Convergent series for quasi-periodically forced strongly dissipative systems

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Abstract

We study the ordinary differential equation $\varepsilon\ddot{x} + \dot{x} + \varepsilon\, g(x) = \varepsilon f(\boldsymbol{\omega} t)$, with f and g analytic and f quasi-periodic in t with frequency vector $\boldsymbol{\omega} \in \mathbb{R}^d$. We show that if there exists $c_0 \in \mathbb{R}$ such that $g(c_0)$ equals the average of f and the first non-zero derivative of g at c_0 is of odd order $\mathfrak{n}$, then, for ε small enough and under very mild Diophantine conditions on $\boldsymbol{\omega}$, there exists a quasi-periodic solution close to c_0, with the same frequency vector as f. In particular if f is a trigonometric polynomial the Diophantine condition on $\boldsymbol{\omega}$ can be completely removed. This extends results previously available in the literature for $\mathfrak{n} = 1$. We also point out that, if $\mathfrak{n} = 1$ and the first derivative of g at c_0 is positive, then the quasi-periodic solution is locally unique and attractive.

1 Introduction

Consider the ordinary differential equation

$$\varepsilon\ddot{x} + \dot{x} + \varepsilon\, g(x) = \varepsilon\, f(\boldsymbol{\omega} t), \tag{1.1}$$

where $x \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, $\boldsymbol{\omega} \in \mathbb{R}^d$, with $d \in \mathbb{N}$, and the functions $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{T}^d \to \mathbb{R}$ are real analytic. From a physical point of view, for $\varepsilon > 0$ the equation describes a one-dimensional system with mechanical force g, subject to a quasi-periodic forcing f with frequency vector $\boldsymbol{\omega}$ and in the presence of friction — with $1/\varepsilon$ being the damping coefficient. Without loss of generality we can (and shall) assume $\boldsymbol{\omega} \cdot \boldsymbol{\nu} \neq 0\ \forall \boldsymbol{\nu} \in \mathbb{Z}^d_* := \mathbb{Z}^d \setminus \{\mathbf{0}\}$; if not, f can be expressed as a quasi-periodic function with frequency vector $\boldsymbol{\omega}' \in \mathbb{R}^{d'}$, $d' < d$, with rationally independent components. Equations like (1.1) also describe electric circuits which are of interest in electronic engineering and theory of circuits; we refer to [18, 2, 16, 8, 3] for physical motivations and more details.

We are interested in studying the existence of quasi-periodic solutions to (1.1) with the same frequency vector as the forcing (response solutions), for small values of ε. Hence the parameter ε plays the role of a perturbation parameter. As we shall see, the existence of such solutions relies on two kinds of conditions: a non-degeneracy assumption on f, g and non-resonance hypothesis on $\boldsymbol{\omega}$. Let us illustrate the conditions. Write

$$f(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} f_{\boldsymbol{\nu}},$$

and set $N(f) = N$ if f is a trigonometric polynomial of degree N and $N(f) = \infty$ otherwise. Define also

$$\alpha_n(\boldsymbol{\omega}) := \min\left\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le 2^n\right\}, \qquad \mathfrak{B}(\boldsymbol{\omega}) := \sum_{n=0}^{\infty} \frac{1}{2^n} \log \frac{1}{\alpha_n(\boldsymbol{\omega})}, \tag{1.2a}$$

$$\beta_n(\boldsymbol{\omega}) := \min\left\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le 2^n, |\boldsymbol{\nu}| \le N(f)\right\}, \qquad \varepsilon_n(\boldsymbol{\omega}) := \frac{1}{2^n} \log \frac{1}{\beta_n(\boldsymbol{\omega})}. \tag{1.2b}$$

The first hypothesis is a non-degeneracy assumption involving the functions f and g.

Hypothesis 1. *There exists a zero c_0 of odd order $\mathfrak{n}$ of the equation $g(c_0) = f_{\mathbf{0}}$.*

The periodic case $d = 1$ is much easier and requires no further assumption. It has been studied in detail [14, 15, 5], with a thorough characterization of the analyticity properties of the response solution for ε in the complex plane. One could study the analyticity properties of the quasi-periodic solution for ε in the complex plane also in the quasi-periodic case — for instance this has been done in [14, 15, 5, 6]. Here we prefer to focus on real ε, both for simplicity and because it represents the interesting case from a physical point of view: $\varepsilon > 0$ small corresponds to a system with large damping coefficient $\gamma = 1/\varepsilon$.

The quasi-periodic case requires a non-resonance assumption on the frequency vector $\boldsymbol{\omega}$. Under the hypothesis that $\boldsymbol{\omega}$ is a Bryuno vector (namely that $\mathcal{B}(\boldsymbol{\omega}) < \infty$ [4]), the existence of a quasi-periodic solution with frequency vector $\boldsymbol{\omega}$ was proved in [15] for $\mathfrak{n} = 1$ and in [11] for any odd $\mathfrak{n}$. The condition on $\boldsymbol{\omega}$ for a response solution to exist can be weakened into the following non-resonance condition.

Hypothesis 2. *One has $\lim_{n\to\infty} \varepsilon_n(\boldsymbol{\omega}) = 0$.*

Such a condition is automatically satisfied either for $d = 1$ (periodic case) or for $d > 1$ and f a trigonometric polynomial. If $\boldsymbol{\omega}$ is a Bryuno vector, then the sequence $\{\varepsilon_n(\boldsymbol{\omega})\}$ is summable, so, for such $\boldsymbol{\omega}$, Hypothesis 2 is obviously satisfied for any f.

Hypothesis 2 was first considered in [5], where it was proved that, under Hypothesis 1 with $\mathfrak{n} = 1$, and Hypothesis 2, there exists a response solution which is jointly analytic in $\boldsymbol{\psi}$ and in ε, for ε in a suitable domain of the complex plane with the boundary tangent to the imaginary axis at the origin (a larger analyticity domain was obtained afterwards in [6]). The proof in [5] follows from the existence of an approximate solution by a fixed point argument. In Section 2 we prove the following variant of the result in [5].

Theorem 1.1. *Assume Hypothesis 1 with* $\mathfrak{n} = 1$, *and Hypothesis 2. There exists* $\varepsilon_0 > 0$ *such that for all* $|\varepsilon| \le \varepsilon_0$ *there is at least one quasi-periodic solution* $x_0(t) = c_0 + u(\boldsymbol{\omega} t, \varepsilon)$ *to (1.1), with* $u(\boldsymbol{\psi}, t) = O(\varepsilon)$ *analytic in* $\boldsymbol{\psi}$ *and* C^∞ *in* ε.

Our proof is different with respect to [5]: we show that, by introducing an auxiliary parameter μ, it is possible to write the solution as a convergent power series in μ with radius of convergence strictly greater than 1. Moreover, strictly speaking, smoothness at $\varepsilon = 0$ does not follows from [5], since the origin is on the boundary of the domain of analyticity in the complex ε-plane constructed therein. However, we want to stress that we borrow from [5] the idea not to estimate all small divisors independently of ε. We refer to Remark 2.9 for a more precise comparison. In fact, the main reason to give explicitly the proof of Theorem 1.1 is to set up notations in an easier case before dealing with a more degenerate case (any odd $\mathfrak{n}$) and prove a stronger result, namely Theorem 1.4 below, which is the main result of the present paper.

From the proof it turns out that, for fixed f, the condition to require on $\boldsymbol{\omega}$ for a quasi-periodic solution to exist can weakened. For instance, for the solution to be L^∞ it is sufficient that

$$\sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \frac{|f_{\boldsymbol{\nu}}|}{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}|} < \infty. \tag{1.3}$$

If we look for solutions with more regularity (as it is natural), a bit more has to be required. For instance, if f is analytic in the complexified torus $\mathbb{T}^d_\xi := \{x \in \mathbb{C} \ : \ \mathrm{Re}\, x \in \mathbb{T}^d,\ |\mathrm{Im}\, x| < \xi\}$, by looking at the proof of Theorem 1.1 in Section 2, it is easy to realise that the following possibilities occur. In order to have an analytic response solution, one may replace Hypothesis 2 with the weaker condition $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \ge \Xi\, \mathrm{e}^{-\xi' |\boldsymbol{\nu}|}$ for all $\boldsymbol{\nu} \neq \mathbf{0}$ and for some $\Xi > 0$ and $\xi' \in (0, \xi)$. To have a solution which is at least differentiable we can further enlarge the set of $\boldsymbol{\omega}$ allowed, by requiring that $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \ge \Xi\, \mathrm{e}^{-\xi |\boldsymbol{\nu}|} |\boldsymbol{\nu}|^p$, provided the exponent p is large enough; moreover the larger the exponent the smoother is the solution. However Hypothesis 2 has the advantage of being independent of the particular f appearing in (1.1) and hence can be formulated only in terms of the frequency vector $\boldsymbol{\omega}$. For further comments of this issue we defer to Section 4.

Both the method used here — and in [14, 15, 11, 13] — and the method of [5] assure the uniqueness of the solution only in a suitable space of smooth functions, so in principle we can neither exclude the existence of other quasi-periodic solutions nor conclude that the solution we construct is attractive. However, under a slightly stronger non-degeneracy condition we can obtain more information. Indeed the following result holds — the proof can be found in [13].

