1)+[G(n-1),X(1)]$$

$$+\sum_{m=1}^{n-2}[G(n-m-1)X(m+1)-\tilde X(m+1)G(n-m-1)], \tag{5.7b}$$

where $G(n)$ (with $G(0)=1$) is a 2×2 matrix.

Second step is to derive the evolution equations for the matrices $\tilde X(n)$. In fact, in the gauge (5.6), this task is particularly simple, as it amounts to formally expand in Taylor series the function $u(1)$ in a parameter η,

$$u(1)=\sum_{n=1}\eta^{n-1}\tilde u(n), \tag{5.8}$$

and to replace the solution $u(1)$ in (5.1) with this series. The resulting evolution equations for $\tilde u(n)$, *f.i.* $\partial_n\tilde u(1)=K_n[\tilde u(1)]$ and

$$\partial_n\tilde u(2)=K_n'[\tilde u(1)]\tilde u(2), \tag{5.9}$$

provide, with $\tilde X(n)=\begin{pmatrix}0 & \tilde u(n)\\ c\tilde u^*(n) & 0\end{pmatrix}$, the evolution equations for the new matrices $\tilde X(n)$. Third, the hierarchy of evolution equations for $X(n)$ is recovered by using the gauge transformation (5.7b). As the actual computations require a computer code because of the long expressions of vectors in the space of differential polynomials, we limit this terse report to merely display the result at the lowest order. Besides the first (trivial) expression, $\tilde{u}(1) = u(1)$, we find, for $X(2)$ given by (5.5),

$$G(1) = -iac\partial_\xi^{-1}|u(1)|^2\sigma_3 \ , \ \tilde{u}(2) = u(2) + ibu_\xi(1) - 2iacu(1)\partial_\xi^{-1}|u(1)|^2 \ . \tag{5.10}$$

At this point we are able to obtain the general expression of the forcing terms $f_n(2)$, see (4.1), by inserting the expression (5.10) of $\tilde{u}(2)$ into the evolution equation (5.9) (see the definition (3.4)), namely

$$f_n(2) = M_n[-ibu_\xi(1) + 2iacu(1)\partial_\xi^{-1}|u(1)|^2], \ n \geq 2, \tag{5.11}$$

the first two of them being (see (2.18))

$$f_2(2) = 4c\omega_2[(a-b)u^2(1)u_1^*(1) + au(1)u_1(1)u^*(1)], \tag{5.12a}$$

$$f_3(2) = 6i\omega_3 c[2(b-a)u(1)u_1(1)u_1^*(1) - a(u(1)u^*(1)u_2(1)$$

$$+u_1^2(1)u^*(1) - cu^3(1)u^{*2}(1))]. \tag{5.12b}$$

In a similar way, though with more computational efforts, one can derive the general expressions of $f_n(3), f_n(4), ..., f_n(m), ...,$, from S-integrability. Therefore, starting with the general expression of $X(n)$, rather than with one obtained via multiple-scale expansion from a particular spectral equation (such as the stationary Schroedinger equation) one can compute the general solution (in the appropriate differential polynomial space) of the compatibility equations (4.2). Moreover, this computational scheme provides also the way to obtain the (approximate) Lax pair in case the PDE (2.1) is only A_N-integrable. Indeed, one can clearly associate with the equation (5.1) and (5.2) the triangular spectral problem (5.4) with $0 \leq n \leq N$. In fact, since we have learned how to go from known $X(n)$, for $1 \leq n \leq N$, to the forcing terms $f_m(n)$, for $m = 2, 3, ...$, and $2 \leq n \leq N$, we can now go backwards and construct the matrices $X(n)$ from the forcing terms found by solving the compatibility conditions (4.2). Thus, for instance, as A_2-integrability implies the expression (4.4) of $f_2(2)$ with $\beta_1 = \beta_2 = 0$, in this case the corresponding spectral equation can be given the form of the 4×4 linear system (as written in 2×2 blocks)

$$\begin{pmatrix} \psi(0) \\ \psi(1) \end{pmatrix}_\xi = \mathcal{M} \begin{pmatrix} \psi(0) \\ \psi(1) \end{pmatrix}, \tag{5.13}$$

with the matrix $\mathcal{M}$ given by

$$\mathcal{M} = \begin{pmatrix} -iq\sigma_3 + U(1) & 0 \\ U(2) + i(\frac{\alpha_1 - \alpha_2}{4\omega_2 c})\sigma_3 U_\xi(1) + \frac{i\alpha_1}{4\omega_2}|u(1)|^2\sigma_3 & -iq\sigma_3 + U(1) \end{pmatrix},$$

where $U(n) = \begin{pmatrix} 0 & u(n) \\ cu^*(n) & 0 \end{pmatrix}$. Quite more complicate expressions obtain for the $2N \times 2N$ linear system in case of A_N-integrability.

