Perhelia Reduction and Global Kolmogorov Tori in the Planetary Problem*

Gabriella Pinzari

Dipartimento di Matematica ed Applicazioni "R. Caccioppoli"
Università di Napoli "Federico II"
Monte Sant'Angelo – Via Cinthia I-80126 Napoli (Italy)
gabriella.pinzari@unina.it

Abstract

We prove the existence of an almost full measure set of $(3n-2)$–dimensional quasi periodic motions in the planetary problem with $(1+n)$ masses, with eccentricities arbitrarily close to the Levi-Civita limiting value and relatively high inclinations. This solves a question posed by V. I. Arnold in the 60s and extends previous results, where smallness of eccentricities and inclinations was assumed. The proof exploits nice parity properties of a new set of coordinates for the planetary problem, which reduces completely the number of degrees of freedom for the system (in particular, its degeneracy due to rotations) and, moreover, is well fitted to its reflection invariance. It allows the explicit construction of an associated close to be integrable system, replacing Birkhoff normal form, common tool of previous literature.

Keywords: Canonical coordinates; Jacobi's reduction; Deprit's reduction; Perihelia reduction; Symmetries; Arnold's theorem on the stability of planetary motions.

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Contents

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1 Background and results

In recent years, substantial progress on a statement by Vladimir Igorevich Arnold concerning the stability of the planetary system has been achieved [22, 2, 23, 33, 20, 14, 27, 9].

It sounds as follows.

"For the majority of initial conditions under which the instantaneous orbits of the planets are close to circles lying in a single plane, perturbation of the planets on one another produces, in the course of an infinite interval of time, little change on these orbits provided the masses of the planets are sufficiently small. [...] *In particular* [...] *in the n-body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities of the bodies belong to this set, the distances of the bodies from each other will remain perpetually bounded."* [2, Chapter III, p. 125].

Solving the differential equations of the motions of the planetary problem, i.e. , n planets interacting among themselves and with a star via gravity is, for $n \geq 2$, a problem with ancient roots. This story goes back to Sir Isaac Newton – who brilliantly solved the case of two bodies and then, switching to the analogue one for three bodies, declared this was a "head ache problem"; passed through investigations by eminent mathematicians like Delaunay, Lagrange, the prize publicly announced by king Oscar II of Sweden and Norway and awarded to Henri Poincaré, but its "solution" is nowadays open. Chaotic and stable regions may coexist [2, 17, 11].

The question approached to a new mathematical description, and a strong modern endorsement, after A. N. Kolmogorov announced, at the International Congress of Mathematicians of 1954, Amsterdam, what is now almost unanimously considered the most important result of the last century for dynamical systems: The theorem of conservation of the invariant torus. This breakthrough result, next enriched of substantial contributions by J. Moser and V. I. Arnold himself [22, 26, 1], states that for a generic Hamiltonian system close to an integrable one, the major part of unperturbed motions survives, after a small perturbation is switched on. In 1962, V. I. Arnold, extending Kolmogorov's ideas, and looking for an application to the planetary problem, at the

International Congress of Mathematicians of Stockholm, announced the theorem of stability of planetary motions quoted above. In 1965 Kolmogorov and Arnold were awarded of the Lenin Prize for their studies on the stability of the planetary problem – but the story was not finished there.

In order to introduce the results of this paper, we highlight basic facts of this story and its continuation, referring the reader to [16, 5, 28, 10, 29] for more notices.

The planetary problem is close to the integrable problem of n uncoupled two–body problems, where each planet interacts separately with the sun. The mutual interactions among planets are regarded as a perturbing function, the smallness of which is ruled by the planets' masses. However, as a perturbed system, the planetary problem has a limiting degeneracy. Its associated integrable system (the two–body problem) is "super–integrable": it has more integrals than degrees of freedom. Systems of this kind share the property that the union of trajectories of the unperturbed problem occupies, in the phase space, a subset of lower dimension, hence, of zero measure. Continuing such trajectories to a *positive measure set* of quasi–periodic trajectories might, in general, be not possible, in absence of further informations on the perturbing function.

Arnold found, for the planetary problem, a brilliant solution to the problem of the limiting degeneracy. This lead him to add, to assumptions and assertions that are proper of perturbation theories (*e.g.*: "the masses of the planets are sufficiently small", "set of initial conditions having a positive Lebesgue measure", "the distances ... will remain perpetually bounded"), a further requirement of smallness of eccentricities and inclinations of the unperturbed Keplerian ellipses ("the instantaneous orbits of the planets are close to circles lying in a single plane"). Let's summarize Arnold's idea.

At a technical level, the limiting degeneracy is exhibited with the disappearance of degrees of freedom in the unperturbed part. Choosing, as Arnold did, Poincaré coordinates [30] (see, also [2, Ch. III, §2], or, *e.g.*, [8, 15]), the system takes the usual close to be integrable form

$$\mathrm{H}_{\mathcal{P}oi} = \mathrm{h}_{\mathrm{Kep}} + \mu\, f_{\mathcal{P}oi} \ ,$$

where μ is a small parameter related to the planetary masses, but the unperturbed "Keplerian" part $\mathrm{h}_{\mathrm{Kep}}(\Lambda)$ depends on only n action variables $\Lambda = (\Lambda_1, \cdots, \Lambda_n)$ (related to the semi–major axes of the instantaneous Keplerian ellipses), out of an overall of $3n$ degrees of freedom. The perturbing function, $f_{\mathcal{P}oi}$, on the other hand, depends on all the coordinates: the actions Λ, their conjugated angles $\ell = (\ell_1, \cdots, \ell_n)$ (proportional to the areas of the elliptic sectors spanned by the planets), and, moreover, on some other coordinates $(\mathrm{p}, \mathrm{q}) = (\mathrm{p}_1, \cdots, \mathrm{p}_{2n}, \mathrm{q}_1, \cdots, \mathrm{q}_{2n})$, $4n$–dimensional, related to those ("secular") quantities (eccentricities, inclinations, nodes and perihelia of the ellipses) that in the unperturbed problem stay fixed, and for this reason do not appear in $\mathrm{h}_{\mathrm{Kep}}$.

It is of great help that the averaged perturbing function (with respect to the angles ℓ) $\overline{f_{\mathcal{P}oi}}(\Lambda, \mathrm{p}, \mathrm{q})$ enjoys several parities in the coordinates (p, q), geometrically related to its invariance by rotations and reflections with respect to the coordinate planes. The "secular origin" $(\mathrm{p}, \mathrm{q}) = 0$, corresponding to all the planets moving on co–centric circles, in the same plane, turns out to be an elliptic equilibrium point for the averaged perturbing function, for any value of Λ.

Arnold brilliantly argued to exploit this circumstance to his purpose. By Birkhoff theory, one might think to switch to another set of canonical coordinates $(\Lambda, \widetilde{\ell}, \widetilde{\mathrm{p}}, \widetilde{\mathrm{q}})$, analogous to Poincaré's coordinates, possibly defined only for $(\widetilde{\mathrm{p}}, \widetilde{\mathrm{q}})$ in a small neighborhood of radius ε around the origin, such that the Hamiltonian of the system, or, more precisely, its $\widetilde{\ell}$–averaged perturbing function $\overline{f_{\mathcal{B}ir}}$, takes a "normalized form" : it is a polynomial of some degree greater or equal than two in the combinations $\tau_i = \frac{\widetilde{\mathrm{p}}_i^2 + \widetilde{\mathrm{q}}_i^2}{2}$, $i = 1, \cdots, 2n$, plus a remainder with a higher order zero in the origin. With these ideas in mind, he proved the following impressive result, and next applied it to the planar three–body problem. It states that stable trajectories occupy a positive measure set of the phase space, and are more and more dense closely to the elliptic equilibrium. Hence, the smaller eccentricities and inclinations are, the larger the number of stable motions is.

'The Fundamental Theorem" (V. I. Arnold, [2]) *If the Hessian matrix of* h *and the matrix of the coefficients of the second–order term in* τ_i *in* $\overline{f_{Bir}}$ *("torsion", or "second-order Birkhoff invariants") do not vanish identically, and if* μ *is suitably small with respect to* ε*, the system affords a positive measure set* $\mathcal{K}_{\mu,\varepsilon}$ *of quasi–periodic motions in phase space such that its density goes to one as* $\varepsilon \to 0$*.*

Arnold perfectly knew that, in order to apply the Fundamental Theorem to the problem in space, one should previously treat an unpleasant fact. One of the first orders Birkhoff invariants vanishes identically. He says that the reason of this is to be sought into the existence of two integrals, the two horizontal components of the total angular momentum of the systems (which, as well as the vertical component are integrals of the motion). If, apparently, a vanishing eigenvalue strongly violates the possibility of the construction of the normalized system (a deeper analysis of the symmetries of the perturbing function [25, 8] however shows that the identically vanishing eigenvalue is not a real difficulty), a major problem definitely prevents the application of the Fundamental Theorem: an infinite number of coefficients of any order of the (formal) Birkhoff series vanishes identically, among which one entire row and a column in the torsion matrix (which so is identically singular). And the reason is again the invariance by rotations. The proof of this is in [8].

Even though (apparently) Arnold was not aware of how this degeneracy was generalized, he suggested a quick solution for the spatial three–body problem, of which he provided very few and somewhat controversial details: to reduce the integrals (hence, the number of degrees of freedom) of the system by switching to a system of canonical coordinates going back to the XIX century, worked out by Jacobi and Radau [21, 32], which in literature go under the name of *Jacobi reduction of the nodes*. The idea was later completely developed by P. Robutel [33], who, in a deeply quantitative study, checked the non–degeneracy assumptions required by the Fundamental Theorem.

Finding a system of canonical coordinates that do the job of Jacobi reduction of the nodes when the number of bodies is more than three, has been a central difficulty for a long time [20, 25]. At this respect, here is a sentence by Arnold, who, after suggesting the utility of Jacobi reduction of the nodes, sadly commented: "In the case of more than three bodies there is no such elegant method of reducing the number of degrees of freedom." [2, Ch. III, §5.5, p. 141].

Exactly twenty years later, F. Boigey and A. Deprit refuted this sentence [3, 12]. They indeed were able to extend Jacobi–Radau reduction to the four, general problem, respectively. It should be remarked, anyway, that, while the works by Jacobi, Radau and Boigey provide canonical coordinates on suitable sub–manifolds of the phase space, the one by Deprit is more general and clarifying, since provides a set of canonical coordinates for the whole phase space, and allows to recover his predecessors by restriction.

The utility of Boigey–Deprit's coordinates was not suddenly clear. Nor Boigey nor Deprit ever provided any motivation of their study, or foresaw applications. The only application that is known to the author up to 2008, concerning indeed Deprit's coordinates, stands in a paper by Ferrer and Osácar, in the 90s, to the three body problem [18]. But this case is not really exhaustive, since for three bodies Deprit's and Jacobi–Radau's coordinates coincide. A reason why Boigey–Deprit's coordinates have been forgotten so long might be that, for more than three bodies, they actually have a less natural aspect, compared to the classical case of Jacobi. A sort of "hierarchical" structure in the geometry of Deprit's coordinates discouraged the author himself, who, at the end of his paper, declared: "Whether the new phase variables are practical in the general theory of perturbation is an open question. At least, for planetary theories, the answer is likely to be in the negative. But finding a natural system of coordinates for eliminating the nodes in a planetary cluster was not the intention of this note." [12, p. 194].

In the meantime, in 2004, the first general proof of Arnold's stability statement appeared. It was by Jacques Féjoz, who completed investigations by the late Michael Herman [14] – but the

different procedure that Herman had in mind did not rely with the necessity of handling, explicitly, good coordinates.

In 2008, Boigey–Deprit's coordinates were rediscovered by the author [27], in a slightly different, "planetary" form. The rediscovery was motivated by the purpose of finding a direct, constructive proof of Arnold's stability statement. The utility of Boigey–Deprit's coordinates became suddenly clear: switching (in order to overcome certain singularities of the chart) to a regularized version, called "RPS" coordinates, (acronym standing for "Regular, Planetary and Symplectic"), allowed to derive the Birkhoff normal form of the planetary problem, to prove its non–degeneracy, and hence completing the application of the Fundamental Theorem to the general problem. These results have been published in [6, 7, 9].

Qualitatively, RPS coordinates are very different from JRBD (Jacobi–Radau–Boigey–Deprit). They rather are more similar to Poincaré coordinates. The mentioned parities and the elliptic equilibrium of the averaged system are still present in the RPS–averaged system. But, as an advantage with respect to Poincaré coordinates, the RPS perform[1] a "partial reduction" of the rotation symmetry – at contrast with JRBD coordinates, which reduce "fully". This way, all the degeneracies of the Birkhoff series mentioned above are removed at once, and the non–degeneracy assumptions of the Fundamental Theorem may be checked. Once again, we underline how deep Arnold's comprehension of the problem had been, since he, more than forty years earlier, had conjectured a system of coordinates like this might exist (suggesting to compute it by series).

The possibility of switching from Delaunay–Poincaré to the more fruitful JRBD, or even RPS coordinates, is an effect of the limiting degeneracy. This gives in fact the opportunity of remixing coordinates related to secular quantities, and, simultaneously, keeping the Keplerian term h_{Kep} unvaried.

Following this idea, in this paper, we show that other systems of coordinates may be determined for the planetary problem which, as well as JRBD, RPS coordinates, are well adapted to overcome the degeneracy due to rotations, and, moreover, enjoy some different properties.

We present a full reduction, which we call $\mathcal{P}$–map, or perihelia reduction. It refines JRBD coordinates in two respects.

Firstly, the $\mathcal{P}$–map is well defined in the case of the planar problem, while JRBD coordinates are not. Everyone knows, in fact, that the starting point for the Radau–Jacobi reduction is the so–called "line of the nodes", the straight line determined by the intersection between the planes of the two orbits. When the orbits of the two planets belong to the same plane, this is not defined. A similar circumstance arises for Boigey–Deprit's coordinates, since their construction relies on certain straight lines in the space, which again loose their meaning in case of co–planarity.

The proof of Arnold's theorem given in [27, 9], is not affected by such singularity, since, as said, it relies on RPS coordinates, which, at expenses of one more degree of freedom, are well defined for co–planar motions – in that case they reduce to the classical Poincaré coordinates.

It has its consequences when one wants to compare results for the fully reduced systems, in the space or in the plane. The singularity of the chart does not allow to state that motions in the spatial problem with minimum number of independent frequencies starting with very small inclinations stay close to the corresponding planar motions. Notwithstanding further studies appeared in [28], where this problem is partially (i.e., via the construction of regular coordinates for co–planar motions defined locally) overcome, it would be nice, in principle, to handle a global system of action–angle coordinates which reduces completely rotations, and is shared simultaneously by the planar and the spatial problem.

[1] In the framework of the study of canonical coordinates for the planetary system, by "partial reduction", we mean a system of canonical coordinates where a couple of conjugated coordinates consists of integrals (*e.g.*, functions of the three components of the total angular momentum). By "full reduction", we mean a partial reduction where also another integral appears among the coordinates. The terms "partial reduction", "full reduction" have been coined in [25].

Secondly, the $\mathcal{P}$–map is well adapted to reflection symmetries of the problem, while JRBD coordinates are not, as discussed in [25, 29].

Reflection symmetries are parities of the Hamiltonian expressed in Cartesian coordinates. As known, this does not change under arbitrary changes of the signs of positions or momenta coordinates. They are not related to integrals. Therefore, it might be a nice fact, and in general useful for applications, to have a system of coordinates that, after integrals are reduced, parities associated to reflections are maintained. Quite often parities are associated to equilibria, and equilibria to stable motions; an example is provided a few lines below.

We shall apply the the $\mathcal{P}$–map by proving a variant of Arnold's stability theorem. We shall face up a question raised again by Arnold in his fantastic paper on the possibility of removing the constraint on eccentricities and inclinations. He indeed proved that, at least for the planar three–body problem, there is no need of assuming their smallness. Rather, it is sufficient that the trajectories of the planets are away enough so as to avoid collisions. He obtained this stronger result by exploiting the convergence of the Birkhoff series associated to the averaged perturbation, a very particular and happy circumstance, due to the few degrees of freedom of the problem.

From the mathematical point of view, the question is whether different strategies for finding stable motions do exist, than the one of exploring the neighborhood of the elliptic equilibrium.

Concerning instead the physical relevance, asteroids, or some trans–Neptunian objects have motions with relatively large eccentricities and inclinations, and an almost continuous spectrum of frequencies.

Besides the mentioned stronger result by Arnold, some other statements in the same direction have been obtained for the case of the spatial three–body problem and of the planar problem, with any number of bodies [28]. Here, the measure of the invariant set has been estimated to be larger and larger as the planetary masses and the semi–axes ratios are small, but no matter the smallness eccentricities and inclinations – the proof relying on an argument of convergence of a significative approximation of the Birkhoff series. Other results in this direction have been publicly announced by J. Féjoz in more than one occasion since (as far as the author knows) September[2] 2013 [13].

Even though the arguments of [2, 28] do not apply to the general spatial problem, since no significative approximation of the Birkhoff series associated to the averaged perturbation is integrable, using the $\mathcal{P}$–map, we shall prove the following

Theorem A *Fix numbers* $0 < \underline{e}_i < \overline{e}_i < 0.6627...$, $i = 1, \cdots, n$. *There exists a number* N *depending only on* n *and* α_0 *depending on* $\underline{e}_i$, $\overline{e}_i$ n *such that, if* $\alpha < \alpha_0$, $\mu \leq \alpha^{\rm N}$, *in a domain of planetary motions where the semi–major axes* $a_1 < a_2 < \cdots < a_n$ *are spaced as follows*

$$a_i^- \leq a_i \leq a_i^+ \qquad \text{with} \qquad a_i^\pm := \frac{a_1^\pm}{\alpha^{\frac{1}{3}(2^{n+1}-2^{n-i+2}+1-i)}} \qquad (*)$$

there exists a positive measure set $\mathcal{K}_{\mu,\alpha}$, *the density of which in phase space can be bounded below as*

$$\text{dens}(\mathcal{K}_{\mu,\alpha}) \geq 1 - (\log \alpha^{-1})^{\rm p} \sqrt{\alpha} \ ,$$

consisting of quasi–periodic motions with $3n-2$ *frequencies where the planets' eccentricities* e_i *verify*

$$\underline{e}_i \leq e_i \leq \overline{e}_i \ .$$

Before we switch to details, a few remarks.

[2] CelMec VI, San Martino al Cimino, Viterbo, Italy.

Firstly, the claimed upper bound 0.6627... on eccentricities was established by T. Levi Civita [24]. It is due to the fact that, as well as in [2, 28], the proof uses the machinery of real–analytic functions.

Secondly, as it may be seen to the choice of $a_j^{\pm}$, the distances among the planets' semi–axes are not of the same order, but grow super–exponentially going towards the sun. This resembles a sort of belt arrangement, observed in nature for asteroids. It is possible to prove an analogous result, with increasing distances in the opposite direction.

Thirdly, the result in Theorem A (especially, the claimed growth of $a_i^{\pm}$) may be regarded as an alternative way of solving the problem of the limiting degeneracy – without Birkhoff normal form.

Acknowledgments I am indebted to Jacques Féjoz, who let me know the work by Harrington [19], without which I had never thought to this application of the $\mathcal{P}$–coordinates.

2 Kepler maps and the Perihelia reduction

We introduce the *Perihelia reduction*, or $\mathcal{P}$–map, in the slightly general context of *Kepler maps*.

Fix a reference frame $\mathrm{G}_0 = (k^{(1)}, k^{(2)}, k^{(3)})$ in the Euclidean space E^3. We identify the three chosen directions $(k^{(1)}, k^{(2)}, k^{(3)})$ with the triples of coordinates with respect of the system of coordinates established by themselves:

$$k^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad k^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad k^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

Definition 2.1 An *ellipse* (with a focus in the origin and non-vanishing eccentricity) is a quadruplet $\mathfrak{E} = (a, e, N, P)$, where $a \in \mathbb{R}_+$ is the *semi–major axis*, $e \in (0, 1)$ is the *eccentricity*, $N \in \mathbb{R}^3 \cap S^2$ is the *normal direction* and $P \in N^{\perp} \cap S^2$ is the *perihelion direction*.

Definition 2.2 (Kepler maps) Given $2n$ positive "mass parameters" $\mathfrak{m}_1, \cdots, \mathfrak{m}_n, \mathfrak{M}_1, \cdots, \mathfrak{M}_n$, a set $\mathfrak{X} \subset \mathbb{R}^{5n}$, we say that

$$\mathcal{K}: \quad \mathrm{K} = (\mathrm{X}_{\mathcal{K}}, \ell) \in \mathcal{D} := \mathfrak{X} \times \mathbb{T}^n \to (y_{\mathcal{K}}, x_{\mathcal{K}}) \in \mathcal{C} := \mathcal{K}(\mathcal{D}) \subset (\mathbb{R}^3)^n \times (\mathbb{R}^3)^n$$

where

$$\ell = (\ell_1, \cdots, \ell_n) , \qquad (y_{\mathcal{K}}, x_{\mathcal{K}}) = (y_{\mathcal{K}}^{(1)}, \cdots, y_{\mathcal{K}}^{(n)}, x_{\mathcal{K}}^{(1)}, \cdots, x_{\mathcal{K}}^{(n)})$$

$$y_{\mathcal{K}}^{(j)} = y_{\mathcal{K}}^{(j)}(\mathrm{X}_{\mathcal{K}}, \ell_j) \qquad x_{\mathcal{K}}^{(j)} = x_{\mathcal{K}}^{(j)}(\mathrm{X}_{\mathcal{K}}, \ell_j) \qquad j = 1, \cdots n ,$$

is a *Kepler map* if there exists an injection

$$\tau_{\mathcal{K}}: \quad \mathrm{X}_{\mathcal{K}} \in \mathfrak{X} \quad \to \quad \mathfrak{E}_{\mathcal{K}} = \left(\mathfrak{E}_{1,\mathcal{K}}, \cdots, \mathfrak{E}_{n,\mathcal{K}}\right)$$

which assigns to any $\mathrm{X}_{\mathcal{K}} \in \mathfrak{X}$ an n–plet $\left(\mathfrak{E}_{1,\mathcal{K}}, \cdots, \mathfrak{E}_{n,\mathcal{K}}\right)$ of (co–focal) ellipses

$$\mathfrak{E}_{j,\mathcal{K}} = \left(a_{j,\mathcal{K}}, e_{j,\mathcal{K}}, N_{\mathcal{K}}^{(j)}, P_{\mathcal{K}}^{(j)}\right) , \quad j = 1, \cdots, n$$

and $\mathcal{K}$ acts in the following way. Letting $Q_{\mathcal{K}}^{(j)} := N_{\mathcal{K}}^{(j)} \times P_{\mathcal{K}}^{(j)}$, then

$$x_{\mathcal{K}}^{(j)} = \mathrm{a}_{j,\mathcal{K}} P_{\mathcal{K}}^{(j)} + \mathrm{b}_{j,\mathcal{K}} Q_{\mathcal{K}}^{(j)} \qquad y_{\mathcal{K}}^{(j)} = \mathrm{a}_{j,\mathcal{K}}^{\circ} P_{\mathcal{K}}^{(j)} + \mathrm{b}_{j,\mathcal{K}}^{\circ} Q_{\mathcal{K}}^{(j)} \tag{1}$$

where, if $\zeta_{j,\mathcal{K}}$, the *eccentric anomaly*, is the solution of Kepler's Equation

$$\zeta_{j,\mathcal{K}} - e_{j,\mathcal{K}} \sin \zeta_{j,\mathcal{K}} = \ell_j \tag{2}$$

then

$$\mathrm{a}_{j,\mathcal{K}} := a_{j,\mathcal{K}}\big(\cos\zeta_{j,\mathcal{K}} - e_{j,\mathcal{K}}\big) \qquad \mathrm{b}_{j,\mathcal{K}} := a_{j,\mathcal{K}}\sqrt{1-e_{j,\mathcal{K}}^2}\,\sin\zeta_{j,\mathcal{K}}$$
$$\mathrm{a}^\circ_{j,\mathcal{K}} := -\mathfrak{m}_j\sqrt{\frac{\mathfrak{M}_j}{a_{j,\mathcal{K}}}}\frac{\sin\zeta_{j,\mathcal{K}}}{1-e_{j,\mathcal{K}}\cos\zeta_{j,\mathcal{K}}} \qquad \mathrm{b}^\circ_{j,\mathcal{K}} := \mathfrak{m}_j\sqrt{\frac{\mathfrak{M}_j(1-e_{j,\mathcal{K}}^2)}{a_{j,\mathcal{K}}}}\frac{\cos\zeta_{j,\mathcal{K}}}{1-e_{j,\mathcal{K}}\cos\zeta_{j,\mathcal{K}}}\ . \qquad \blacksquare \tag{3}$$

Remark 2.1 The definition implies that

(i) $\mathcal{K}$ is a bijection of the sets $\mathcal{D}$ and $\mathcal{C}$;

(ii) the *angular momenta* and the *energies*[3]

$$\mathrm{C}^{(j)}_{\mathcal{K}} := x^{(j)}_{\mathcal{K}} \times y^{(j)}_{\mathcal{K}} \qquad \mathrm{H}^{(j)}_{\mathcal{K}} := \frac{\|y^{(j)}_{\mathcal{K}}\|^2}{2\mathfrak{m}_j} - \frac{\mathfrak{m}_j\mathfrak{M}_j}{\|x^{(j)}_{\mathcal{K}}\|}\ . \tag{4}$$

do not depend on ℓ_j and are given by

$$\mathrm{C}^{(j)}_{\mathcal{K}} = \mathfrak{m}_j\sqrt{\mathfrak{M}_j a_{j,\mathcal{K}}(1-e_{j,\mathcal{K}}^2)}N^{(j)}_{\mathcal{K}}\ , \quad \mathrm{H}^{(j)}_{\mathcal{K}} = -\frac{\mathfrak{m}_j\mathfrak{M}_j}{2a_{j,\mathcal{K}}}\ ; \tag{5}$$

(iii) the couples $(y^{(j)}_{\mathcal{K}}, x^{(j)}_{\mathcal{K}})$ verify the system of ODEs

$$\begin{cases} \mathfrak{m}_j\sqrt{\dfrac{\mathfrak{M}_j}{a^3_{j,\mathcal{K}}}}\partial_{\ell_j}x^{(j)}_{\mathcal{K}} = y^{(j)}_{\mathcal{K}} \\ \sqrt{\dfrac{\mathfrak{M}_j}{a^3_{j,\mathcal{K}}}}\partial_{\ell_j}y^{(j)}_{\mathcal{K}} = -\mathfrak{m}_j\mathfrak{M}_j\dfrac{x^{(j)}_{\mathcal{K}}}{\|x^{(j)}_{\mathcal{K}}\|^3}\ . \end{cases} \tag{6}$$

(iv) Even though *canonical* maps (with respect to the standard two–form) have a pre-eminent rôle in Hamiltonian Mechanics, Kepler maps are used also in different contexts in Astronomy, where to be canonical is not required. For example, one can consider the Kepler map associated to the "elliptic elements" injection

$$\tau_{\mathcal{E}e\ell\ell}: \qquad (a,e,P,i,\Omega) \to \mathfrak{E}_{\mathcal{E}e\ell\ell}$$

where $a = (a_1,\cdots,a_n)$ are the *semi–major axes*, $e = (e_1,\cdots,e_n)$ are the *eccentricities*, $P = (P^{(1)},\cdots,P^{(n)})$ are the *perihelia*, $i = (i_1,\cdots,i_n)$ are the *inclinations*, $\Omega = (\Omega_1,\cdots,\Omega_n)$ are the *nodes' longidudes*.

The only known examples up to now of *canonical Kepler maps* are the classical *Delaunay map* $\mathcal{D}e\ell$ (its definition is recalled in the next Definition 2.5) and the map $\mathcal{D}ep$ [27, 7] related to Deprit's coordinates [12], which is recalled in Appendix E. Below, we introduce a new canonical Kepler map.

Definition 2.3 (perihelia reduction, or $\mathcal{P}$–map) We denote as $\mathcal{P}$, and call *perihelia reduction*, or $\mathcal{P}$–map, the Kepler map

$$\mathcal{P}: \quad \mathrm{P} = (\mathrm{X}_{\mathcal{P}},\ell) \in \mathcal{D}_{\mathcal{P}} = \mathfrak{X}_{\mathcal{P}} \times \mathbb{T}^n \to (y,x) \in \mathbb{R}^{3n}\times\mathbb{R}^{3n} \tag{7}$$

associated to the bijection

$$\tau_{\mathcal{P}}: \quad \mathrm{X}_{\mathcal{P}} = (\Theta,\chi,\Lambda,\vartheta,\kappa) \in \mathfrak{X}_{\mathcal{P}} \to (\mathfrak{E}_1,\cdots,\mathfrak{E}_n) \in \mathcal{E}_{\mathcal{P}} = \tau_{\mathcal{P}}(\mathfrak{X}_{\mathcal{P}}) \subset E^{3n}$$

defined by means of Definition 2.4 and Proposition 2.1 below.

[3] Here, $\|v\| := \sqrt{v_1^2+v_2^2+v_3^2}$ denotes the usual Euclidean norm of $v = (v_1,v_2,v_3) \in \mathbb{R}^3$.