Theorem 1.2. *Consider (1.1) with* $\varepsilon > 0$. *Assume Hypothesis 1 with* $\mathfrak{n} = 1$ *and* $a := g'(c_0) > 0$. *If there is a quasi-periodic solution to (1.1) of the form* $x_0(t) = c_0 + O(\varepsilon)$, *then it is a local attractor.*

By existence of local attractor we mean that there is a simply connected open set containing the solution such that all trajectories starting inside that set tend to the solution as time goes to infinity. In particular this yields that the quasi-periodic solution is locally unique. Combining Theorems 1.1 and 1.2 we deduce the following result.

Theorem 1.3. *Consider (1.1) with $\varepsilon > 0$ and f a trigonometric polynomial. If Hypothesis 1 holds with $\mathfrak{n} = 1$ and $a > 0$, then for any $\boldsymbol{\omega}$ there is a quasi-periodic local attractor with the same frequency vector as f.*

On physical grounds we could expect a result of this kind to hold for any analytic function f. However Theorem 1.1 requires some Diophantine condition on $\boldsymbol{\omega}$ — however mild it may be. It would be interesting to see whether the Diophantine condition on $\boldsymbol{\omega}$ can be removed completely for f analytic, as in the case of trigonometric polynomials. Another interesting question is whether results of the kind of Theorems 1.1 to 1.3 could be obtained when $\mathfrak{n} \geq 3$.

As far as the second question is concerned, the first remark in order is that the assumption that the zero is of odd order cannot be removed: if there is a zero c_0 of even order, then there is no quasi-periodic solution to (1.1) reducing to c_0 as $\varepsilon \to 0$ (as it has been shown in [11]). For odd $\mathfrak{n}$ we shall prove a result analogous to Theorem 1.1.

Theorem 1.4. *Assume Hypothesis 1 with $\mathfrak{n} \geq 3$, and Hypothesis 2. There exist $\varepsilon_0 > 0$ such that for all $|\varepsilon| \leq \varepsilon_0$ there is at least one quasi-periodic solution $x_0(t) = c_0 + u(\boldsymbol{\omega}t, \varepsilon)$ to (1.1), with $u(\boldsymbol{\psi}, t) = O(\varepsilon)$ analytic in $\boldsymbol{\psi}$ and C^∞ in ε.*

If we require for $\boldsymbol{\omega}$ to be a Bryuno vector, then the existence of an analytic quasi-periodic solution of the form $c_0 + O(\varepsilon)$ for all $\mathfrak{n}$ odd follows from [11, 12]. The proof of Theorem 1.4 follows the same lines of the proof of Theorem 1.1, after a first step of perturbation theory in order to modify the linear operator, and with a more careful use of the irrationality of the frequency vector $\boldsymbol{\omega}$. In Section 3 we shall discuss the case of trigonometric polynomials (with the proof of a somewhat more technical lemma worked out in Appendix A). We shall see in Appendix B how to generalise the proof to any analytic f.

As in the case of Theorem 1.1, it is an open problem whether and how far the non-resonance condition on $\boldsymbol{\omega}$ can be weakened so as to yield the same result for any analytic f. On the contrary, Hypothesis 1 is optimal: if the equation $g(c_0) = f_{\mathbf{0}}$ either has no zero or has a zero c_0 of even order, then no response solution of the form $c_0 + O(\varepsilon)$ exists [11]. It would be worth investigating whether some analogues to Theorems 1.2 and 1.3 could be obtained for $\mathfrak{n} \geq 3$.

2 Proof of Theorem 1.1

Let us denote by $\mathbb{T}^d_\xi$ the complexified torus, i.e. $\mathbb{T}^d_\xi := \{x \in \mathbb{C} \,:\, \mathrm{Re}(x) \in \mathbb{T}^d,\ |\mathrm{Im}(x)| < \xi\}$ and by $\Delta(c_0, \rho)$ the disk of center c_0 and radius ρ in the complex plane. By the assumptions on f and g, for any $c_0 \in \mathbb{R}$ there exist $\xi_0 > 0$ and $\rho_0 > 0$ such that f is analytic in $\mathbb{T}^d_{\xi_0}$ and

g is analytic in $\Delta(c_0, \rho_0)$. Then for all $\xi < \xi_0$ and all $\rho < \rho_0$ one has

$$f(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} f_{\boldsymbol{\nu}}, \qquad |f_{\boldsymbol{\nu}}| \leq \Phi\, \mathrm{e}^{-\xi|\boldsymbol{\nu}|}, \tag{2.1a}$$

$$g(x) = \sum_{p=0}^{\infty} a_p (x - c_0)^p, \qquad a_p := \frac{1}{p!} \frac{\mathrm{d}^p g}{\mathrm{d}x^p}(c_0), \qquad |a_p| \leq \Gamma\, \rho^p, \tag{2.1b}$$

where Φ is the maximum of $f(\boldsymbol{\psi})$ for $\boldsymbol{\psi} \in \mathbb{T}^d_\xi$ and Γ is the maximum of $g(x)$ for $x \in \Delta(c_0, \rho)$. Of course both ρ_0 and Γ depend on c_0.

Hypothesis 1 implies that $a := a_{\mathfrak{n}} \neq 0$ and $a_p = 0$ for $\mathfrak{n} > 1$ and $p = 1, \ldots, \mathfrak{n} - 1$. Here we assume both Hypothesis 2 and Hypothesis 1, with $\mathfrak{n} = 1$. Let us rewrite (1.1) as

$$\varepsilon \ddot{x} + \dot{x} + \varepsilon\, a\, (x - c_0) + \varepsilon\, G(x) = \varepsilon\, \widetilde{f}(\boldsymbol{\omega} t), \tag{2.2}$$

where

$$G(x) := g(x) - g(c_0) - a\, (x - c_0) = \sum_{p=2}^{\infty} a_p (x - c_0)^p, \tag{2.3a}$$

$$\widetilde{f}(\boldsymbol{\psi}) := f(\boldsymbol{\psi}) - f_{\mathbf{0}} = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} f_{\boldsymbol{\nu}}, \tag{2.3b}$$

and introduce the auxiliary parameter μ by modifying (2.2) into

$$\varepsilon \ddot{x} + \dot{x} + \varepsilon\, a\, (x - c_0) + \mu\, \varepsilon\, G(x) = \mu\, \varepsilon\, \widetilde{f}(\boldsymbol{\omega} t). \tag{2.4}$$

Then we look for a quasi-periodic solution to (2.4) of the form

$$x(t, \varepsilon, \mu) = c_0 + u(\boldsymbol{\omega} t, \varepsilon, \mu), \qquad u(\boldsymbol{\psi}, \varepsilon, \mu) = \sum_{k=1}^{\infty} \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \mu^k \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} u^{(k)}_{\boldsymbol{\nu}}(\varepsilon). \tag{2.5}$$

We shall show that there exists $\mu_0 > 0$ such that there exists a solution of the form (2.5), analytic in μ for $|\mu| < \mu_0$. Since the original equation is recovered when $\mu = 1$ we need $\mu_0 > 1$. This will be obtained by showing that the coefficients $u^{(k)}_{\boldsymbol{\nu}}(\varepsilon)$ are bounded as $|u^{(k)}_{\boldsymbol{\nu}}(\varepsilon)| \leq A B^k \mathrm{e}^{-\xi'|\boldsymbol{\nu}|} |\varepsilon|^{\alpha k}$, for suitable positive constants A, B, ξ', α.

By inserting (2.5) into (2.4) we obtain a recursive definition for the coefficients $u^{(k)}_{\boldsymbol{\nu}}(\varepsilon)$. By defining

$$D(\varepsilon, s) := -\varepsilon s^2 + \mathrm{i} s + \varepsilon\, a, \tag{2.6}$$

one has, formally,

$$D(\varepsilon, \boldsymbol{\omega} \cdot \boldsymbol{\nu})\, u^{(1)}_{\boldsymbol{\nu}}(\varepsilon) = \varepsilon\, f_{\boldsymbol{\nu}} \tag{2.7a}$$

$$D(\varepsilon, \boldsymbol{\omega} \cdot \boldsymbol{\nu})\, u^{(k)}_{\boldsymbol{\nu}}(\varepsilon) = -\varepsilon \sum_{p=2}^{\infty} a_p \sum_{\substack{k_1, \ldots, k_p \geq 1 \\ k_1 + \ldots + k_p = k-1}} \sum_{\substack{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_p \in \mathbb{Z}^d \\ \boldsymbol{\nu}_1 + \ldots + \boldsymbol{\nu}_p = \boldsymbol{\nu}}} u^{(k_1)}_{\boldsymbol{\nu}_1}(\varepsilon) \ldots u^{(k_p)}_{\boldsymbol{\nu}_p}(\varepsilon), \qquad k \geq 2, \tag{2.7b}$$

for $\boldsymbol{\nu} \neq 0$, and

$$a\, u_{\mathbf{0}}^{(k)}(\varepsilon) = -\sum_{p=2}^{\infty} a_p \sum_{\substack{k_1,\ldots,k_p \geq 1 \\ k_1+\ldots+k_p=k-1}} \sum_{\substack{\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_p \in \mathbb{Z}^d \\ \boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_p=\mathbf{0}}} u_{\boldsymbol{\nu}_1}^{(k_1)}(\varepsilon) \ldots u_{\boldsymbol{\nu}_p}^{(k_p)}(\varepsilon), \qquad k \geq 1. \tag{2.8}$$

Here and henceforth the sums over the empty set are meant as zero.