We finally conclude with observing that the multiple-scale analysis, as formulated here, provides both a test of approximate integrability of a PDE as (2.1), as well as a tool to investigate by spectral methods higher order effects in the propagation of quasi-monochromatic waves as far as the nonlinear terms in the perturbation expansion satisfy the asymptotic integrability conditions.

References

[1] C.S.Gardner, J.M.Greene, M.D.Kruskal and R.M.Miura, "Method for solving the Korteweg-de Vries equation," Phys. Rev. Lett. **19**, 1095-1097 (1967).

[2] V.E.Zakharov and A.B.Shabat, "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media," Soviet Phys. JETP **34**, 62-69 (1972) [Russian original: Zh. Eksp. Teor. Fiz. **61**, 118-134 (1971)].

[3] A.Degasperis, S.V.Manakov and P.M.Santini, "Multiple-scale perturbation beyond the Nonlinear Schroedinger Equation. I," Physica D**100**, 187-211 (1997).

[4] F.Calogero, "Why are certain nonlinear PDEs both widely applicable and integrable?." In: *What is integrability?*, edited by V.E.Zakharov, Springer, Heidelberg, 1990, pp. 1-62.

[5] T.Taniuti and N.Yajima, "Perturbation method for a nonlinear wave modulation. I, "J. Math. Phys. **10**, 1369-1372 (1969).

[6] Suppl. Prog. Theor. Phys. **55**, 1974 (special issue devoted to the "Reductive Perturbation Method for Nonlinear Wave Propagation").

[7] A.Jeffrey and T.Kawahara, *Asymptotic Methods in Nonlinear Wave Theory*, Pitman, London, 1982.

[8] T.Taniuti and K.Nishihara, *Nonlinear Waves*, Pitman, London, 1983.

[9] F.Calogero and W.Eckhaus, (a) "Nonlinear evolution equations, rescalings, model PDEs and their integrability. I & II," Inverse Problems **3**, 229-262 (1987) & **4**, 11-33 (1988); (b) "Necessary conditions for integrability of PDEs, "Inverse Problems **3**, L27-L32 (1987).

[10] F.Calogero, A.Degasperis and Ji Xiaoda, "Nonlinear Schroedinger type equations from multiscale reduction of PDEs. II. Necessary conditions for integrability of PDEs," (in preparation).

[11] Y.Kodama and A.V.Mikhailov, "Obstacle to asymptotic integrability." In: *Algebraic Aspects of Integrable Systems: in memory of Irene Dorfman*, edited by A.S.Fokas and I.M.Gelfand, Birkhäuser, Boston, 1996, pp. 173-204.

[12] A.Degasperis and M.Procesi, "Multiple-scale perturbation beyond the Nonlinear Schroedinger Equation. II:Asymptotic Integrability" (in preparation).

[13] F.Calogero and A.Degasperis, *Spectral Transform and Solitons. I*, North-Holland, Amsterdam (1982).

[14] A.S.Fokas and Q.M.Liu, "Asymptotic Integrability of Water Waves," Phys. Rev. Lett **77**, 2347-2351 (1996).

[15] M.Wadati, "The exact solution of the modified Korteweg-de Vries equation," J.Phys.Soc.Japan **32**, 1681 (1972).

[16] F.Calogero and A.Degasperis, "Reduction Technique for Matrix Nonlinear Evolution Equations Solvable by the Spectral Transform," J.Math.Phys. **22**, 23-31 (1981).

[17] M.J.Ablowitz and P.A.Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge (1991).

[18] M.J.Ablowitz, D.J.Kaup, A.C.Newell and H.Segur, "The inverse scattering transform - Fourier analysis for nonlinear problems," Stud. Appl. Math. **53**, 249-315 (1974).

[19] V.E.Zakharov and E.A.Kuznetsov, "Multiscale expansion in the Theory of Systems Integrable by the Inverse Scattering Transform," Physica D**18**, 455 (1986).