Definition 2.4 For a given $(\mathfrak{E}_1, \cdots, \mathfrak{E}_n) \subset E^3 \times \cdots \times E^3$, with $\mathfrak{E}_j = (a_j, e_j, N^{(j)}, P^{(j)})$, and masses $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$, $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$, define

$$\mathrm{C}_{\mathcal{E}}^{(j)} := \mathfrak{m}_j \sqrt{\mathfrak{M}_j a_j (1 - e_j^2)} N^{(j)} \qquad \mathrm{S}_{\mathcal{E}}^{(j)} := \sum_{i=j}^{n} \mathrm{C}_{\mathcal{E}}^{(i)} \qquad 1 \leq j \leq n \tag{8}$$

be the *angular momenta* associated to $\mathfrak{E}_j$ and the j^{th} *partial angular momenta*, so that

$$\mathrm{S}_{\mathcal{E}}^{(1)} = \sum_{i=1}^{n} \mathrm{C}_{\mathcal{E}}^{(i)} \qquad \mathrm{S}_{\mathcal{E}}^{(n)} = \mathrm{C}_{\mathcal{E}}^{(n)} \tag{9}$$

are the *total angular momentum* and the angular momentum of the last ellipse, respectively. Define the $\mathcal{P}$*–nodes*

$$\nu_j := \begin{cases} k^{(3)} \times \mathrm{S}_{\mathcal{E}}^{(1)} & j = 1 \\ P^{(j-1)} \times \mathrm{S}_{\mathcal{E}}^{(j)} & j = 2, \cdots, n \end{cases} \qquad \mathrm{n}_j := \mathrm{S}_{\mathcal{E}}^{(j)} \times P^{(j)} \qquad j = 1, \cdots, n\,. \tag{10}$$

Finally, define

$$\mathcal{E}_{\mathcal{P}} := \left\{ ((\mathfrak{E}_1, \cdots, \mathfrak{E}_n) \subset E^3 \times \cdots \times E^3) : \quad 0 < e_j < 1\,, \quad \nu_j \neq 0 \quad \mathrm{n}_j \neq 0 \quad \forall\, j = 1, \cdots, n \right\},$$

and, on this set, the map

$$\tau_{\mathcal{P}}^{-1} : \quad (\mathfrak{E}_1, \cdots, \mathfrak{E}_n) \in \mathcal{E}_{\mathcal{P}} \to \mathrm{X}_{\mathcal{P}} \in \mathfrak{X}_{\mathcal{P}} = \tau_{\mathcal{P}}^{-1}(\mathcal{E}_{\mathcal{P}})$$

where

$$\mathrm{X}_{\mathcal{P}} = (\Theta, \chi, \Lambda, \vartheta, \kappa) \in \mathbb{R}^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{T}^n \times \mathbb{T}^n$$

with

$$\Theta = (\Theta_0, \cdots, \Theta_{n-1})\,, \quad \vartheta = (\vartheta_0, \cdots, \vartheta_{n-1})$$

$$\chi = (\chi_0, \cdots, \chi_{n-1})\,, \quad \kappa = (\kappa_0, \cdots, \kappa_{n-1})$$

$$\Lambda = (\Lambda_1, \cdots, \Lambda_n)$$

are defined via the following formulae

$$\Theta_{j-1} := \begin{cases} Z := \mathrm{S}_{\mathcal{E}}^{(1)} \cdot k^{(3)} \\ \mathrm{S}_{\mathcal{E}}^{(j)} \cdot P^{(j-1)} \end{cases} \qquad \vartheta_{j-1} := \begin{cases} \zeta := \alpha_{k^{(3)}}(k^{(1)}, \nu_1) & j = 1 \\ \alpha_{P^{(j-1)}}(\mathrm{n}_{j-1}, \nu_j) & 2 \leq j \leq n \end{cases}$$

$$\chi_{j-1} := \begin{cases} \mathrm{G} := \|\mathrm{S}_{\mathcal{E}}^{(1)}\| \\ \|\mathrm{S}_{\mathcal{E}}^{(j)}\| \end{cases} \qquad \kappa_{j-1} := \begin{cases} \mathfrak{g} := \alpha_{\mathrm{S}_{\mathcal{E}}^{(1)}}(\nu_1, \mathrm{n}_1) & j = 1 \\ \alpha_{\mathrm{S}_{\mathcal{E}}^{(j)}}(\nu_j, \mathrm{n}_j) & 2 \leq j \leq n \end{cases} \tag{11}$$

$$\Lambda_j := \mathfrak{M}_j \sqrt{\mathfrak{m}_j a_j}\,.$$

Proposition 2.1 *Let $\mathfrak{X}_{\mathcal{P}}$ be the subset of $\mathbb{R}^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{T}^n \times \mathbb{T}^n$ defined by the following inequalities*

$$\begin{aligned} &\sqrt{\chi_{i-1}^2 + \chi_i^2 - 2\Theta_i^2 + 2\sqrt{(\chi_i^2 - \Theta_i^2)(\chi_{i-1}^2 - \Theta_i^2)}\cos\vartheta_i} < \Lambda_i \\ &(\chi_{i-1} - \chi_i, \vartheta_i) \neq (0, \pi) \qquad 0 < \chi_n < \Lambda_n \quad i = 1, \cdots, n-1 \end{aligned} \tag{12}$$

and

$$|\Theta_0| < \chi_0 \qquad |\Theta_i| < \min(\chi_{i-1}, \chi_i) \qquad i = 1, \cdots, n-1 \ . \tag{13}$$

The map $\tau_{\mathcal{P}}^{-1}$ is a bijection of $\mathcal{E}_{\mathcal{P}}$ onto $\mathfrak{X}_{\mathcal{P}}$. The formulae of the inverse map

$$\tau_{\mathcal{P}} : \quad \mathrm{X}_{\mathcal{P}} = (\Theta, \chi, \Lambda, \vartheta, \kappa) \in \mathcal{D}_{\mathcal{P}} \to \mathfrak{E}_{\mathcal{P}} = (\mathfrak{E}_{1,\mathcal{P}}, \cdots, \mathfrak{E}_{n,\mathcal{P}}) \in \mathcal{E}_{\mathcal{P}} \quad \mathfrak{E}_{j,\mathcal{P}} = (a_{j,\mathcal{P}}, e_{j,\mathcal{P}}, N_{\mathcal{P}}^{(j)}, P_{\mathcal{P}}^{(j)})$$

are as follows. Let $\iota_1, \cdots, \iota_n, \mathrm{i}_1, \cdots, \mathrm{i}_n \in (0, \pi)$ be defined via

$$\cos \iota_j = \frac{\Theta_{j-1}}{\chi_{j-1}} \ , \quad \cos \mathrm{i}_j := \frac{\Theta_j}{\chi_{j-1}} \ , \quad 1 \leq j \leq n \tag{14}$$

(with $\Theta_n := 0$, so that $\mathrm{i}_n = \frac{\pi}{2}$) and $\mathcal{T}_1, \cdots, \mathcal{T}_n, \mathcal{S}_1, \cdots, \mathcal{S}_n \in \mathrm{SO}(3)$ via

$$\mathcal{T}_j := \mathcal{R}_3(\vartheta_j)\mathcal{R}_1(\iota_j) \qquad \mathcal{S}_j := \mathcal{R}_3(\kappa_j)\mathcal{R}_1(\mathrm{i}_j) \ , \quad 1 \leq j \leq n \tag{15}$$

and let

$$\mathrm{C}_{\mathcal{P}}^{(j)} := \mathcal{T}_1\mathcal{S}_1 \cdots \mathcal{T}_{j-1}\mathcal{S}_{j-1}\mathcal{T}_j\big(\chi_{j-1}k^{(3)} - \chi_j\mathcal{S}_j\mathcal{T}_{j+1}k^{(3)}\big) \tag{16}$$

with $\chi_n := 0$, so that

$$\|\mathrm{C}_{\mathcal{P}}^{(j)}\| = \begin{cases} \sqrt{\chi_{j-1}^2 + \chi_j^2 - 2\Theta_j^2 + 2\sqrt{(\chi_j^2 - \Theta_j^2)(\chi_{j-1}^2 - \Theta_j^2)}\cos\vartheta_j} & j = 1, \cdots, n-1 \\ \chi_{n-1} & j = n \ . \end{cases} \tag{17}$$

Then $\mathrm{C}_{\mathcal{P}}^{(j)} = \mathrm{C}_{\mathcal{E}}^{(j)} \circ \tau_{\mathcal{P}}$ and

$$\begin{aligned} a_{j,\mathcal{P}} &= \frac{1}{\mathfrak{M}_j}\Big(\frac{\Lambda_j}{\mathfrak{m}_j}\Big)^2 \qquad & e_{j,\mathcal{P}} &= \sqrt{1 - \frac{\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2}{\Lambda_j^2}} \\ N_{\mathcal{P}}^{(j)} &= \frac{\mathrm{C}_{\mathcal{P}}^{(j)}}{\|\mathrm{C}_{\mathcal{P}}^{(j)}\|} \qquad & P_{\mathcal{P}}^{(j)} &= \mathcal{T}_1\mathcal{S}_1 \cdots \mathcal{T}_j\mathcal{S}_j k^{(3)} \ . \end{aligned} \tag{18}$$

Remark 2.2

(i) From $\mathrm{C}_{\mathcal{P}}^{(j)} = \mathrm{C}_{\mathcal{E}}^{(j)} \circ \tau_{\mathcal{P}}$, (4), (5) and (25), there follows that $\mathrm{C}_{\mathcal{P}}^{(j)} = x_{\mathcal{P}}^{(j)} \times y_{\mathcal{P}}^{(j)}$.

(ii) $P_{\mathcal{P}}^{(j)} \perp N_{\mathcal{P}}^{(j)}$. Indeed, using the definitions,

$$\begin{aligned} \mathrm{C}_{\mathcal{P}}^{(j)} \cdot P_{\mathcal{P}}^{(j)} &= \chi_{j-1}k^{(3)} \cdot \big(\mathcal{S}_j k^{(3)}\big) - \mathcal{T}_{j+1}\chi_j k^{(3)} \cdot \big(k^{(3)}\big) \\ &= \chi_{j-1}\cos\iota_j - \chi_j \cos \mathrm{i}_{j+1} = 0 \end{aligned}$$

(iii) $\mathrm{S}_{\mathcal{P}}^{(j)} := \mathrm{S}_{\mathcal{E}}^{(j)} \circ \tau_{\mathcal{P}} = \sum_{i=j}^n \mathrm{C}_{\mathcal{P}}^{(i)} = \chi_{j-1}\mathcal{T}_1\mathcal{S}_1 \cdots \mathcal{T}_{j-1}\mathcal{S}_{j-1}\mathcal{T}_j k^{(3)}$.

We shall prove that

Theorem 2.1 *The $\mathcal{P}$–map preserves the standard 2–form*

$$\sum_{j=1}^n dy_{\mathcal{P}}^{(j)} \wedge dx_{\mathcal{P}}^{(j)} = \sum_{i=1}^n \big(d\Theta_{i-1} \wedge d\vartheta_{i-1} + d\chi_{i-1} \wedge d\kappa_{i-1} + d\Lambda_i \wedge d\ell_i\big) \ .$$

Remark 2.3 Actually, we shall prove a finer result: the change $\phi_{\mathcal{D}e\ell}^{\mathcal{P}} := \mathcal{D}e\ell^{-1} \circ \mathcal{P}$ which relates the $\mathcal{P}$–coordinates to the classical Delaunay coordinates (see the Definition 2.5) is homogeneous–canonical (compare Lemma 2.6).

Proof of Proposition 2.1 The formula for $a_{j,\mathcal{P}}$ in (16) is immediate from the definition of Λ_j. Postponing to below that $\mathrm{C}^{(j)}_{\mathcal{P}} := \mathrm{C}^{(j)}_{\mathcal{E}} \circ \tau_{\mathcal{P}}$ has the expression in (16) (In turn this implies (17), the formula for $N^{(j)}$ and the one for $e_{j,\mathcal{P}}$ in (18)), we check that the image set $\tau^{-1}_{\mathcal{P}}(\mathcal{E}_{\mathcal{P}})$ is included in the domain $\mathfrak{X}_{\mathcal{P}}$ defined by inequalities (12), (13). From the formula for $e_{j,\mathcal{P}}$ in (18), we have that conditions $0 < e_{j,\mathcal{P}} < 1$ for all $j = 1, \cdots, n$ corresponds to relations in (12). Note that the first condition in the second line of (12) is equivalent to $e_{j,\mathcal{P}} \neq 1$, as one sees rewriting

$$\|\mathrm{C}^{(j)}_{\mathcal{P}}\|^2 \;=\; \big(\sqrt{\chi^2_{j-1} - \Theta^2_j} - \sqrt{\chi^2_j - \Theta^2_j}\big)^2 + 2\sqrt{(\chi^2_j - \Theta^2_j)(\chi^2_{j-1} - \Theta^2_j)}(1 + \cos\vartheta_j)\ . \quad (19)$$

Next, recalling the definitions of Θ_0, χ_0 in (11), and noticing the relations

$$\Theta_j = \mathrm{S}^{(j+1)}_{\mathcal{E}} \cdot P^{(j)} = (\mathrm{S}^{(j)}_{\mathcal{E}} - \mathrm{C}^{(j)}_{\mathcal{E}}) \cdot P^{(j)} = \mathrm{S}^{(j)}_{\mathcal{E}} \cdot P^{(j)} \qquad j = 1, \cdots, n-1\ ,$$

we immediately see that conditions $\nu_i \neq 0 \neq \mathrm{n}_i$ imply (13). We have so checked what we wanted. Now it remains to check the formula for $\mathrm{C}^{(j)}_{\mathcal{P}}$ in (16) and the one for $P^{(j)}_{\mathcal{P}}$ in (18), for any $\mathrm{X}_{\mathcal{P}} \in \mathfrak{X}_{\mathcal{P}}$. To this end, we consider the following chain of vectors

$$\begin{array}{ccccccccccccccc}
k^{(3)} & \to & \mathrm{S}^{(1)}_{\mathcal{E}} & \to & P^{(1)} & \to & \cdots & \to & \mathrm{S}^{(j)}_{\mathcal{E}} & \to & P^{(j)} & \to & \mathrm{S}^{(j+1)}_{\mathcal{E}} & \to & \cdots & \to & P^{(n)} \\
 & & \Downarrow & & \Downarrow & & \vdots & & \Downarrow & & \Downarrow & & \Downarrow & & \vdots & & \Downarrow \\
 & & \nu_1 & & \mathrm{n}_1 & & \vdots & & \nu_j & & \mathrm{n}_j & & \nu_{j+1} & & \vdots & & \mathrm{n}_n
\end{array} \quad (20)$$

where $\nu_1, \mathrm{n}_1, \cdots, \nu_n, \mathrm{n}_n$ are the $\mathcal{P}$*–nodes* in (10), given by the skew–product of the two consecutive vectors in the chain.

We associate to this chain of vectors the following chain of frames

$$\mathrm{G}_0 \;\to\; \mathrm{F}_1 \;\to\; \mathrm{G}_1 \;\to\; \cdots \;\to\; \mathrm{F}_j \;\to\; \mathrm{G}_j \;\to\; \mathrm{F}_{j+1} \;\to\; \cdots \;\to\; \mathrm{G}_n \quad (21)$$

where $\mathrm{G}_0 = (k^{(1)}, k^{(2)}, k^{(3)})$ is the initial prefixed frame and the frames, while F_i, G_i are frames defined via

$$\mathrm{F}_j = (\nu_j,\ \cdot\ , \mathrm{S}^{(j)}) \quad \mathrm{G}_j = (\mathrm{n}_j,\ \cdot\ , P^{(j)}) \qquad j = 1, \cdots, n\ . \quad (22)$$

By construction, each frame in the chain has its first axis coinciding with the intersection of the its horizontal plane with the horizontal plane of the previous frame (hence, in particular, $\nu_j \perp \mathrm{S}^{(j)}$ and $\mathrm{n}_j \perp P^{(j)}$). Denote as $\mathcal{T}_j$ the rotation matrix which describes the change of coordinates from G_{j-1} to F_j and as $\mathcal{S}_j$ the the one from F_j to G_j. The matrices $\mathcal{T}_j$, $\mathcal{S}_j$ have just the expressions claimed in (14)–(15). This follows from the definitions of $(\Theta, \chi, \vartheta, \kappa)$ in (11). Then we have the following sequence of transformations

$$\begin{array}{ccccccccccccccc}
 & & \mathcal{T}_1 & & \mathcal{S}_1 & & \cdots & & \mathcal{S}_j & & \mathcal{T}_{j+1} & & & & \cdots & & \mathcal{S}_n \\
\mathrm{G}_0 & \to & \mathrm{F}_1 & \to & \mathrm{G}_1 & \to & \cdots & \to & \mathrm{F}_j & \to & \mathrm{G}_j & \to & \mathrm{F}_{j+1} & \to & \cdots & \to & \mathrm{G}_n
\end{array}$$

connecting G_0 to any other frame in the chain. From this, and the definitions of the frames (22), the formulae for $P^{(j)}_{\mathcal{P}}$ in (18) and

$$\mathrm{S}^{(j)}_{\mathcal{P}} = \chi_{j-1}\mathcal{T}_1\mathcal{S}_1 \cdots \mathcal{T}_{j-1}\mathcal{S}_{j-1}\mathcal{T}_j k^{(3)}$$

follow at once. Hence, also the ones for $\mathrm{C}_{\mathcal{P}}^{(j)}$, which is given by $\mathrm{C}_{\mathcal{P}}^{(j)} = \mathrm{S}_{\mathcal{P}}^{(j)} - \mathrm{S}_{\mathcal{P}}^{(j+1)}$, with $\mathrm{S}_{\mathcal{P}}^{(n+1)} \equiv 0$. ∎

For the proof of Theorem 2.1, we shall use three auxiliary maps, that we shall denote as $\widetilde{\mathcal{P}}$, $\widetilde{\mathcal{D}e\ell}$ and $\mathcal{D}e\ell$. The map $\widetilde{\mathcal{P}}$ is very closely related to $\mathcal{P}$; $\widetilde{\mathcal{D}e\ell}$ and $\mathcal{D}e\ell$ are well known: in the literature they are often referred to as *Delaunay* maps (two variants of).

The map $\widetilde{\mathcal{P}}$ Define the set

$$\mathcal{C}_{\widetilde{\mathcal{P}}} := \left\{ (y,x) \in \mathbb{R}^{3n} \times \mathbb{R}^{3n} : \quad x^{(j)} \neq 0 \ , \quad \widetilde{\mathrm{n}}_j :\neq 0 \ , \quad \widetilde{\nu}_j \neq 0 \quad \forall\, j = 1, \cdots, n \right\} ,$$

where, for $(y,x) \in \mathbb{R}^{3n} \times \mathbb{R}^{3n}$, with $y = (y^{(1)}, \cdots, y^{(n)})$, $x = (x^{(1)}, \cdots, x^{(n)})$, $x^{(j)} \neq 0$, we have let

$$\widetilde{\nu}_j := \begin{cases} k^{(3)} \times \mathrm{S}_{\mathcal{C}}^{(1)} & j = 1 \\ \dfrac{x^{(j-1)}}{\|x^{(j-1)}\|} \times \mathrm{S}_{\mathcal{C}}^{(j)} & j = 2, \cdots, n \end{cases} \qquad \widetilde{\mathrm{n}}_j := \mathrm{S}_{\mathcal{C}}^{(j)} \times \frac{x^{(j)}}{\|x^{(j)}\|}$$

with $j = 1, \cdots, n$ and

$$\mathrm{C}_{\mathcal{C}}^{(j)} := x^{(j)} \times y^{(j)} \ , \quad \mathrm{S}_{\mathcal{C}}^{(j)} := \sum_{i=j}^{n} \mathrm{C}^{(i)} \ . \tag{23}$$

Define a map

$$\widetilde{\mathcal{P}}^{-1} : \quad (y,x) \in \mathcal{C}_{\widetilde{\mathcal{P}}} \to (\widetilde{\Theta}, \widetilde{\chi}, \widetilde{\mathrm{R}}, \widetilde{\vartheta}, \widetilde{\kappa}, \widetilde{\mathrm{r}}) \in \mathbb{R}^n \times \mathbb{R}_+^n \times \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{T}^n \times \mathbb{R}_+^n$$

with

$$\widetilde{\Theta} = (\widetilde{\Theta}_0, \cdots, \widetilde{\Theta}_{n-1}) \qquad \widetilde{\vartheta} = (\widetilde{\vartheta}_0, \cdots, \widetilde{\vartheta}_{n-1})$$

$$\widetilde{\chi} = (\widetilde{\chi}_0, \cdots, \widetilde{\chi}_{n-1}) \qquad \widetilde{\kappa} = (\widetilde{\kappa}_0, \cdots, \widetilde{\kappa}_{n-1})$$

$$\widetilde{\mathrm{R}} = (\widetilde{\mathrm{R}}_1, \cdots, \widetilde{\mathrm{R}}_n) \qquad \widetilde{\mathrm{r}} = (\widetilde{\mathrm{r}}_1, \cdots, \widetilde{\mathrm{r}}_n)$$

via the following formulae

$$\widetilde{\mathrm{R}}_j = \frac{y^{(j)} \cdot x^{(j)}}{\|x^{(j)}\|} \qquad \widetilde{\mathrm{r}}_j = \|x^{(j)}\| \qquad j = 1, \cdots, n$$

$$\widetilde{\chi}_{j-1} = \|\mathrm{S}_{\mathcal{C}}^{(j)}\| \qquad \widetilde{\kappa}_{j-1} = \alpha_{\mathrm{S}_{\mathcal{C}}^{(j)}}(\widetilde{\nu}_j, \widetilde{\mathrm{n}}_j) \qquad j = 1, \cdots, n$$

$$\widetilde{\Theta}_{j-1} = \begin{cases} \mathrm{S}_{\mathcal{C}}^{(1)} \cdot k^{(3)} \\ \mathrm{S}_{\mathcal{C}}^{(j)} \cdot \dfrac{x^{(j-1)}}{\|x^{(j-1)}\|} \end{cases} \qquad \widetilde{\vartheta}_{j-1} = \begin{cases} \alpha_{k^{(3)}}(k^{(1)}, \widetilde{\nu}_1) & j = 1 \\ \alpha_{\frac{x^{(j-1)}}{\|x^{(j-1)}\|}}(\widetilde{\mathrm{n}}_{j-1}, \widetilde{\nu}_j) & j = 2, \cdots, n \ . \end{cases}$$

Lemma 2.1 *Let $\mathcal{D}_{\widetilde{\mathcal{P}}}$ be the set of $(\widetilde{\Theta}, \widetilde{\chi}, \widetilde{\mathrm{R}}, \widetilde{\vartheta}, \widetilde{\kappa}, \widetilde{\mathrm{r}}) \in \mathbb{R}^n \times \mathbb{R}_+^n \times \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{T}^n \times \mathbb{R}_+^n$ such that $(\widetilde{\Theta}, \widetilde{\chi}, \widetilde{\vartheta}, \widetilde{\kappa})$ satisfies (13), and let $\widetilde{\mathcal{T}}_j$, $\widetilde{\mathcal{S}}_j$ and $\mathrm{C}_{\widetilde{\mathcal{P}}}^{(j)}$ the functions of $(\widetilde{\Theta}, \widetilde{\chi}, \widetilde{\vartheta}, \widetilde{\kappa})$ defined in (14)–(16), with $(\widetilde{\Theta}, \widetilde{\chi}, \widetilde{\vartheta}, \widetilde{\kappa})$ replacing $(\Theta, \chi, \vartheta, \kappa)$.*

The map $\widetilde{\mathcal{P}}^{-1}$ is a bijection from $\mathcal{C}_{\widetilde{\mathcal{P}}}$ onto the set $\mathcal{D}_{\widetilde{\mathcal{P}}}$. Its inverse map

$$\widetilde{\mathcal{P}}: \qquad (\widetilde{\Theta}, \widetilde{\chi}, \widetilde{\mathrm{R}}, \widetilde{\vartheta}, \widetilde{\kappa}, \widetilde{\mathrm{r}}) \in \mathcal{D}_{\widetilde{\mathcal{P}}} \to (y_{\widetilde{\mathcal{P}}}, x_{\widetilde{\mathcal{P}}}) \in \mathbb{R}^n \times \mathbb{R}^n$$

has the following analytical expression:

$$\begin{cases} x^{(j)}_{\widetilde{\mathcal{P}}} := \widetilde{\mathrm{r}}_j \widetilde{\mathcal{T}}_1 \widetilde{\mathcal{S}}_1 \cdots \widetilde{\mathcal{T}}_j \widetilde{\mathcal{S}}_j k^{(3)} \\ \\ y^{(j)}_{\widetilde{\mathcal{P}}} := \dfrac{\widetilde{\mathrm{R}}_j}{\widetilde{\mathrm{r}}_j} x^{(j)}_{\widetilde{\mathcal{P}}} + \dfrac{1}{\widetilde{\mathrm{r}}_j^2} \mathrm{C}^{(j)}_{\widetilde{\mathcal{P}}} \times x^{(j)}_{\widetilde{\mathcal{P}}} \qquad 1 \le j \le n \end{cases} \tag{24}$$

Moreover, the following relation holds

$$\mathrm{C}^{(j)}_{\widetilde{\mathcal{P}}} = \mathrm{C}^{(j)}_{\mathcal{C}} \circ \widetilde{\mathcal{P}} = x^{(j)}_{\mathcal{P}} \times y^{(j)}_{\mathcal{P}} \ . \tag{25}$$

Proof With similar arguments as the ones of the proof of Proposition 2.1, but replacing, in the diagram (20), $\mathrm{S}^{(j)}_{\mathcal{E}}$ with $\mathrm{S}^{(j)}_{\mathcal{C}}$, $P^{(j)}_{\mathcal{P}}$ with $\frac{x^{(j)}}{\|x^{(j)}\|}$ and the nodes ν_k, n_k with $\widetilde{\nu}_k$, $\widetilde{\mathrm{n}}_k$, one finds the formula for $x^{(j)}_{\widetilde{\mathcal{P}}}$ in (24), the formula for

$$\mathrm{S}^{(j)}_{\widetilde{\mathcal{P}}} := \mathrm{S}^{(j)}_{\mathcal{C}} \circ \widetilde{\mathcal{P}} = \widetilde{\chi}_{j-1} \widetilde{\mathcal{T}}_1 \widetilde{\mathcal{S}}_1 \cdots \widetilde{\mathcal{T}}_{j-1} \widetilde{\mathcal{S}}_{j-1} \widetilde{\mathcal{T}}_j k^{(3)}$$

and hence the formula for

$$\mathrm{C}^{(j)}_{\mathcal{C}} \circ \widetilde{\mathcal{P}} = \mathrm{S}^{(j)}_{\widetilde{\mathcal{P}}} - \mathrm{S}^{(j+1)}_{\widetilde{\mathcal{P}}} = \mathrm{C}^{(j)}_{\widetilde{\mathcal{P}}}$$

being just the formula for $\mathrm{C}^{(j)}_{\mathcal{P}}$ in (16), with $(\Theta, \chi, \vartheta, \kappa)$ replaced by $(\widetilde{\Theta}, \widetilde{\chi}, \widetilde{\vartheta}, \widetilde{\kappa})$. With the same argument as in Remark 2.2, (ii), we see that $x^{(j)}_{\widetilde{\mathcal{P}}} \perp \mathrm{C}^{(j)}_{\widetilde{\mathcal{P}}}$. Finally, the formula for $y^{(j)}_{\widetilde{\mathcal{P}}}$ is found taking for $y^{(j)}_{\widetilde{\mathcal{P}}}$ the unique vector verifying

$$y^{(j)}_{\mathcal{P}} \cdot \frac{x^{(j)}_{\mathcal{P}}}{\|x^{(j)}_{\mathcal{P}}\|} = \mathrm{R}_j \qquad x^{(j)}_{\mathcal{P}} \times y^{(j)}_{\mathcal{P}} = \mathrm{C}^{(j)}_{\mathcal{P}} \ . \quad \blacksquare$$

Lemma 2.2 *$\widetilde{\mathcal{P}}$ preserves the standard Liouville 1–form:*

$$\sum_{j=1}^{n} y^{(j)}_{\widetilde{\mathcal{P}}} \cdot dx^{(j)}_{\widetilde{\mathcal{P}}} = \sum_{j=1}^{n} \left(\widetilde{\Theta}_{j-1} d\widetilde{\vartheta}_{j-1} + \widetilde{\chi}_{j-1} d\widetilde{\kappa}_{j-1} + \widetilde{\mathrm{R}}_j d\widetilde{\mathrm{r}}_j \right) \ . \tag{26}$$

The proof of Lemma 2.2 uses the flowing easy

Lemma 2.3 ([7]) *Let*

$$x = \mathcal{R}_3(\theta)\mathcal{R}_1(i)\bar{x} \ , \quad y = \mathcal{R}_3(\theta)\mathcal{R}_1(i)\bar{y} \ , \quad \mathrm{C} := x \times y \ , \quad \bar{\mathrm{C}} := \bar{x} \times \bar{y} \ ,$$

with $x, \bar{x}, y, \bar{y} \in \mathbb{R}^3$. Then,

$$y \cdot dx = \mathrm{C} \cdot k^{(3)} d\theta + \bar{\mathrm{C}} \cdot k^{(1)} di + \bar{y} \cdot d\bar{x} \ .$$

Proof of Lemma 2.2 We may write

$$x^{(j)}_{\widetilde{\mathcal{P}}} = \widetilde{\mathcal{T}}_1 \widetilde{\mathcal{S}}_1 \cdots \widetilde{\mathcal{T}}_j \widetilde{\mathcal{S}}_j \tilde{x}^{(j)} \ , \quad y^{(j)}_{\widetilde{\mathcal{P}}} = \widetilde{\mathcal{T}}_1 \widetilde{\mathcal{S}}_1 \cdots \widetilde{\mathcal{T}}_j \widetilde{\mathcal{S}}_j \tilde{y}^{(j)} \ , \quad \mathrm{C}^{(j)}_{\widetilde{\mathcal{P}}} = \widetilde{\mathcal{T}}_1 \widetilde{\mathcal{S}}_1 \cdots \widetilde{\mathcal{T}}_j \widetilde{\mathcal{S}}_j \tilde{\mathrm{C}}^{(j)}$$

where

$$\begin{aligned}\tilde{x}^{(j)} &:= \widetilde{\mathrm{r}}_j k^{(3)} \qquad j=1,\cdots,n-1\\ \tilde{y}^{(j)} &:= \widetilde{\mathrm{R}}_j k^{(3)} + \frac{1}{\widetilde{\mathrm{r}}_j}\tilde{\mathrm{C}}^{(j)}\times k^{(3)}\\ \tilde{\mathrm{C}}^{(j)} &:= \widetilde{\chi}_{j-1}\widetilde{\mathcal{S}}_j^{-1}k^{(3)} - \widetilde{\chi}_j\widetilde{\mathcal{T}}_{j+1}k^{(3)} = \tilde{x}^{(j)}\times\tilde{y}^{(j)}\end{aligned} \tag{27}$$

with $\widetilde{\chi}_n := 0$, $\widetilde{\mathcal{S}}_n := \mathrm{id}$. We also let, for $1\le k\le j\le n$ and $1\le i\le n-1$,

$$\hat{\mathrm{C}}_k^{(j)} = \widetilde{\mathcal{S}}_k(\widetilde{\mathcal{T}}_{k+1}\widetilde{\mathcal{S}}_{k+1}\cdots\widetilde{\mathcal{T}}_j\widetilde{\mathcal{S}}_j)\tilde{\mathrm{C}}^{(j)}\ ,\quad \check{\mathrm{C}}_k^{(j)} = \widetilde{\mathcal{T}}_k\widetilde{\mathcal{S}}_k\cdots\widetilde{\mathcal{T}}_j\widetilde{\mathcal{S}}_j\tilde{\mathrm{C}}^{(j)}\ ,\quad \check{\mathrm{C}}_{j+1}^{(j)} := \tilde{\mathrm{C}}^{(j)}$$

$$\hat{\mathrm{S}}_k^{(j)} := \sum_{m=j}^n \hat{\mathrm{C}}_k^{(m)}\ ,\quad \check{\mathrm{S}}_k^{(j)} := \sum_{m=j}^n \check{\mathrm{C}}_k^{(m)}\ ,\quad \check{\mathrm{S}}_{i+1}^{(i)} := \tilde{\mathrm{C}}^{(i)} + \check{\mathrm{S}}_{i+1}^{(i+1)}$$

where the product $\widetilde{\mathcal{T}}_{k+1}\widetilde{\mathcal{S}}_{k+1}\cdots\widetilde{\mathcal{T}}_j\widetilde{\mathcal{S}}_j$ is to be replaced with the identity when $k=j$. We have the following identities (implied by $\mathrm{S}^{(j)} = \sum_{k=j}^n \mathrm{C}^{(k)}$)

$$\check{\mathrm{S}}_j^{(j)} = \sum_{k=j}^n \check{\mathrm{C}}_j^{(k)} = \widetilde{\chi}_{j-1}\widetilde{\mathcal{T}}_j k^{(3)}\ ,\quad \hat{\mathrm{S}}_j^{(j)} = \sum_{k=j}^n \hat{\mathrm{C}}_j^{(k)} = \widetilde{\chi}_{j-1}k^{(3)}\ ,\quad \check{\mathrm{S}}_{i+1}^{(i)} = \widetilde{\chi}_{j-1}\widetilde{\mathcal{S}}_i^{-1}k^{(3)}\ . \tag{28}$$

Applying Lemma 2.3 repeatedly and using (as it follows from (27)),

$$\tilde{y}^{(j)}\cdot d\tilde{x}^{(j)} = \widetilde{\mathrm{R}}_j d\widetilde{\mathrm{r}}_j$$

we have, for $1\le j\le n$,

$$\begin{aligned}y_{\widetilde{\mathcal{P}}}^{(j)}\cdot x_{\widetilde{\mathcal{P}}}^{(j)} &= \sum_{k=1}^{j}\big(\check{\mathrm{C}}_k^{(j)}\cdot k^{(3)}d\widetilde{\vartheta}_{k-1} + \hat{\mathrm{C}}_k^{(j)}\cdot k^{(1)}d\widetilde{\iota}_k + \hat{\mathrm{C}}_k^{(j)}\cdot k^{(3)}d\widetilde{\kappa}_{k-1} + \check{\mathrm{C}}_{k+1}^{(j)}\cdot k^{(1)}d\widetilde{\mathrm{i}}_k\big)\\ &\quad + \widetilde{\mathrm{R}}_j d\widetilde{\mathrm{r}}_j\end{aligned}$$

where, as in the proof of Lemma 2.1, $\widetilde{\iota}_j$, $\widetilde{\mathrm{i}}_j$ denote the functions ι_j, i_j in (14), with Θ_i, χ_i replaced by $\widetilde{\Theta}_i$, $\widetilde{\chi}_i$. Note that we have used $d\widetilde{\mathrm{i}}_n \equiv 0$, since, by definition, $\widetilde{\mathrm{i}}_n = \frac{\pi}{2}$. Taking the sum over $j=1,\cdots,n$,

$$\begin{aligned}\sum_{j=1}^n y_{\widetilde{\mathcal{P}}}^{(j)}\cdot dx_{\widetilde{\mathcal{P}}}^{(j)} &= \sum_{j=1}^n \check{\mathrm{S}}_j^{(j)}\cdot k^{(3)}d\widetilde{\vartheta}_{j-1} + \hat{\mathrm{S}}_j^{(j)}\cdot k^{(1)}d\widetilde{\iota}_j + \hat{\mathrm{S}}_j^{(j)}\cdot k^{(3)}d\widetilde{\kappa}_{j-1} + \check{\mathrm{S}}_{j+1}^{(j)}\cdot k^{(1)}d\widetilde{\mathrm{i}}_j\\ &\quad + \sum_{j=1}^n \widetilde{\mathrm{R}}_j d\widetilde{\mathrm{r}}_j\ .\end{aligned}$$