Remark 2.1. For $k = 1$ (2.8) yields $u_{\mathbf{0}}^{(1)} = 0$. For $k = 2$ one has $u_{\boldsymbol{\nu}}^{(2)} = 0\ \forall \boldsymbol{\nu} \in \mathbb{Z}^d$.

By iterating the definition one obtains an explicit expression for the coefficients $u_{\boldsymbol{\nu}}^{(k)}$, which can be represented in terms of trees.

A *rooted tree* θ is a graph with no cycle, such that all the lines are oriented toward a unique point (*root*) which has only one incident line (root line). All the points in θ except the root are called *nodes*. The orientation of the lines in θ induces a partial ordering relation ($\preceq$) between the nodes. Given two nodes v and w, we shall write $w \prec v$ every time v is along the path (of lines) which connects w to the root; we shall write $w \prec \ell$ if $w \preceq v$, where v is the unique node that the line ℓ exits. For any node v denote by p_v the number of lines entering v.

Given a rooted tree θ we denote by $N(\theta)$ the set of nodes, by $E(\theta)$ the set of *end nodes*, i.e. nodes v with $p_v = 0$, by $V(\theta)$ the set of *internal nodes*, i.e. nodes v with $p_v \geq 1$, and by $L(\theta)$ the set of lines; by definition $N(\theta) = E(\theta) \amalg V(\theta)$.

We associate with each end node $v \in E(\theta)$ a *mode* label $\boldsymbol{\nu}_v \in \mathbb{Z}^d_*$ and with each internal node a *degree* label $d_v \in \{0, 1\}$. With each line $\ell \in L(\theta)$ we associate a *momentum* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d$ with the constraint

$$\boldsymbol{\nu}_\ell = \sum_{\substack{w \in E(\theta) \\ w \prec \ell}} \boldsymbol{\nu}_w. \tag{2.9}$$

We add the two following further constraints: (1) $p_v \geq 2\ \forall v \in V(\theta)$ and (2) if $d_v = 0$ then the line ℓ exiting v has $\boldsymbol{\nu}_\ell = \mathbf{0}$. We shall write $V(\theta) = V_0(\theta) \amalg V_1(\theta)$, where $V_0(\theta) := \{v \in V(\theta) : d_v = 0\}$. For any discrete set A we denote by $|A|$ its cardinality. Define the *degree* and the *order* of θ as $d(\theta) := |E(\theta)| + |V_1(\theta)|$ and $k(\theta) := |N(\theta)|$, respectively.

We call *equivalent* two labelled rooted trees which can be transformed into each other by continuously deforming the lines in such a way that they do not cross each other. In the following we shall consider only inequivalent labelled rooted trees, and we shall call them trees *tout court*, for simplicity.

We associate with each node $v \in N(\theta)$ a *node factor*

$$F_v := \begin{cases} -\varepsilon^{d_v}\, a_{p_v}, & v \in V(\theta), \\ \varepsilon\, f_{\boldsymbol{\nu}_v}, & v \in E(\theta), \end{cases} \tag{2.10}$$

and with each line $\ell \in L(\theta)$ a *propagator*

$$\mathcal{G}_\ell := \begin{cases} 1/D(\varepsilon, \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell), & \boldsymbol{\nu}_\ell \neq \mathbf{0}, \\ 1/a, & \boldsymbol{\nu}_\ell = \mathbf{0}. \end{cases} \tag{2.11}$$

Then, by defining

$$\mathscr{V}(\theta, \varepsilon) := \Bigg(\prod_{v \in N(\theta)} F_v \Bigg) \Bigg(\prod_{\ell \in L(\theta)} \mathcal{G}_\ell \Bigg) \tag{2.12}$$

one has

$$u_{\boldsymbol{\nu}}^{(k)}(\varepsilon) = \sum_{\theta \in \mathcal{T}_{k,\boldsymbol{\nu}}} \mathscr{V}(\theta, \varepsilon), \quad \boldsymbol{\nu} \in \mathbb{Z}^d, \tag{2.13}$$

where $\mathcal{T}_{k,\boldsymbol{\nu}}$ is the set of trees of order k and momentum $\boldsymbol{\nu}$ associated with the root line. The tree expansion (2.13) for the coefficients $u_{\boldsymbol{\nu}}^{(k)}(\varepsilon)$ can be easily checked by induction on the order k; see for instance [11, 13, 14] for details.

Lemma 2.2. *One has $|D(\varepsilon, s)| \geq \max\{|a\varepsilon|, |s|\}$ for ε small enough and all $s \in \mathbb{R}$.*

Proof. One has $|D(\varepsilon, s)| \geq |\mathrm{Im}\, D(\varepsilon, s)|$ and $|D(\varepsilon, s)| \geq |D(\varepsilon, 0)|$ for ε small enough. ■

Lemma 2.3. *For any tree θ one has $|E(\theta)| \geq |V(\theta)| + 1$.*

Proof. By induction on the order of the tree. ■

Remark 2.4. Equality $|E(\theta)| = |V(\theta)| + 1$ holds when $|N(\theta)| = 2^p + 1$, with $p \geq 1$, and $p_v = 2$ for all $v \in V(\theta)$.

Corollary 2.5. *For any tree θ one has $|E(\theta)| \geq \dfrac{1}{2}\,(k(\theta) + 1)$.*

Lemma 2.6. *For any $k \geq 1$, any $\boldsymbol{\nu} \in \mathbb{Z}^d$ and any tree $\theta \in \mathcal{T}_{k,\boldsymbol{\nu}}$ one has*

$$|\mathscr{V}(\theta, \varepsilon)| \leq B^k |\varepsilon|^{(k+1)/2} \prod_{v \in E(\theta)} \mathrm{e}^{-3\xi |\boldsymbol{\nu}_v|/4}$$

where ξ is as in (1.2a), with B a positive constant depending on Φ, Γ and ρ.

Proof. One bounds (2.12) as

$$|\mathscr{V}(\theta, \varepsilon)| \leq |\varepsilon|^{d(\theta)} \Bigg(\prod_{v \in V(\theta)} |a_{p_v}| \Bigg) \Bigg(\prod_{v \in E(\theta)} \frac{|f_{\boldsymbol{\nu}_v}|}{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_v|} \Bigg) \Bigg(\prod_{v \in V_0(\theta)} \frac{1}{|a|} \Bigg) \Bigg(\prod_{v \in V_1(\theta)} \frac{1}{|a\varepsilon|} \Bigg),$$

where we have used the bound $|D(\varepsilon, s)| \geq |s|$ for the propagators of the lines exiting the end nodes and the bound $|D(\varepsilon, s)| \geq |a\varepsilon|$ for the propagators of the lines exiting the nodes in $V_1(\theta)$. For each end node we bound $f_{\boldsymbol{\nu}_v}$ as in (1.2): then we extract a factor $\mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}$ and use Hypothesis 2 to bound $\mathrm{e}^{-\xi|\boldsymbol{\nu}|/4}|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|^{-1} \le C_0$, for a suitable constant C_0, for all $\boldsymbol{\nu}$ such that $f_{\boldsymbol{\nu}} \neq 0$. Moreover, by Corollary 2.5,

$$d(\theta) - |V_1(\theta)| = |E(\theta)| \ge \frac{k(\theta)+1}{2},$$

so that we obtain

$$|\mathscr{V}(\theta,\varepsilon)| \le \Gamma^{|V(\theta)|}\rho^{|N(\theta)|}(C_0\Phi)^{|E(\theta)|}a^{-|V(\theta)|}\left(\prod_{v\in E(\theta)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}\right)|\varepsilon|^{(k(\theta)+1)/2}.$$

Therefore, by bounding $\max\{|E(\theta)|,|V(\theta)|\} \le k(\theta)$, the assertion follows. ∎

Lemma 2.7. *For any $k \ge 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d$ one has*

$$\left|u^{(k)}_{\boldsymbol{\nu}}(\varepsilon)\right| \le C^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2}|\varepsilon|^{(k+1)/2},$$

where ξ is as in (1.2), with C a positive constant depending on Φ, Γ, ξ and ρ.

Proof. The coefficients $u^{(k)}_{\boldsymbol{\nu}}$ are defined by (2.13): we have to use the bounds of Lemma 2.6 and sum over all trees in $\mathcal{T}_{k,\boldsymbol{\nu}}$. The sum over the Fourier labels $\{\boldsymbol{\nu}_v\}_{v\in E(\theta)}$ is performed thanks to the factors $\mathrm{e}^{-3\xi|\boldsymbol{\nu}_v|/4}$ associated with the end nodes that we have not used to control the denominators $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v|$ — see the proof of Lemma 2.6 — and produces an overall factor $C_1^{|E(\theta)|}\mathrm{e}^{-\xi|\boldsymbol{\nu}|/2}$, for some positive constant C_1. The sum over the other labels produces a factor $C_2^{|N(\theta)|}$, with C_2 a suitable positive constant. Then the assertion follows by taking $C = BC_1C_2$. ∎

Remark 2.8. The main idea in the proof is to bound in a different way the propagators, depending on whether or not the lines exit end nodes. Eventually the propagators of the lines exiting the end nodes have a "gain" factor ε with respect to the propagators of the other lines: together with the fact that each internal node has at least two entering lines — so that the number of "bad" propagators turns out to be less than the number of "good" propagators —, this leads to bound the product of the propagators of any tree of order k proportionally to $|\varepsilon|^{-k/2}$. Note that a similar feature has been exploited in [10] in a rather different context, i.e. the problem of synchronisation in chaotic systems. As in that case — and as in [5] — no small divisor problem arises: of course this makes easier to study the convergence of the series.

Remark 2.9. The crucial property described in Remark 2.8, which allows to require only Hypothesis 2 on $\boldsymbol{\omega}$, has been already pointed out and used in [5]: in our proof we simply adapted that idea to our formalism. Smoothness in ε at $\varepsilon = 0$ is not discussed in [5], but very likely could be derived also with the method used therein.

The function (2.5), with the coefficients given by (2.13), solves (2.4) order by order. Since the series converges uniformly, then it is also a solution *tout court* of (2.4) — and hence of (2.2) for $\mu = 1$. Analyticity in $\boldsymbol{\psi} \in \mathbb{T}^d_{\xi'}$ for any $\xi' < \xi/2$ follows from the bound on the Fourier coefficients given by Lemma 2.7. To prove smoothness in ε one can reason as follows. Each value $\mathscr{V}(\theta, \varepsilon)$ is a polynomial in ε with coefficients depending on ε through the propagators. If one computes the n-th derivative of $\mathscr{V}(\theta, \varepsilon)$ with respect to ε, one can bound it in a different way depending on whether one has or not $(k(\theta) + 1)/2 \leq n$. If $(k(\theta) + 1)/2 \leq n$, then all the propagators and their derivatives are bounded by using the inequality $|D(\varepsilon, s)| \geq |s|$ of Lemma 2.2. If $(k(\theta) + 1)/2 > n$ one can reason as done above to arrive at the bounds in Lemma 2.7: one obtains the same bounds, up to a multiplicative constant A depending on n only, and with a power of ε decreased by n, so that the sum over $k(\theta)$ can still be performed.

3 Proof of Theorem 1.4

Assume Hypothesis 2 and Hypothesis 1 with $\mathfrak{n} \geq 3$. We look for a solution $x(t)$ to (1.1) of the form

$$x(t) = c_0 + \varepsilon\, x_1(t) + \xi(t), \tag{3.1}$$

where $\varepsilon\, x_1(t)$ it the solution to the first-order truncation of (1.1), i.e.

$$\varepsilon \ddot{x}_1 + \dot{x}_1 = \widetilde{f}(\boldsymbol{\omega} t),$$

with $\widetilde{f}$ as in (2.3). An easy computation gives $x_1(t) = \zeta + u^{[1]}(\boldsymbol{\omega} t, \varepsilon)$, where

$$u^{[1]}(\boldsymbol{\psi}, \varepsilon) := \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} u^{[1]}_{\boldsymbol{\nu}}(\varepsilon), \qquad u^{[1]}_{\boldsymbol{\nu}}(\varepsilon) := \frac{f_{\boldsymbol{\nu}}}{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}(1 + \mathrm{i}\varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\nu})}, \tag{3.2}$$

and ζ is a real parameter that will be fixed later on. Note that Hypothesis 2 guarantees that the function (3.2) is well-defined for any analytic f. Therefore the problem is reduced to finding a zero-average quasi-periodic solution $\xi(t)$ to the equation

$$\varepsilon\ddot{\xi} + \dot{\xi} + \varepsilon\, \widetilde{G}(\varepsilon\, x_1(t) + \xi) = 0, \qquad \widetilde{G}(x) := \sum_{p=\mathfrak{n}}^{\infty} a_p x^p, \tag{3.3}$$

which can be rewritten as

$$\varepsilon\ddot{\xi} + \dot{\xi} + b\, \varepsilon^{\mathfrak{n}}\, \xi + \mu\, \varepsilon\, \widehat{G}(\mu\varepsilon x_1(t), \xi) = 0, \tag{3.4}$$

where $\mu = 1$ and

$$b := \sum_{p=\mathfrak{n}}^{\infty} p\, a_p\, \varepsilon^{p-\mathfrak{n}} \big[\big(x_1(t)\big)^{p-1}\big]_{\mathbf{0}}, \tag{3.5a}$$

$$\widehat{G}(x, \xi) := \sum_{p=\mathfrak{n}}^{\infty} a_p \sum_{s=0}^{p} \binom{p}{s} \xi^s \left(x^{p-s} - \delta_{s,1} \left[x^{p-s}\right]_{\mathbf{0}}\right), \tag{3.5b}$$

with $[\cdot]_{\mathbf{0}}$ denoting — here and henceforth — the average on $\mathbb{T}^d$.

Remark 3.1. By setting $b_0 := \mathfrak{n}\, a_{\mathfrak{n}}\, [(x_1(t))^{\mathfrak{n}-1}]_{\mathbf{0}}$, one has $b_0 \neq 0$, because $a_{\mathfrak{n}} = a \neq 0$ and $\mathfrak{n}$ is odd, and hence $b = b_0(1 + O(\varepsilon))$ does not vanish for ε small enough.

As in Section 2 we first ignore the constraint $\mu = 1$ and treat it as a parameter: we shall look for a solution which can be written as a power series in μ, with coefficients which still admit a tree expansion.

Let us assume here that f is a trigonometric polynomial of degree N, i.e. that $f_{\boldsymbol{\nu}} = 0$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d$ such that $|\boldsymbol{\nu}| > N$. In such a case it is more convenient to redefine $\Phi = \max\{|f_{\boldsymbol{\nu}}| : |\boldsymbol{\nu}| \leq N\}$. We shall see in Appendix B how to extend the proof to the case of f analytic. Define

$$\alpha = \min\{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \leq (\mathfrak{n} + 1)\, N\}. \tag{3.6}$$

One has $\alpha > 0$ by the assumption of irrationality on $\boldsymbol{\omega}$.

With respect to Section 2 we modify the tree expansion as follows. Rooted trees and the sets $N(\theta)$, $E(\theta)$, $V(\theta)$ and $L(\theta)$ are defined as previously. If p_v denotes the number of lines entering $v \in V(\theta)$ we impose the constraint $p_v \geq \mathfrak{n}$. We associate with each end node v a mode label $\boldsymbol{\nu}_v \in \mathbb{Z}^d$ and with each line a momentum $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d$, still satisfying (2.9) and with the further constraint that $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ if ℓ exits an internal node. We split $E(\theta) = E_0(\theta) \amalg E_1(\theta)$, with $E_0(\theta) := \{v \in E(\theta) : \boldsymbol{\nu}_v = \mathbf{0}\}$. The order of θ is still defined as $k(\theta) = |N(\theta)|$. Let us define also $L_0(\theta) := \{\ell \in L(\theta) : |\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell| < \alpha/2\}$ and $V_0(\theta) := \{v \in V(\theta) : \text{the line } \ell \text{ exiting } v \text{ belongs to } L_0(\theta)\}$, and set $V_1(\theta) := V(\theta) \setminus V_0(\theta)$.

We call *excluded* a node v such that $p_v - 1$ lines entering v exit end nodes, and the other line entering v exits an internal node and has the same momentum as the line exiting v. Let $\mathfrak{T}_{k,\boldsymbol{\nu}}$ be the set of inequivalent labelled rooted trees, which do not contain any excluded nodes, of order k and momentum $\boldsymbol{\nu}$ associated with the root line. In the following we shall call simply trees the elements of $\mathfrak{T}_{k,\boldsymbol{\nu}}$.

We associate with each node $v \in N(\theta)$ a *node factor*

$$F_v := \begin{cases} -\varepsilon\, a_{p_v}, & v \in V(\theta), \\ \varepsilon\, f_{\boldsymbol{\nu}}, & v \in E_1(\theta) \\ \varepsilon\, \zeta, & v \in E_0(\theta), \end{cases} \tag{3.7}$$

where $\zeta \in \mathbb{R}$ is the parameter introduced before (3.2), and with each line $\ell \in L(\theta)$ a *propagator*

$$\mathcal{G}_\ell := \begin{cases} \mathcal{G}_E(\varepsilon, \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell), & \boldsymbol{\nu}_\ell \neq \mathbf{0} \text{ and } \ell \text{ exits an end node}, \\ \mathcal{G}_V(\varepsilon, \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell), & \boldsymbol{\nu}_\ell \neq \mathbf{0} \text{ and } \ell \text{ exits an internal node}, \\ 1, & \boldsymbol{\nu}_\ell = \mathbf{0}, \end{cases} \tag{3.8}$$

with

$$\mathcal{G}_E(\varepsilon, s) := \frac{1}{\mathrm{i}s(1 + \mathrm{i}\varepsilon s)}, \qquad \mathcal{G}_V(\varepsilon, s) := \frac{1}{\mathrm{i}s(1 + \mathrm{i}\varepsilon s) + b\, \varepsilon^{\mathfrak{n}}}, \tag{3.9}$$

where $b \in \mathbb{R}_+$ is defined in (3.5a) — and hence is a function of ζ.