In view of (28) and of the definitions in (14)–(15), we then find (26). ∎

The map $\widetilde{\mathcal{D}e\ell}$ The map

$$\widetilde{\mathcal{D}e\ell}:\qquad (\widetilde{\mathrm{H}},\widetilde{\Gamma},\widetilde{\mathrm{R}},\widetilde{\mathrm{h}},\widetilde{\mathrm{g}},\widetilde{\mathrm{r}})\in\mathcal{D}_{\widetilde{\mathcal{D}e\ell}}\to(y_{\widetilde{\mathcal{D}e\ell}},x_{\widetilde{\mathcal{D}e\ell}})\in\mathbb{R}^{3n}\times\mathbb{R}^{3n}$$

is defined on the set

$$\begin{aligned}\mathcal{D}_{\widetilde{\mathcal{D}e\ell}} := \Big\{&(\widetilde{\mathrm{H}},\widetilde{\Gamma},\widetilde{\mathrm{R}},\widetilde{\mathrm{h}},\widetilde{\mathrm{g}},\widetilde{\mathrm{r}}) = (\widetilde{\mathrm{H}}_1,\cdots,\widetilde{\mathrm{H}}_n,\widetilde{\Gamma}_1,\cdots,\widetilde{\Gamma}_n,\widetilde{\mathrm{R}}_1,\cdots,\widetilde{\mathrm{R}}_n,\widetilde{\mathrm{h}}_1,\cdots,\widetilde{\mathrm{h}}_n,\\ &\widetilde{\mathrm{g}}_1,\cdots,\widetilde{\mathrm{g}}_n,\widetilde{\mathrm{r}}_1,\cdots,\widetilde{\mathrm{r}}_n)\in\mathbb{R}^{3n}\times\mathbb{T}^{2n}\times\mathbb{R}_+^n:\quad \widetilde{\mathrm{r}}_j>0\ ,\quad \widetilde{\Gamma}_j>0\ ,\quad \frac{|\widetilde{\mathrm{H}}_j|}{\widetilde{\Gamma}_j}<1\\ &\forall\ j=1,\cdots,n\Big\}\end{aligned}$$

via the following formulae

$$x^{(j)}_{\widetilde{\mathcal{D}e\ell}} := \mathcal{R}_3(\widetilde{\mathrm{h}}_\mathrm{j})\mathcal{R}_1(\widetilde{i}_j)\overline{x}^{(j)}_{\widetilde{\mathcal{D}e\ell}} \ , \qquad y^{(j)}_{\widetilde{\mathcal{D}e\ell}} := \mathcal{R}_3(\widetilde{\mathrm{h}}_\mathrm{j})\mathcal{R}_1(\widetilde{i}_j)\overline{y}^{(j)}_{\widetilde{\mathcal{D}e\ell}}$$

where

$$\begin{aligned}
\widetilde{i}_j &:= \cos^{-1}\frac{\widetilde{\mathrm{H}}_j}{\widetilde{\Gamma}_j} \in (0,\pi) \\
\overline{x}^{(j)}_{\widetilde{\mathcal{D}e\ell}} &:= \widetilde{\mathrm{r}}_j\cos\widetilde{\mathrm{g}}_j k^{(1)} + \widetilde{\mathrm{r}}_j\sin\widetilde{\mathrm{g}}_j k^{(2)} \\
\overline{y}^{(j)}_{\widetilde{\mathcal{D}e\ell}} &:= \big(\widetilde{\mathrm{R}}_j\cos\widetilde{\mathrm{g}}_j - \frac{\widetilde{\Gamma}_j}{\widetilde{\mathrm{r}}_j}\sin\widetilde{\mathrm{g}}_j\big)k^{(1)} + \big(\widetilde{\mathrm{R}}_j\sin\widetilde{\mathrm{g}}_j + \frac{\widetilde{\Gamma}_j}{\widetilde{\mathrm{r}}_j}\cos\widetilde{\mathrm{g}}_j\big)k^{(2)} \ .
\end{aligned}$$

Lemma 2.4 (Delaunay) *$\widetilde{\mathcal{D}e\ell}$ is a bijection from the domain $\mathcal{D}_{\widetilde{\mathcal{D}e\ell}}$ onto the set*

$$\begin{aligned}
\mathcal{C}_{\widetilde{\mathcal{D}e\ell}} := \Big\{&(y,x) = (y^{(1)},\cdots,y^{(n)},x^{(1)},\cdots,x^{(n)}) \in \mathbb{R}^{3n}\times\mathbb{R}^{3n} : \\
&\widetilde{\mathfrak{n}}_j := k^{(3)}\times \mathrm{C}^{(j)}_{\mathcal{C}} \neq 0 \ , \quad x^{(j)} \neq 0 \qquad \forall\ j = 1,\cdots,n\Big\}
\end{aligned}$$

where $\mathrm{C}^{(j)}_{\mathcal{C}}$ is as in (23). The formulae for the inverse map

$$\widetilde{\mathcal{D}e\ell}^{\,-1} : \qquad (y,x) \in \mathcal{C}_{\widetilde{\mathcal{D}e\ell}} \to (\widetilde{\mathrm{H}},\widetilde{\Gamma},\widetilde{\mathrm{R}},\widetilde{\mathrm{h}},\widetilde{\mathrm{g}},\widetilde{\mathrm{r}}) \in \mathcal{D}_{\widetilde{\mathcal{D}e\ell}}$$

are

$$\begin{cases} \widetilde{\mathrm{H}}_j = \mathrm{C}^{(j)}_{\mathcal{C}}\cdot k^{(3)} \\ \widetilde{\mathrm{h}}_j := \alpha_{k^{(3)}}(k^{(1)},\widetilde{\mathfrak{n}}_j) \end{cases} \qquad \begin{cases} \widetilde{\Gamma}_j := \|\mathrm{C}^{(j)}_{\mathcal{C}}\| \\ \widetilde{\mathrm{g}}_j := \alpha_{\mathrm{C}^{(j)}_{\mathcal{C}}}(\widetilde{\mathfrak{n}}_j, x^{(j)}) \end{cases} \qquad \begin{cases} \widetilde{\mathrm{R}}_j = \dfrac{y^{(j)}\cdot x^{(j)}}{\|x^{(j)}\|} \\ \widetilde{\mathrm{r}}_j = \|x^{(j)}\| \end{cases} \tag{29}$$

Finally, $\widetilde{\mathcal{D}e\ell}$ preserves the standard Liouville 1–form

$$\sum_{i=1}^n y^{(i)}_{\widetilde{\mathcal{D}e\ell}}\cdot dx^{(i)}_{\widetilde{\mathcal{D}e\ell}} = \sum_{i=1}^n \big(\widetilde{\mathrm{H}}_i d\widetilde{\mathrm{h}}_i + \widetilde{\Gamma}_i d\widetilde{\mathrm{g}}_i + \widetilde{\mathrm{R}}_i d\widetilde{\mathrm{r}}_i\big) \ .$$

We omit the proof of Lemma 2.4, which may be found in classical textbooks.

The map $\mathcal{D}e\ell$

Definition 2.5 (Delaunay map) Let

$$\begin{aligned}
\mathfrak{X}_{\mathcal{D}e\ell} := \Big\{&\mathrm{X}_{\mathcal{D}e\ell} := (\mathrm{H},\Gamma,\Lambda,\mathrm{h},\mathrm{g}) = (\mathrm{H}_1,\cdots,\mathrm{H}_n,\Gamma_1,\cdots,\Gamma_n,\Lambda_1,\cdots,\Lambda_n,\mathrm{h}_1,\cdots,\mathrm{h}_n, \\
&\mathrm{g}_1,\cdots,\mathrm{g}_n) \in \mathbb{R}^{3n}\times\mathbb{T}^{2n} : \qquad \Gamma_j > 0 \ , \quad \frac{|\mathrm{H}_j|}{\Gamma_j} < 1 \ , \quad \Lambda_j > 0 \\
&\forall\ j = 1,\cdots,n\Big\}
\end{aligned}$$

and let $\mathcal{E}_{\mathcal{D}e\ell}$ be the set of n–plets $(\mathfrak{E}_1,\cdots,\mathfrak{E}_n)$ where $\mathfrak{E}_j = (a_j,e_j,N^{(j)},P^{(j)})$ satisfies

$$0 < e_j < 1 \ , \qquad \mathfrak{n}_j := k^{(3)}\times N^{(j)} \neq 0 \ , \quad \forall\ j = 1,\cdots,n \ .$$

Fix positive numbers $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$, $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$. Denote as

$$\tau_{\mathcal{D}e\ell}: \ \mathrm{X}_{\mathcal{D}e\ell} := (\mathrm{H}, \Gamma, \Lambda, \mathrm{h}, \mathrm{g}) \in \mathfrak{X}_{\mathcal{D}e\ell} \to \mathfrak{E}_{\mathcal{D}e\ell} = (\mathfrak{E}_{1,\mathcal{D}e\ell}, \cdots, \mathfrak{E}_{n,\mathcal{D}e\ell})$$

defined by $\mathfrak{E}_{j,\mathcal{D}e\ell} = (a_{j,\mathcal{D}e\ell}, e_{j,\mathcal{D}e\ell}, N^{(j)}_{\mathcal{D}e\ell}, P^{(j)}_{\mathcal{D}e\ell})$ and

$$a_{j,\mathcal{D}e\ell} = \frac{1}{\mathfrak{M}_j}(\frac{\Lambda_j}{\mathfrak{m}_j})^2 \ , \qquad e_{j,\mathcal{D}e\ell} = \sqrt{1 - (\tfrac{\Gamma_j}{\Lambda_j})^2}$$

$$N^{(j)}_{\mathcal{D}e\ell} = \mathcal{R}_3(\mathrm{h_j})\mathcal{R}_1(i_j)k^{(3)} \quad P^{(j)}_{\mathcal{D}e\ell} = \mathcal{R}_3(\mathrm{h_j})\mathcal{R}_1(i_j)\mathcal{R}_3(\mathrm{g}_j)k^{(1)}$$

where $i_j := \cos^{-1}\frac{\mathrm{H}_j}{\Gamma_j}$.
We call *Delaunay map* the map

$$\mathcal{D}e\ell: \qquad \mathrm{Del} = (\mathrm{H}, \Gamma, \Lambda, \mathrm{h}, \mathrm{g}, \ell) \in \mathcal{D}_{\mathcal{D}e\ell} \to (y_{\mathcal{D}e\ell}, x_{\mathcal{D}e\ell}) \in \mathbb{R}^{3n} \times \mathbb{R}^{3n} \tag{30}$$

which is defined on the domain

$$\mathcal{D}_{\mathcal{D}e\ell} \ := \ \mathfrak{X}_{\mathcal{D}e\ell} \times \mathbb{T}^n$$

as the Kepler map associated to $\tau_{\mathcal{D}e\ell}$ via the following lemma (the proof of which may be found in classical textbooks).

Lemma 2.5 (Delaunay) *$\tau_{\mathcal{D}e\ell}$ is a bijection of $\mathfrak{X}_{\mathcal{D}e\ell}$ onto $\mathcal{E}_{\mathcal{D}e\ell}$. Its inverse map*

$$\tau^{-1}_{\mathcal{D}e\ell}: \qquad \mathfrak{E}_{\mathcal{D}e\ell} = (\mathfrak{E}_{1,\mathcal{D}e\ell}, \cdots, \mathfrak{E}_{n,\mathcal{D}e\ell}) \in \mathcal{E}_{\mathcal{D}e\ell} \to \mathrm{X}_{\mathcal{D}e\ell} \in \mathfrak{X}_{\mathcal{D}e\ell}$$

is defined by equations

$$\begin{cases} \mathrm{H}_j = \mathrm{C}^{(j)}_{\mathcal{E}} \cdot k^{(3)} \\ \mathrm{h}_j := \alpha_{k^{(3)}}(k^{(1)}, \mathfrak{n}_j) \end{cases} \qquad \begin{cases} \Gamma_j = \|\mathrm{C}^{(j)}_{\mathcal{E}}\| \\ \mathrm{g}_j := \alpha_{\mathrm{C}^{(j)}_{\mathcal{E}}}(\mathfrak{n}_j, P^{(j)}) \end{cases} \qquad \Lambda_j = \mathfrak{m}_j\sqrt{\mathfrak{M}_j a_j} \ , \tag{31}$$

where $\mathrm{C}^{(j)}_{\mathcal{E}}$ is as in (9). Furthermore, $\mathcal{D}e\ell$ preserves the standard 2–form:

$$\sum_{j=1}^{n} dy^{(j)}_{\mathcal{D}e\ell} \wedge dx^{(j)}_{\mathcal{D}e\ell} = \sum_{j=1}^{n} \left(d\mathrm{H}_j \wedge d\mathrm{h}_j + d\Gamma_j \wedge d\mathrm{g}_j + d\Lambda_j \wedge d\ell_j \right) \ .$$

Now we are ready to complete the

Proof of Theorem 2.1 Let

$$\mathcal{D}^*_{\mathcal{P}} := \left\{ \mathrm{P} = (\Theta, \chi, \Lambda, \vartheta, \kappa, \ell) \in \mathcal{D}_{\mathcal{P}}: \qquad \mathcal{P}(\mathrm{P}) \in \mathcal{C}_{\mathcal{D}e\ell} \right\} .$$

It is enough to prove Theorem 2.1 on $\mathcal{D}^*_{\mathcal{P}}$, since indeed the $\mathcal{P}$–map is regular on $\mathcal{D}_{\mathcal{P}} = \overline{\mathcal{D}^*_{\mathcal{P}}}$. On $\mathcal{D}^*_{\mathcal{P}}$, we consider the map

$$\begin{aligned} &\phi^{\mathcal{P}}_{\mathcal{D}e\ell} := \mathcal{D}e\ell^{-1} \circ \mathcal{P}: \\ &\mathrm{P} = (\Theta, \chi, \Lambda, \vartheta, \kappa, \ell) \in \mathcal{D}^*_{\mathcal{P}} \to \mathrm{Del} = (\mathrm{H}, \Gamma, \Lambda, \mathrm{h}, \mathrm{g}, \ell) \in \mathcal{D}^*_{\mathcal{D}e\ell} := \phi^{\mathcal{P}}_{\mathcal{D}e\ell}(\mathcal{D}^*_{\mathcal{P}}) \subset \mathcal{D}_{\mathcal{D}e\ell} \ . \end{aligned}$$

$\phi^{\mathcal{P}}_{\mathcal{D}e\ell}$ gives the Delaunay coordinates at left hand side in (30) in terms of the P–coordinates at left hand side of (7) in the subset $\mathcal{D}^*_{\mathcal{P}}$ of $\mathcal{D}_{\mathcal{P}}$ the $\mathcal{P}$–image of which lies in the $\mathcal{D}e\ell$–image of $\mathcal{D}_{\mathcal{D}e\ell}$.

Clearly, $\phi^{\mathcal{P}}_{\mathcal{D}e\ell}$ leaves the (Λ,ℓ) unvaried. More precisely, $\phi^{\mathcal{P}}_{\mathcal{D}e\ell}$ decouples into two disjoint maps: the identity on the (Λ,ℓ), and a $4n$–dimensional map

$$\widehat{\phi^{\mathcal{P}}_{\mathcal{D}e\ell}}: \qquad (\Theta,\chi,\vartheta,\kappa)\in\widehat{\mathcal{D}^*_{\mathcal{P}}}\to(\mathrm{H},\Gamma,\mathrm{h},\mathrm{g})\in\widehat{\mathcal{D}^*_{\mathcal{D}e\ell}}=\phi^{\mathcal{P}}_{\mathcal{D}e\ell}(\widehat{\mathcal{D}^*_{\mathcal{P}}})\subset\widehat{\mathcal{D}_{\mathcal{D}e\ell}}$$

on the remaining coordinates, which turns out to be a bijection of the sets $\widehat{\mathcal{D}^*_{\mathcal{P}}}$ and $\widehat{\mathcal{D}^*_{\mathcal{D}e\ell}}$. Here, the map $\widehat{\phi^{\mathcal{P}}_{\mathcal{D}e\ell}}$ and the sets $\widehat{\mathcal{D}^*_{\mathcal{P}}}$ and $\widehat{\mathcal{D}_{\mathcal{D}e\ell}}$ do not depend on (Λ,ℓ). Indeed, the explicit expressions of $\widehat{\phi^{\mathcal{P}}_{\mathcal{D}e\ell}}$, $\widehat{\mathcal{D}^*_{\mathcal{P}}}$ in terms of $\mathrm{P}=(\Theta,\chi,\Lambda,\vartheta,\kappa,\ell)$; or of of $\widehat{\mathcal{D}_{\mathcal{D}e\ell}}$ in terms of $\mathrm{Del}=(\mathrm{H},\Gamma,\Lambda,\mathrm{h},\mathrm{g},\ell)$ involve only the $\mathrm{C}^{(j)}_{\mathcal{P}}$, $P^{(j)}_{\mathcal{P}}$; the $\mathrm{C}^{(j)}_{\mathcal{D}e\ell}$, $P^{(j)}_{\mathcal{D}e\ell}$, that do not depend on (Λ,ℓ): (31) (where one has to replace $\mathcal{C}$ with $\mathcal{P}$), (15) and (18).

In view of the previous consideration and of Lemma 2.5, Theorem 2.1 is implied by

Lemma 2.6 *The map $\widehat{\phi^{\mathcal{P}}_{\mathcal{D}e\ell}}$ preserves that standard 1–form:*

$$\sum_{j=1}^{n}\left(\mathrm{H}_j d\mathrm{h}_j+\Gamma_j d\mathrm{g}_j\right)=\sum_{j=1}^{n}\left(\Theta_{j-1}d\vartheta_{j-1}+\chi_{j-1}d\kappa_{j-1}\right)\,.$$

Proof We look at the analogue map

$$\widehat{\phi^{\widetilde{\mathcal{P}}}_{\widetilde{\mathcal{D}e\ell}}}: \qquad (\widetilde{\Theta},\widetilde{\chi},\widetilde{\vartheta},\widetilde{\kappa})\in\widehat{\mathcal{D}^*_{\widetilde{\mathcal{P}}}}\to(\widetilde{\mathrm{H}},\widetilde{\Gamma},\widetilde{\mathrm{h}},\widetilde{\mathrm{g}})\in\widehat{\mathcal{D}^*_{\widetilde{\mathcal{D}e\ell}}}=\phi^{\widetilde{\mathcal{P}}}_{\widetilde{\mathcal{D}e\ell}}(\widehat{\mathcal{D}^*_{\widetilde{\mathcal{P}}}})\subset\widehat{\mathcal{D}_{\widetilde{\mathcal{D}e\ell}}}\,.$$

The analytical expression of this map is identical to the one of $\widehat{\phi^{\mathcal{P}}_{\mathcal{D}e\ell}}$. This follows from the fact that $\widehat{\phi^{\widetilde{\mathcal{P}}}_{\widetilde{\mathcal{D}e\ell}}}$ depends on the coordinates $(\widetilde{\Theta},\widetilde{\chi},\widetilde{\vartheta},\widetilde{\kappa})$ only via $\mathrm{C}^{(j)}_{\widetilde{\mathcal{P}}}$ and $\frac{x^{(j)}_{\widetilde{\mathcal{P}}}}{\|x^{(j)}_{\widetilde{\mathcal{P}}}\|}$ exactly as $\widehat{\phi^{\mathcal{P}}_{\mathcal{D}e\ell}}$ depends on $(\Theta,\chi,\vartheta,\kappa)$ only via $\mathrm{C}^{(j)}_{\mathcal{P}}$ and $\Pi^{(j)}_{\mathcal{P}}$, that $\mathrm{C}^{(j)}_{\widetilde{\mathcal{P}}}$ and $\frac{x^{(j)}_{\widetilde{\mathcal{P}}}}{\|x^{(j)}_{\widetilde{\mathcal{P}}}\|}$ have exactly the same expressions of $\mathrm{C}^{(j)}_{\mathcal{P}}$ and $P^{(j)}$, apart for replacing $(\Theta,\chi,\vartheta,\kappa)$ with $(\widetilde{\Theta},\widetilde{\chi},\widetilde{\vartheta},\widetilde{\kappa})$: Compare (29) (where one has to replace $\mathrm{C}^{(j)}_{\mathcal{C}}$ with $\mathrm{C}^{(j)}_{\widetilde{\mathcal{P}}}$), (31) (where one has to replace $\mathrm{C}^{(j)}_{\mathcal{E}}$ with $\mathrm{C}^{(j)}_{\mathcal{P}}$), (15), (18), (24) and (25).

But Lemmata 2.2 and 2.4 imply that $\widehat{\phi^{\widetilde{\mathcal{P}}}_{\widetilde{\mathcal{D}e\ell}}}$ preserves that standard 1–form:

$$\sum_{j=1}^{n}\left(\widetilde{\mathrm{H}}_j d\widetilde{\mathrm{h}}_j+\widetilde{\Gamma}_j d\widetilde{\mathrm{g}}_j\right)=\sum_{j=1}^{n}\left(\widetilde{\Theta}_{j-1}d\widetilde{\vartheta}_{j-1}+\widetilde{\chi}_{j-1}d\widetilde{\kappa}_{j-1}\right)\,.$$

Then $\widehat{\phi^{\mathcal{P}}_{\mathcal{D}e\ell}}$ does. ∎

∎

2.1 The $\mathcal{P}$–map vs rotations and reflections

Now we discuss how the $\mathcal{P}$–map behaves in presence of symmetries in the Hamiltonian due to rotations or reflections.

Let $\mathrm{H}=\mathrm{H}(y,x)$ be the Hamiltonian governing the motion of n particles, where such particles are expressed in the canonical coordinates $(y^{(1)},x^{(1)}),\cdots,(y^{(n)},x^{(n)})$. Assume that H is left unvaried by rotations and reflections. Namely, if

$$\phi_{\mathcal{R},\mathcal{S}}: \qquad (y^{(j)},x^{(j)})\to(\mathcal{R}y^{(j)},\mathcal{S}x^{(j)})\,, \qquad j=1,\cdots,n$$

where $\mathcal{R}$, $\mathcal{S}$ are real a 3×3 matrices, then rotation invariance is

$$\mathrm{H}\circ\phi_{\mathcal{R},\mathcal{R}}=\mathrm{H}\qquad\forall\,\mathcal{R}:\ \mathcal{R}\mathcal{R}^{\mathrm{t}}=\mathrm{id}$$

while reflection invariance is

$$\mathrm{H}\circ\phi_{\mathcal{S}_\sigma,\mathcal{S}_\tau}=\mathrm{H}\quad\text{for some}\quad\mathcal{S}_\sigma=\begin{pmatrix}\sigma_1&0&0\\0&\sigma_2&0\\0&0&\sigma_3\end{pmatrix}\quad\mathcal{S}_\tau=\begin{pmatrix}\tau_1&0&0\\0&\tau_2&0\\0&0&\tau_3\end{pmatrix}\quad\sigma_i,\ \tau_i=\pm1\ .$$

Rotation invariance is associated to the conservation, through the motion, of the total angular momentum $\mathrm{S}^{(1)}_{\mathcal{C}}$ is (23). Reflection invariance is not associated to integrals.

The Hamiltonian $\mathrm{H}_{\mathrm{hel}}$ in (33) is rotation and reflection invariant, and reflection invariance holds with any choice of σ, τ.

Let

$$\mathrm{H}_{\mathcal{P}}:=\mathrm{H}\circ\mathcal{P}\ .$$

The fact that $\mathrm{S}^{(1)}_{\mathcal{C}}$ is preserved along the motions of H implies that the coordinates

$$\Theta_0=Z\ ,\quad\vartheta_0=\zeta\ ,\quad\kappa_0=\mathfrak{g}$$

do not appear in $\mathrm{H}_{\mathcal{P}}$. Indeed, Z and ζ are integrals, while $\mathfrak{g}$ is conjugated to $\mathrm{G}=\|\mathrm{S}^{(1)}_{\mathcal{P}}\|$, which is an integral for $\mathrm{H}_{\mathcal{P}}$. Thus, the number of degrees of freedom is naturally reduced by two units, once one regards G as a prefixed external parameter. Namely, for any fixed $\chi_0=\mathrm{G}$, $\mathrm{H}_{\mathcal{K}}$ may be regarded as a function of the $2(3n-1)$ dimensional coordinates

$$\overline{\mathrm{P}}:=(\overline{\Theta},\chi,\Lambda,\overline{\vartheta},\kappa,\ell)$$

which does not depend on κ_0. Here,

$$\bar\Theta=(\Theta_1,\cdots,\Theta_{n-1})\ ,\quad\bar\vartheta=(\vartheta_1,\cdots,\vartheta_{n-1})\ .$$

This fact is completely specular to what happens using the action–angle coordinates (Ψ, Γ, Λ, ψ, γ, ℓ), in turn related to a set of coordinates discovered by A. Deprit [12] in the 80s (compare [27, 7, 9, 34], or the Appendix E).

The main novelty introduced by the $\mathcal{P}$–coordinates (that does not hold for the coordinates of [7]) is how $\mathcal{P}$ behaves relatively to reflections.

We denote as

$$\mathcal{R}_2^-:=\phi_{\mathcal{S}_{\sigma^{(2)}},\mathcal{S}_{\sigma^{(2)}}}\qquad\sigma^{(2)}=(1,-1,1)$$

the reflection of the second coordinate both for the $y^{(j)}$'s and the $x^{(j)}$'s and we let

$$\mathcal{S}^-(\Theta,\chi,\Lambda,\vartheta,\kappa,\ell):=(-\Theta,\chi,\Lambda,-\vartheta,\kappa,\ell)\ .$$

Proposition 2.2

$$\mathcal{R}_2^-\circ\mathcal{P}=\mathcal{P}\circ\mathcal{S}^-\ .\tag{32}$$

Therefore, if $\mathrm{H}=\mathrm{H}(y,x)$ *satisfies*

$$\mathrm{H}\circ\mathcal{R}_2^-=\mathrm{H}$$

then $\mathrm{H}_{\mathcal{P}}:=\mathrm{H}\circ\mathcal{P}$ *satisfies*

$$\mathrm{H}_{\mathcal{P}}\circ\mathcal{S}^-=\mathrm{H}_{\mathcal{P}}\ .$$

Hence, any of the the points

$$\Theta_0=\cdots=\Theta_{n-1}=0\ ,\quad(\vartheta_0,\cdots,\vartheta_{n-1})=(k_0,\cdots,k_{n-1})\pi\qquad\mathrm{mod}\ 2\pi\mathbb{Z}^n$$

is an equilibrium point for $\mathrm{H}_{\mathcal{P}}$*, for any* $(\chi,\Lambda,\kappa,\ell)$.

Proof Defining $\mathcal{R}^{(j)} := \mathcal{T}_j \mathcal{S}_j$, $s^{(j)} := \mathcal{T}_j k^{(3)}$, we write the vectors $P_{\mathcal{P}}^{(j)}$ and $\mathrm{S}_{\mathcal{P}}^{(j)}$ (compare Eq. (18) and Remark 2.2, (iii)) as

$$P_{\mathcal{P}}^{(j)} = \mathcal{R}^{(1)} \cdots \mathcal{R}^{(j)} k^{(3)} \ , \quad \mathrm{S}_{\mathcal{P}}^{(j)} = \chi_{j-1} \mathcal{R}^{(1)} \cdots \mathcal{R}^{(j)} s^{(j)} \ .$$

The explicit expressions of $\mathcal{R}^{(j)}$ and $s^{(j)}$ are

$$\begin{aligned}
\mathcal{R}_{11}^{(j)} &= \cos\kappa_{j-1}\cos\vartheta_{j-1} - \sin\kappa_{j-1}\cos\iota_j\sin\vartheta_{j-1} \\
\mathcal{R}_{21}^{(j)} &= \cos\kappa_{j-1}\sin\vartheta_{j-1} + \sin\kappa_{j-1}\cos\iota_j\cos\vartheta_{j-1} \\
\mathcal{R}_{31}^{(j)} &= \sin\kappa_{j-1}\sin\iota_j \\
\mathcal{R}_{12}^{(j)} &= -\cos\mathrm{i}_j\sin\kappa_{j-1}\cos\vartheta_{j-1} + \sin\vartheta_{j-1}(-\cos\mathrm{i}_j\cos\iota_j\cos\kappa_{j-1} + \sin\iota_j\sin\mathrm{i}_j) \\
\mathcal{R}_{22}^{(j)} &= -\cos\mathrm{i}_j\sin\kappa_{j-1}\sin\vartheta_{j-1} - \cos\vartheta_{j-1}(-\cos\mathrm{i}_j\cos\iota_j\cos\kappa_{j-1} + \sin\iota_j\sin\mathrm{i}_j) \\
\mathcal{R}_{32}^{(j)} &= \cos\mathrm{i}_j\cos\kappa_{j-1}\sin\iota_j + \sin\mathrm{i}_j\cos\iota_j \\
\mathcal{R}_{13}^{(j)} &= \sin\mathrm{i}_j\sin\kappa_{j-1}\cos\vartheta_{j-1} + \sin\vartheta_{j-1}(\sin\mathrm{i}_j\cos\iota_j\cos\kappa_{j-1} + \sin\iota_j\cos\mathrm{i}_j) \\
\mathcal{R}_{23}^{(j)} &= \sin\mathrm{i}_j\sin\kappa_{j-1}\sin\vartheta_{j-1} - \cos\vartheta_{j-1}(\sin\mathrm{i}_j\cos\iota_j\cos\kappa_{j-1} + \sin\iota_j\cos\mathrm{i}_j) \\
\mathcal{R}_{33}^{(j)} &= -\sin\mathrm{i}_j\cos\kappa_{j-1}\sin\iota_j + \cos\mathrm{i}_j\cos\iota_j \\
s_1^{(j)} &= \sin\iota_j\sin\vartheta_{j-1} \\
s_2^{(j)} &= -\sin\iota_j\cos\vartheta_{j-1} \\
s_3^{(j)} &= \cos\iota_j \ .
\end{aligned}$$

Then $\mathcal{S}^-$ lets $P_{\mathcal{P}}^{(j)}$ and $\mathrm{S}_{\mathcal{P}}^{(j)}$ respectively, into

$$(P_{\mathcal{P}}^{(j)})^- := \mathcal{R}_2^- P_{\mathcal{P}}^{(j)} \quad \text{and} \quad (\mathrm{S}_{\mathcal{P}}^{(j)})^- := -\mathcal{R}_2^- \mathrm{S}_{\mathcal{P}}^{(j)} \ .$$

Therefore, $\mathrm{C}_{\mathcal{P}}^{(j)} = \mathrm{S}_{\mathcal{P}}^{(j)} - \mathrm{S}_{\mathcal{P}}^{(j+1)}$ (with $\mathrm{S}_{\mathcal{P}}^{(n+1)} := 0$) and $Q_{\mathcal{P}}^{(j)} = \frac{\mathrm{C}_{\mathcal{P}}^{(j)}}{\|\mathrm{C}_{\mathcal{P}}^{(j)}\|} \times P_{\mathcal{P}}^{(j)}$ are transformed, respectively, into

$$(\mathrm{C}_{\mathcal{P}}^{(j)})^- := -\mathcal{R}_2^- \mathrm{C}_{\mathcal{P}}^{(j)} \ , \qquad (Q_{\mathcal{P}}^{(j)})^- := \mathcal{R}_2^- Q_{\mathcal{P}}^{(j)} \ .$$

On the other hand, $a_{j,\mathcal{P}}$ and $e_{j,\mathcal{P}}$ are left unvaried by $\mathcal{S}^-$. In view of Definition 2.2 and Definition 2.3, the thesis (32) follows. ∎

3 The $\mathcal{P}$–map and the planetary problem

After the reduction of the invariance by translations, a Hamiltonian governing the motions of n planets with masses $\mu m_1, \cdots, \mu m_n$ interacting among themselves and with a star with mass m_0 can be taken to be the "heliocentric" one

$$\mathrm{H}_{\mathrm{hel}} := \sum_{1\le i\le n} \Big(\frac{\|y^{(i)}\|^2}{2\mathfrak{m}_i} - \frac{\mathfrak{m}_i \mathfrak{M}_i}{\|x^{(i)}\|} \Big) + \mu \sum_{1\le i<j\le n} \big(\frac{y^{(i)}\cdot y^{(j)}}{m_0} - \frac{m_i m_j}{\|x^{(i)} - x^{(j)}\|} \big) \tag{33}$$

where $(y,\, x) = (y^{(1)}, \cdots, y^{(n)},\, x^{(1)}, \cdots, x^{(n)})$ are "Cartesian coordinates" taking values on the "collision–less" phase space $\mathbb{R}^{3n} \times \mathbb{R}^{3n} \setminus \Delta$, where

$$\Delta = \Big\{ x = (x^{(1)}, \cdots, x^{(n)}) \in \mathbb{R}^3 \times \cdots \times \mathbb{R}^3 : \quad 0 \neq x^{(i)} \neq x^{(j)} \quad \forall\ 1 \le i < j \le n \Big\}$$

endowed with the standard 2– form

$$\Omega := dy \wedge dx := \sum_{i=1}^{n} \sum_{j=1}^{3} dy_j^{(i)} \wedge dx_j^{(i)}$$

and with

$$\mathfrak{M}_i = m_0 + \mu m_i \qquad \mathfrak{m}_i = \frac{m_0 m_i}{m_0 + \mu m_i} \tag{34}$$

being the so–called "reduced masses".