Setting

$$\mathscr{V}(\theta,\varepsilon) := \left(\prod_{v\in N(\theta)} F_v\right)\left(\prod_{\ell\in L(\theta)} \mathcal{G}_\ell\right) \tag{3.10}$$

and

$$u^{[k]}_{\boldsymbol{\nu}}(\varepsilon) := \sum_{\theta\in\mathfrak{T}_{k,\boldsymbol{\nu}}} \mathscr{V}(\theta,\varepsilon), \quad \boldsymbol{\nu}\neq\mathbf{0}, \qquad k\geq 2, \tag{3.11}$$

we define (formally) the series

$$\overline{\xi}(\boldsymbol{\psi},\varepsilon,\mu) := \sum_{k=2}^{\infty} \mu^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} u^{[k]}_{\boldsymbol{\nu}}(\varepsilon). \tag{3.12}$$

and set $\overline{u}(\boldsymbol{\psi},\varepsilon,\mu) = \mu\varepsilon\big(\zeta + u^{[1]}(\boldsymbol{\psi},\varepsilon)\big) + \overline{\xi}(\boldsymbol{\psi},\varepsilon,\mu)$.

Remark 3.2. The constraint $p_v \geq \mathfrak{n}$ implies $u^{[k]}_{\boldsymbol{\nu}}(\varepsilon) = 0$ $\forall \boldsymbol{\nu}\in\mathbb{Z}^d_*$ and $2\leq k\leq\mathfrak{n}$.

Remark 3.3. The coefficients (3.10) depend on ζ, which so far is still a free parameter.

The definition (3.12) is formal not only in the sense that it may fail to converge. In fact the very definition of the coefficients $u^{[k]}_{\boldsymbol{\nu}}(\varepsilon)$ involves quantities — the propagators — for which we have not yet any estimate. The latter problem is easily solved as follows. Define

$$D_V(\varepsilon,s) := \frac{1}{\mathcal{G}_V(\varepsilon,s)} = \mathrm{i}s\,(1+\mathrm{i}\varepsilon s) + b\,\varepsilon^{\mathfrak{n}}. \tag{3.13}$$

Lemma 3.4. *There exist $\varepsilon_1 > 0$ such that $|D_N(\varepsilon,s)| \geq \max\{|s|, |b\varepsilon^{\mathfrak{n}}|\}$ for all $s\in\mathbb{R}$ and $|\varepsilon| < \varepsilon_1$.*

Proof. Reason as in the proof of Lemma 2.2. ∎

Thanks to Lemma 3.4 we deduce that the coefficients (3.11) are well defined for all $k\geq 2$ and all $\boldsymbol{\nu}\in\mathbb{Z}^d_*$. Now we want to find conditions for the series (3.12) to converge. We shall prove that, for any $\zeta\in\mathbb{R}$, under the assumption that ε is small enough, depending on ζ, the series (3.12) converges to a well-defined function analytic in $\boldsymbol{\psi}$ and C^∞ in ε, with a radius of convergence $\mu_0 > 1$: this will allow us to take $\mu = 1$ in (3.4). Moreover we shall show that, for any fixed $\overline{\zeta}\in\mathbb{R}$, the coefficients $u^{[k]}_{\boldsymbol{\nu}}(\varepsilon)$ admit uniform bounds for $|\zeta|\leq\overline{\zeta}$.

Lemma 3.5. *For any tree θ one has $|E(\theta)| \geq (\mathfrak{n}-1)\,|V(\theta)| + 1$.*

Proof. The bound is proved by induction by using that $p_v\geq\mathfrak{n}$ $\forall v\in V(\theta)$. ∎

Corollary 3.6. *For any tree θ one has $\mathfrak{n}\,|E(\theta)| \geq (\mathfrak{n}-1)\,k(\theta) + 1$.*

Lemma 3.7. *For any tree θ one has $\mathfrak{n}\,|V_0(\theta)| \le |E(\theta)| - 2$.*

The proof of Lemma 3.7 is in Appendix A.

Lemma 3.8. *For any $k \ge 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d_*$ and any tree $\theta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$ one has*

$$|\mathscr{V}(\theta,\varepsilon)| \le B^k |\zeta|^{|E_0(\theta)|} |b|^{-|V_0(\theta)|} |\varepsilon|^{1+\frac{\mathfrak{n}-1}{\mathfrak{n}^2}k},$$

with B a positive constant depending on Φ, Γ, ρ and α.

Proof. One bounds (3.10) as

$$|\mathscr{V}(\theta,\varepsilon)| \le |\varepsilon|^{k(\theta)} |\zeta|^{|E_0(\theta)|} \Bigg(\prod_{v\in V(\theta)} |a_{p_v}| \Bigg) \Bigg(\prod_{v\in E_1(\theta)} \frac{|f_{\boldsymbol{\nu}_v}|}{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v|} \Bigg) \Bigg(\prod_{v\in V_0(\theta)} \frac{1}{|b\varepsilon^{\mathfrak{n}}|} \Bigg) \Bigg(\prod_{v\in V_1(\theta)} \frac{2}{\alpha} \Bigg),$$

where, by relying on Lemma 3.4, we have used the bound $|D(\varepsilon,s)| \ge |s|$ for the propagators of the lines exiting either the nodes $v \in E_1(\theta)$ or the nodes $v \in V_1(\theta)$ and the bound $|D(\varepsilon,s)| \ge |b\varepsilon^{\mathfrak{n}}|$ for the propagators of the lines exiting the nodes in $V_0(\theta)$. For each end node we bound $|f_{\boldsymbol{\nu}_v}|\,|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v|^{-1} \le 2\Phi/\alpha$. Then

$$|\mathscr{V}(\theta,\varepsilon)| \le |\varepsilon|^{k(\theta)-\mathfrak{n}|V_0(\theta)|} \Gamma^{|V(\theta)|} \rho^{|N(\theta)|} |\zeta|^{|E_0(\theta)|} \Phi^{|E_1(\theta)|} |b|^{-|V_0(\theta)|} (2/\alpha)^{|V_1(\theta)|+|E_1(\theta)|},$$

where we can bound, by using Lemma 3.7 and Corollary 3.6,

$$\begin{aligned} k(\theta) - \mathfrak{n}|V_0(\theta)| &= |E(\theta)| + |V(\theta)| - \mathfrak{n}|V_0(\theta)| \ge |E(\theta)| - (\mathfrak{n}-1)|V_0(\theta)|, \\ &\ge |E(\theta)| - (\mathfrak{n}-1)\frac{E(\theta)-2}{\mathfrak{n}} = 2 - \frac{2}{\mathfrak{n}} + \frac{|E(\theta)|}{\mathfrak{n}} \ge 1 + \frac{\mathfrak{n}-1}{\mathfrak{n}^2}k(\theta), \end{aligned}$$

so that, by using that $\max\{|V(\theta)|,|E(\theta)|\} \le |N(\theta)| = k(\theta)$, the assertion follows. ∎

Remark 3.9. The bounds in Lemma 3.8 depend on ζ. However, for any given $\overline{\zeta} > 0$, there exists $\overline{b} > 0$ such that $|b| \ge \overline{b}$ (by continuity), and hence for $|\zeta| \le \overline{\zeta}$ we can obtain uniform bounds

$$|\mathscr{V}(\theta,\varepsilon)| \le B^k \zeta_*^k b_*^{-k/\mathfrak{n}} |\varepsilon|^{1+\frac{\mathfrak{n}-1}{\mathfrak{n}^2}k},$$

where $\zeta_* = \max\{1,\overline{\zeta}\}$ and $b_* = \min\{1,\overline{b}\}$.

Lemma 3.10. *For any $k \ge 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d_*$ one has*

$$\left| u^{[k]}_{\boldsymbol{\nu}}(\varepsilon) \right| \le C^k |\varepsilon|^{1+\frac{\mathfrak{n}-1}{\mathfrak{n}^2}k},$$

with C a positive constant depending on Φ, Γ, ρ, ζ_*, b_*, N and α.

Proof. Reason as in the proof of Lemma 2.7 and use Remark 3.9. Now the sum over the mode labels can be bounded by $(2N+1)^{dk}$. ■

We have proved that the series (3.12) converges to a well-defined function for $|\mu|<\mu_0$, with $\mu_0>1$, provided (1) $|\zeta|\le\overline{\zeta}$ for some $\overline{\zeta}>0$ and (2) ε is small enough. Moreover, by construction, the function is periodic and analytic in $\boldsymbol{\psi}$ and C^∞ in both ε and ζ (this can be seen as in Section 2). In the remaining part of this section we shall prove that ζ can be fixed in such a way that $|\zeta|\le\overline{\zeta}$ and the function $\overline{\xi}(\boldsymbol{\omega}t,\varepsilon,\mu)$ solves the equation (3.4) for $|\mu|<\mu_0$ — and hence in particular $\overline{\xi}(\boldsymbol{\omega}t,\varepsilon,1)$ solves the equation (3.3).

We can write (3.4) in Fourier space, if we expand (formally)

$$\xi(t)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}\xi_{\boldsymbol{\nu}},$$

so as to obtain

$$\left(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}(1+\mathrm{i}\varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\nu})+b\,\varepsilon^{\mathrm{n}}\right)\xi_{\boldsymbol{\nu}}=-\left[\mu\,\varepsilon\,\widehat{G}(\mu\,\varepsilon\,x_1(t),\xi)\right]_{\boldsymbol{\nu}},\qquad\boldsymbol{\nu}\neq\mathbf{0},\tag{3.14a}$$

$$0=-\left[\mu\,\varepsilon\,\widehat{G}(\mu\,\varepsilon\,x_1(t),\xi)\right]_{\mathbf{0}},\tag{3.14b}$$

where $[A]_{\boldsymbol{\nu}}$ means that we expand the function A in Fourier series in $\boldsymbol{\psi}$ and keep the Fourier coefficient with label $\boldsymbol{\nu}$. If we expand further (again formally) $\xi_{\boldsymbol{\nu}}$ as a Taylor series in μ, by writing

$$\xi_{\boldsymbol{\nu}}=\sum_{k=2}^{\infty}\mu^k\xi_{\boldsymbol{\nu}}^{[k]},$$

we can write (3.14a) order by order,

$$\left(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}(1+\mathrm{i}\varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\nu})+b\,\varepsilon^{\mathrm{n}}\right)\xi_{\boldsymbol{\nu}}^{[k]}=-\left[\varepsilon\,\widehat{G}(\mu\,\varepsilon\,x_1(t),\xi)\right]_{\boldsymbol{\nu}}^{[k-1]},\quad\boldsymbol{\nu}\neq\mathbf{0},\quad k\ge2,\tag{3.15}$$

where $[A]_{\boldsymbol{\nu}}^{[k]}$ means that we expand the function $[A]_{\boldsymbol{\nu}}$ in powers of μ and keep the Taylor coefficient to order k.

Lemma 3.11. *For any $\zeta\in\mathbb{R}$ the coefficients $\xi_{\boldsymbol{\nu}}^{[k]}=u_{\boldsymbol{\nu}}^{[k]}(\varepsilon)$ solve (3.15).*

Proof. Expand the right hand side of (3.15) in powers of εx_1 and ξ, and write $\varepsilon\,x_1$ according to (3.1), with the coefficients $u_{\boldsymbol{\nu}}^{[1]}(\varepsilon)$ as in (3.2), and ξ according to (3.12), with the coefficients $u_{\boldsymbol{\nu}}^{[k]}(\varepsilon)$ as in (3.11). Then (3.15) reduces to (3.11) itself. ■

Therefore, for any $\zeta\in\mathbb{R}$, the function (3.12) formally solves (3.14a). If $|\zeta|\le\overline{\zeta}$ and ε is small enough then the series (3.12) converges uniformly and therefore solves (3.14a). Moreover the function $\widehat{G}(\mu\,\varepsilon\,x_1(t),\xi)$ is well defined and hence it makes sense to consider its average. So we are left with the equation (3.14b): we shall show that it is possible to fix ζ in such a way that such an equation is satisfied.

Consider the implicit function problem

$$F_2(\zeta,\varepsilon) := \frac{1}{\varepsilon^{\mathfrak{n}}}\big[\varepsilon\,\widehat{G}(\mu\,\varepsilon\,x_1(t),\xi)\big]_{\mathbf{0}} = 0. \tag{3.16}$$

If we are able to find a solution to (3.16) then (3.14b) is also satisfied.

Lemma 3.12. *Let ε be small enough. There is $\overline{\zeta} > 0$ such that there exists a unique real solution $\zeta = \zeta(\varepsilon)$ to (3.16), with $|\zeta(\varepsilon)| < \overline{\zeta}$. Moreover $\zeta(\varepsilon)$ is C^∞ in ε.*

Proof. One has

$$F_2(\zeta,\varepsilon) = \overline{F}_2(\zeta) + F_3(\zeta,\varepsilon), \qquad \overline{F}_2(\zeta) := \big[\, \Big(\zeta + u^{[1]}(\boldsymbol{\omega} t,\varepsilon)\Big)^{\mathfrak{n}} \,\big]_{\mathbf{0}},$$

where $F_3(\zeta,\varepsilon)$ is a function which goes to zero when ε goes to zero. The function $\overline{F}_2(\zeta)$ is a polynomial of order $\mathfrak{n}$ in ζ. The equation $\overline{F}_2(\zeta) = 0$ admits a unique real root ζ_0 — see Lemma 2.4 in [12]. Since $\mathrm{d}\overline{F}_2(\zeta)/\mathrm{d}\zeta = \mathfrak{n}\big[\, \big(\zeta + u^{[1]}(\boldsymbol{\omega} t,\varepsilon)\big)^{\mathfrak{n}-1} \,\big]_{\mathbf{0}} = b_0/a$, the root is simple. Therefore, by the implicit function theorem, for ε small enough there is a unique $\zeta(\varepsilon)$ such that $\zeta(0) = \zeta_0$ and $F_2(\zeta(\varepsilon),\varepsilon) = 0$. ■

In particular, as the proof of Lemma 3.12 shows, one can take $\overline{\zeta} = 2\zeta_0$, where ζ_0 is the root of $\overline{F}_2(\zeta) = 0$. The proof of Theorem 1.4 is complete, in the case of trigonometric polynomials f.

4 Comments

By looking at the proofs of Theorems 1.1 and 1.4 given in Sections 2 and 3, respectively, we see that, in both cases, Hypothesis 2 ensures that the first order is well-defined. Indeed we used Hypothesis 2 to bound the propagators of the lines ℓ exiting the end nodes as $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|^{-1}$, i.e. to control $D(\varepsilon,\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ in (2.7a) and the denominators $\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}(1+\mathrm{i}\varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ in (3.3). Therefore we can rephrase the two theorems by saying that the condition for a quasi-periodic solution to exist is the same condition for the first order to be well-defined.

We note also that, under Hypothesis 2, the formal expansion in powers of ε is well defined to all orders. This can be easily checked, for instance, by looking for a solution to (1.1) in the form of a formal power series in ε: one finds a recursive definition for the coefficients of the series and writes down a tree expansion for such coefficients. Then one shows that the coefficients are well defined to all orders under the only Hypothesis 2 — see [7] for more details in a similar case. In our case, the existence of the formal power series is enough to conclude about the existence of a solution: Theorems 1.1 and 1.4 imply that the conditions for the existence of a solution are the same conditions required for the existence of a formal solution, i.e. a solution in the form of a formal power series in ε. This is a quite non-general feature. Usually, one cannot infer that a solution to an ordinary differential equation exists simply from the fact that a formal solution exists: a classical example are elliptic lower-dimensional tori (see however [17] and references quoted therein for germs of vector fields in a neighborhood of a fixed point with hyperbolic linear part).

From the perspective outlined above it is not clear whether the theorems could be generalised to any vector $\boldsymbol{\omega} \in \mathbb{R}^d$ for any analytic forcing f: for the first order or the formal expansion to be well defined, some Diophantine condition on $\boldsymbol{\omega}$ seems necessary. On the other hand, for instance when $\mathfrak{n} = 1$ and $a > 0$, from a physical point of view one expects for a local attractor to exist, and it is not unlikely that a synchronisation phenomenon occurs. Of course it could also happen that the conjugation exists but is not smooth (think of Denjoy's theorem for diffeomorphisms of the circle [9, 1]): in that case it would not possible to construct it with the techniques used here.

We conclude with two technical comments.

1. The smallness assumptions of both Theorems 1.1 and 1.4 can be weakened. Indeed, by looking at the proofs of the Theorems — in particular the bounds on $\mathscr{V}(\theta, \varepsilon)$ given in the proofs of Lemmas 2.6 and 3.8 —, one sees that the parameter that must be small is $\varepsilon\Phi$. Therefore a large forcing is still allowed as far as $\varepsilon\Phi$ remains small.

2. A property like Lemma 3.7 — or Lemma B.5 in the case of analytic forcing — can be found to hold also for the case $\mathfrak{n} = 1$ (for a suitably defined set $V_0(\theta)$ — or $L_0(\theta)$ in Appendix B). The argument of Section 2 shows that this is not necessary to prove the existence of a quasi-periodic solution. However, a property of this kind can be used to enlarge, with respect to the results found in [5], the domain of analyticity for ε in the complex plane; see [6] for results in that direction.

A Proof of Lemma 3.7

The proof is by induction on the order of the tree. First of all note that $k(\theta) \geq \mathfrak{n} + 1$ by construction (see Remark 3.2). If $k(\theta) = \mathfrak{n} + 1$, then the root line has momentum $\boldsymbol{\nu} = \boldsymbol{\nu}_1 + \ldots + \boldsymbol{\nu}_{\mathfrak{n}}$, where $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_{\mathfrak{n}}$ are the mode labels of the $\mathfrak{n}$ end nodes of θ, so that $|\boldsymbol{\nu}| \leq \mathfrak{n}N$ and hence $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq \alpha > \alpha/2$. Therefore $|V_0(\theta)| = 0$ in such a case and the bound holds.