In the following Section 3.1 we describe a general property of Kepler maps, in relation to their application to the Hamiltonian $\mathrm{H}_{\mathrm{hel}}$. Then (in Section 3.2) we shall specialize to the case of the $\mathcal{P}$–map.

3.1 A general property of Kepler maps

For a general Kepler map $\mathcal{K}$, we denote

$$\mathrm{H}_{\mathcal{K}}(\mathrm{K}) := \mathrm{H}_{\mathrm{hel}} \circ \mathcal{K} = -\sum_{j=1}^{n} \frac{\mathfrak{m}_j \mathfrak{M}_j}{2a_{j,\mathcal{K}}(\mathrm{X}_{\mathcal{K}})} + \mu f_{\mathcal{K}}(\mathrm{K}) \ ,$$

where

$$f_{\mathcal{K}}(\mathrm{K}) := \sum_{1 \le i < j \le n} \left(\frac{y_{\mathcal{K}}^{(i)} \cdot y_{\mathcal{K}}^{(j)}}{m_0} - \frac{m_i m_j}{\| x_{\mathcal{K}}^{(i)} - x_{\mathcal{K}}^{(j)} \|} \right)$$

and $y_{\mathcal{K}}^{(j)}$, $x_{\mathcal{K}}^{(j)}$ are as in Definition 2.2.

We denote as

$$\overline{f_{\mathcal{K}}}(\mathrm{X}_{\mathcal{K}}) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_{\mathcal{K}}(\mathrm{X}_{\mathcal{K}}, \ell) d\ell \ , \tag{35}$$

so that

$$f_{\mathcal{K}} = \sum_{1 \le i < j \le n} f_{\mathcal{K}}^{ij} \ , \qquad \overline{f_{\mathcal{K}}} = \sum_{1 \le i < j \le n} \overline{f_{\mathcal{K}}^{ij}}$$

$$f_{\mathcal{K}}^{ij} := \frac{y_{\mathcal{K}}^{(i)} \cdot y_{\mathcal{K}}^{(j)}}{m_0} - \frac{m_i m_j}{\| x_{\mathcal{K}}^{(i)} - x_{\mathcal{K}}^{(j)} \|} \ , \quad \overline{f_{\mathcal{K}}^{ij}} := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_{\mathcal{K}} d\ell_1 \cdots d\ell_n \ .$$

For a general Kepler map, one always has, as a consequence of (6),

$$\begin{aligned} -\frac{1}{2\pi} \int_{\mathbb{T}} T_{\mathcal{K}}^{(j)} d\ell_j &= \frac{1}{2\pi} \int_{\mathbb{T}} \frac{V_{\mathcal{K}}^{(j)}}{2} d\ell_j = T_{\mathcal{K}}^{(j)} + V_{\mathcal{K}}^{(j)} = -\frac{\mathfrak{m}\mathfrak{M}}{2a_{j,\mathcal{K}}} \\ \frac{1}{2\pi} \int_{\mathbb{T}} y_{\mathcal{K}}^{(j)} d\ell_j &= 0 \qquad \frac{1}{2\pi} \int_{\mathbb{T}} \frac{x_{\mathcal{K}}^{(j)}}{\| x_{\mathcal{K}}^{(j)} \|^3} d\ell_j = 0 \ , \end{aligned} \tag{36}$$

where we have denoted as

$$T_{\mathcal{K}}^{(j)} := \frac{\| y_{\mathcal{K}}^{(j)} \|^2}{2\mathfrak{m}_j} \qquad V_{\mathcal{K}}^{(j)} := -\frac{\mathfrak{m}_j \mathfrak{M}_j}{\| x_{\mathcal{K}}^{(j)} \|}$$

the kinetic, potential part of $\mathrm{H}_{\mathcal{K}}^{(j)}$ in (4), respectively.

Consider the average $\overline{f_{\mathcal{K}}}(\mathrm{X}_{\mathcal{K}})$ in (35). Due to the fact that $y^{(j)}_{\mathcal{K}}$ has zero–average, one has that only the Newtonian part contributes to $\overline{f_{\mathcal{K}}}(\mathrm{X}_{\mathcal{K}})$:

$$\overline{f_{\mathcal{K}}} = -\sum_{1\leq i<j\leq n} \frac{m_i m_j}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{d\ell_i d\ell_j}{\|x^{(i)}_{\mathcal{K}} - x^{(j)}_{\mathcal{K}}\|} \ .$$

We now consider any of the contributions to this sum

$$\overline{f^{ij}_{\mathcal{K}}} = -\frac{m_i m_j}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{d\ell_i d\ell_j}{\|x^{(i)}_{\mathcal{K}} - x^{(j)}_{\mathcal{K}}\|} \qquad 1\leq i<j\leq n$$

and expand any of such terms

$$\overline{f^{ij}_{\mathcal{K}}} = \overline{f^{ij}_{\mathcal{K}}}^{(0)} + \overline{f^{ij}_{\mathcal{K}}}^{(1)} + \overline{f^{ij}_{\mathcal{K}}}^{(2)} + \cdots$$

where

$$\overline{f^{ij}_{\mathcal{K}}}^{(h)} := -\frac{m_i m_j}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{1}{h!}\frac{d^h}{d\varepsilon^h}\frac{1}{\|\varepsilon x^{(i)}_{\mathcal{K}} - x^{(j)}_{\mathcal{K}}\|}\Big|_{\varepsilon=0} d\ell_i d\ell_j$$

is proportional to $\frac{1}{a_j}(\frac{a_i}{a_j})^h$. Then the formulae in (36) imply that the two first terms of this expansion are given by

$$\overline{f^{ij}_{\mathcal{K}}}^{(0)} = -\frac{m_i m_j}{a_{j,\mathcal{K}}} \ , \qquad \overline{f^{ij}_{\mathcal{K}}}^{(1)} = 0 \ .$$

Namely, whatever is the Kepler map that is used, the first term that depends on the secular coordinates $\mathrm{X}_{\mathcal{K}}$ is the double average of the second order term

$$\overline{f^{ij}_{\mathcal{K}}}^{(2)}(\mathrm{X}_{\mathcal{K}}) = -\frac{m_i m_j}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{3(x^{(i)}_{\mathcal{K}}\cdot x^{(j)}_{\mathcal{K}})^2 - \|x^{(i)}_{\mathcal{K}}\|^2\|x^{(j)}_{\mathcal{K}}\|^2}{\|x^{(j)}_{\mathcal{K}}\|^5} d\ell_i d\ell_j \ .$$

Now we specialize to the case of the $\mathcal{P}$–map.

3.2 The case of the $\mathcal{P}$–map

We denote as

$$\mathrm{H}_{\mathcal{P}}(\mathrm{X}_{\mathcal{P}},\ell) = \mathrm{h}^0_{\mathfrak{fast}}(\Lambda) + \mu f_{\mathcal{P}}(\mathrm{X}_{\mathcal{P}},\ell) \qquad \mathrm{X}_{\mathcal{P}} := (\Theta,\chi,\Lambda,\vartheta,\kappa) \tag{37}$$

where

$$\mathrm{h}^0_{\mathfrak{fast}}(\Lambda) := -\sum_{j=1}^{n}\frac{\mathfrak{m}_j^3\mathfrak{M}_j^2}{2\Lambda_j^2} \ , \tag{38}$$

the Hamiltonian (33) expressed in $\mathcal{P}$–coordinates.
Using the definitions, it not difficult to see that

Lemma 3.1 *$\overline{f^{ij}_{\mathcal{P}}}$, $f^{ij}_{\mathcal{P}}$ depend, respectively, only on the coordinates*

$$\begin{aligned}
\overline{\mathrm{X}^{ij}_{\mathcal{P}}} \ &:= \ \big(\Theta_i \ , \ \cdots \ , \ \Theta_{j\wedge(n-1)} \ , \ \chi_{i-1} \ , \ \cdots \ , \ \chi_{j\wedge(n-1)} \ , \ \Lambda_i \ , \ \Lambda_j \ , \\
&\qquad \vartheta_i \ , \ \cdots \ , \ \vartheta_{j\wedge(n-1)} \ , \ \kappa_i \ , \ \cdots \ , \ \kappa_{j-1}\big) \\
\overline{\mathrm{P}^{ij}} \ &:= \ (\overline{\mathrm{X}^{ij}_{\mathcal{P}}},\ell_i,\ell_j)
\end{aligned}$$

with $a\wedge b$ denoting the minimum of a and b.

Accordingly to the previous lemma, the "nearest–neighbor" terms $\overline{f_{\mathcal{P}}^{i,i+1}}$, with $i = 1, \cdots, n-1$, depend only on

$$\overline{\mathrm{X}_{\mathcal{P}}^{i,i+1}} = \begin{cases} \left(\Theta_i,\ \Theta_{i+1},\ \chi_{i-1},\ \chi_i,\ \chi_{i+1},\ \Lambda_i,\ \Lambda_{i+1},\ \vartheta_i,\ \vartheta_{i+1},\ \kappa_i\right) & n \geq 3\ \&\ i = 1,\ \cdots,\ n-2 \\ \left(\Theta_{n-1},\ \chi_{n-2},\ \chi_{n-1},\ \Lambda_{n-1},\ \Lambda_n\ ,\ \vartheta_{n-1},\ \kappa_{n-1}\right) & i = n-1\ . \end{cases}$$

However, for the terms $\overline{f_{\mathcal{P}}^{i,i+1}}^{(2)}$, we have a special rule. Indeed, Harrington [19], using Jacobi reduction of the nodes, observed that, in the three–body case, the Euclidean length $\|\mathrm{C}^{(2)}\|$ of the angular momentum of the exterior planets is an integral for the averaged sencond order term of the perturbation $\overline{f_{\mathcal{J}ac}^{12}}^{(2)}$. Therefore, using $\mathcal{P}$–coordinates, we shall have that $\|\mathrm{C}_{\mathcal{P}}^{(i+1)}\|$ is an integral for $\overline{f_{\mathcal{P}}^{i,i+1}}^{(2)}$, for $i = 1, \cdots, n-1$. But since

$$\|\mathrm{C}_{\mathcal{P}}^{(n)}\| = \chi_{n-1}\ , \qquad \|\mathrm{C}_{\mathcal{P}}^{(i+1)}\|\Big|_{(\Theta_{i+1},\vartheta_{i+1})=(0,\pi)} = \chi_i - \chi_{i+1} \qquad i = 1,\ \cdots,\ n-2,\ n \geq 3\ ,$$

and, in the latter case, $\overline{f_{\mathcal{P}}^{i,i+1}}^{(2)}$ does not depend on κ_{i+1}, then,

Lemma 3.2 *$\overline{f_{\mathcal{P}}^{n-1,n}}^{(2)}$ does not depend on κ_{n-1} and , for $n \geq 3$ and $1 \leq i \leq n-2$, the functions*

$$\overline{\overline{f_{\mathcal{P}}^{i,i+1}}}^{(2)} := \overline{f_{\mathcal{P}}^{i,i+1}}^{(2)}\Big|_{(\Theta_{i+1},\vartheta_{i+1})=(0,\pi)}$$

do not depend on κ_i.

In the $\mathcal{P}$–coordinates integrability of $\overline{f_{\mathcal{P}}^{n-1,n}}^{(2)}$ is exhibited via the dependence on only one angle, ϑ_{n-1}, which, together with its conjugated action, Θ_{n-1}, provides the an equilibrium point at $(0,\pi)$ for $\overline{f_{\mathcal{P}}^{n-1,n}}^{(2)}$. A similar fact holds for $\overline{\overline{f_{\mathcal{P}}^{i,i+1}}}^{(2)}$, for $i = 1, \cdots, n-2$. Indeed, we have:

Lemma 3.3 *The function $\overline{f_{\mathcal{P}}^{n-1,n}}^{(2)}$ and , for $n \geq 3$ and $1 \leq i \leq n-2$, the functions $\overline{\overline{f_{\mathcal{P}}^{i,i+1}}}^{(2)}$ have the following expressions*

$$\begin{aligned} \overline{f_{\mathcal{P}}^{n-1,n}}^{(2)} &= m_{n-1}m_n\frac{a_{n-1}^2}{4a_n^3}\frac{\Lambda_n^3}{\chi_{n-1}^5}\Big[\frac{5}{2}(3\Theta_{n-1}^2 - \chi_{n-1}^2) \\ &- \frac{3}{2}\frac{4\Theta_{n-1}^2 - \chi_{n-1}^2}{\Lambda_{n-1}^2}\Big(\chi_{n-2}^2 + \chi_{n-1}^2 - 2\Theta_{n-1}^2 + 2\sqrt{(\chi_{n-1}^2 - \Theta_{n-1}^2)(\chi_{n-2}^2 - \Theta_{n-1}^2)}\cos\vartheta_{n-1}\Big) \\ &+ \frac{3}{2}\frac{(\chi_{n-1}^2 - \Theta_{n-1}^2)(\chi_{n-2}^2 - \Theta_{n-1}^2)}{\Lambda_{n-1}^2}\sin^2\vartheta_{n-1}\Big] \end{aligned} \tag{39}$$

and

$$\begin{aligned} \overline{\overline{f_{\mathcal{P}}^{i,i+1}}}^{(2)} &= m_i m_{i+1}\frac{a_i^2}{4a_{i+1}^3}\frac{\Lambda_{i+1}^3}{\chi_i^2(\chi_i - \chi_{i+1})^3}\Big[\frac{5}{2}(3\Theta_i^2 - \chi_i^2) \\ &- \frac{3}{2}\frac{4\Theta_i^2 - \chi_i^2}{\Lambda_i^2}\Big(\chi_{i-1}^2 + \chi_i^2 - 2\Theta_i^2 + 2\sqrt{(\chi_i^2 - \Theta_i^2)(\chi_{i-1}^2 - \Theta_i^2)}\cos\vartheta_i\Big) \\ &+ \frac{3}{2}\frac{(\chi_i^2 - \Theta_i^2)(\chi_{i-1}^2 - \Theta_i^2)}{\Lambda_i^2}\sin^2\vartheta_i\Big]\ . \end{aligned} \tag{40}$$

We note that the formula in (40) holds also for complex values of the coordinates, provided $\arg(\chi_i - \chi_{i+1}) \in (-\frac{\pi}{2},\ \frac{\pi}{2}] \bmod 2\pi$.
Lemma 3.3 is proved in Appendix B. In the next sections, we shall use it in the following form.

Lemma 3.4 *It is possible to find complex domains $\overline{\mathfrak{B}_i}$ with non–empty real part and a canonical, real–analytic change of coordinates*

$$\overline{\phi^i_{\mathtt{int}}}: \qquad (p_i, q_i, \mathrm{y}^*_i, \mathrm{x}^*_i) \in \overline{\mathfrak{B}_i} \to (\Theta_i, \vartheta_i, \mathrm{y}_i, \mathrm{x}_i) \tag{41}$$

where

$$\begin{aligned}
\mathrm{y}^*_i &:= \begin{cases} (\chi^*_{n-2}, \chi^*_{n-1}, \Lambda^*_{n-1}, \Lambda^*_n) & i = n-1 \\ (\chi^*_{i-1}, \chi^*_i, \chi^*_{i+1}, \Lambda^*_i, \Lambda^*_{i+1}) & i = 1, \cdots, n-2 \ \&\ n \geq 3 \end{cases} \\
\mathrm{x}^*_i &:= \begin{cases} (\kappa^*_{n-2}, \kappa^*_{n-1}, \ell^*_{n-1}, \ell^*_n) & i = n-1 \\ (\kappa^*_{i-1}, \kappa^*_i, \kappa^*_{i+1}, \ell^*_i, \ell^*_{i+1}) & i = 1, \cdots, n-2 \ \&\ n \geq 3 \end{cases} \\
\mathrm{y}_i &:= \begin{cases} (\chi_{n-2}, \chi_{n-1}, \Lambda_{n-1}, \Lambda_n) & i = n-1 \\ (\chi_{i-1}, \chi_i, \chi_{i+1}, \Lambda_i, \Lambda_{i+1}) & i = 1, \cdots, n-2 \ \&\ n \geq 3 \end{cases} \\
\mathrm{x}_i &:= \begin{cases} (\kappa_{n-2}, \kappa_{n-1}, \ell_{n-1}, \ell_n) & i = n-1 \\ (\kappa_{i-1}, \kappa_i, \kappa_{i+1}, \ell_i, \ell_{i+1}) & i = 1, \cdots, n-2 \ \&\ n \geq 3 \end{cases}
\end{aligned} \tag{42}$$

such that

$$\overline{\mathrm{h}^i_{\mathtt{sec}}} := \begin{cases} \overline{f^{n-1,n}_{\mathcal{P}}}^{(2)} \circ \overline{\phi^{n-1}_{\mathtt{int}}} & i = n-1 \\ \overline{\overline{f^{i,i+1}_{\mathcal{P}}}}^{(2)} \circ \overline{\phi^i_{\mathtt{int}}} & i = 1, \cdots, n-2 \ \ \& \ \ n \geq 3 \end{cases} \tag{43}$$

depends only on

$$\mathrm{Y}^*_i := \begin{cases} (\dfrac{p^2_{n-1} + q^2_{n-1}}{2}, \Lambda^*_{n-1}, \Lambda^*_n, \chi^*_{n-2}, \chi^*_{n-1}) & i = n-1 \\ (\dfrac{p^2_i + q^2_i}{2}, \Lambda^*_i, \Lambda^*_{i+1}, \chi^*_{i-1}, \chi^*_i, \chi^*_{i+1}) & i = 1, \cdots, n-2 \ \&\ n \geq 3\ . \end{cases}$$

The transformation $\overline{\phi^i_{\mathtt{int}}}$ may be chosen so as to verify

$$\begin{aligned}
&\mathrm{y}^*_i = \mathrm{y}_i\ , \quad (\Theta_i,\ \vartheta_i\ ,\ \mathrm{x}_i - \mathrm{x}^*_i) = \mathcal{F}_i(p_i, q_i, \mathrm{y}^*_i) \\
&\overline{\phi^i_{\mathtt{int}}}(-p_i, -q_i, \mathrm{y}^*_i, \mathrm{x}^*_i) = (-\Theta_i, -\vartheta_i, \mathrm{y}_i, \mathrm{x}_i)
\end{aligned} \tag{44}$$

if

$$\overline{\phi^i_{\mathtt{int}}}(p_i, q_i, \mathrm{y}^*_i, \mathrm{x}^*_i) = (\Theta_i, \vartheta_i, \mathrm{y}_i, \mathrm{x}_i)\ .$$

Lemma 3.4 is proved in the following Section 5.2.1.

4 Global Kolmogorov tori in the planetary problem

In this section we show how the $\mathcal{P}$–map can be used to prove Theorem A. We defer to the next Section 5 more technical parts.

4.1 A domain of holomorphy

A typical practice, in order to use perturbation theory techniques, is to extend Hamiltonians governing dynamical systems to the complex field, and then to study their holomorphy properties. In this section we aim to discuss a domain of holomorphy for the perturbing function $f_{\mathcal{P}}$ in (37), regarded as a function of complex coordinates. We shall choose it of the following form

$$\mathfrak{D}_{\mathcal{P}} := \mathcal{T}_{\Theta^+,\vartheta^+} \times \left(\mathcal{X}_\theta \times \overline{\mathbb{T}}_s^n\right) \times \left(\mathcal{A}_\theta \times \overline{\mathbb{T}}_s^n\right) ,$$

where, for given positive numbers

$$\Theta_j^+ \ , \quad \vartheta_j^+ \ , \quad \mathrm{G}_i^\pm \ , \quad \Lambda_i^\pm \ , \quad \theta_i \ , \quad s$$

with $i = 1, \cdots, n$, $j = 1, \cdots, n-1$,

$$\begin{aligned}
\mathcal{T}_{\Theta^+,\vartheta^+} &:= \Big\{(\overline{\Theta}, \overline{\vartheta}) = (\Theta_1, \cdots, \Theta_{n-1}, \vartheta_1, \cdots, \vartheta_{n-1}) \in \mathbb{C}^{n-1} \times \mathbb{T}_{\mathbb{C}}^{n-1} : \\
&\quad |\vartheta_j - \pi| \le \vartheta_j^+ \ , \quad |\Theta_j| \le \Theta_j^+ \ , \quad \forall\ j = 1, \cdots, n-1\Big\} \\
\mathcal{X}_\theta &:= \Big\{\chi = (\chi_0, \cdots, \chi_{n-1}) \in \mathbb{C}^n : \ \mathrm{G}_j^- \le |\chi_{j-1} - \chi_j| \le \mathrm{G}_j^+ \ , \quad |\operatorname{Im}(\chi_{j-1} - \chi_j)| \le \theta_j \\
&\quad \forall\ j = 1, \cdots, n\Big\} \\
\mathcal{A}_\theta &:= \Big\{\Lambda = (\Lambda_1, \cdots, \Lambda_n) \in \mathbb{C}^n : \quad \Lambda_j^- \le |\Lambda_j| \le \Lambda_j^+ \ , \quad |\operatorname{Im}\Lambda_j| \le \theta_j \\
&\quad \forall\ j = 1, \cdots, n\Big\} \\
\overline{\mathbb{T}}_s &:= \mathbb{T} + \mathrm{i}[-s, s]
\end{aligned} \tag{45}$$

with $\chi_n := 0$.

The domain $\mathfrak{D}_{\mathcal{P}}$ will be determined as the intersection of the "collision–less" set, where, as functions of complex variables, the mutual distances of the planets

$$d_{j,\mathcal{P}} := \|x_{\mathcal{P}}^{(j)} - x_{\mathcal{P}}^{(j+1)}\|$$

are far away from zero, with with the holomorphy domain of $\mathcal{P}$, where, again as as functions of complex variables, the absolute values $|e_{j,\mathcal{P}}|$ of eccentricities in (18) are bounded away from 0 and 1, those of the inclinations $|\iota_j|$, $|\mathrm{i}_j|$ in (14) are away from 0 and, finally, *Kepler equation* (2) provides a holomorphic solution.

The latter issue is not a peculiarity of this problem, since it naturally arises in the context of the two–body problem's equations. In the early XX century, T. Levi Civita [24] studied the holomorphy of the solution of Kepler's Equation with respect to the eccentricity. The holomorphy with respect to the mean anomaly has been investigated, using similar arguments as in [24], in [4]. Here, we address the problem of determining the holomorphy with respect to both the arguments.

Proposition 4.1 *Let $\widehat{e} = 0,6627...$ be the solution of*

$$0 \le \rho \le 1 \quad \& \quad \frac{\rho\, e^{\sqrt{1+\rho^2}}}{1 + \sqrt{1+\rho^2}} = 1 \ . \tag{46}$$

Then one can find a positive number $\bar{\ell}$ depending on $\overline{e}$ such that, for any $\mathrm{e} = \mathrm{e}_1 + \mathrm{i}\mathrm{e}_2 \in \mathbb{C}$, with $|\mathrm{e}| \le \overline{e}$, the complex Kepler's equation

$$\zeta - \mathrm{e}\sin\zeta = \ell$$

has a unique solution $\zeta(\ell, \mathrm{e})$ which turns out to be real–analytic for $\ell \in \overline{\mathbb{T}}_{\bar{\ell}}$.

The following result completes the study of the holomorphy of $f_{\mathcal{P}}$.

Proposition 4.2 *Let $\widehat{e}$ be as in Proposition 4.1. For any given $\underline{e}_i$, $\overline{e}_i$, with*

$$0 < \underline{e}_i < \overline{e}_i < \widehat{e} \qquad i = 1, \cdots, n$$

it is possible to find positive numbers

$$\mathcal{A}_j \ , \quad \mathcal{B}_j \ , \quad \overline{\mathcal{C}}_i > \underline{\mathcal{C}}_i \ , \quad \bar{d}_j \ , \quad s \in (0,1) \ , \quad \sigma \in (0,1)$$

such that, if the following inequalities are satisfied

$$\begin{aligned}
&\underline{\mathcal{C}}_i \Lambda_i^+ < \mathrm{G}_i^- < \mathrm{G}_i^+ < \overline{\mathcal{C}}_i \Lambda_i^- \ ; \\
&\max\Big\{ \frac{\theta_i}{\Lambda_i^-} \ , \quad \frac{\theta_i}{\mathrm{G}_i^-} \ , \quad \sum_{i=1}^{n-1} |\sin^{-1}(\frac{\mathrm{G}_i^+}{\mathrm{G}_{i+1}^-})| \ , \quad \frac{\Theta_j^+}{\mathrm{G}_n^-} \ , \quad \sum_{i=1}^{n-1} \frac{\mathrm{G}_i^+}{\mathrm{G}_n^-} \ , \quad \vartheta_j^+ \\
&|\operatorname{Im} \kappa_j| \ , \quad |\operatorname{Im} \ell_i| \Big\} \le s \\
&\vartheta_j^+ \le \min\Big\{ \frac{\mathcal{A}}{\mathrm{G}_n^+} \sqrt{(\mathrm{G}_j^-)^2 - (\underline{\mathcal{C}}_j \Lambda_j^+)^2} \ , \quad \frac{\mathcal{B}}{\mathrm{G}_n^+} \sqrt{(\overline{\mathcal{C}}_j \Lambda_j^-)^2 - (\mathrm{G}_j^+)^2} \Big\} \ ,
\end{aligned} \tag{47}$$

then the eccentricities $e_{i,\mathcal{P}}$, inclinations ι_i, i_i and the mutual distances $d_{i,\mathcal{P}}$ verify

$$\underline{e}_i \le |e_{i,\mathcal{P}}| \le \overline{e}_i \ , \quad \max_{i,j} \Big\{ |\cos \iota_i|, \ |\cos \mathrm{i}_j| \Big\} \le \sigma \ , \quad |d_{i,\mathcal{P}}| \ge \bar{d} \tag{48}$$

Propositions 4.1 and Proposition 4.2 are proved in Appendix A.1 and A.2, respectively. We shall use them in the form below. We remark that the super–exponential decay of the semi–major axes ratio will be used only in Section 5.2 below.

Corollary 4.1 (choice of parameters) *Fix $\underline{e}_i < \overline{e}_i$, $c \in (0,1)$. Let $\underline{\mathcal{C}}_i < \underline{\mathcal{C}}_i^* < \overline{\mathcal{C}}_i^* < \overline{\mathcal{C}}_i$, $\mathcal{D}_i := \min\{\mathcal{A}\sqrt{(\underline{\mathcal{C}}_i^*)^2 - (\underline{\mathcal{C}}_i)^2}, \mathcal{B}\sqrt{(\overline{\mathcal{C}}_i)^2 - (\overline{\mathcal{C}}_i^*)^2}\}$, $\mathcal{D} := \min_{1 \le j \le n-1} \frac{\mathcal{D}_j}{\overline{\mathcal{C}}_n^*} \frac{\mathfrak{m}_j \sqrt{\mathfrak{M}_j}}{\mathfrak{m}_n \sqrt{\mathfrak{M}_n}}$, $\alpha < \frac{s}{\mathcal{D}}$. Define, for $i = 1, \cdots, n$ and $j = 1, \cdots, n-1$,*

$$\begin{aligned}
\Lambda_i^\pm &:= \mathfrak{m}_i \sqrt{\mathfrak{M}_i a_i^\pm} \ , \quad \mathrm{G}_i^+ := \overline{\mathcal{C}}_i^* \Lambda_i^- \ , \qquad \mathrm{G}_i^- := \underline{\mathcal{C}}_i^* \Lambda_i^+ \ , \quad \Theta_j^+ := s \mathrm{G}_n^- \ , \qquad \vartheta_j^+ := \mathcal{D}_i \frac{\Lambda_i^-}{\mathrm{G}_n^+} \\
\theta_i &:= s \sqrt{\Lambda_i^-}
\end{aligned} \tag{49}$$

where $a_i^\pm$ is as in $(*)$. Then, $f_{\mathcal{P}}$ is real–analytic in the domain $\mathfrak{D}_{\mathcal{P}}$.

4.2 A normal form for the planetary problem

Definition 4.1 ([2]) Given m, ν_1, $\cdots$, $\nu_m \in \mathbb{N}$, $\nu := \nu_1 + \cdots + \nu_m$, let

$$\mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \mathfrak{L}_2 \supset \cdots \supset \mathfrak{L}_m = \{0\}$$

be a decreasing sequence of sub–lattices of $\mathbb{Z}^\nu$ defined by

$$\mathfrak{L}_0 := \mathbb{Z}^\nu \ , \quad \mathfrak{L}_i := \Big\{ k = (k_1, \cdots, k_m) \in \mathbb{Z}^\nu \ , \quad k_j \in \mathbb{Z}^{\nu_j} : \quad k_1 = \cdots = k_i = 0 \Big\} \tag{50}$$

with $i = 1, \cdots, n$. Next, given γ, γ_1, $\cdots$, γ_m, $\tau \in \mathbb{R}_+$, we define the set $\mathcal{D}^\nu_{\gamma_1 \cdots \gamma_m ; \tau}$ of the $(\gamma_1 \cdots \gamma_m ; \tau)$–*diophantine numbers* via the following formulae

$$\begin{aligned}
\mathcal{D}^{\nu,K,i}_{\gamma;\tau} &:= \Big\{ \omega \in \mathbb{R}^\nu : \ |\omega \cdot k| \ge \frac{\gamma}{|k|^\tau} \quad \forall k \in \mathfrak{L}_{i-1} \setminus \mathfrak{L}_i, \ |k|_1 \le K \Big\} \\
\mathcal{D}^{\nu,K}_{\gamma_1 \cdots \gamma_m ; \tau} &:= \bigcap_{i=1}^m \mathcal{D}^{\nu,K,i}_{\gamma_i;\tau} \qquad \mathcal{D}^\nu_{\gamma_1 \cdots \gamma_m ; \tau} := \bigcap_{K \in \mathbb{N}} \mathcal{D}^{\nu,K}_{\gamma_1 \cdots \gamma_m ; \tau} \ .
\end{aligned}$$

In other words $\omega = (\omega_1, \cdots, \omega_m) \in \mathcal{D}^{\nu}_{\gamma_1 \cdots \gamma_m; \tau}$ if, for any $k = (k_1, \cdots, k_m) \in \mathbb{Z}^{\nu} \setminus \{0\}$, with $k_j \in \mathbb{Z}^{\nu_j}$,

$$|\omega \cdot k| = |\sum_{j=1}^{m} \omega_j \cdot k_j| \geq \begin{cases} \dfrac{\gamma_1}{|k|^{\tau}} \quad \text{if} \quad k_1 \neq 0 \ ; \\ \dfrac{\gamma_2}{|k|^{\tau}} \quad \text{if} \quad k_1 = 0 \ , \quad k_2 \neq 0 \ ; \\ \cdots \\ \dfrac{\gamma_m}{|k_m|^{\tau}} \quad \text{if} \quad k_1 = \cdots = k_{m-1} = 0, \ \cdots, \ k_m \neq 0 \ . \end{cases} \tag{51}$$

Remark 4.1 The choice $m = 1$, $\gamma_1 := \gamma$ gives the usual Diophantine set $\mathcal{D}^{\nu}_{\gamma,\tau}$. The $m = 2$-case, $\mathcal{D}^{\nu}_{\gamma_1,\gamma_2,\tau}$, with $\gamma_1 = \mathrm{O}(1)$ and $\gamma_2 = \mathrm{O}(\mu)$, where μ is the strength of the planetary masses has been considered in [2] for the proof of the Fundamental Theorem, mentioned in the introduction.

The following result in proven in the next Section 5. It is unavoidably detailed.

Proposition 4.3 *Let $\mathfrak{m}_j$, $\mathfrak{M}_j$ be as in (34) and $\mathrm{m}_j := \sum_{i=1}^{j-1} m_i$, with $j = 2, \cdots, n$. There exists a number $\mathfrak{c}$, depending only on n, m_0, $\cdots$, m_n, $a_n^{\pm}$, $\underline{e}_j$, $\overline{e}_j$, and a number $0 < \overline{\mathfrak{c}} < 1$, depending only on n such that, for any fixed positive numbers $\overline{\gamma} < 1 < \bar{K}$, $\alpha > 0$ verifying*

$$\bar{K} \leq \frac{\mathfrak{c}}{\alpha^{3/2}} \tag{52}$$

and

$$\frac{1}{\mathfrak{c}} \max \left\{ \mu (\frac{a_n^+}{a_1^-})^5 \frac{\bar{K}^{2\bar{\tau}+2}}{\bar{\gamma}^2}, \ \frac{\bar{K}^{2(\bar{\tau}+1)}\alpha}{\bar{\gamma}^2} \right\} < 1 \tag{53}$$

there exist natural numbers ν_1, $\cdots$, ν_{2n-1}, with $\sum_j \nu_j = 3n - 2$, positive real numbers $\gamma_1 > \cdots > \gamma_{2n-1}$ ε_1, $\cdots$, ε_{n-1}, $\overline{r}_1$, $\cdots$, $\overline{r}_{n-1}$, $\widetilde{r}_1$, $\cdots$, $\widetilde{r}_n$, open sets $B_j^* \subset B^2_{\varepsilon_j}$, $\mathcal{X}^* \subset \mathcal{X}$, a domain

$$\mathfrak{D}_{\mathrm{n}} := B_{\sqrt{2\overline{r}}} \times \mathcal{X}_{\overline{r}} \times \mathcal{A}_{\widetilde{r}} \times \mathbb{T}^n_{\overline{\mathfrak{c}}s} \times \mathbb{T}^n_{\overline{\mathfrak{c}}s}$$

a sub-domain of the form

$$\mathfrak{D}^*_{\mathrm{n}} := B^*_{\sqrt{2\overline{r}}} \times \mathcal{X}^*_{\overline{r}} \times \mathcal{A}_{\widetilde{r}} \times \mathbb{T}^n_{\overline{\mathfrak{c}}s} \times \mathbb{T}^n_{\overline{\mathfrak{c}}s}$$

verifying

$$\operatorname{meas} \mathfrak{D}^*_{\mathrm{n}} \geq \big(1 - \frac{\bar{\gamma}}{\overline{\mathfrak{c}}}\big) \operatorname{meas} \mathfrak{D}_{\mathrm{n}} \tag{54}$$

a real–analytic transformation

$$\phi_{\mathrm{n}} : \qquad (p, q, \chi, \Lambda, \kappa, \ell) \in \mathfrak{D}^*_{\mathrm{n}} \to \mathfrak{D}_{\mathcal{P}}$$

which conjugates $\mathrm{H}_{\mathcal{P}}$ to

$$\mathrm{H}_{\mathrm{n}}(p, q, \chi, \Lambda, \kappa, \ell) := \mathrm{H}_{\mathcal{P}} \circ \phi_{\mathrm{n}} = \mathrm{h}_{\mathfrak{fast}, \mathfrak{sec}}(p, q, \chi, \Lambda) + \mu \, f_{\mathfrak{exp}}(p, q, \chi, \Lambda, \kappa, \ell)$$

where $f_{\mathfrak{exp}}(p, q, \chi, \Lambda, \kappa, \ell)$ is independent of κ_0, and the following holds.

1. *The function* $\mathrm{h}_{\mathfrak{fast},\mathfrak{sec}}(p,q,\chi,\Lambda)$ *is a sum*

$$\mathrm{h}_{\mathfrak{fast},\mathfrak{sec}}(p,q,\chi,\Lambda)=\mathrm{h}_{\mathfrak{fast}}(\Lambda)+\mu\,\mathrm{h}_{\mathfrak{sec}}(p,q,\chi,\Lambda)$$

where, if

$$\hat{\mathrm{y}}_i:=\big(\frac{p_i^2+q_i^2}{2},\ \cdots,\ \frac{p_{n-1}^2+q_{n-1}^2}{2},\ \chi_{i-1},\ \cdots,\ \chi_{n-1},\ \Lambda_i,\ \cdots,\ \Lambda_n\big)$$

then $\mathrm{h}_{\mathfrak{fast}}$ *and* $\mathrm{h}_{\mathfrak{sec}}$ *are given by*

$$\mathrm{h}_{\mathfrak{fast}}(\Lambda)=-\sum_{j=1}^{n}\frac{\mathfrak{m}_j^3\mathfrak{M}_j^2}{2\Lambda_j^2}-\mu\sum_{j=2}^{n}\frac{\mathfrak{M}_j\mathfrak{m}_j^2 m_j\mathrm{m}_j}{\Lambda_j^2},\qquad \mathrm{h}_{\mathfrak{sec}}(p,q,\chi,\Lambda)=\sum_{i=1}^{n-1}\mathrm{h}_{\mathfrak{sec}}^i(\hat{\mathrm{y}}_i)$$

where the functions $\mathrm{h}_{\mathfrak{sec}}^i$ *have an analytic extension on* $\mathfrak{D}_{\mathrm{n}}$ *and verify*

$$\mathfrak{c}\frac{(a_{n-j}^+)^2}{(a_{n-j+1}^-)^3}\le|\mathrm{h}_{\mathfrak{sec}}^j(\hat{\mathrm{y}}_j)|\le\frac{1}{\mathfrak{c}}\frac{(a_{n-j}^+)^2}{(a_{n-j+1}^-)^3}\ .$$

2. *The function* $f_{\mathfrak{exp}}$ *satisfies*

$$|f_{\mathfrak{exp}}|\le\frac{1}{\mathfrak{c}}\frac{e^{-\mathfrak{c}\bar{K}}}{a_1^-}\ .$$

3. *If* ζ *is* $\hat{\mathrm{y}}_1$ *deprived of* χ_0*, the frequency–map*

$$\zeta\to\omega_{\mathfrak{fast},\mathfrak{sec}}(\zeta):=\partial_\zeta\mathrm{h}_{\mathfrak{fast},\mathfrak{sec}}(\zeta)$$

is a diffeomorphism of $\Pi_\zeta(B^*_{\sqrt{2\bar{\tau}}}\times\mathcal{X}^*_{\bar{\tau}}\times\mathcal{A}^*_{\bar{\tau}})$ *and, moreover, it satisfies* (51)*, with* $m=2n-1$*,* $\tau=\bar{\tau}>2$*, and*

$$
\nu_j \;:=\; \begin{cases} 1 & j=1,\cdots,n \\ 2 & j=3\ ,\ n=2 \\ 3 & j=n+1\ ,\ n\geq 3 \\ 2 & n+2\leq j\leq 2n-2\ ,\ n\geq 4 \\ 1 & j=2n-1\ ,\ n\geq 3 \end{cases}
$$

$$
\omega_j \;:=\; \begin{cases} \partial_{\Lambda_j} \mathrm{h}_{\mathfrak{fast},\mathfrak{sec}} & j=1,\cdots,n \\ \partial_{(\frac{p_1^2+q_1^2}{2},\chi_1)} \mathrm{h}_{\mathfrak{fast},\mathfrak{sec}} & j=3\ ,\ n=2 \\ \partial_{(\frac{p_{n-1}^2+q_{n-1}^2}{2},\chi_{n-2},\chi_{n-1})} \mathrm{h}_{\mathfrak{fast},\mathfrak{sec}} & j=n+1\ ,\ n\geq 3 \\ \partial_{(\frac{p_{2n-j}^2+q_{2n-j}^2}{2},\chi_{2n-j-1})} \mathrm{h}_{\mathfrak{fast},\mathfrak{sec}} & n+2\leq j\leq 2n-2\ ,\ n\geq 4 \\ \partial_{\frac{p_1^2+q_1^2}{2}} \mathrm{h}_{\mathfrak{fast},\mathfrak{sec}} & j=2n-1\ ,\ n\geq 3 \end{cases}
$$

$$
\gamma_j \;:=\; \begin{cases} \dfrac{1}{a_j^-}\dfrac{\overline{\gamma}}{\theta_j} & 1\leq j\leq n \\ \dfrac{\mu(a_{j-n}^+)^2}{(a_{j+1-n}^-)^3}\dfrac{\overline{\gamma}}{\theta_{j-n}} & n+1\leq j\leq 2n-1 \end{cases} \tag{55}
$$

4. *The mentioned constants are*

$$
\varepsilon_j := \mathfrak{c}\sqrt{\theta_j}\ ,\quad \overline{r}_j := \frac{\theta_j\overline{\gamma}}{\bar{K}^{\bar{\tau}+1}}\ ,\quad \widetilde{r}_i := \mathfrak{c}\,\theta_j
$$

with $\bar{\tau}>2$.

4.3 A "multi–scale" KAM Theorem and proof of Theorem A

In this section we state a "multi–scale" KAM Theorem and next we show how this theorem applies to the Hamiltonian $\mathrm{H_n}$ so as to obtain the proof of Theorem A.

Theorem 4.1 (Multi–scale KAM Theorem) *Let* m, ℓ, $\nu_1,\cdots,\nu_m\in\mathbb{N}$, $\nu:=\nu_1+\cdots+\nu_m\geq\ell$, $\tau_*>\nu$, $\gamma_1\geq\cdots\geq\gamma_m>0$, $0<4s\leq\bar{s}<1$, $\rho_1,\cdots,\rho_\ell$, $r_1,\cdots,r_{\nu-\ell}$, $\varepsilon_1,\cdots,\varepsilon_\ell>0$, $B_1,\cdots,B_\ell\subset\mathbb{R}^2$, $D_j:=\{\frac{x^2+y^2}{2}\in\mathbb{R}:\ (x,y)\in B_j\}\subset\mathbb{R}$, $B:=B_1\times\cdots\times B_\ell\subset\mathbb{R}^{2\ell}$, $D:=D_1\times\cdots\times D_\ell\subset\mathbb{R}^\ell$, $C\subset\mathbb{R}^{\nu-\ell}$, $A:=D_\rho\times C_r$. *Let*

$$
\mathrm{H}(p,q,I,\psi)=\mathrm{h}(p,q,I)+f(p,q,I,\psi)
$$

be real–analytic on $B_{\sqrt{2\rho}}\times C_r\times\mathbb{T}^{\nu-\ell}_{\bar{s}+s}$, *where* $\mathrm{h}(p,q,I)$ *depends on* (p,q) *only via*

$$
J(p,q):=\Big(\frac{p_1^2+q_1^2}{2}\ ,\ \cdots\ ,\ \frac{p_\ell^2+q_\ell^2}{2}\Big)\ .
$$

Assume that $\omega_0 := \partial_{(J(p,q),I)}\mathrm{h}$ *is a diffeomorphism of* A *with non singular Hessian matrix* $U := \partial^2_{(J(p,q),I)}\mathrm{h}$ *and let* U_k *denote the* $(\nu_k + \cdots + \nu_m) \times \nu$ *submatrix of* U, i.e., *the matrix with entries* $(U_k)_{ij} = U_{ij}$, *for* $\nu_1 + \cdots + \nu_{k-1} + 1 \le i \le \nu$, $1 \le j \le \nu$, *where* $2 \le k \le m$. *Let*

$$\mathrm{M} \ge \sup_A \|U\| \ , \quad \mathrm{M}_k \ge \sup_A \|U_k\| \ , \quad \bar{\mathrm{M}} \ge \sup_A \|U^{-1}\| \ , \quad E \ge \|f\|_{\rho,\bar{\mathfrak{s}}+s}$$

$$\bar{\mathrm{M}}_k \ge \sup_A \|T_k\| \quad \text{if} \quad U^{-1} = \begin{pmatrix} T_1 \\ \vdots \\ T_m \end{pmatrix} \qquad 1 \le k \le m \ .$$

Define

$$K := \frac{6}{s} \ \log_+ \left(\frac{E\mathrm{M}_1^2 \, L}{\gamma_1^2} \right)^{-1} \quad \text{where} \quad \log_+ a := \max\{1, \log a\}$$

$$\hat\rho_k := \frac{\gamma_k}{3\mathrm{M}_k K^{\tau_*+1}} \ , \quad \hat\rho := \min\left\{\hat\rho_1 \ , \ \cdots \ , \ \hat\rho_m, \ \rho_1 \ , \ \cdots \ , \ \rho_\ell, \ r_1, \ \cdots \ , \ r_{\nu-\ell}\right\}$$

$$L := \max\ \left\{\bar{\mathrm{M}}, \ \mathrm{M}_1^{-1}, \ \cdots \ , \ \mathrm{M}_m^{-1}\right\}$$

$$\hat E := \frac{EL}{\hat\rho^2} \ .$$

Then one can find two numbers $\hat c_\nu > c_\nu$ *depending only on* ν *such that, if the perturbation* f *so small that the following "KAM condition" holds*

$$\hat c_\nu \hat E < 1 \ ,$$

for any $\omega \in \Omega_* := \omega_0(D) \cap \mathcal{D}_{\gamma_1,\cdots,\gamma_m,\tau_*}$, *one can find a unique real–analytic embedding*

$$\begin{aligned} \phi_\omega : \quad \vartheta = (\hat\vartheta, \bar\vartheta) \in \mathbb{T}^\nu \quad &\to \quad (\hat v(\vartheta;\omega), \hat\vartheta + \hat u(\vartheta;\omega), \mathcal{R}_{\bar\vartheta+\bar u(\vartheta;\omega)} w_1, \ \cdots \ , \ \mathcal{R}_{\bar\vartheta+\bar u(\vartheta;\omega)} w_\ell) \\ &\in \ \mathrm{Re}\, C_r \times \mathbb{T}^{\nu-\ell} \times \mathrm{Re}\, B^{2\ell}_{\sqrt{2}r} \end{aligned}$$

where $r := c_\nu \hat E \hat\rho$ *such that* $\mathrm{T}_\omega := \phi_\omega(\mathbb{T}^\nu)$ *is a real–analytic* ν*–dimensional* H*–invariant torus, on which the* H*–flow is analytically conjugated to* $\vartheta \to \vartheta + \omega\, t$. *Furthermore, the map* $(\vartheta;\omega) \to \phi_\omega(\vartheta)$ *is Lipschitz and one–to–one and the invariant set* $\mathrm{K} := \bigcup_{\omega\in\Omega_*} \mathrm{T}_\omega$ *satisfies the following measure estimate*

$$\begin{aligned} &\mathrm{meas}\left(\mathrm{Re}\,(D_r) \times \mathbb{T}^n \setminus \mathrm{K} \right) \\ \le & c_\nu \Big(\mathrm{meas}\,(D \setminus D_{\gamma_1,\cdots,\gamma_m,\tau_*} \times \mathbb{T}^n) + \mathrm{meas}\,(\mathrm{Re}\,(D_r) \setminus D) \times \mathbb{T}^n \Big), \end{aligned}$$

where $D_{\gamma_1,\cdots,\gamma_m,\tau_*}$ *denotes the* ω_0*–pre-image of* $\mathcal{D}_{\gamma_1,\cdots,\gamma_m,\tau_*}$ *in* D. *Finally, on* $\mathbb{T}^\nu \times \Omega_*$, *the following uniform estimates hold*

$$|v_k(\cdot;\omega) - I^0_k(\omega)| \le c_\nu \Big(\frac{\bar{\mathrm{M}}_k}{\bar{\mathrm{M}}} + \frac{\mathrm{M}_k}{\mathrm{M}_1} \Big) \hat E \, \hat\rho$$

$$|u(\cdot;\omega)| \le c_\nu \hat E \, s$$

where v_k *denotes the projection of* $v = (\hat v, \bar v) \in \mathbb{R}^{\nu_1} \times \cdots \times \mathbb{R}^{\nu_m}$ *over* $\mathbb{R}^{\nu_k}$, $\bar v_k := \dfrac{|w_k|^2}{2}$ *and* $I^0(\omega) = (I^0_1(\omega), \cdots, I^0_\nu(\omega)) \in D$ *is the* ω_0*– pre–image of* $\omega \in \Omega_*$.

Theorem 4.1 generalizes [6, Proposition 3] in two respects. The former generalization concerns the fact of considering of $m \ge 2$ scales (in [6] the case $m = 2$ was only treated). The latter consists of taking H depending also on the rectangular variables $(p, q) \in B^{2\ell}$. Such generalizations can be easily obtained, and hence will be not discussed here.

Proof of Theorem A Let

$$\bar\gamma := \overline{\mathfrak{c}}\sqrt{\alpha}(\log\alpha^{-1})^{\bar\tau+1}\ , \quad \bar K = \frac{1}{\widetilde{\mathfrak{c}}}\log\frac{1}{\alpha}$$

where $\overline{\mathfrak{c}}$ is as in (54) and $\widetilde{\mathfrak{c}}$ will be fixed later. We aim apply Theorem 4.1 to the Hamiltonian H_{n} of Proposition 4.3, with these choices of $\bar\gamma$ and $\bar K$. To this end, we take

$$\mathrm{M}_j = \begin{cases} \dfrac{1}{\mathfrak{c}_1 a_j^- \theta_j^2} & 1\le j\le n \\ \dfrac{\mu(a_j^+)^2}{\mathfrak{c}_1(a_{j+1}^-)^3\theta_j^2} & n+1\le j\le 2n-1 \end{cases} \qquad L=\bar{\mathrm{M}} = \frac{1}{\mathfrak{c}_2}\frac{\theta_1^2(a_2^+)^3}{\mu(a_1^-)^2}$$

$$E = \frac{1}{\mathfrak{c}_3}\frac{\mu}{a_1^-}e^{-\mathfrak{c}\bar K} \qquad K = \frac{1}{\mathfrak{c}_4}\log_+\Big(\frac{1}{\overline{\gamma}^2}\frac{(a_2)^3}{(a_1^-)^3}e^{-\mathfrak{c}\bar K}\Big)^{-1}$$

$$\hat\rho_j = \begin{cases} \mathfrak{c}_5\dfrac{\overline{\gamma}\theta_j}{K^{\tau_*+1}} & 1\le j\le n \\ \mathfrak{c}_5\dfrac{\overline{\gamma}\theta_{j-n}}{K^{\tau_*+1}} & n+1\le j\le 2n-1 \end{cases} \qquad \hat\rho := \frac{\theta_1\overline{\gamma}}{\hat K^{\tau_*+1}} \quad \tau_*>3n-2$$

$$\hat E = \frac{1}{\mathfrak{c}_6}\frac{1}{\overline{\gamma}^2}\frac{(a_2)^3}{(a_1^-)^3}e^{-\mathfrak{c}\bar K}\hat K^{2(\tau_*+1)}$$

where $\hat K := \max\{K,\bar K\}$. The number $\frac{1}{\overline{\gamma}^2}\frac{(a_2)^3}{(a_1^-)^3}$ can be bounded by $\frac{1}{\alpha^N}$ for a sufficiently large N depending only on n. Hence, if $\widetilde{\mathfrak{c}}<\frac{\mathfrak{c}}{N}$ and $\alpha<\mathfrak{c}_6$, we have $\hat E<1$ and the theorem is proved. ∎

5 Proofs

In this section we provide the proof of Proposition 4.3. This is divided in two steps: normalization of fast angles and of secular coordinates.

5.1 Normalization of fast angles

Let $\overline{f_{\mathcal{P}}^{ij}}$, $\overline{f_{\mathcal{P}}^{ij}}^{(k)}$ as in Lemma 3.1, and let

$$\overline{f_{\mathcal{P}}^{ij}}^{(\ge 2)} := \overline{f_{\mathcal{P}}^{ij}} - \overline{f_{\mathcal{P}}^{ij}}^{(0)}\ . \tag{56}$$

Proposition 5.1 *There exist two small numbers $\widehat{\mathfrak{c}}$, $\mathfrak{c}_1$, where $\widehat{\mathfrak{c}}$ depends only on n, while $\mathfrak{c}_1$ depends only on n, $m_1,\cdots,m_n$, such that, if the inequality in (52) and*

$$\frac{1}{\mathfrak{c}}\mu\bar K\Big(\frac{a_n^+}{a_1^-}\Big)^{\frac{3}{2}}<1 \tag{57}$$

hold, one can find a real–analytic and symplectic transformation

$$\phi_{\mathfrak{fast}}: \qquad (\overline{\Theta},\overline{\vartheta},\chi,\Lambda,\kappa,\ell)\in\mathfrak{D}_{\mathfrak{fast}} := \mathcal{T}_{\widehat{\mathfrak{c}}\Theta^+,\widehat{\mathfrak{c}}\vartheta^+}\times\mathcal{X}_{\widehat{\mathfrak{c}}\theta}\times\mathcal{A}_{\widehat{\mathfrak{c}}\theta}\times\mathbb{T}^n_{\widehat{\mathfrak{c}}s}\times\mathbb{T}^n_{\widehat{\mathfrak{c}}s}\to\mathfrak{D}_{\mathcal{P}}$$

which conjugates $\mathrm{H}_{\mathcal{P}}$ to

$$\mathrm{H}_{\mathfrak{fast},\mathfrak{exp}}(\overline{\Theta},\chi,\Lambda,\overline{\vartheta},\kappa,\ell) := \mathrm{H}_{\mathcal{P}}\circ\phi_{\mathfrak{fast}} = \mathrm{h}_{\mathfrak{fast}}(\Lambda)+\mu\, f_{\mathfrak{fast}}(\overline{\Theta},\chi,\Lambda,\overline{\vartheta},\kappa)+\mu\, f_{\mathfrak{fast},\mathfrak{exp}}(\overline{\Theta},\chi,\Lambda,\overline{\vartheta},\kappa,\ell) \tag{58}$$

where $\mathrm{h}_{\mathfrak{fast}}$ *is as in Proposition 4.3, and*

$$f_{\mathfrak{fast}} \;:=\; \sum_{i=1}^{n-1} f^i_{\mathfrak{fast}}\ , \qquad f_{\mathfrak{fast},\mathfrak{exp}} := \sum_{i=1}^{n-1} f^i_{\mathfrak{fast},\mathfrak{exp}}\ . \tag{59}$$

Here,

1. *The "fast frequency–map"*

$$\omega_{\mathfrak{fast}} := \partial \mathrm{h}_{\mathfrak{fast}}$$

is a diffeomorphism of $\mathcal{A}$ *with non–vanishing Jacobian matrix on* $\mathcal{A}_{\hat{\mathfrak{c}}\theta}$ *and, moreover,*

$$\omega_{\mathfrak{fast}} \in \mathcal{D}^{\bar K,\nu_{\mathfrak{fast}}}_{\gamma_{\mathfrak{fast}},\tau} \qquad \forall\ \Lambda \in \mathcal{A}\ ,$$

with

$$\gamma_{\mathfrak{fast}} := (\gamma_1\ ,\ \cdots\ ,\ \gamma_n) \qquad \nu_{\mathfrak{fast}} := (\nu_1\ ,\ \cdots\ ,\ \nu_n)$$

and ν_i, γ_i *as in (55);*

2. *the functions* $f^i_{\mathfrak{fast}}$, $f^i_{\mathfrak{fast},\mathfrak{exp}}$ *do not depend on* κ_0*; the* $f^i_{\mathfrak{fast}}$*'s are given by*

$$f^i_{\mathfrak{fast}} = f^i_{\mathfrak{fast}}(\mathrm{t}_i,\mathrm{y}_i,\mathrm{x}_i) = \overline{f^i_{\mathcal{P}}}^{(\geq 2)}(\mathrm{t}_i,\mathrm{y}_i,\mathrm{x}_i) + \widetilde{f^i_{\mathfrak{fast}}}(\mathrm{t}_i,\mathrm{y}_i,\mathrm{x}_i)\ , \qquad i=1,\ \cdots\ ,\ n-1\ , \tag{60}$$

with

$$\begin{aligned}
&\overline{f^i_{\mathcal{P}}}^{(\geq 2)} := \sum_{j=i+1}^{n} \overline{f^{ij}_{\mathcal{P}}}^{(\geq 2)}\\
&\mathrm{t}_i := \big(\Theta_i,\ \cdots\ ,\ \Theta_{n-1},\ \vartheta_i,\ \cdots\ ,\ \vartheta_{n-1}\big)\ , \qquad \mathrm{y}_i := (\chi_{i-1},\ \cdots,\chi_{n-1},\ \Lambda_i,\ \cdots\ ,\ \Lambda_n)\\
&\mathrm{x}_i := (\kappa_i,\cdots,\kappa_{n-1})\ .
\end{aligned}$$

In particular, $\widetilde{f^i_{\mathfrak{fast}}}$ *do not depend on* ℓ_1, $\cdots$, ℓ_n*;*

3. *finally,* $\widetilde{f^i_{\mathfrak{fast}}}$, $f^i_{\mathrm{exp},\mathfrak{fast}}$ *satisfy the following bounds*

$$\|\widetilde{f^i_{\mathfrak{fast}}}\|_{\mathfrak{D}_{\mathfrak{fast}}} \leq \frac{1}{\mathfrak{c}_1}\mu\bar K\big(\frac{a^+_n}{a^-_1}\big)^{\frac{3}{2}}\frac{1}{a^-_{i+1}}\ , \qquad \|f^i_{\mathfrak{fast},\mathfrak{exp}}\|_{\mathfrak{D}_{\mathfrak{fast}}} \leq \frac{1}{\mathfrak{c}_1}\frac{e^{-\hat{\mathfrak{c}}\bar K s}}{a^-_{i+1}}\ . \tag{61}$$

Let L_0, $\cdots$, L_n be defined as $\mathfrak{L}_i$ in (50), with

$$\nu = m = n\ , \qquad \nu_1 = \cdots = \nu_n = 1\ .$$

Lemma 5.1 *If* $\bar K$ *verifies the inequality in (52), then one can find a number* $\mathfrak{c}_3$*, depending only on* m_0, $\cdots$, m_n*, such that*

$$|\omega_{\mathrm{k},\mathfrak{fast}}(\Lambda)\cdot k| \geq \frac{\mathfrak{c}_3}{(a^+_j)^{3/2}} \quad \forall\ k\in \mathrm{L}_{j-1}\setminus \mathrm{L}_j\ ,\quad |k|\leq \bar K\ ,\quad \forall\ \Lambda\in\mathcal{A}_\theta\ ,\quad \forall\ j=1,\cdots,n\ .$$

Proof For $\Lambda\in\mathcal{A}_\theta$, $\omega_{\mathrm{k},\mathfrak{fast},\mathrm{j}} := \frac{\mathfrak{M}_j^2\mathfrak{m}_j^3}{\Lambda_j^3}$ verifies $\frac{\sqrt{\mathfrak{M}_j}}{(a^+_j)^{3/2}} \leq |\omega_{\mathrm{k},\mathfrak{fast},\mathrm{j}}| \leq \frac{\sqrt{\mathfrak{M}_j}}{(a^-_j)^{3/2}}$. In the case $j=n$, we find $|\omega_{\mathrm{k},\mathfrak{fast}}\cdot k| = |\omega_{\mathrm{k},\mathfrak{fast},\mathrm{n}}k_n| \geq \frac{\sqrt{\mathfrak{M}_j}}{(a^+_n)^{3/2}}$, since $k_n\neq 0$. Let then $j\neq n$. For $k\in \mathrm{L}_{j-1}\setminus\mathrm{L}_j$, $k_j\neq 0$, so, inequality (52), with $\mathfrak{c}_2 \leq \frac{\min_j\sqrt{\mathfrak{M}_j}}{\max_j\sqrt{\mathfrak{M}_j}}$, and (49) imply

$$\bar K \leq \frac{\min_j\sqrt{\mathfrak{M}_j}}{\max_j\sqrt{\mathfrak{M}_j}}\min_{1\leq j\leq n-1}\Big(\frac{a^-_{j+1}}{a^+_j}\Big)^{3/2}$$

and hence

$$\begin{aligned} |\omega_{\mathrm{k},\mathfrak{fast}}\cdot k| &= |\sum_{i=j}^{n}\omega_{\mathrm{k},\mathfrak{fast},\mathrm{i}}k_i| \geq \inf_{\mathcal{A}_\theta}|\omega_{\mathrm{k},\mathfrak{fast},\mathrm{j}}| - \bar{K}\max_{j<i\leq n}\sup_{\mathcal{A}_\theta}|\omega_{\mathrm{k},\mathfrak{fast},\mathrm{i}}| \\ &\geq \frac{\sqrt{\mathfrak{M}}_j}{(a_j^+)^{3/2}} - \bar{K}\frac{\max_{i>j}\sqrt{\mathfrak{M}_i}}{(a_{j+1}^-)^{3/2}} \geq \frac{\sqrt{\mathfrak{M}}_j}{2(a_j^+)^{3/2}} \quad \blacksquare \end{aligned}$$

Proof of Proposition 5.1 The proof proceeds by recursion, in n steps. We describe the h^{th} step of this recursion, with $h = 1, \cdots, n$. We start with an Hamiltonian of the form

$$\mathrm{H}_{h-1} = \mathrm{h}^0_{\mathfrak{fast}} + \mu\, f_{h-1} \tag{62}$$

where $\mathrm{h}^0_{\mathfrak{fast}}$ is as in (38), and a domain

$$\mathfrak{D}_{h-1} = \mathcal{T}_{\Theta^{+(h-1)},\vartheta^{+(h-1)}} \times \mathcal{X}_{\theta^{(h-1)}} \times \mathcal{A}_{\theta^{(h-1)}} \times \mathbb{T}^n_{s^{(h-1)}} \times \mathbb{T}^n_{s^{(h-1)}} \ .$$

When $h = 1$, we take $\mathrm{H}_0 := \mathrm{H}_{\mathcal{P}}$, $\Theta^{(0)}_+ := \Theta^+$, $\vartheta^{(0)}_+ := \vartheta^+$, $\theta^{(0)} := \theta$, $s^{(0)} := s$, $f_0 := f_{\mathcal{P}}$ and we decompose

$$f_0 := \widehat{f}_0 := \sum_{i=1}^{n-1}\widehat{f}^i_0 \qquad \text{with} \qquad f^i_0 := \sum_{j=i+1}^{n} f^{ij}_{\mathcal{P}} \ .$$

We observe that $\widehat{f^i_0}$ depends on the coordinates

$$\begin{array}{cccccccccc} \Theta_i\ , & \cdots\ , & \Theta_{n-1}\ , & \chi_{i-1}\ , & \cdots, & \chi_{n-1}\ , & \Lambda_i\ , & \cdots\ , & \Lambda_n \\ \vartheta_i\ , & \cdots\ , & \vartheta_{n-1}\ , & \kappa_i\ , & \cdots\ , & \kappa_{n-1}\ , & \ell_i\ , & \cdots\ , & \ell_n\ . \end{array}$$

For $n \geq 3$ and $2 \leq h \leq n-1$, we assume, inductively, that f_{h-1} is a sum

$$f_{h-1} = \widehat{f_{h-1}} + f_{\mathfrak{exp},h-1} = \sum_{1\leq i\leq n}\widehat{f^i_{h-1}} + \sum_{1\leq i\leq n} f^i_{\mathfrak{exp},h-1}\ , \tag{63}$$

where, in turn,

$$\widehat{f^i_{h-1}} = \overline{f^i_{h-1}} + \widetilde{f^i_{h-1}}$$

with $\overline{f^i_{h-1}}$, $\widetilde{f^i_{h-1}}$ depending only on the coordinates

$$\begin{array}{cccccccccc} \Theta_i\ , & \cdots\ , & \Theta_{n-1}\ , & \chi_{i-1}\ , & \cdots, & \chi_{n-1}\ , & \Lambda_i\ , & \cdots\ , & \Lambda_n \\ \vartheta_i\ , & \cdots\ , & \vartheta_{n-1}\ , & \kappa_i\ , & \cdots\ , & \kappa_{n-1}\ , & \ell_{i\vee h}\ , & \cdots\ , & \ell_n \end{array}$$

and $\overline{f^i_{h-1}}$, $\widetilde{f^i_{h-1}}$, $f_{\mathfrak{exp},h-1}$ verifying the following bounds and identities

$$\begin{aligned} \overline{f^i_{h-1}} &= \Pi_{\mathrm{L}_{h-1}} T_{\bar{K}} \widehat{f^i_{h-2}} \\ \|\widetilde{f^i_{h-1}}\|_{\mathfrak{D}_{h-1}} &\leq \mathfrak{C}_{1,h-1}\mu\bar{K}\big(\frac{a_n^+}{a_1^-}\big)^{\frac{3}{2}}\|\widehat{f^i_{h-2}}\|_{\mathfrak{D}_{h-2}} \\ \|f^i_{\mathfrak{exp},h-1}\|_{\mathfrak{D}_{h-1}} &\leq \mathfrak{C}_{2,h-1}e^{-Ks^{(h)}}\|\widehat{f^i_{h-2}}\|_{\mathfrak{D}_{h-2}}\ . \end{aligned} \tag{64}$$

Here Π_{L_h} denotes the projection over the module L_h. In any case, $h = 1$, or $2 \leq h \leq n-1$, we focus on the Hamiltonian

$$\widehat{\mathrm{H}_{h-1}} = \mathrm{h}^0_{\mathfrak{fast}} + \mu\,\widehat{f_{h-1}} = \mathrm{h}^0_{\mathfrak{fast}} + \mu\sum_{i=1}^{n-1}\widehat{f^i_{h-1}}\ . \tag{65}$$

Our purpose is to apply Proposition D.1 to this Hamiltonian, in the case that the abstract system (132) does not depend on the coordinates (p,q). To this end, we take the coordinates

$$I := \Lambda \ , \quad \varphi := \ell \ , \quad \eta := (\overline{\Theta}, \chi) \ , \quad \xi := (\overline{\vartheta}, \kappa) \ ,$$

the functions f_i in (134) to be the $\widehat{f^{n-i}_{h-1}}$, and

$$\begin{aligned}
&N = n-1 \ , \qquad \nu = n \ , \qquad m_i := 2i \\
&(I_1, \cdots, I_\nu) := (\Lambda_n, \cdots, \Lambda_1) \\
&(\varphi_1, \cdots, \varphi_{\nu_i}) := (\ell_n, \cdots, \ell_{\max\{n-i,h\}}) \\
&(\eta_1 \ , \cdots, \eta_{m_i}) := (\Theta_{n-1}, \cdots, \Theta_{n-i}, \chi_{n-1}, \cdots, \chi_{n-i-1}) \\
&(\xi_1 \ , \cdots, \xi_{m_i}) := (\vartheta_{n-1}, \cdots, \vartheta_{n-i}, \kappa_{n-1}, \cdots, \kappa_{n-i}) \\
&u_i := (\Lambda_n, \cdots, \Lambda_1, \Theta_{n-1}, \cdots, \Theta_{n-i}, \chi_{n-1}, \cdots, \chi_{n-i-1}, \vartheta_{n-1}, \cdots, \vartheta_{n-i}, \kappa_{n-1}, \cdots, \kappa_{n-i}) \ .
\end{aligned}$$

The non–resonance assumption (133) for $\omega = \omega_{\mathrm{k,fast}} = \partial_\Lambda \mathrm{h}_{\mathrm{k,fast}}$, with

$$\mathfrak{Z}_i = \mathrm{L}_{h-1} \ , \qquad \mathfrak{Z} = \cup_i \mathfrak{Z}_i = \mathrm{L}_{h-1} \ , \qquad \mathfrak{L} = \mathrm{L}_h \qquad K = \bar{K}$$

is ensured by Lemma 5.1, with

$$\mathfrak{a} = \frac{\mathfrak{c}_3}{(a^+_h)^{3/2}} \ , \quad A = \mathcal{A} \ , \quad r = \theta^{(h-1)}_1 \ .$$

Now we have to check condition (138). In the case $2 \le h \le n-1$ the inductive assumptions (64) and assumption (57) imply

$$\begin{aligned}
\|\widehat{f^i_{h-1}}\|_{\mathfrak{D}_{h-1}} \ &\le \ \|\overline{f^i_{h-1}}\|_{\mathfrak{D}_{h-1}} + \|\widetilde{f^i_{h-1}}\|_{\mathfrak{D}_{h-1}} \le \Big(1 + \mathfrak{C}_1 \mu \bar{K} \big(\frac{a^+_n}{a^-_1}\big)^{\frac{3}{2}}\Big) \|\widehat{f^i_{h-2}}\|_{\mathfrak{D}_{h-2}} \\
&\le \ \cdots \le (1 + \mathfrak{C}_{1,h-1}\mathfrak{c}_1)^{h-1} \|\widehat{f^i_0}\|_{\mathfrak{D}_0} \le \frac{\mathfrak{C}_{4,h-1}}{a^-_i} =: E_i \ . \qquad (66)
\end{aligned}$$

An analogue bound holds also for $h = 1$. The numbers c_i and d_i in (137) may be evaluated as

$$c_i = e(1 + 2ie)/2 \qquad d_i = \min\{\theta^{(h-1)}_1 s^{(h-1)}, \Theta^{+(h-1)}_i \vartheta^{+(h-1)}_i\} = \mathfrak{c}_2 \theta^{(h-1)}_1 \ .$$

From these bounds it is immediate to see that inequality (138) is implied by (57), provided $\mathfrak{c}_1 < 2^{-7}\frac{6}{7}(\frac{8}{9})^{n-2}\mathfrak{c}_2/(\mathfrak{C}_4 c_n)$. Then Proposition D.1 applies. Its thesis implies that $\widehat{\mathrm{H}_{h-1}}$ in (65) can be conjugated to a suitable $\mathrm{H}^*_h = \mathrm{h}_{\mathrm{k,fast}} + \mu f^*_h$, where f^*_h verifies equalities and inequalities in (63)–(64) with h replaced by $h+1$ and $\mathfrak{C}_{1,h-1}$, $\mathfrak{C}_{2,h-1}$ replaced by suitable $\mathfrak{C}^*_{1,h}$, $\mathfrak{C}^*_{2,h}$. Then, applying the same transformation to H_{h-1} in (62), we shall conjugate H_{h-1} to $\mathrm{H}_h = \mathrm{h}_{\mathfrak{fast}} + \mu f_h$, where f_h satisfies the same equalities and inequalities as of f^*_h, with suitable $\mathfrak{C}_{1,h} \ge \mathfrak{C}^*_{1,h}$, $\mathfrak{C}_{2,h} \ge \mathfrak{C}^*_{2,h}$.