If $k(\theta) \geq \mathfrak{n} + 2$, call ℓ_0 the root line of θ and v_0 the node which ℓ_0 exits. Let r be the number of end nodes whose exiting lines enter v_0 and set $s = p_{v_0} - r$: there will be s trees $\theta_1, \ldots, \theta_s$ such that the respective root lines $\ell_1, \ldots, \ell_s$ enter v_0. Note that $|E(\theta)| = |E(\theta_1)| + \ldots + |E(\theta_s)| + r$. If $s = 0$ the bound holds: indeed if $k(\theta) = \mathfrak{n} + 2$, then $|E(\theta)| = \mathfrak{n} + 1$ and $|V_0(\theta)| = 0$, because the momentum $\boldsymbol{\nu}$ of the root line is such that $|\boldsymbol{\nu}| \leq (\mathfrak{n} + 1)N$ and hence $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq \alpha$, while if $k(\theta) \geq \mathfrak{n} + 3$, then $|E(\theta)| \geq \mathfrak{n} + 2$ and $|V_0(\theta)| \leq 1$. Therefore in the following we assume $s \geq 1$.

If $\ell_0 \notin L_0(\theta)$, then $|V_0(\theta)| = |V_0(\theta_1)| + \ldots + |V_0(\theta_s)|$, so that, by the inductive hypothesis,

$$|V_0(\theta)| \leq \sum_{k=1}^{s} \frac{|E(\theta_k)| - 2}{\mathfrak{n}} = \frac{|E(\theta)| - r - 2s}{\mathfrak{n}} \leq \frac{|E(\theta)| - 2}{\mathfrak{n}},$$

and the bound follows.

If $\ell_0 \in L_0(\theta)$, then $|V_0(\theta)| = 1 + |V_0(\theta_1)| + \ldots + |V_0(\theta_s)|$ and, again by the inductive hypothesis,

$$|V_0(\theta)| \leq 1 + \sum_{k=1}^{s} \frac{|E(\theta_k)| - 2}{\mathfrak{n}} = \frac{|E(\theta)| - 2}{\mathfrak{n}} + \left[1 - \frac{r + 2\,(s-1)}{\mathfrak{n}}\right].$$

If $s + r \geq \mathfrak{n} + 1$, then $r + 2\,(s-1) \geq \mathfrak{n} + (s-1) \geq \mathfrak{n}$. If $s + r = \mathfrak{n}$ and $s \geq 2$, then $r + 2\,(s-1) \geq \mathfrak{n} + (s-2) \geq \mathfrak{n}$. Thus in both cases the last term in square brackets is non-positive and the bound follows.

If $s + r = \mathfrak{n}$ and $s = 1$, then the line ℓ_1 must be in $L_1(\theta)$. This can be seen by reductio ad absurdum. Suppose that $\ell_1 \in L_0(\theta)$. Then $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell_1}| < \alpha/2$. Moreover $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell_0}| < \alpha/2$ because $\ell_0 \in L_0(\theta)$ by hypothesis. On the other hand one has $\boldsymbol{\nu}_{\ell_0} = \boldsymbol{\nu}_{\ell_1} + \boldsymbol{\nu}_1 + \ldots + \boldsymbol{\nu}_r$, where $r = \mathfrak{n} - 1$ and $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_r$ are the mode labels of the r end nodes whose exiting lines enter v_0. Therefore, if we use that $|\boldsymbol{\nu}_1 + \ldots + \boldsymbol{\nu}_r| \leq (\mathfrak{n}-1)N$ and $\boldsymbol{\nu}_1 + \ldots + \boldsymbol{\nu}_r \neq \mathbf{0}$ (otherwise v_0 would be an excluded node), we obtain

$$\alpha > |\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell_0}| + |\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell_1}| \geq |\boldsymbol{\omega} \cdot (\boldsymbol{\nu}_{\ell_0} - \boldsymbol{\nu}_{\ell_1})| = |\boldsymbol{\omega} \cdot (\boldsymbol{\nu}_1 + \ldots + \boldsymbol{\nu}_r)| \geq \alpha,$$

so arriving at a contradiction. Let v_1 be the node which ℓ_1 exits: there will be r' end nodes whose exiting lines enter v_1 and s' trees $\theta'_1, \ldots, \theta'_{s'}$ whose root lines $\ell'_1, \ldots, \ell'_s$ enter v_1. One has $|E(\theta)| = |E(\theta'_1)| + \ldots + |E(\theta'_{s'})| + r + r'$ and $|V_0(\theta)| = 1 + |V_0(\theta'_1)| + \ldots + |V_0(\theta'_{s'})|$, where $s = 1$, $r = \mathfrak{n} - 1$ and $r' + s' \geq \mathfrak{n}$. By the inductive hypothesis one has

$$|V_0(\theta)| \leq 1 + \sum_{k'=1}^{s'} \frac{|E(\theta'_{k'})| - 2}{\mathfrak{n}} = \frac{|E(\theta)| - 2}{\mathfrak{n}} + \left[1 - \frac{r + r' + 2s' - 2}{\mathfrak{n}}\right],$$

where $r + r' + 2s' - 2 \geq 2\mathfrak{n} + s' - 3 \geq \mathfrak{n} + (\mathfrak{n} - 3) \geq \mathfrak{n}$, so that the last term in square bracket is non-positive. Therefore the bound follows once more.

B Proof of Theorem 1.4 for analytic forcing

In the analytic case the trees are constructed as in Section 3: in particular the definition of the coefficients (3.11) of the series (3.12) is the same. The only difference is how to bound the values of the trees in (3.11).

First of all we need some notations. We shall not introduce the sets $V_0(\theta)$ and $V_1(\theta)$ of Section 3. Instead, we shall proceed as follows. For any node $v \in V(\theta)$ define $E(\theta, v) := \{w \in E(\theta) :$ the line exiting w enters $v\}$, $r_v := |E(\theta, v)|$, $s_v := p_v - r_v$ and

$$\boldsymbol{\mu}_v := \sum_{w \in E(\theta, v)} \boldsymbol{\nu}_w, \qquad \mu_v := |\boldsymbol{\mu}_v|.$$

Set $V_0(\theta) := \{v \in V(\theta) : s_v = 0\}$ and $V_1(\theta) := \{v \in V(\theta) : s_v = 1\}$. If $v \in V_0(\theta)$ we call ℓ_v the line exiting v, while if $v \in V_1(\theta)$ we call ℓ_v the line exiting v and ℓ'_v the line entering v which does not exits an end node.

Remark B.1. By Hypothesis 2 there exists $C_0 > 0$ such that $C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| \geq \mathrm{e}^{-\xi|\boldsymbol{\nu}|/8}$ $\forall \boldsymbol{\nu} \in \mathbb{Z}^d_*$.

Lemma B.2. *If $v \in V_0(\theta)$ one has $C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_v}| \geq \mathrm{e}^{-\xi\mu_v/8}$.*

Proof. For $v \in V_0(\theta)$ one has $\boldsymbol{\nu}_{\ell_v} = \boldsymbol{\mu}_v$, so that the bound follows from Remark B.1. ■

Lemma B.3. *If $v \in V_1(\theta)$ one has $C_0 \max\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_v}|, |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_v}|\} \geq \mathrm{e}^{-\xi\mu_v/8}/2$.*

Proof. By contradiction: if the bound does not hold then

$$\mathrm{e}^{-\xi\mu_v/8} > C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_v}| + C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_v}| \geq C_0|\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_{\ell_v} - \boldsymbol{\nu}_{\ell'_v})| = C_0|\boldsymbol{\omega}\cdot\boldsymbol{\mu}_v| \geq \mathrm{e}^{-\xi\mu_v/8}, \tag{B.1}$$

where we have used that $\boldsymbol{\mu}_v \neq \mathbf{0}$, otherwise v would be an excluded node. ■

Define $L_1(\theta, v) := \{\ell_v\}$ for $v \in V_0(\theta)$ and $L_1(\theta, v) := \{\ell \in \{\ell_v, \ell'_v\} : C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| \geq \mathrm{e}^{-\xi\mu_v/8}/2\}$ for $v \in V_1(\theta)$. By Lemmas B.2 and B.3 one has $L_1(\theta, v) \neq \emptyset$ for all $v \in V_0(\theta) \cup V_1(\theta)$. Set also $L_1(\theta) := \{\ell \in L(\theta) : \exists v \in V_0(\theta) \cup V_1(\theta)$ such that $\ell \in L_1(\theta, v)\}$, $L_{\rm int}(\theta) := \{\ell \in L(\theta) : \ell$ exits a node $v \in V(\theta)\}$ and $L_0(\theta) := L_{\rm int}(\theta) \setminus L_1(\theta)$.

Lemma B.4. *For any tree θ one has $\mathfrak{n}\,|L_0(\theta)| \leq |E(\theta)| - 2$.*

Proof. One proceeds by induction on $V(\theta)$. If $|V(\theta)| = 1$ then $V(\theta) = V_0(\theta)$ and hence $|L_0(\theta)| = 0$, while $|E(\theta)| - 2 > 0$, so that the bound holds. If $|V(\theta)| \geq 2$ the root line ℓ_0 of θ exits a node $v_0 \in V(\theta)$ with $s_{v_0} + r_{v_0} \geq \mathfrak{n}$ and $s_{v_0} \geq 1$. Call $\theta_1, \ldots, \theta_{s_{v_0}}$ the trees whose respective root lines $\ell_1, \ldots, \ell_{s_{v_0}}$ enter v_0: one has $|E(\theta)| = |E(\theta_1)| + \ldots + |E(\theta_{s_{v_0}})| + r_{v_0}$. If $\ell_0 \notin L_0(\theta)$ then $|L_0(\theta)| = |L_0(\theta_1)| + \ldots + |L_0(\theta_{s_{v_0}})|$ and the bound follows from the inductive hypothesis.

If $\ell_0 \in L_0(\theta)$ then one has $|L_0(\theta)| = 1 + |L_0(\theta_1)| + \ldots + |L_0(\theta_{s_{v_0}})|$, so that, again by the inductive hypothesis,

$$|L_0(\theta) \leq \frac{|E(\theta)| - 2}{\mathfrak{n}} + \left[1 - \frac{r_{v_0} + 2\,(s_{v_0} - 1)}{\mathfrak{n}}\right],$$

so that, if either $r_{v_0} + s_{v_0} \geq \mathfrak{n} + 1$ or $r_{v_0} + s_{v_0} = \mathfrak{n}$ and $s_{v_0} \geq 2$, the bound follows.

If $r_{v_0} + s_{v_0} = \mathfrak{n}$ and $s_{v_0} = 1$, then $v_0 \in V_1(\theta)$ and, since $C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_0}| < \mathrm{e}^{-\xi\mu_{v_0}/8}/2$ (because $\ell \in L_0(\theta)$), then $C_0|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}| \geq \mathrm{e}^{-\xi\mu_{v_0}/8}/2$ by Lemma B.3. Therefore $\ell_1 \notin L_0(\theta)$. If v_1 is the line which ℓ_1 exits, call $\theta'_1, \ldots, \theta'_{s_{v_1}}$ the trees whose root lines enter v_1: one has $|L_0(\theta)| = 1 + |L(\theta'_1)| + \ldots + |L_0(\theta'_{s_{v_1}})|$ and hence, by the inductive hypothesis,

$$|L_0(\theta)| \leq 1 + \frac{|E(\theta)| - r_{v_0} - r_{v_1} - 2s_{v_1}}{\mathfrak{n}} \leq \frac{|E(\theta)| - 2}{\mathfrak{n}} + \left[1 - \frac{r_{v_0} + r_{v_1} + 2\,(s_{v_1} - 1)}{\mathfrak{n}}\right],$$

where $r_{v_0} + r_{v_1} + 2s_{v_1} - 2 \geq \mathfrak{n}$, so that the bound follows in this case too. ■

Lemma B.5. *For any $k \geq 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d_*$ and any tree $\theta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$ one has*

$$|\mathscr{V}(\theta,\varepsilon)| \leq B^k |\zeta|^{|E_0(\theta)|} |b|^{-|L_0(\theta)|} |\varepsilon|^{1+\frac{\mathfrak{n}-1}{\mathfrak{n}^2}k} \prod_{v \in E_1(\theta)} \mathrm{e}^{-5\xi|\boldsymbol{\nu}_v|/8},$$

where ξ is as in (1.2), and B is a positive constant depending on Φ, Γ and ρ.

Proof. One bounds (3.10) as

$$|\mathscr{V}(\theta,\varepsilon)| \leq |\varepsilon|^{k(\theta)} |\zeta|^{|E_0(\theta)|} \Bigg(\prod_{v \in V(\theta)} |a_{p_v}| \Bigg) \Bigg(\prod_{v \in E_1(\theta)} |f_{\boldsymbol{\nu}_v}| \Bigg) \Bigg(\prod_{\ell \in L(\theta)} |\mathcal{G}_\ell| \Bigg).$$

We deal with the propagators as follows. If ℓ exits a node $v \in V_0(\theta)$, then we have

$$|\mathcal{G}_\ell| \prod_{w \in E_1(\theta,v)} |f_{\boldsymbol{\nu}_w}|\, |\mathcal{G}_{\ell_w}| \leq \frac{1}{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|} \prod_{w \in E_1(\theta,v)} \frac{|f_{\boldsymbol{\nu}_w}|}{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_w|} \leq C_0 (\Phi C_0)^{|E_1(\theta,v)|} \prod_{w \in E_1(\theta,v)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_w|/4},$$

where ℓ_w is the line exiting the end node w and we have defined $E_1(\theta,v) := \{w \in E(\theta,v) : \boldsymbol{\nu}_w \neq \mathbf{0}\}$. For the lines in $L_1(\theta)$ which do not exit nodes $v \in V_0(\theta)$ we distinguish three cases: given a node $v \in V_1(\theta)$ and denoting by v' the node ℓ'_v exits, (1) if one has $\ell_v \in L_1(\theta,v)$ and either $\ell'_v \notin L_1(\theta,v)$ or $\ell'_v \in L_1(\theta,v')$, we proceed as for the nodes $v \in V_0(\theta)$, so as to obtain

$$|\mathcal{G}_{\ell_v}| \prod_{w \in E_1(\theta,v)} |f_{\boldsymbol{\nu}_w}|\, |\mathcal{G}_{\ell_w}| \leq C_0 (\Phi C_0)^{|E_1(\theta,v)|} \prod_{w \in E_1(\theta,v)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_w|/4};$$

(2) if $L_1(\theta,v) = \{\ell'_v\}$ and $\ell_{v'} \notin L_1(\theta,v')$, we bound

$$\big|\mathcal{G}_{\ell'_v}\big| \prod_{w \in E_1(\theta,v)} |f_{\boldsymbol{\nu}_w}|\, |\mathcal{G}_{\ell_w}| \leq C_0 (\Phi C_0)^{|E_1(\theta,v)|} \prod_{w \in E_1(\theta,v)} \mathrm{e}^{-3\xi|\boldsymbol{\nu}_w|/4};$$

(3) if both lines ℓ_v, ℓ'_v belong to $L_1(\theta,v)$ and $\ell'_v \notin L_1(\theta,v')$, we bound

$$\big|\mathcal{G}_{\ell_v}\mathcal{G}_{\ell'_v}\big| \prod_{w \in E_1(\theta,v)} \frac{|f_{\boldsymbol{\nu}_w}|}{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_w|} \leq C_0^2 (\Phi C_0)^{|E_1(\theta,v)|} \prod_{w \in E_1(\theta,v)} \mathrm{e}^{-5\xi|\boldsymbol{\nu}_w|/8}.$$

For all the other propagators we bound (1) $|\mathcal{G}_\ell| \leq 1$ if ℓ exits an end node v with $\boldsymbol{\nu}_v = \mathbf{0}$, (2) $|\mathcal{G}_\ell| \leq |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|^{-1}$ if ℓ exits an end node v with $\boldsymbol{\nu}_v \neq \mathbf{0}$ and has not been already used in the bounds above for the lines $\ell \in L_1(\theta)$, and (3) $|\mathcal{G}_\ell| \leq |b\varepsilon^{\mathfrak{n}}|^{-1}$ if $\ell \in L_0(\theta)$. Then we obtain

$$|\mathscr{V}(\theta,\varepsilon)| \leq |\varepsilon|^{k(\theta)-\mathfrak{n}|L_0(\theta)|} \Gamma^{|V(\theta)|} \rho^{|N(\theta)|} |\zeta|^{|E_0(\theta)|} C_0^{|L_1(\theta)|} (C_0\Phi)^{|E_1(\theta)|} |b|^{-|L_0(\theta)|} \mathrm{e}^{-5\xi|\boldsymbol{\nu}|/8},$$

where we can bound, by using Corollary 3.6 and Lemma B.5,

$$k(\theta) - \mathfrak{n}|L_0(\theta)| = |E(\theta)| + |V(\theta)| - \mathfrak{n}|L_0(\theta)| \geq |E(\theta)| - (\mathfrak{n}-1)|L_0(\theta)| \geq 1 + \frac{\mathfrak{n}-1}{\mathfrak{n}^2} k(\theta),$$

so that the assertion follows. ∎

Fix $\overline{\zeta}$ and $\overline{b}$, and define ζ_* and b_* as in Section 3.

Lemma B.6. *For any $k \geq 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d$ one has*

$$\left| u_{\boldsymbol{\nu}}^{(k)}(\varepsilon) \right| \leq C^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2} |\varepsilon|^{1+\frac{\mathfrak{n}-1}{\mathfrak{n}^2}k},$$

where ξ is as in (1.2), and C is a positive constant depending on Φ, Γ, ρ, ξ, ζ_* and b_*.

Proof. Reason as in the proof of Lemma 2.7. ∎

From this point onward the proof proceeds as in the case of a trigonometric polynomial, so we skip the details.

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