After we have performed n steps, we let $\mathfrak{D}_{\mathfrak{fast}} := \mathfrak{D}_n$, $\mathrm{H}_{\mathfrak{fast},\mathfrak{exp}} := \mathrm{H}_n$, $f^i_{\mathfrak{fast}} := \widehat{f^i_n}$, $\widetilde{f^i_{\mathfrak{fast}}} := f^i_{\mathfrak{fast}} - \overline{f^i_{\mathcal{P}}}$, $f^i_{\mathfrak{fast},\mathfrak{exp}} := f^i_{\mathrm{exp,n}}$, $\widehat{f}_{\mathfrak{fast}} := \sum_{i=1}^{n-1} \widehat{f^i}$, $\widetilde{f}_{\mathfrak{fast}} := \sum_{i=1}^{n-1} \widetilde{f^i}$, $f_{\mathfrak{fast},\mathfrak{exp}} := \sum_{i=1}^{n-1} f^i_{\mathfrak{fast},\mathfrak{exp}}$, with $\overline{f^i_{\mathcal{P}}} := \sum_{j=i+1}^{n} \overline{f^{ij}_{\mathcal{P}}}$. Therefore,

$$\mathrm{H}_{\mathfrak{fast}} = \mathrm{h}^{(0)}_{\mathfrak{fast}} + \mu\big(\widehat{f}_{\mathfrak{fast}} + f_{\mathfrak{exp},\mathfrak{fast}}\big) = \mathrm{h}^{(0)}_{\mathfrak{fast}} + \mu\Big(\sum_{1 \le i < j \le n} \overline{f^{ij}_{\mathcal{P}}} + \widetilde{f}_{\mathfrak{fast}} + f_{\mathrm{exp},\mathfrak{fast}}\Big)$$

reduces to (58) and the formulae given below, using (56).

It remains to check the bound on the left in (61) (the one on the right follows by construction). This follows by telescopic arguments. Indeed,

$$
\begin{aligned}
\|\widehat{f^i_{\mathfrak{fast}}}\|_{\mathfrak{D}_n} &= \|f^i_{\mathfrak{fast}} - \overline{f^i_{\mathcal{P}}}\|_{\mathfrak{D}_n} = \|\widehat{f^i_n} - \overline{f^i_{\mathcal{P}}}\|_{\mathfrak{D}_n} = \|\Pi_{\mathrm{L}_n}\widehat{f^i_n} - \Pi_{\mathrm{L}_n} f^i_{\mathcal{P}}\|_{\mathfrak{D}_n} \\
&\leq \sum_{h=1}^{n} \|\Pi_{\mathrm{L}_n}\widehat{f^i_h} - \Pi_{\mathrm{L}_n} T_{\bar K}\widehat{f^i_{h-1}}\|_{\mathfrak{D}_n} \\
&= \sum_{h=1}^{n} \|\Pi_{\mathrm{L}_n}\widehat{f^i_h} - \Pi_{\mathrm{L}_n}\Pi_{\mathrm{L}_h} T_{\bar K}\widehat{f^i_{h-1}}\|_{\mathfrak{D}_n} \\
&\leq \sum_{h=1}^{n} \|\widehat{f^i_h} - \Pi_{\mathrm{L}_h} T_{\bar K}\widehat{f^i_{h-1}}\|_{\mathfrak{D}_n} \\
&\leq \sum_{h=1}^{n} \|\widehat{f^i_h} - \Pi_{\mathrm{L}_h} T_{\bar K}\widehat{f^i_{h-1}}\|_{\mathfrak{D}_h} \\
&\leq \mu\bar K \big(\frac{a^+_n}{a^-_1}\big)^{\frac{3}{2}} \frac{\sum_{h=1}^{n} \mathfrak{C}_{1,h}\mathfrak{C}_{4,h-1}}{a^-_{i+1}} \ .
\end{aligned}
$$

Here, we have used the second bound in (64), (66), that $\widehat{f^i_n}$ does not depend on $\ell_1, \cdots, \ell_n$, and, finally, $\Pi_{\mathrm{L}_n} = \Pi_{\mathrm{L}_n} T_{\bar K} = \Pi_{\mathrm{L}_n}\Pi_{\mathrm{L}_h}$, for all $1 \leq h \leq n$. ∎

5.2 Secular normalizations

Consider the following truncation

$$
\mathrm{H}_{\mathfrak{fast}}(\overline{\Theta}, \chi, \Lambda, \overline{\vartheta}, \kappa) := \mathrm{h}_{\mathfrak{fast}}(\Lambda) + \mu\, f_{\mathfrak{fast}}(\overline{\Theta}, \chi, \Lambda, \overline{\vartheta}, \kappa)
$$

of the Hamiltonian $\mathrm{H}_{\mathfrak{fast},\mathfrak{exp}}$ in (58). The purpose of this section is to describe an iterative scheme which, after $(n-1)$ steps, conjugates $\mathrm{H}_{\mathfrak{fast}}$ to a close-to be integrable system, with an arbitrarily small remainder.

Let us firstly establish the following notation.

- Given a Taylor–Fourier expansion of the form

$$
g(p,q,\kappa) = \sum_{\substack{(a,b)\in\mathbb{N}^{2m_1} \\ k\in\mathbb{Z}^{m_2}}} g_{a,b,k} \big(\frac{p-\mathrm{i}q}{\sqrt{2}}\big)^a \big(\frac{p+\mathrm{i}q}{\sqrt{2}\mathrm{i}}\big)^b e^{\mathrm{i}k\cdot\kappa} \qquad (p,q,\kappa) \in B^{2m_1}(0)\times\mathbb{T}^{m_2} \ .
$$

  we denote as

$$
\Pi_{p,q,\kappa} g := \sum_{a\mathbb{N}^{m_1}} g_{0,a,a} \big(\frac{p^2+q^2}{2\mathrm{i}}\big)^a \ .
$$

Proposition 5.2 *There exists number $\overline{\mathfrak{c}}_h$, depending only on n, $m_0, \cdots, m_n$, $a^\pm_n$ such that, for any $h = 1, \cdots, n-1$ and any $\bar K$, $\bar\gamma > 0$ such that (53) hold with $\mathfrak{c}$ replaced by $\overline{\mathfrak{c}}_h$, one finds open sets*

$$
B^*_j \subset B^2_{\varepsilon_j} \ , \qquad \mathcal{G}^*_j \subset \mathcal{G}_j := \Big[\mathrm{G}^+_j,\ \mathrm{G}^+_j\Big] \ , \qquad j = n-h,\ \cdots,\ n-1
$$

verifying

$$
\mathrm{meas}\,\big(B^*_j \times \mathcal{G}^*_j\big) \geq \big(1 - \frac{\bar\gamma}{\overline{\mathfrak{c}}_h}\big)\,\mathrm{meas}\,\big(B^2_{\varepsilon_j} \times \mathcal{G}_j\big) \tag{67}
$$

such that, such that, defining

$$\mathcal{T}^h_{\overline{\mathfrak{c}}_h\theta} := \begin{cases} \Big\{(\Theta_1,\cdots,\Theta_{n-h-1},\vartheta_1,\cdots,\vartheta_{n-h-1}) \in \mathbb{C}^{n-1}\times\mathbb{T}^{n-1}_{\mathbb{C}} : \\ \qquad |\vartheta_j-\pi| \le \overline{\mathfrak{c}}_h \frac{\theta_j}{\mathrm{G}_n^-}\ , \quad |\Theta_j| \le \overline{\mathfrak{c}}_h \mathrm{G}_n^+ \\ \qquad \forall\ j=1,\cdots,n-h-1\Big\} & n\ge 3,\ 1\le h<n-2 \\ \\ \emptyset & \text{otherwise} \end{cases}$$

$$B^{*h}_{\overline{\mathfrak{c}}_h\overline{r}} := (B^*_{n-h})_{\overline{\mathfrak{c}}_h\sqrt{\overline{r}_{n-h}}} \times\cdots\times (B^*_{n-1})_{\overline{\mathfrak{c}}_h\sqrt{\overline{r}_{n-1}}}$$

$$\mathcal{X}^{*h}_{\overline{\mathfrak{c}}_h\theta,\overline{\mathfrak{c}}_h\bar{r}} := \Big\{\chi=(\chi_0,\cdots,\ \chi_{n-1}) : \quad \chi_{i-1}-\chi_i \in (\mathcal{G}^*_i)_{\overline{\mathfrak{c}}_h\theta_i}\ ,\ \chi_{j-1}-\chi_j\in(\mathcal{G}_j)_{\overline{\mathfrak{c}}_h\bar{r}_j} \\ \forall\ i=1,\cdots,n-h-1,\ j=n-h,\cdots,n\ , \quad \chi_n:=0\Big\}$$

$$\mathfrak{D}^h_{\mathfrak{sec}} := \mathcal{T}^h_{\overline{\mathfrak{c}}\theta}\times B^{*h}_{\overline{\mathfrak{c}}\varepsilon}\times\mathcal{X}^{*h}_{\overline{\mathfrak{c}}\theta,\overline{\mathfrak{c}}r}\times\mathcal{A}_{\overline{\mathfrak{c}}_h\tilde{r}}\times\mathbb{T}^n_{\overline{\mathfrak{c}}_h s}\times\mathbb{T}^n_{\overline{\mathfrak{c}}_h s} \tag{68}$$

a real–analytic transformation

$$\Phi_{\mathfrak{sec},h}: \qquad \mathfrak{D}^h_{\mathfrak{sec}}\to\mathfrak{D}_{\mathfrak{fast}}\ ,$$

may be found, which conjugates $f_{\mathfrak{fast}}$ to a new function

$$f_{\mathfrak{sec},h} := f_{\mathfrak{fast}}\circ\Phi_{\mathfrak{sec},h}$$

enjoying the following properties.

1. *Denoting* $(\mathrm{t}^{(h)},\mathrm{z}^{(h)},\mathrm{y}^{(h)},\mathrm{x}^{(h)})$*, where*

$$\begin{aligned} \mathrm{t}^{(h)} &= (\Theta^{(h)},\vartheta^{(h)}) = (\Theta^{(h)}_1,\ \cdots,\ \Theta^{(h)}_{n-h-1},\ \vartheta^{(h)}_1,\ \cdots,\ \vartheta^{(h)}_{n-h-1}) \\ \mathrm{z}^{(h)} &= (p^{(h)},q^{(h)}) = (p^{(h)}_{n-h},\ \cdots,\ p^{(h)}_{n-1},\ q^{(h)}_{n-h},\ \cdots,\ q^{(h)}_{n-1}) \\ \mathrm{y}^{(h)} &= (\chi^{(h)},\Lambda^{(h)}) = (\chi^{(h)}_0,\ \cdots,\ \chi^{(h)}_{n-1},\ \Lambda^{(h)}_1,\ \cdots,\ \Lambda^{(h)}_n) \\ \mathrm{x}^{(h)} &= (\kappa^{(h)},\ell^{(h)}) = (\kappa^{(h)}_0,\ \cdots,\ \kappa^{(h)}_{n-1},\ \ell^{(h)}_1,\ \cdots,\ \ell^{(h)}_n)\ , \end{aligned} \tag{69}$$

coordinates on $\mathfrak{D}^h_{\mathfrak{sec}}$ then $\Phi_{\mathfrak{sec},h}$ is co–variant with the symmetry:

$$\Phi_{\mathfrak{sec},h}(-\mathrm{t}^{(h)},-\mathrm{z}^{(h)},\mathrm{y}^{(h)},\mathrm{x}^{(h)}) = (-\mathrm{t}^{(0)},\mathrm{y}^{(0)},\mathrm{x}^{(0)}) \qquad \text{if}$$

$$\Phi_{\mathfrak{sec},h}(\mathrm{t}^{(h)},\mathrm{z}^{(h)},\mathrm{y}^{(h)},\mathrm{x}^{(h)}) = (\mathrm{t}^{(0)},\mathrm{y}^{(0)},\mathrm{x}^{(0)})$$

and hence, $f_{\mathfrak{sec},h}$ is even around

$$\mathrm{t}^{(h)} = (0,k\pi)\ , \qquad \mathrm{z}^{(h)}=0 \qquad k\in\{0,\ 1\}^{n-h-1}$$

2. *Defining*

$$
\begin{aligned}
\mathrm{t}_i^{(h)} &:= \begin{cases} \big(\Theta_i^{(h)}, \cdots, \Theta_{n-h-1}^{(h)}, \vartheta_i^{(h)}, \cdots, \vartheta_{n-h-1}^{(h)}\big) & i \le n-h-1 \\ \emptyset \qquad \text{otherwise} \end{cases} \\
\hat{\mathrm{y}}_i^{(h)} &= \begin{cases} \big(\frac{(p_i^{(h)})^2+(q_i^{(h)})^2}{2},\ \cdots,\ \frac{(p_{n-1}^{(h)})^2+(q_{n-1}^{(h)})^2}{2},\ \chi_{i-1}^{(h)},\ \cdots,\ \chi_{n-1},\ \Lambda_i^{(h)},\ \cdots,\ \Lambda_n^{(h)}\big) & i \ge n-h \\ \big(\frac{(p_{n-h}^{(h)})^2+(q_{n-h}^{(h)})^2}{2},\ \cdots,\ \frac{(p_{n-1}^{(h)})^2+(q_{n-1}^{(h)})^2}{2},\ \chi_{i-1}^{(h)},\ \cdots,\ \chi_{n-1},\ \Lambda_i^{(h)},\ \cdots,\ \Lambda_n^{(h)}\big) & \text{otherwise} \end{cases} \\
\hat{\mathrm{x}}_i^{(h)} &= \begin{cases} \big(\kappa_i^{(h)},\ \cdots,\ \kappa_{n-h-2}^{(h)}\big) & n \ge 4\ \&\ 1 \le h \le n-3\ \&\ 1 \le i \le n-h-2 \\ \emptyset \qquad \text{otherwise} \end{cases}
\end{aligned}
\tag{70}
$$

and $\hat{\mathrm{y}} := \hat{\mathrm{y}}_1$, $\hat{\mathrm{x}} := \hat{\mathrm{x}}_1$, $f_{\mathfrak{sec},h}$ *has the form*

$$
\begin{aligned}
f_{\mathfrak{sec},h}(\mathrm{t}^{(h)}, \mathrm{z}^{(h)}, \mathrm{y}^{(h)}, \mathrm{x}^{(h)}) &= \mathrm{h}_{\mathfrak{sec},h}(\hat{\mathrm{y}}_{n-h}^{(h)}) + f_{\mathfrak{norm},h}(\mathrm{t}^{(h)}, \hat{\mathrm{y}}^{(h)}, \hat{\mathrm{x}}^{(h)}) \\
&+ f_{\mathfrak{exp},\mathfrak{sec},h}(\mathrm{t}^{(h)}, \mathrm{z}^{(h)}, \mathrm{y}^{(h)}, \mathrm{x}^{(h)})
\end{aligned}
\tag{71}
$$

with

$$
\begin{aligned}
\mathrm{h}_{\mathfrak{sec}}(\hat{\mathrm{y}}_{n-h}^{(h)}) &= \sum_{i=n-h}^{n-1} \mathrm{h}_{\mathfrak{sec}}^i(\hat{\mathrm{y}}_i^{(h)}) \\
f_{\mathfrak{norm},h}(\mathrm{t}^{(h)}, \hat{\mathrm{y}}^{(h)}, \hat{\mathrm{x}}^{(h)}) &= \sum_{i=1}^{n-h-1} f_{\mathfrak{norm},h}^i(\mathrm{t}_i^{(h)}, \hat{\mathrm{y}}_i^{(h)}, \hat{\mathrm{x}}_i^{(h)})
\end{aligned}
\tag{72}
$$

where

3. *the functions* $\mathrm{h}_{\mathfrak{sec}}^i$ $f_{\mathfrak{norm},h}^i$ *may be decomposed as*

$$
\begin{aligned}
\mathrm{h}_{\mathfrak{sec}}^i(\hat{\mathrm{y}}_i^{(h)}) &= \overline{\mathrm{h}_{\mathfrak{sec}}^i}(\hat{\mathrm{y}}_i^{(h)}) + \widetilde{\mathrm{h}_{\mathfrak{sec}}^i}i(\hat{\mathrm{y}}_i^{(h)}) \\
f_{\mathfrak{norm},h}^i(\mathrm{t}_i^{(h)}, \hat{\mathrm{y}}_i^{(h)}, \hat{\mathrm{x}}_i^{(h)}) &= \overline{f_{\mathfrak{norm},h}^i}(\mathrm{t}_i^{(h)}, \hat{\mathrm{y}}_i^{(h)}, \hat{\mathrm{x}}_i^{(h)}) + \widetilde{f_{\mathfrak{norm},h}^i}(\mathrm{t}_i^{(h)}, \hat{\mathrm{y}}_i^{(h)}, \hat{\mathrm{x}}_i^{(h)})
\end{aligned}
\tag{73}
$$

where

$$
\overline{f_{\mathfrak{norm},h}^i} = \sum_{j=i+1}^{n} \Pi_h\big(\overline{f_{\mathcal{P}}^{ij}}^{(\ge 2)} \circ \overline{\phi_{\mathfrak{int}}^{n-1}} \circ \cdots \circ \overline{\phi_{\mathfrak{int}}^{n-h}}\big)
\tag{74}
$$

and $\overline{\mathrm{h}_{\mathfrak{sec}}^i}$, $\overline{\phi_{\mathfrak{int}}^i}$ *as in Lemma 3.4. The functions* $\widetilde{\mathrm{h}_{\mathfrak{sec},h}}$, $\widetilde{f_{\mathfrak{norm},h}}$, $f_{\mathfrak{exp},\mathfrak{sec},h}$ *in (71) may be bounded as*

$$
\begin{aligned}
|\widetilde{\mathrm{h}_{\mathfrak{sec},h}^i}| &\le \frac{1}{\mathfrak{c}_h} \max\Big\{ \mu \bar{K} \big(\frac{a_n}{a_1}\big)^{3/2} \frac{1}{a_{i+1}^-}\ ,\quad \frac{\bar{K}^{\bar{\tau}+1}\sqrt{\alpha}}{\bar{\gamma}} \frac{(a_i^+)^2}{(a_{i+1}^+)^3}\ ,\quad \frac{\varepsilon_{i+1}^2}{\theta_{i+1}} \frac{(a_i^+)^2}{(a_{i+1}^-)^3} \Big\} \\
|\widetilde{f_{\mathfrak{norm},h}^i}| &\le \frac{1}{\mathfrak{c}_h} \max\Big\{ \mu \bar{K} \big(\frac{a_n}{a_1}\big)^{3/2} \frac{1}{a_{i+1}^-}\ ,\quad \frac{\bar{K}^{\bar{\tau}+1}\sqrt{\alpha}}{\bar{\gamma}} \frac{(a_i^+)^2}{(a_{i+1}^-)^3} \Big\} \\
|f_{\mathfrak{exp},\mathfrak{sec},h}| &\le \frac{1}{\mathfrak{c}_h} \frac{(a_{n-1}^+)^2}{(a_n^-)^3} e^{-\mathfrak{c}_h \bar{K}}
\end{aligned}
\tag{75}
$$

4. *Defining*

$$\zeta^{(h)} := \Big(\frac{(p^{(h)}_{n-h})^2 + (q^{(h)}_{n-h})^2}{2}, \ \cdots \ , \ \frac{(p^{(h)}_{n-1})^2 + (q^{(h)}_{n-1})^2}{2}, \ \chi^{(h)}_{i-1}, \ \cdots \ , \ \chi_{n-1} \Big)$$

so that

$$\hat{\mathrm{y}}^{(h)}_{n-h} = (\zeta^{(h)}, \Lambda^{(h)}_{n-h}, \ \cdots \ , \ \Lambda^{(h)}_{n})$$

for any $\Lambda^{(h)}_{n-h}$, $\cdots$, $\Lambda^{(h)}_{n}$, *the map*

$$\zeta^{(h)} \to \omega_{\mathfrak{sec},h} := \partial_{\zeta^{(h)}} \mathrm{h}_{\mathfrak{sec},h}(\zeta^{(h)}, \Lambda^{(h)})$$

is a diffeomorphism of $D_{\overline{r}} \times \mathcal{X}_{\overline{r}}$, *with non–vanishing Jacobian matrix. The set* $D^*_{\overline{r}} \times \mathcal{X}^*_{\overline{r}}$ *consists of the subset of* $D_{\overline{r}} \times \mathcal{X}_{\overline{r}}$ *such that* $\omega_{\mathfrak{fast},\mathfrak{sec}} \in \mathcal{D}^{\bar{K},\nu_{\mathfrak{sec}}}_{\gamma_{\mathfrak{sec}};\overline{r}}$, *where, if* ν_j, γ_j *are as in* (55),

$$\nu_{\mathfrak{sec}} := (\nu_{n+1} \ , \ \cdots \ , \ \nu_{2n-1}) \qquad \gamma_{\mathfrak{sec}} := (\gamma_{n+1} \ , \cdots \ , \ \gamma_{2n-1}) \ .$$

We shall give the complete details of the proof of Proposition 5.2 along the following sections 5.2.1–5.2.4. In this section we just provide main ideas.

Scheme of proof The proof is by recursion. The h^{th} step of this recursion starts with

$$f_{\mathfrak{sec},h-1} = \mathrm{h}_{\mathfrak{sec},h-1} + f_{\mathfrak{norm},h-1} + f_{\mathfrak{exp},\mathfrak{sec},h-1} \ ,$$

where, for $h = 1$

$$\mathrm{h}_{\mathfrak{sec},0} \equiv 0, \qquad f_{\mathfrak{exp},\mathfrak{sec},0} \equiv 0, \qquad f_{\mathfrak{sec},0} := f_{\mathfrak{norm},0} := f_{\mathfrak{fast}} \ , \tag{76}$$

while, for $n \geq 3$ and $h = 2, \cdots, n-1$, we assume, inductively, that $\mathrm{h}_{\mathfrak{sec},h-1}$, $f_{\mathfrak{sec},h-1}$ and $f_{\mathfrak{exp},\mathfrak{sec},h-1}$ satisfy the theses of Proposition 5.2, with h replaced by $(h-1)$.
The transformation $\phi^{n-h}_{\mathfrak{sec}}$ conjugating $f_{\mathfrak{sec},h-1}$ to $f_{\mathfrak{sec},h}$ will be constructed as a product $\phi^{n-h}_{\mathfrak{sec}} = \phi^{n-h}_{\mathfrak{int}} \circ \phi^{n-h}_{\mathfrak{norm}}$ of an "integrating" and a "normalizing" transformation.
Due to the bound on $f_{\mathfrak{exp},\mathfrak{sec},h-1}$, it is enough to focus on the truncation

$$\widehat{f_{\mathfrak{sec},h-1}} := \mathrm{h}_{\mathfrak{sec},h-1} + f_{\mathfrak{norm},h-1} = \mathrm{h}_{\mathfrak{sec},h-1} + \sum_{i=1}^{n-h} f^i_{\mathfrak{norm},h-1}(\mathrm{t}^{(h-1)}_i, \hat{\mathrm{y}}^{(h-1)}_i, \hat{\mathrm{x}}^{(h-1)}_i) \tag{77}$$

of $f_{\mathfrak{sec},h-1}$. We split

$$f_{\mathfrak{norm},h-1} = f^{n-h}_{\mathfrak{norm},h-1}(\mathrm{t}^{(h-1)}_{n-h}, \hat{\mathrm{y}}^{(h-1)}_{n-h}, \hat{\mathrm{x}}^{(h-1)}_{n-h}) + \sum_{i=1}^{n-h-1} f^i_{\mathfrak{norm},h-1}(\mathrm{t}^{(h-1)}_i, \hat{\mathrm{y}}^{(h-1)}_i, \hat{\mathrm{x}}^{(h-1)}_i)$$

and we distinguish two cases.

Case $n \geq 3$, $h = 2, \cdots, n-1$ By the inductive assumption (see (70) with h replaced by $(h-1)$), the function $f^{n-h}_{\mathfrak{norm},h-1}$ depends only on

$$\mathrm{t}^{(h-1)}_{n-h} = (\Theta^{(h-1)}_{n-h}, \ \vartheta^{(h-1)}_{n-h}) \quad \text{and} \quad \hat{\mathrm{y}}^{(h-1)}_{n-h}$$

therefore, it is integrable. In Section 5.2.2, we shall construct a canonical, real–analytic change of coordinates

$$\begin{aligned} \phi^{n-h}_{\mathfrak{int}} : \qquad & \mathfrak{D}^h_{\mathfrak{int}} \to \mathfrak{D}^{h-1}_{\mathfrak{sec}} \\ & (\mathrm{t}^{(h)}_*, \mathrm{z}^{(h)}_*, \mathrm{y}^{(h)}_*, \mathrm{x}^{(h)}_*) \to (\mathrm{t}^{(h-1)}, \mathrm{z}^{(h-1)}, \mathrm{y}^{(h-1)}, \mathrm{x}^{(h-1)}) \end{aligned}$$

$$\mathfrak{D}^h_{\mathtt{int}} := \mathcal{T}^h_{\hat{\mathfrak{c}}_h\theta} \times B^2_{\hat{\mathfrak{c}}_h\varepsilon_{n-h}} \times B^{*,h-1}_{\hat{\mathfrak{c}}_h\varepsilon} \times \mathcal{X}^{*,h-1}_{\hat{\mathfrak{c}}_h\theta,\hat{\mathfrak{c}}_h\overline{r}} \times \mathcal{A}_{\hat{\mathfrak{c}}_h\widetilde{r}} \times \mathbb{T}^n_{\hat{\mathfrak{c}}_h s} \times \mathbb{T}^n_{\hat{\mathfrak{c}}_h s} \tag{78}$$

such that

$$f^{n-h}_{\mathtt{norm},h-1} \circ \phi^{n-h}_{\mathtt{int}} = \mathrm{h}^{n-h}_{\mathfrak{sec}}(\hat{\mathrm{y}}^{(h)}_{*,n-h}) \tag{79}$$

depends only on $\hat{\mathrm{y}}^{(h)}_{*,n-h}$, where $\hat{\mathrm{y}}^{(h)}_{*,i}$ is defined analogously to $\hat{\mathrm{y}}^{(h)}_i$ in (70). Here,

$$\begin{cases} \mathrm{t}^{(h)}_* := \big(\Theta^{(h)}_*,\ \vartheta^{(h)}_*\big) \\ \mathrm{z}^{(h)}_* := \big(p^{(h)}_*,\ q^{(h)}_*\big) \\ \mathrm{y}^{(h)}_* := \big(\chi^{(h)}_*,\ \Lambda^{(h)}_*\big) \\ \mathrm{x}^{(h)}_* := \big(\kappa^{(h)}_*,\ \ell^{(h)}_*\big) \end{cases} \qquad \begin{cases} \mathrm{t}^{(h-1)} := \big(\Theta^{(h-1)},\ \vartheta^{(h-1)}\big) \\ \mathrm{z}^{(h-1)} := \big(p^{(h-1)},\ q^{(h-1)}\big) \\ \mathrm{y}^{(h-1)} := \big(\chi^{(h-1)},\ \Lambda^{(h-1)}\big) \\ \mathrm{x}^{(h-1)} := \big(\kappa^{(h-1)},\ \ell^{(h-1)}\big) \end{cases}$$

are defined analogously to (69).

We shall construct $\phi^{n-h}_{\mathtt{int}}$ such in a way it involves only the coordinates

$$\phi^{n-h}_{\mathtt{int}} :\ (\mathrm{z}^{(h)}_{*,n-h},\ \mathrm{y}^{(h)}_{*,n-h},\ \mathrm{x}^{(h)}_{*,n-h}) \to (\mathrm{t}^{(h-1)}_{n-h},\ \mathrm{z}^{(h-1)}_{n-h+1},\ \mathrm{y}^{(h-1)}_{n-h},\ \mathrm{x}^{(h-1)}_{n-h})$$

with

$$\begin{aligned} \mathrm{z}^{(h)}_{*,n-h} &:= \big(p^{(h)}_{*,n-h},\ \cdots,\ p^{(h)}_{n-1},\ q^{(h)}_{*,n-h},\ \cdots,\ q^{(h)}_{n-1}\big) \\ \mathrm{y}^{(h)}_{*,n-h} &:= \big(\chi^{(h)}_{*,n-h-1},\ \cdots,\ \chi_{*,n-1},\ \Lambda^{(h)}_{*,n-h},\ \cdots,\ \Lambda^{(h)}_n\big) \\ \mathrm{x}^{(h)}_{*,n-h} &:= \big(\kappa^{(h)}_{*,n-h-1},\ \cdots,\ \kappa^{(h)}_{*,n-1},\ \ell^{(h)}_{*,n-h},\ \cdots,\ \ell^{(h)}_n\big) \end{aligned}$$

$$\begin{aligned} \mathrm{t}^{(h-1)}_{n-h} &:= \big(\Theta^{(h-1)}_{n-h},\ \vartheta^{(h-1)}_{n-h}\big) \\ \mathrm{z}^{(h-1)}_{n-h+1} &:= \big(p^{(h-1)}_{n-h+1},\ \cdots,\ p^{(h-1)}_{n-1},\ q^{(h-1)}_{n-h+1},\ \cdots,\ q^{(h-1)}_{n-1}\big) \\ \mathrm{y}^{(h-1)}_{n-h} &:= \big(\chi^{(h-1)}_{n-h-1},\ \cdots,\ \chi_{n-1},\ \Lambda^{(h-1)}_{n-h},\ \cdots,\ \Lambda^{(h-1)}_n\big) \\ \mathrm{x}^{(h-1)}_{n-h} &:= \big(\kappa^{(h-1)}_{n-h-1},\ \cdots,\ \kappa^{(h-1)}_{n-1},\ \ell^{(h-1)}_{n-h},\ \cdots,\ \ell^{(h-1)}_n\big) \end{aligned} \tag{80}$$

and has the form

$$\phi^{n-h}_{\mathtt{int}} :\ \begin{cases} \Theta^{(h-1)}_{n-h} = \mathcal{F}^{(h)}_{\mathtt{int}}(p^{(h)}_{*,n-h},\ q^{(h)}_{*,n-h},\ \tilde{\mathrm{y}}^{(h)}_*) \\ \vartheta^{(h-1)}_{n-h} - \pi = \mathcal{G}^{(h)}_{\mathtt{int}}(p^{(h)}_{*,n-h},\ q^{(h)}_{*,n-h},\ \tilde{\mathrm{y}}^{(h)}_*) \\ \hat{\mathrm{z}}^{(h-1)}_j = \hat{\mathrm{z}}^{(h)}_{*,j} e^{\mathrm{i}\psi^{(h)}_{\mathtt{int},j}(p^{(h)}_{*,n-h},\ q^{(h)}_{*,n-h},\ \tilde{\mathrm{y}}^{(h)}_*)} \\ \mathrm{y}^{(h-1)}_{n-h} = \mathrm{y}^{(h)}_{*,n-h} \\ \mathrm{x}^{(h-1)}_{n-h} = \mathrm{x}^{(h)}_{*,n-h} + \varphi^{(h)}_{\mathtt{int}}(p^{(h)}_{*,n-h},\ q^{(h)}_{*,n-h},\ \tilde{\mathrm{y}}^{(h)}_*) \end{cases} \tag{81}$$

with $\mathcal{F}^{(h)}_{\mathtt{int}}$, $\mathcal{G}^{(h)}_{\mathtt{int}}$ odd, $\psi^{(h)}_{\mathtt{int},j}$, $\varphi^{(h)}_{\mathtt{int}}$ even in $(p^{(h)}_{*,n-h},\ q^{(h)}_{*,n-h})$,

$$\begin{aligned} \tilde{\mathrm{y}}^{(h)}_* &:= \Big(\frac{(p^{(h)}_{*,n-h+1})^2 + (q^{(h)}_{*,n-h+1})^2}{2}, \cdots \frac{(p^{(h)}_{*,1})^2 + (q^{(h)}_{*,1})^2}{2},\ \mathrm{y}^{(h)}_{*,n-h}\Big) \\ \hat{\mathrm{z}}^{(h-1)}_j &:= \big(p^{(h-1)}_j,\ q^{(h-1)}_j\big) := p^{(h-1)}_j + \mathrm{i}q^{(h-1)}_j \\ \hat{\mathrm{z}}^{(h)}_{*,j} &:= \big(p^{(h)}_{*,j},\ q^{(h)}_{*,j}\big) := p^{(h)}_{*,j} + \mathrm{i}q^{(h)}_{*,j} \end{aligned} \tag{82}$$

with $j = n-h+1, \cdots, n-1$, for $n \geq 3$, $h \geq 2$ and $\mathrm{y}^{(h)}_{*,n-h}$ as in (80).

In particular, observe that $\phi^{n-h}_{\mathtt{int}}$ enjoys the following properties:

- it is co–variant with the symmetry: if

$$\phi_{\mathfrak{int}}^{n-h}(\mathrm{t}_*^{(h)},\mathrm{z}_*^{(h)},\ \mathrm{y}_*^{(h)},\ \mathrm{x}_*^{(h)}) = (\mathrm{t}^{(h-1)},\ \mathrm{z}^{(h-1)},\ \mathrm{y}^{(h-1)},\ \mathrm{x}^{(h-1)})\ ,$$

  then

$$\phi_{\mathfrak{int}}^{n-h}(-\mathrm{t}_*^{(h)},-\mathrm{z}_*^{(h)},\ \mathrm{y}_*^{(h)},\ \mathrm{x}_*^{(h)}) = (-\mathrm{t}^{(h-1)},\ -\mathrm{z}^{(h-1)},\ \mathrm{y}^{(h-1)},\ \mathrm{x}^{(h-1)})\ ;$$

- leaves the "actions"

$$\tilde{\mathrm{y}}_*^{(h)} = \tilde{\mathrm{y}}^{(h-1)}$$

  unvaried, where $\tilde{\mathrm{y}}_*^{(h)}$ is as in (82), and

$$\tilde{\mathrm{y}}^{(h-1)} \ := \ (\frac{(p_{n-h+1}^{(h-1)})^2+(q_{n-h+1}^{(h-1)})^2}{2},\cdots\frac{(p_1^{(h-1)})^2+(q_1^{(h-1)})^2}{2},\ \mathrm{y}_{n-h}^{(h-1)})$$

  is defined analogously;

- leaves the averages with respect to the x–coordinates unvaried. Namely, for any real–analytic function g on $\mathfrak{D}_{\mathfrak{sec}}^{h-1}$,

$$\Pi_{\mathrm{x}_*^{(h)}}\big(g\circ\phi_{\mathfrak{int}}^{n-h}\big) = \big(\Pi_{\mathrm{x}^{(h-1)}}g\big)\circ\phi_{\mathfrak{int}}^{n-h}\ .$$

Applying $\phi_{\mathfrak{int}}^{n-h}$ to $\widehat{f_{\mathfrak{sec},h-1}}$ in (77), we obtain

$$\begin{aligned} f_{\mathfrak{sec},\mathfrak{int},h-1} \ &:= \ \widehat{f_{\mathfrak{sec},h-1}}\circ\phi_{\mathfrak{int}}^{n-h} = \mathrm{h}_{\mathfrak{sec},h-1}+\mathrm{h}_{\mathfrak{sec}}^{n-h}+\sum_{i=1}^{n-h-1} f^i_{\mathfrak{norm},\mathfrak{int},h-1}(\mathrm{t}_{*,i}^{(h)},\tilde{\hat{\mathrm{y}}}_{*,i}^{(h)},\tilde{\hat{\mathrm{x}}}_{*,i}^{(h)}) \\ &= \ \sum_{i=n-h}^{n-1}\mathrm{h}^i_{\mathfrak{sec},h}(\hat{\mathrm{y}}_{*,i}^{(h)})+\sum_{i=1}^{n-h-1} f^i_{\mathfrak{norm},\mathfrak{int},h-1}(\mathrm{t}_{*,i}^{(h)},\tilde{\hat{\mathrm{y}}}_{*,i}^{(h)},\tilde{\hat{\mathrm{x}}}_{*,i}^{(h)}) \end{aligned}$$

with

$$\mathrm{h}_{\mathfrak{sec},h} \ := \ \mathrm{h}_{\mathfrak{sec},h-1}+\mathrm{h}_{\mathfrak{sec}}^{n-h}\ ,\qquad f^i_{\mathfrak{norm},\mathfrak{int},h-1} := f^i_{\mathfrak{norm},h-1}\circ\phi_{\mathfrak{int}}^{n-h} \tag{83}$$

and (as it follows from (70) with $h-1$ replacing h and (81)) $f^i_{\mathfrak{norm},\mathfrak{int},h-1}$ depends only on the arguments

$$\begin{aligned} \mathrm{t}_{*,i}^{(h)} \ &:= \ (\Theta_{*,i}^{(h)},\cdots,\Theta_{*,n-h-1}^{(h)},\vartheta_{*,i}^{(h)},\cdots,\vartheta_{*,n-h-1}^{(h)}) \\ \tilde{\hat{\mathrm{y}}}_{*,i}^{(h)} \ &:= \ (p_{*,n-h}^{(h)},\ q_{*,n-h}^{(h)},\ \frac{(p_{*,n-h+1}^{(h)})^2+(q_{*,n-h+1}^{(h)})^2}{2},\ \cdots,\ \frac{(p_{*,n-1}^{(h)})^2+(q_{*,n-1}^{(h)})^2}{2}, \\ &\qquad \chi_{*,i-1}^{(h)},\ \cdots,\ \chi_{*,n-1},\Lambda_{*,i}^{(h)},\ \cdots,\ \Lambda_{*,n}^{(h)}) \\ \tilde{\hat{\mathrm{x}}}_{*,i}^{(h)} \ &:= \ \begin{cases} (\kappa_{*,i}^{(h)},\ \cdots,\ \kappa_{*,n-h-1}^{(h)}) & n\geq 4\ \&\ 1\leq h-1\leq n-3 \\ \\ \emptyset & \text{otherwise} \end{cases} \end{aligned} \tag{84}$$

The next step will be to retain the dependence on $(p_{n-h}^{(h)},q_{n-h}^{(h)})$ only via $\frac{(p_{n-h}^{(h)})^2+(q_{n-h}^{(h)})^2}{2}$ and, for $h<n-1$, to to eliminate from $f_{\mathfrak{sec},\mathfrak{int},h-1}$ the dependence upon the angle $k_{*,n-h-1}^{(h)}$, up to an exponential remainder. Namely, we look for another canonical, real–analytic change of coordinates

$$\begin{aligned} \phi_{\mathfrak{norm}}^{n-h}: \qquad & \mathfrak{D}_{\mathfrak{sec}}^h\to\mathfrak{D}_{\mathfrak{int}}^h \\ & (\mathrm{t}^{(h)},\ \mathrm{z}^{(h)},\ \mathrm{y}^{(h)},\ \mathrm{x}^{(h)})\to(\mathrm{t}_*^{(h)},\ \mathrm{z}_*^{(h)},\ \mathrm{y}_*^{(h)},\ \mathrm{x}_*^{(h)}) \end{aligned} \tag{85}$$

so as to conjugate $f_{\mathfrak{sec},\mathfrak{int},h-1}$ to a new Hamiltonian

$$\widehat{f_{\mathfrak{sec},h}} := f_{\mathfrak{sec},\mathfrak{int},h-1} \circ \phi^{n-h}_{\mathfrak{norm}} = \mathrm{h}_{\mathfrak{sec},h} + \sum_{i=1}^{n-h-1} f^i_{\mathfrak{norm},h}(\mathrm{t}^{(h)}_i, \hat{\mathrm{y}}^{(h)}_i, \hat{\mathrm{x}}^{(h)}_i) + \widehat{f_{\mathfrak{exp},\mathfrak{sec},h}} \tag{86}$$

where $f^i_{\mathfrak{norm},h}$ and $\widehat{f_{\mathfrak{exp},\mathfrak{sec},h}}$ satisfy (73)–(75). We choose $\mathfrak{D}^h_{\mathfrak{sec}}$ as the subset of $\mathfrak{D}^h_{\mathfrak{int}}$ where the map

$$\omega_{\mathfrak{sec},h} := \begin{cases} \partial_{\frac{(p^{(h)}_{n-h})^2+(q^{(h)}_{n-h})^2}{2}, \chi^{(h)}_{n-h-1}} \mathrm{h}_{\mathfrak{sec},h} & h = 2, \cdots, n-2 \ \& \ n \geq 4 \\ \partial_{\frac{(p^{(n-1)}_1)^2+(q^{(n-1)}_1)^2}{2}} \mathrm{h}_{\mathfrak{sec},n-1} & h = n-1 \end{cases}$$

does not verifies resonances up to order $\bar{K}$, and next we apply a suitable normal form theory (Proposition D.1). We shall choose $\phi^{n-h}_{\mathfrak{norm}}$ such in a way that

- it is co–variant with the symmetry: if

$$\phi^{n-h}_{\mathfrak{norm}}(\mathrm{t}^{(h)}, \mathrm{z}^{(h)}, \mathrm{y}^{(h)}, \mathrm{x}^{(h)}) = (\mathrm{t}^{(h)}_*, \mathrm{z}^{(h)}_*, \mathrm{y}^{(h)}_*, \mathrm{x}^{(h)}_*) \ ,$$

  then

$$\phi^{n-h}_{\mathfrak{norm}}(-\mathrm{t}^{(h)}, -\mathrm{z}^{(h)}, \mathrm{y}^{(h)}, \mathrm{x}^{(h)}) = (-\mathrm{t}^{(h)}_*, -\mathrm{z}^{(h)}_*, \mathrm{y}^{(h)}_*, \mathrm{x}^{(h)}_*) \ , \tag{87}$$

- leaves the "actions"

$$\mathrm{y}^{(h)}_{*,n-h} = \mathrm{y}^{(h)}_{n-h}$$

  unvaried, where

$$\begin{aligned} \mathrm{y}^{(h)}_{n-h} &:= \big(\frac{(p^{(h)}_{n-h+1})^2 + (q^{(h)}_{n-h+1})^2}{2}, \cdots \frac{(p^{(h)}_1)^2 + (q^{(h)}_1)^2}{2}, \ \chi^{(h)}_{n-h}, \ \cdots, \chi^{(h)}_{n-1}, \\ &\quad \Lambda^{(h)}_1, \ \cdots, \ \Lambda^{(h)}_n\big) \ ; \\ \mathrm{y}^{(h)}_{*,n-h} &:= \big(\frac{(p^{(h)}_{*,n-h+1})^2 + (q^{(h)}_{*,n-h+1})^2}{2}, \cdots \frac{(p^{(h)}_{*,1})^2 + (q^{(h)}_{*,1})^2}{2}, \ \chi^{(h)}_{*,n-h}, \ \cdots, \chi^{(h)}_{*,n-1}, \\ &\quad \Lambda^{(h)}_{*,1}, \ \cdots, \ \Lambda^{(h)}_{*,n}\big) \ ; \end{aligned} \tag{88}$$

- verifies

$$\Pi_{\mathrm{z}^{(h)}_{*,n-h+1}, \mathrm{x}^{(h)}_{*,n-h+1}} \big(g \circ \phi^{n-h}_{\mathfrak{norm}}\big) = \big(\Pi_{\mathrm{z}^{(h)}_{n-h+1}, \mathrm{x}^{(h)}_{n-h+1}} g\big) \circ \phi^{n-h}_{\mathfrak{norm}} \ . \tag{89}$$

The thesis of Proposition 5.2 at rank h follows, with

$$f_{\mathfrak{sec},h} := \widehat{f_{\mathfrak{sec},h}} + f_{\mathfrak{exp},\mathfrak{sec},h-1} \circ \phi^{n-h}_{\mathfrak{sec}}, \ f_{\mathfrak{exp},\mathfrak{sec},h} := \widehat{f_{\mathfrak{exp},\mathfrak{sec},h}} + f_{\mathfrak{exp},\mathfrak{sec},h-1} \circ \phi^{n-h}_{\mathfrak{sec}} \ .$$

Case $h = 1$ The proof of this case uses similar ideas as the proof of the case $2 \leq h \leq n-1$ for $n \geq 3$. However, due to subtle differences between the two cases (compare, *e.g.*, the inductive assumption on $f^{n-h}_{\mathfrak{norm},h-1}$ in (70) for $h \geq 2$ with Eq. (90); the definition of $\mathrm{h}^{n-h}_{\mathfrak{sec}}$, $\phi^{n-h}_{\mathfrak{int}}$ for $h \geq 2$ in (79), with the definition of $\mathrm{h}^{n-1}_{\mathfrak{sec}}$, $\phi^{n-1}_{\mathfrak{int}}$ in (92) and (95)), for sake of precision, we briefly discuss also this case.

Let $f_{\mathfrak{sec},0}$ be as in (76). In view of (59) and (60), we can split

$$f_{\mathfrak{sec},0} = \overline{f^{n-1,n}_{\mathcal{P}}}^{(2)} + \overline{f^{n-1}_{\mathcal{P}}}^{(\geq 3)} + \widetilde{f^{n-1}_{\mathfrak{fast}}} + \sum_{i=1}^{n-2} f^i_{\mathfrak{fast}} \tag{90}$$

where

$$\overline{f_{\mathcal{P}}^{n-1}}^{(\geq 3)} := \overline{f_{\mathcal{P}}^{n-1}}^{(\geq 2)} - \overline{f_{\mathcal{P}}^{n-1}}^{(2)}$$

and the summand appears only when $n \geq 3$. As for $\overline{f_{\mathcal{P}}^{n-1,n}}^{(2)}$, by Lemmata 3.4 and (see also Lemma 5.2), we find a domain $\overline{\mathfrak{B}_{n-1}}$ (defined in Eq. (96) below), a real–analytic and canonical transformation

$$\phi_{\mathtt{int}}^{n-1} : \ \big(\mathrm{z}^{(1)}_{*,n-1}, \mathrm{y}^{(1)}_{*,n-1}, \mathrm{x}^{(1)}_{*,n-1}\big) \in \overline{\mathfrak{B}_{n-1}} \to \big(\mathrm{z}^{(0)}_{n-1}, \mathrm{y}^{(0)}_{n-1}, \mathrm{x}^{(0)}_{n-1}\big) \in \overline{\mathfrak{D}_{n-1}} := \phi_{\mathtt{int}}^{n-1}\big(\overline{\mathfrak{B}_{n-1}}\big) \tag{91}$$

of the form (81), with $h = 1$ (but neglecting the coordinates $\hat{\mathrm{z}}^{(0)}_j$, $\hat{\mathrm{z}}^{(1)}_{j,*}$) such that

$$\overline{f_{\mathcal{P}}^{n-1}}^{(2)} \circ \phi_{\mathtt{int}}^{n-1} = \overline{\mathrm{h}_{\mathtt{sec}}^{n-1}}(\hat{\mathrm{y}}^{(1)}_{*,n-1}) \tag{92}$$

depends only on

$$\hat{\mathrm{y}}^{(1)}_{*,n-1} = \Big(\frac{(p^{(1)}_{*,n-1})^2 + (q^{(1)}_{*,n-1})^2}{2},\ \chi^{(1)}_{*,n-2},\ \chi^{(1)}_{*,n-1},\ \Lambda^{(1)}_{*,n-1},\ \Lambda^{(1)}_{*,n}\Big)\ . \tag{93}$$

In (91), we have let

$$\begin{cases} \mathrm{z}^{(1)}_{*,n-1} := \big(p^{(1)}_{*,n-1},\ q^{(1)}_{*,n-1}\big) \\ \mathrm{y}^{(1)}_{*,n-1} := \big(\chi^{(1)}_{*,n-2},\ \chi^{(1)}_{*,n-1},\ \Lambda^{(1)}_{*,n-1},\ \Lambda^{(1)}_{*,n}\big) \\ \mathrm{x}^{(1)}_{*,n-1} := \big(\kappa^{(1)}_{*,n-2},\ \kappa^{(1)}_{*,n-1}\big) \end{cases} \qquad \begin{cases} \mathrm{t}^{(0)}_{n-1} := \big(\Theta^{(0)}_{n-1},\ \vartheta^{(0)}_{n-1}\big) \\ \mathrm{y}^{(0)}_{n-1} := \big(\chi^{(0)}_{n-2},\ \chi^{(0)}_{n-1},\ \Lambda^{(0)}_{n-1},\ \Lambda^{(0)}_{n}\big) \\ \mathrm{x}^{(0)}_{n-1} := \big(\kappa^{(0)}_{n-2},\ \kappa^{(0)}_{n-1}\big)\ . \end{cases}$$

We let

$$\begin{cases} \mathrm{t}^{(0)} := \big(\Theta^{(0)},\ \vartheta^{(0)}\big) \\ \mathrm{y}^{(0)} := \big(\chi^{(0)}, \Lambda^{(0)}\big) \\ \mathrm{x}^{(0)} := \big(\kappa^{(0)}, \ell^{(0)}\big) \end{cases} \qquad \begin{cases} \mathrm{t}^{(1)}_{*} := \big(\Theta^{(1)}_{*},\ \vartheta^{(1)}_{*}\big) \\ \mathrm{z}^{(1)}_{*} := \big(p^{(1)}_{*},\ q^{(1)}_{*}\big) \\ \mathrm{y}^{(1)}_{*} := \big(\chi^{(1)}_{*}, \Lambda^{(1)}_{*}\big) \\ \mathrm{x}^{(1)}_{*} := \big(\kappa^{(1)}_{*}, \ell^{(1)}_{*}\big) \end{cases}$$

analogously to (69), with $h = 0, 1$, and then we regard the map in (91) as a map

$$\phi_{\mathtt{int}}^{n-1} : \ \big(\mathrm{t}^{(1)}_{*},\ \mathrm{z}^{(1)}_{*},\ \mathrm{y}^{(1)}_{*},\ \mathrm{x}^{(1)}_{*}\big) \in \mathfrak{D}^{1}_{\mathtt{int}} \to \big(\mathrm{t}^{(0)},\ \mathrm{y}^{(0)},\ \mathrm{x}^{(0)}\big)$$

on the set

$$\mathfrak{D}^{1}_{\mathtt{int}} \ := \ \Big\{\big(\mathrm{t}^{(1)}_{*},\ \mathrm{z}^{(1)}_{*},\ \mathrm{y}^{(1)}_{*},\ \mathrm{x}^{(1)}_{*}\big) : \quad \big(\mathrm{z}^{(1)}_{*,n-1},\ \mathrm{y}^{(1)}_{*,n-1},\ \mathrm{x}^{(1)}_{*,n-1}\big) \in \overline{\mathfrak{B}_{n-1}}\Big\}$$

where $\phi_{\mathtt{int}}^{n-1}$ is defined on the extra–coordinates via the identity. $\mathfrak{D}^{1}_{\mathtt{int}}$ has the form in (78), with $h = 1$. Applying this extension to $f_{\mathtt{sec},0}$ in (90) we obtain

$$f_{\mathtt{sec},\mathtt{int},0} := f_{\mathtt{sec},0} \circ \overline{\phi_{\mathtt{int}}^{n-1}} = \overline{\mathrm{h}_{\mathtt{sec}}^{n-1}}(\hat{\mathrm{y}}^{(1)}_{*,n-1}) + \sum_{i=1}^{n-1} f^{i}_{\mathtt{norm},\mathtt{int},0}(\mathrm{t}^{(1)}_{*,i}, \tilde{\hat{\mathrm{y}}}^{(1)}_{*,i}, \tilde{\hat{\mathrm{x}}}^{(1)}_{*,i})$$

where

$$f^{n-1}_{\mathtt{norm},\mathtt{int},0} := \big(\overline{f_{\mathcal{P}}^{n-1}}^{(\geq 3)} + \widetilde{f^{n-1}_{\mathtt{fast}}}\big) \circ \overline{\phi_{\mathtt{int}}^{n-1}}\ , \qquad f^{i}_{\mathtt{norm},\mathtt{int},0} := \widehat{f^{i}_{\mathtt{fast}}} \circ \overline{\phi_{\mathtt{int}}^{n-1}}$$

and, as a consequence of (60) and of (81), with $h=1$, $f^{i}_{\mathfrak{norm},\mathfrak{int},0}$ depends only on the arguments

$$\begin{aligned}
&\mathrm{t}^{(1)}_{*,i} := \left(\Theta^{(1)}_{*,i},\cdots,\Theta^{(1)}_{*,n-2},\vartheta^{(1)}_{*,i},\cdots,\vartheta^{(1)}_{*,n-2},\right)\\
&\tilde{\mathrm{y}}^{(1)}_{*,i} := \left(p^{(1)}_{*,n-1},\ q^{(1)}_{*,n-1},\ \chi^{(1)}_{*,i-1},\ \cdots,\ \chi_{*,n-1},\ \Lambda^{(1)}_{*,i},\ \cdots,\ \Lambda^{(1)}_{*,n}\right)\\
&\tilde{\mathrm{x}}^{(1)}_{*,i} := \left(\kappa^{(1)}_{*,i},\ \cdots,\ \kappa^{(1)}_{*,n-1}\right) .
\end{aligned}$$

Note, in particular, that $f^{n-1}_{\mathfrak{norm},\mathfrak{int},0}$ is a function of

$$(\mathrm{t}_{*,n-1},\mathrm{y}_{*,n-1},\mathrm{x}_{*,n-1}) = (p^{(1)}_{*n-1}\ ,\ q^{(1)}_{*n-1}\ ,\chi_{*,n-2}\ ,\ \chi_{*,n-1}\ ,\ \Lambda_{*,n-1}\ ,\ \Lambda_{*,n},\ \kappa_{*n-1})\ . \tag{94}$$

In view of the fact that $\overline{\mathrm{h}^{n-1}_{\mathfrak{sec}}}$ depends on the actions in (93), we aim to eliminate from $f_{\mathfrak{sec},\mathfrak{int},0}$ the dependence on the following angles

$$\begin{cases}
\kappa_{*,1} & \text{if} \quad n=2\\
\\
\kappa_{*,n-2},\ \kappa_{*,n-1} & \text{if} \quad n\geq 3
\end{cases}$$

and to retain the dependence on $(p^{(1)}_{*,n-1},\ q^{(1)}_{*,n-1})$ only via $\frac{(p^{(1)}_{*,n-1})^2+(q^{(1)}_{*,n-1})^2}{2}$. Then we choose a domain $\mathfrak{D}^1_{\mathfrak{sec}}\subset\mathfrak{D}^1_{\mathfrak{int}}$ as in (68) where the frequency

$$\omega_{\mathfrak{sec},1} := \begin{cases}
\partial_{\frac{p^{(1)}_{*,n-1})^2+(q^{(1)}_{*,n-1})^2}{2},\chi^{(1)}_{*,n-1}}\ \overline{\mathrm{h}^{n-1}_{\mathfrak{sec}}} & n=2\\
\\
\partial_{\frac{p^{(1)}_{*,n-1})^2+(q^{(1)}_{*,n-1})^2}{2},\chi^{(1)}_{*,n-2},\chi^{(1)}_{*,n-1}}\ \overline{\mathrm{h}^{n-1}_{\mathfrak{sec}}} & n\geq 3
\end{cases}$$

is non–resonant up to the order $\bar{K}$ and on this domain we construct a real–analytic transformation $\phi^{n-1}_{\mathfrak{norm}}$ as in (85) which conjugates $\overline{\mathrm{f}_{\mathfrak{sec},1}}$ to a Hamiltonian

$$f_{\mathfrak{sec},1} := f_{\mathfrak{sec},\mathfrak{int},0}\circ\phi^{n-1}_{\mathfrak{norm}} = \overline{\mathrm{h}^{n-1}_{\mathfrak{sec}}}(\hat{\mathrm{y}}^{(1)}_{n-1}) + \sum_{i=1}^{n-1} f^{i}_{\mathfrak{norm},1}(\mathrm{t}^{(1)}_{i},\hat{\mathrm{y}}^{(1)}_{i},\hat{\mathrm{x}}^{(1)}_{i}) + f_{\mathfrak{exp},\mathfrak{sec},1}$$

Now, since (as it follows from (94)), $f^{n-1}_{\mathfrak{norm},1}$ is actually a function of $\hat{\mathrm{y}}^{(1)}_{n-1}$ only, this step is proved, with

$$\mathrm{h}^{n-1}_{\mathfrak{sec}}(\hat{\mathrm{y}}^{(1)}_{n-1}) := \overline{\mathrm{h}^{n-1}_{\mathfrak{sec}}}(\hat{\mathrm{y}}^{(1)}_{n-1}) + f^{n-1}_{\mathfrak{norm},1}(\hat{\mathrm{y}}^{(1)}_{n-1})\ . \quad \blacksquare \tag{95}$$

5.2.1 Construction of $\phi^{n-1}_{\mathfrak{int}}$

The following lemma completes Lemma 3.4. In particular, it provides the transformation $\phi^{n-1}_{\mathfrak{int}} = \overline{\phi^{n-1}_{\mathfrak{int}}}$ in (92).

Lemma 5.2 *Let $i=1,\cdots,n-1$. Let $\mathcal{A}$, $\mathcal{X}$, θ in (45) be chosen such in a way that*

$$\begin{aligned}
&\inf_{\mathfrak{D}_{\mathcal{P}}}|g|>0\ ,\quad \sup_{\mathfrak{D}_{\mathcal{P}}}|\arg g|<\frac{\pi}{4}\\
&\forall\ g\in\left\{\chi_{i-1}\ ,\quad \chi_{i}\ ,\quad \chi_{i-1}+\chi_{i}\ ,\quad 5\chi_{i-1}\Lambda_i^2-(\chi_{i-1}-\chi_i)^2(2\chi_{i-1}-\chi_i)\right\}\ .
\end{aligned}\tag{96}$$

Then, the domains $\overline{\mathfrak{B}_i}$ *in* (41)*, the functions* $\overline{\mathrm{h}^i_{\mathfrak{sec}}}$ *and the transformations* $\overline{\phi^i_{\mathfrak{int}}}$ *can be taken as follows*

$$\overline{\mathfrak{B}_i} \;=\; \begin{cases} B^2_{\varepsilon_i}\times\mathcal{A}^i_{\bar\theta^i}\times\chi^i_{\bar\theta^i}\times\mathbb{T}^4_{\bar s^i} & i=n-1\\ B^2_{\varepsilon_i}\times\mathcal{A}^i_{\bar\theta^i}\times\chi^i_{\bar\theta^i}\times\mathbb{T}^5_{\bar s^i} & i=1,\cdots,n-2 \;\;\&\;\; n\geq 3\end{cases}$$

$$\overline{\phi^i_{\mathfrak{int}}} \;:\; \begin{cases} \Theta_i=\dfrac{p_i}{\beta_i}+f_i(p_i,q_i,\mathrm{y}^*_i)\\ \vartheta_i-\pi=\beta_i q_i+g_i(p_i,q_i,\mathrm{y}^*_i)\\ \mathrm{y}_i=\mathrm{y}^*_i\\ \mathrm{x}_i=\mathrm{x}^*_i+\varphi_i(p_i,q_i,\mathrm{y}^*_i)\end{cases}$$

$$\overline{\mathrm{h}^i_{\mathfrak{sec}}} \;=\; \mathcal{A}_i\Big[\mathrm{E}_i+\Omega_i\frac{p_i^2+q_i^2}{2}+\tau_i(\frac{p_i^2+q_i^2}{2})^2+\mathrm{O}(p_i,q_i)^6\Big] \tag{97}$$

where $\mathcal{X}^i_{\bar\theta^i}\times\mathcal{A}^i_{\bar\theta^i}$ *denote the projection of the set* $\mathcal{X}_{\bar\theta}\times\mathcal{A}_{\bar\theta}$ *over the coordinates* y_i *in* (42)*,* $\bar\theta:=\theta/2$*,* $\bar s:=s/2$*,* f_i*,* g_i *are* $\mathrm{O}(p_i,q_i)^3$*, odd in* (p_i,q_i)*,* φ_i *is* $\mathrm{O}(p_i,q_i)^2$*, and*

$$\begin{aligned}
\varepsilon_i &= \mathfrak{c}_i\sqrt{\theta_i}\\
\beta_i &:= \sqrt[4]{\frac{5\chi_{i-1}\Lambda_i^2-(\chi_{i-1}-\chi_i)^2(2\chi_{i-1}-\chi_i)}{\chi_{i-1}^2\chi_i^2(\chi_{i-1}+\chi_i)}}\\
\mathcal{A}_i &:= m_im_{i+1}\frac{a_i^2}{4a_{i+1}^3}\\
\mathrm{E}_i &:= -\frac{\Lambda_{i+1}^3}{2(\chi_i-\chi_{i+1})^3}\Big(5-3\frac{(\chi_{i-1}-\chi_i)^2}{\Lambda_i^2}\Big)\\
\Omega_i &:= \frac{3\Lambda_{i+1}^3}{\chi_i\Lambda_i^2(\chi_i-\chi_{i+1})^3}\sqrt{(5\chi_{i-1}\Lambda_i^2-(\chi_{i-1}-\chi_i)^2(2\chi_{i-1}-\chi_i))(\chi_{i-1}+\chi_i)}\\
\tau_i &:= \frac{\Lambda_{i+1}^3}{\chi_i^2(\chi_i-\chi_{i+1})^3}\Big[-\frac{9}{16}\frac{(\chi_{i-1}-\chi_i)^2(3\chi_{i-1}-\chi_i)(5\chi_{i-1}+\chi_i)}{\chi_{i-1}^3\chi_i\Lambda_i^2\beta_i^4}\\
&\quad -\frac{3}{8}\frac{2\chi_{i-1}^3+9\chi_{i-1}^2\chi_i+2\chi_{i-1}\chi_i^2+\chi_i^3}{\chi_{i-1}\Lambda_i^2}-\frac{3}{16}\frac{\chi_{i-1}\chi_i^2}{\Lambda_i^2}(4\chi_{i-1}+\chi_i)\beta_i^4\Big]
\end{aligned} \tag{98}$$

with $\chi_n\equiv 0$*,* $\bar{\mathfrak{c}}_i$ *depending at most on the ratios* a_i^+/a_i^-*, the masses* m_1*,* $\cdots$*,* m_n *and, as usual,* $\sqrt[m]{z}$ *denoting the principal determination of the* m^{th} *root of a complex number* z*.*

Proof Since the formula for $\overline{\overline{f^{n-1,n}_{\mathcal{P}}}}$ coincides with the one for $\overline{f^{n-1,n}_{\mathcal{P}}}$ taking $\chi_n\equiv 0$, we shall only work on the terms $\overline{\overline{f^{i,i+1}_{\mathcal{P}}}}$'s.
Let y_i be as in (42), and let

$$\mathfrak{D}_i:\quad (\Theta_i,\vartheta_i)\in\mathcal{T}^i_{\Theta_i^+,\vartheta_i^+}\qquad \mathrm{y}_i\in\mathcal{A}^i_{\theta_i}\times\mathcal{X}^i_{\theta_i}\qquad \mathrm{x}_i\in\mathbb{T}^{m_i}_s \tag{99}$$

where $\mathcal{T}^i_{\Theta_i^+,\vartheta_i^+}$ is the projection of $\mathcal{T}_{\Theta^+,\vartheta^+}$ over the coordinates (Θ_i,ϑ_i), while m_i is 4 or 5, accordingly to (97). We shall obtain the transformation $\overline{\phi^i_{\mathfrak{int}}}$ in (41) as a product $\overline{\phi^i_{\mathfrak{int}}}=\overline{\phi^i_{\mathfrak{diag}}}\circ\overline{\phi^i_{\mathfrak{bir}}}$, where $\overline{\phi^i_{\mathfrak{diag}}}$ and $\overline{\phi^i_{\mathfrak{bir}}}$ are described below.

A Taylor expansion of $\overline{\overline{f_{\mathcal{P}}^{i,i+1}}}$ around $(\Theta_i, \vartheta_i) = (0, \pi)$ gives

$$\overline{\overline{f_{\mathcal{P}}^{i,i+1}}} \;=\; \mathcal{A}_i\Big[\mathrm{E}_i + \Omega_i \frac{\beta_i^2\Theta_i^2 + \frac{(\vartheta_i-\pi)^2}{\beta_i^2}}{2} + \mathcal{R}_i\Big] \tag{100}$$

where $\mathcal{A}_i$, E_i, β_i, Ω_i are as in (98). Note that β_i, Ω_i are well defined under the assumption (96). The expansion in (100) shows that $(\Theta_i, \vartheta_i) = (0, \pi)$ is an *elliptic* equilibrium point for $\overline{\overline{f_{\mathcal{P}}^{i,i+1}}}$. The remainder $\mathcal{R}_i$ is given by

$$\begin{aligned}\mathcal{R}_i \;=\; & \mathcal{F}\Big[-\frac{3}{2}\frac{4\Theta_i^2-\chi_i^2}{\Lambda_i^2}\Big(\frac{(\chi_i^2-\chi_{i-1}^2)^2}{(\sqrt{\chi_i^2-\Theta_i^2}+\sqrt{\chi_{i-1}^2-\Theta_i^2})^2} \\ & + 2\sqrt{(\chi_i^2-\Theta_i^2)(\chi_{i-1}^2-\Theta_i^2)}(1+\cos\vartheta_i)\Big) + \frac{1}{2}\frac{(\chi_i^2-\Theta_i^2)(\chi_{i-1}^2-\Theta_i^2)}{\Lambda_i^2}\sin^2\vartheta_i\Big]\end{aligned}$$

where the symbol $\mathcal{F}$ on the left means that only terms of the fourth order in $(\Theta_i, \vartheta_i - \pi)$ have to be included. The lower order expansion of $\mathcal{R}_i$ is

$$\mathcal{R}_i = \tau_{1,i}\Theta_i^4 + \tau_{2,i}(\vartheta_i-\pi)^2\Theta_i^2 + \tau_{3,i}(\vartheta_i-\pi)^4 + \mathrm{O}(\Theta_i, \vartheta_i-\pi)^6$$

with

$$\begin{aligned}\tau_{1,i} \;&:=\; \tau_1(\mathrm{y}_i) := -\frac{3(\chi_{i-1}-\chi_i)^2(3\chi_{i-1}-\chi_i)(5\chi_{i-1}+\chi_i)}{8\chi_{i-1}^3\chi_i\Lambda_i^2} \\ \tau_{2,i} \;&:=\; \tau_2(\mathrm{y}_i) := -\frac{3(2\chi_{i-1}^3+9\chi_{i-1}^2\chi_i+2\chi_{i-1}\chi_i^2+\chi_i^3)}{4\chi_{i-1}\Lambda_i^2} \\ \tau_{3,i} \;&:=\; \tau_3(\mathrm{y}_i) := -\frac{\chi_{i-1}\chi_i^2}{8\Lambda_i^2}(4\chi_{i-1}+\chi_i)\ .\end{aligned}$$

We introduce the generating function

$$S_{\mathfrak{diag},i}(\tilde{p}_i, \tilde{\mathrm{y}}_i, \vartheta_i, \mathrm{x}_i) = \frac{\tilde{p}_i(\vartheta_i-\pi)}{\tilde{\beta}_i} + \tilde{\mathrm{y}}_i\mathrm{x}_i\ .$$

It generates the canonical transformation

$$\overline{\phi_{\mathfrak{diag}}^i}: \quad \Theta_i = \frac{\tilde{p}_i}{\tilde{\beta}_i} \qquad \vartheta_i - \pi = \tilde{\beta}_i\tilde{q}_i\ , \quad \mathrm{y}_i = \tilde{\mathrm{y}}_i\ , \quad \mathrm{x}_i = \tilde{\mathrm{x}}_i + \frac{\partial_{\mathrm{y}_i}\beta_i(\tilde{\mathrm{y}}_i)}{\beta_i(\tilde{\mathrm{y}}_i)}\tilde{p}_i\tilde{q}_i$$

which transforms $\overline{\overline{f_{\mathcal{P}}^{i,i+1}}}$ into

$$\overline{\overline{f_{\mathfrak{diag},i}}} = \overline{\overline{f_{\mathcal{P}}^{i,i+1}}} \circ \overline{\phi_{\mathfrak{diag}}^i} \;=\; \tilde{\mathcal{A}}_i\Big[\tilde{\mathrm{E}}_i + \tilde{\Omega}_i\frac{\tilde{p}_i^2+\tilde{q}_i}{2} + \tilde{\mathcal{R}}_i\Big] \tag{101}$$

with

$$\begin{aligned}\tilde{\beta}_i \;&:=\; \beta(\tilde{\mathrm{y}}_i)\ , \quad \tilde{\mathcal{A}}_i := \mathcal{A}_i(\tilde{\mathrm{y}}_i)\ , \quad \tilde{\mathrm{E}}_i := C(\tilde{\mathrm{y}}_i)\ , \quad \tilde{\Omega}_i := \Omega(\tilde{\mathrm{y}}_i)\ , \\ \tilde{\mathcal{R}}_i \;&:=\; \mathcal{R}_i \circ \phi_{\mathfrak{diag}}^i = \tilde{\tau}_{1,i}\tilde{p}_i^4 + \tilde{\tau}_{2,i}\tilde{p}_i^2\tilde{q}_i^2 + \tilde{\tau}_{3,i}\tilde{q}_i^4 + \mathrm{O}(\tilde{p}_i, \tilde{q}_i)^6 \\ \tilde{\tau}_{1,i} \;&:=\; \frac{\tau_1(\tilde{\mathrm{y}}_i)}{\tilde{\beta}_i^4}\ , \qquad \tilde{\tau}_{2,i} := \tau_2(\tilde{\mathrm{y}}_i)\ , \qquad \tilde{\tau}_{3,i} := \tau_3(\tilde{\mathrm{y}}_i)\tilde{\beta}_i^4\end{aligned}$$

To compute the domain of $\overline{\phi^i_{\mathfrak{diag}}}$, we use the following inequalities, which readily follow from the definitions:

$$\hat{\mathfrak{c}}_i \frac{\sqrt{\theta_i}}{\mathrm{G}_n^+} \leq |\beta_i| \leq \frac{1}{\hat{\mathfrak{c}}_i} \frac{\sqrt{\theta_i}}{\mathrm{G}_n^-}$$

and

$$|\frac{\partial_{\mathrm{y}_i} \beta_i(\tilde{\mathrm{y}}_i)}{\beta_i(\tilde{\mathrm{y}}_i)}| \leq \frac{1}{\hat{\mathfrak{c}}_i \theta_i} \ .$$

We then see that, choosing a suitable $\tilde{\mathfrak{c}}_i \leq \hat{\mathfrak{c}}_i$, and the domain

$$\widetilde{\overline{\mathfrak{B}_i}} : \qquad |(\tilde{p}_i, \tilde{q}_i)| \leq \tilde{\varepsilon}_i = \tilde{\mathfrak{c}}_i \sqrt{\theta_i} \qquad \tilde{\mathrm{y}}_i \in \mathcal{A}^i_{\theta_i} \times \mathcal{X}^i_{\theta_i} \qquad \tilde{\mathrm{x}}_i \in \mathbb{T}^{m_i}_{\frac{3}{4}s}$$

inequalities[4] (99) are verified, as desired. Now we look for another canonical transformation

$$\overline{\phi^i_{\mathfrak{bir}}} : \quad (p^*_i, q^*_i, \mathrm{y}^*_i, \mathrm{x}^*_i) \to (\tilde{p}_i, \tilde{q}_i, \tilde{\mathrm{y}}_i, \tilde{\mathrm{x}}_i) \qquad (\mathrm{y}^*_i = \tilde{\mathrm{y}}_i)$$

defined in a analogous domain

$$\overline{\mathfrak{B}_i}^* := \overline{\mathfrak{B}_i} : \qquad |(p^*_i, q^*_i)| \leq \varepsilon_i = \mathfrak{c}^*_i \sqrt{\theta_i} \qquad \mathrm{y}^*_i \in \mathcal{A}^i_{\theta_i} \times \mathcal{X}^i_{\theta_i} \qquad \mathrm{x}^*_i \in \mathbb{T}^{m_i}_{\frac{s}{2}}$$

with $\mathfrak{c}^*_i =: \mathfrak{c}_i \leq \tilde{\mathfrak{c}}_i/2$, such that

$$\overline{\overline{f_{\mathfrak{diag},i}} \circ \phi^i_{\mathfrak{bir}}} = \overline{\mathrm{h}^i_{\mathfrak{sec}}}$$

satisfies the thesis of the lemma. We aim to apply Theorem D.1, with

$$\mathrm{h} = \tilde{\mathrm{E}}_i + \tilde{\Omega}_i \frac{\tilde{p}_i^2 + \tilde{q}_i}{2} \ , \qquad f = \mathcal{R}_i \ , \quad \varepsilon = 2\mathfrak{c}^*_i \sqrt{\theta_i} \ , \quad \bar{\varepsilon} = \mathfrak{c}^*_i \sqrt{\theta_i} \ .$$

We have to check that inequalities (151) are satisfied. We can take $\mathfrak{a}$ and $\mathfrak{e}$ as it follows from the following inequalities, which, in turn, are easily implied by the definitions

$$\begin{aligned}
\inf_{\mathfrak{B}^*_i} |\partial \mathrm{h}| \quad &= \quad \inf_{\mathfrak{B}^*_i} |\tilde{\Omega}_i| \geq \frac{\check{\mathfrak{c}}_i |\mathrm{G}_n^-|^2}{\theta_i} =: \mathfrak{a} \\
\sup_{\mathfrak{B}^*_i} |\tilde{\mathcal{R}}_i| \quad &\leq \quad \sup_{\mathfrak{D}^*_i} |\mathcal{R}_i| \leq \frac{1}{\check{\mathfrak{c}}_i} \max \sup_{\mathfrak{D}^*_i} \{ \frac{(\Theta^*_i)^4}{(\mathrm{G}_n^-)^2} \ , \ (\vartheta^*_i - \pi)^2 \Theta_i^2 \ , \ (\mathrm{G}_n^+)^2 (\vartheta^*_i - \pi)^4 \} \\
&\leq \quad \frac{(\mathfrak{c}^*)^4 (\mathrm{G}_n^+)^2}{\bar{\mathfrak{c}}_i} =: \mathfrak{e}
\end{aligned}$$

Here, we have used that, for $|(p^*_i, q^*_i)| \leq 2\mathfrak{c}^* \sqrt{\theta_i}$, $(\Theta^*_i, \vartheta^*_i) := (\overline{\phi^i_{\mathfrak{diag}}})^{-1}(p^*_i, q^*_i)$ verifies

$$\begin{aligned}
|\Theta^*_i| = \frac{|p^*_i|}{|\beta_i|} &\leq 2\mathfrak{c}^* \sqrt{\theta_i} \frac{\mathrm{G}_n^+}{\hat{\mathfrak{c}}_i \sqrt{\theta_i}} = 2 \frac{\mathfrak{c}^* \mathrm{G}_n^+}{\hat{\mathfrak{c}}_i} \\
|\vartheta^*_i - \pi| = |q^*_i||\beta_i| &\leq \frac{2\mathfrak{c}^* \sqrt{\theta_i}}{\mathfrak{c}_1} \frac{\sqrt{\theta_i}}{\mathrm{G}_n^-} = 2 \frac{\mathfrak{c}^*}{\hat{\mathfrak{c}}_i} \frac{\theta_i}{\mathrm{G}_n^-} \ .
\end{aligned}$$

We then have that condition (151) holds, provided one takes

$$\mathfrak{c}^* := \min \{ \frac{\mathrm{G}_n^-}{\mathrm{G}_n^+} \sqrt{\check{\mathfrak{c}}_i \bar{\mathfrak{c}}_i} \ , \ \frac{\tilde{\mathfrak{c}}_i}{2} \} \ .$$

From (101), one easily computes that the fourth order term of $\overline{\mathrm{h}^i_{\mathfrak{sec}}}$ corresponds to be as in (97), with

$$\tau_i = \frac{3}{2} \tau^*_{1,i} + \frac{1}{2} \tau^*_{2,i} + \frac{3}{2} \tau^*_{3,i} \qquad \tau^*_{j,i} := \tilde{\tau}_{j,i}(\mathrm{y}^*_i) \ .$$

Finally, properties (44) easily follow from the construction. ∎

[4] Compare (49).

5.2.2 Construction of $\phi^1_{\mathtt{int}}, \cdots, \phi^{n-2}_{\mathtt{int}}$ $(n \geq 3)$

We have to solve (79), assuming that Proposition 5.2 holds, up to rank $h-1$. Accordingly to (73), (74) and letting

$$\overline{\Phi^{n-h+1}_{\mathtt{int}}} := \overline{\phi^{n-h+1}_{\mathtt{int}}} \circ \cdots \circ \overline{\phi^{n-1}_{\mathtt{int}}}$$

we may split

$$\begin{aligned}
f^{n-h}_{\mathtt{norm},h-1} &= \sum_{j=n-h+1}^{n} \Pi_{h-1}\big(\overline{f^{n-h,j}_{\mathcal{P}}}^{(\geq 2)} \circ \overline{\Phi^{n-h+1}_{\mathtt{int}}}\big) + \widetilde{f^{n-h}_{\mathfrak{sec},h-1}} \\
&= \Pi_{h-1}\big(\overline{f^{n-h,n-h+1}_{\mathcal{P}}}^{(2)} \circ \overline{\Phi^{n-h+1}_{\mathtt{int}}}\big) + \Pi_{h-1}\big(\overline{f^{n-h,n-h+1}_{\mathcal{P}}}^{(\geq 3)} \circ \overline{\Phi^{n-h+1}_{\mathtt{int}}}\big) \\
&+ \sum_{j=n-h+2}^{n} \Pi_{h-1}\big(\overline{f^{n-h,j}_{\mathcal{P}}}^{(\geq 2)} \circ \overline{\Phi^{n-h+1}_{\mathtt{int}}}\big) + \widetilde{f^{n-h}_{\mathfrak{sec},h-1}} \\
&= \overline{\overline{f^{n-h,n-h+1}_{\mathcal{P}}}}^{(2)} + \Pi_{h-1}\big(\widetilde{\overline{f^{n-h,n-h+1}_{\mathcal{P}}}}^{(2)} \circ \overline{\Phi^{n-h+1}_{\mathtt{int}}}\big) \\
&+ \Pi_{h-1}\big(\overline{f^{n-h,n-h+1}_{\mathcal{P}}}^{(\geq 3)} \circ \overline{\Phi^{n-h+1}_{\mathtt{int}}}\big) + \sum_{j=n-h+2}^{n} \Pi_{h-1}\big(\overline{f^{n-h,j}_{\mathcal{P}}}^{(\geq 2)} \circ \overline{\Phi^{n-h+1}_{\mathtt{int}}}\big) \\
&+ \widetilde{f^{n-h}_{\mathfrak{sec},h-1}}
\end{aligned}$$

where

$$\begin{aligned}
\overline{f^{n-h,n-h+1}_{\mathcal{P}}}^{(\geq 3)} &:= \overline{f^{n-h,n-h+1}_{\mathcal{P}}}^{(\geq 2)} - \overline{f^{n-h,n-h+1}_{\mathcal{P}}}^{(2)} \\
\widetilde{\overline{f^{n-h,n-h+1}_{\mathcal{P}}}}^{(2)} &:= \overline{f^{n-h,n-h+1}_{\mathcal{P}}}^{(2)} - \overline{\overline{f^{n-h,n-h+1}_{\mathcal{P}}}}^{(2)}
\end{aligned}$$

and $\overline{\overline{f^{n-h,n-h+1}_{\mathcal{P}}}}^{(2)}$ as in Lemma 3.4. Note that we have used that $\overline{\overline{f^{n-h,n-h+1}_{\mathcal{P}}}}^{(2)}$ is left unvaried by $\overline{\Phi^{n-h+1}_{\mathtt{int}}}$. Let $\overline{\mathfrak{B}_{n-h}}$, $\overline{\phi^{n-h}_{\mathtt{int}}}$ be as in Lemmata 3.4, with the symbols $(\Theta_{n-h}, \vartheta_{n-h})$, y_{n-h}, x_{n-h} of that lemma corresponding to

$$\begin{aligned}
\overline{\mathrm{t}}^{(h-1)}_{n-h} &:= \big(\Theta^{(h-1)}_{n-h},\ \vartheta^{(h-1)}_{n-h}\big) \\
\overline{\mathrm{y}}^{(h-1)}_{n-h} &:= \big(\chi^{(h-1)}_{n-h-1}\ ,\ \chi^{(h-1)}_{n-h}\ ,\ \chi^{(h-1)}_{n-h+1}\ ,\ \Lambda^{(h-1)}_{n-h}\ ,\ \Lambda^{(h-1)}_{n-h+1}\big) \\
\overline{\mathrm{x}}^{(h-1)}_{n-h} &:= \big(\kappa^{(h-1)}_{n-h-1}\ ,\ \kappa^{(h-1)}_{n-h}\ ,\ \kappa^{(h-1)}_{n-h+1}\ ,\ \ell^{(h-1)}_{n-h}\ ,\ \ell^{(h-1)}_{n-h+1}\big)
\end{aligned}$$

and the symbols (p_{n-h}, q_{n-h}), y^*_{n-h}, x^*_{n-h} to

$$\begin{aligned}
\overline{\mathrm{z}}^{*(h)}_{n-h} &:= \big(p^{*(h)}_{n-h},\ q^{*(h)}_{n-h}\big) \\
\overline{\mathrm{y}}^{*(h)}_{n-h} &:= \big(\chi^{*(h)}_{n-h-1}\ ,\ \chi^{*(h)}_{n-h}\ ,\ \chi^{*(h)}_{n-h+1}\ ,\ \Lambda^{*(h)}_{n-h}\ ,\ \Lambda^{*(h)}_{n-h+1}\big) \\
\overline{\mathrm{x}}^{*(h)}_{n-h} &:= \big(\kappa^{*(h)}_{n-h-1}\ ,\ \kappa^{*(h)}_{n-h}\ ,\ \kappa^{*(h)}_{n-h+1}\ ,\ \ell^{*(h)}_{n-h}\ ,\ \ell^{*(h)}_{n-h+1}\big)\ .
\end{aligned}$$

Defining

$$\begin{aligned}
\mathrm{t}^{*(h)} &:= \big(\Theta^{*(h)},\ \vartheta^{*(h)}\big) \\
\mathrm{z}^{*(h)} &:= \big(p^{*(h)},\ q^{*(h)}\big) \\
\mathrm{y}^{*(h)} &:= \big(\chi^{*(h)}, \Lambda^{*(h)}\big) \\
\mathrm{x}^{*(h)} &:= \big(\kappa^{*(h)}, \ell^{*(h)}\big)
\end{aligned}$$

in an alagous way as in (69), we regard $\overline{\phi^{n-h}_{\mathfrak{int}}}$ as a map on the set

$$\overline{\mathfrak{D}^h_{\mathfrak{int}}} \;:=\; \Big\{\big(\mathrm{t}^{*(h)},\ \mathrm{z}^{*(h)},\ \mathrm{y}^{*(h)},\ \mathrm{x}^{*(h)}\big):\ \big(\overline{\mathrm{z}}^{*(h)}_{n-h},\ \overline{\mathrm{y}}^{*(h)}_{n-h},\ \overline{\mathrm{x}}^{*(h)}_{n-h}\big)\in\overline{\mathfrak{B}_{n-h}}\Big\}$$

extended via the identity on the extra–coordinates. We then have that $\overline{\phi^{n-h}_{\mathfrak{int}}}$ transforms $f^{n-h}_{\mathfrak{sec},h-1}$ into

$$\overline{f^{n-h}_{\mathfrak{sec},h-1}} := f^{n-h}_{\mathfrak{sec},h-1}\circ\overline{\phi^{n-h}_{\mathfrak{int}}} \;=\; \overline{\mathrm{h}^{n-h}_{\mathfrak{sec}}}+\overline{\mathrm{f}^{n-h}_{\mathfrak{sec}}}$$

where

$$\begin{aligned}\overline{\mathrm{f}^{n-h}_{\mathfrak{sec}}} \;:=\;& \Pi_{h-1}\big(\widetilde{\overline{f^{n-h,n-h+1}_{\mathcal{P}}}}^{(2)}\circ\overline{\Phi^{n-h}_{\mathfrak{int}}}\big)\\ &+\Pi_{h-1}\big(\overline{f^{n-h,n-h+1}_{\mathcal{P}}}^{(\geq 3)}\circ\overline{\Phi^{n-h}_{\mathfrak{int}}}\big)+\sum_{j=n-h+2}^{n}\Pi_{h-1}\big(\overline{f^{n-h,j}_{\mathcal{P}}}^{(\geq 2)}\circ\overline{\Phi^{n-h}_{\mathfrak{int}}}\big)\\ &+\widetilde{f^{n-h}_{\mathfrak{sec},h-1}}\circ\overline{\phi^{n-h}_{\mathfrak{int}}}\ .\end{aligned}\tag{102}$$

Here, we have used $\overline{\Phi^{n-h}_{\mathfrak{int}}}=\overline{\Phi^{n-h+1}_{\mathfrak{int}}}\circ\overline{\phi^{n-h}_{\mathfrak{int}}}$; that Π_{h-1} and $\overline{\phi^{n-h}_{\mathfrak{int}}}$ commute and observe that $\overline{\mathfrak{D}^h_{\mathfrak{int}}}$ has the form of $\mathfrak{D}^h_{\mathfrak{int}}$ in (78), with $\hat{\mathfrak{c}}_h$ replaced by a suitable $\hat{\mathfrak{c}}'_h$ of the same form. The function $\overline{\mathrm{f}^{n-h}_{\mathfrak{sec}}}$ satisfies the following two properties:

- It depends only on
$$\big(p^{*(h)}_{n-h},\ q^{*(h)}_{n-h},\ \tilde{\mathrm{y}}^{*(h)}\big)$$
where $\tilde{\mathrm{y}}^{*(h)}$ is defined analogously to (82);
- is uniformly bounded by the right hand side of the first inequality in (75) (this follows from the definition in (102));
- is even for
$$(p^{*(h)}_{n-h},q^{*(h)}_{n-h})\to-(p^{*(h)}_{n-h},q^{*(h)}_{n-h})\ .$$

Proceeding in a similar way as we did for the construction of $\overline{\phi^i_{\mathfrak{bit}}}$ in the proof of Lemma 5.2, we may apply Theorem D.1, with

$$\begin{aligned}&\mathrm{h}=\overline{\mathrm{h}^{n-h}_{\mathfrak{sec}}}\ ,\quad f=\overline{\mathrm{f}^{n-h}_{\mathfrak{sec}}}\ ,\quad (P,Q)=(p^{*(h)}_{n-h},q^{*(h)}_{n-h})\\ &(P',Q')=\hat{\mathrm{z}}^{*(h)}_{n-h}\ ,\quad \mathrm{y}=\mathrm{y}^{*(h)}_{n-h}\ ,\quad \mathrm{x}=\mathrm{x}^{*(h)}_{n-h}\ .\end{aligned}$$

with $\mathrm{y}^{*(h)}_{n-h}$, $\mathrm{x}^{*(h)}_{n-h}$ defined analogously to $\mathrm{y}^{(h)}_{*,n-h}$, $\mathrm{x}^{(h)}_{*,n-h}$ in (80) and $\hat{\mathrm{z}}^{*(h)}_{n-h}$ defined analogously to $\hat{\mathrm{z}}^{(h)}_{*,n-h}$ in (82). We then find another domain $\mathfrak{D}^h_{\mathfrak{int}}$ as in (78) and another real–analytic transformation

$$\phi^{n-h}_{*,\mathfrak{int}}:\qquad \big(\mathrm{t}^{(h)}_*,\mathrm{z}^{(h)}_*,\mathrm{y}^{(h)}_*,\mathrm{x}^{(h)}_*\big)\in\mathfrak{D}^h_{\mathfrak{int}}\to\big(\mathrm{t}^{*(h)},\mathrm{z}^{*(h)},\mathrm{y}^{*(h)},\mathrm{x}^{*(h)}\big)\in\overline{\mathfrak{D}_{\mathfrak{int},h}}$$

such that

$$\widetilde{f^{n-h}_{\mathfrak{sec},h-1}}:=\overline{f^{n-h}_{\mathfrak{sec},h-1}}\circ\phi^{n-h}_{*,\mathfrak{int}}=f^{n-h}_{\mathfrak{sec},h-1}\circ\overline{\phi^{n-h}_{\mathfrak{int}}}\circ\phi^{n-h}_{*,\mathfrak{int}}\quad=\quad\mathrm{h}^{n-h}_{\mathfrak{sec}}$$

as desired, depends only on $\hat{\mathrm{y}}^{(h)}_{*,n-h}$ in (70), and hence (79) is satisfied. That $\phi^{n-h}_{*,\mathfrak{int}}$ may be also chosen of a form analogue to (81), with $\Theta^{(h-1)}_{n-h}$, $\vartheta^{(h-1)}_{n-h}$, $\hat{\mathrm{z}}^{(h-1)}$, $\mathrm{y}^{(h-1)}_{n-h}$, $\mathrm{x}^{(h-1)}_{n-h}$ replaced by $p^{*(h)}$, $q^{*(h)}$, $\hat{\mathrm{z}}^{*(h)}$, $\mathrm{y}^{*(h)}$, $\mathrm{x}^{*(h)}$ also easily follows from the properties bove. Therefore the composition

$$\phi^{n-h}_{\mathfrak{int}}:=\overline{\phi^{n-h}_{\mathfrak{int}}}\circ\phi^{n-h}_{*,\mathfrak{int}}$$

has again the form in (81) and satisfies (79), as wanted. ∎

5.2.3 Construction of $\phi^1_{\mathtt{norm}}, \cdots, \phi^{n-2}_{\mathtt{norm}}$ $(n \geq 3)$

In this section we aim to determine, for $n \geq 3$ and $1 \leq h \leq n-2$, a transformation $\phi^{n-h}_{\mathtt{norm}}$ solving (85)–(86), assuming the Proposition 5.2 holds up to rank $(h-1)$ and that $\phi^{n-h}_{\mathtt{int}}$ has been constructed.

We switch from the coordinates $(\chi^{(h)}_*, \kappa^{(h)}_*)$ defined implicitly via the right hand side of (85) to the auxiliary coordinates

$$\mathrm{G}^{(h)}_{\mathtt{aux}} = (\mathrm{G}^{(h)}_{\mathtt{aux},1}, \cdots, \mathrm{G}^{(h)}_{\mathtt{aux},n}) \ , \quad \mathrm{g}^{(h)}_{\mathtt{aux}} = (\mathrm{g}^{(h)}_{\mathtt{aux},1}, \cdots, \mathrm{g}^{(h)}_{\mathtt{aux},n})$$

defined via the linear transformation

$$\phi^{n-h}_{\mathtt{aux}} : \quad \begin{cases} \chi^{(h)}_{*,i-1} = \mathrm{G}^{(h)}_{\mathtt{aux},i} + \cdots + \mathrm{G}^{(h)}_{\mathtt{aux},n} \\ \\ \kappa^{(h)}_{*,i-1} = \mathrm{g}^{(h)}_{\mathtt{aux},i} - \mathrm{g}^{(h)}_{\mathtt{aux},i-1} \end{cases} \tag{103}$$

with $1 \leq i \leq n$ and $\mathrm{g}_{\mathtt{aux},0} := 0$. We regard $\phi^{n-h}_{\mathtt{aux}}$ as a transformation on all the coordinates, extending it as the identity on the remaining ones. We denote the new coordinates as

$$\begin{aligned} \mathrm{t}^{(h)}_{\mathtt{aux}} &:= \begin{cases} (\Theta^{(h)}_{\mathtt{aux},1}, \ \cdots, \ \Theta^{(h)}_{\mathtt{aux},n-h-1}, \ \vartheta^{(h)}_{\mathtt{aux},1}, \ \cdots, \ \vartheta^{(h)}_{\mathtt{aux},n-h-1}) \\ \qquad n \geq 4, \ 2 \leq h \leq n-2 \\ \emptyset \qquad \text{otherwise} \end{cases} \\ \mathrm{z}^{(h)}_{\mathtt{aux}} &:= (p^{(h)}_{\mathtt{aux},n-h}, \ \cdots, \ p^{(h)}_{\mathtt{aux},n-1}, \ q^{(h)}_{\mathtt{aux},n-h}, \ \cdots, \ q^{(h)}_{\mathtt{aux},n-1}) \\ \mathrm{y}^{(h)}_{\mathtt{aux}} &:= (\mathrm{G}^{(h)}_{\mathtt{aux},1}, \ \cdots, \ \mathrm{G}^{(h)}_{\mathtt{aux},n}, \ \Lambda^{(h)}_{\mathtt{aux},1}, \ \cdots, \ \Lambda^{(h)}_{\mathtt{aux},n}) \\ \mathrm{x}^{(h)}_{\mathtt{aux}} &:= (\mathrm{g}^{(h)}_{\mathtt{aux},1}, \ \cdots, \ \mathrm{g}^{(h)}_{\mathtt{aux},n}, \ \ell^{(h)}_{\mathtt{aux},1}, \ \cdots, \ \ell^{(h)}_{\mathtt{aux},n}) \end{aligned}$$

the new Hamiltonian as

$$f_{\mathtt{sec},\mathtt{int},\mathtt{aux},h-1}(\mathrm{t}^{(h)}_{\mathtt{aux}}, \mathrm{z}^{(h)}_{\mathtt{aux}}, \mathrm{y}^{(h)}_{\mathtt{aux}}, \mathrm{x}^{(h)}_{\mathtt{aux}}) := f_{\mathtt{sec},\mathtt{int},h-1} \circ \phi^{n-h}_{\mathtt{aux}}(\mathrm{t}^{(h)}_{\mathtt{aux}}, \mathrm{z}^{(h)}_{\mathtt{aux}}, \mathrm{y}^{(h)}_{\mathtt{aux}}, \mathrm{x}^{(h)}_{\mathtt{aux}}) \ . \tag{104}$$

Now we define the domain where we want to consider $f_{\mathtt{sec},\mathtt{int},\mathtt{aux},h-1}$. Firstly, we let

$$\mathfrak{D}^h_{\mathtt{int},\mathtt{aux}} := \Big\{ (\mathrm{t}^{(h)}_{\mathtt{aux}}, \mathrm{z}^{(h)}_{\mathtt{aux}}, \mathrm{y}^{(h)}_{\mathtt{aux}}, \mathrm{x}^{(h)}_{\mathtt{aux}}) : \quad (\mathrm{t}^{(h)}_*, \mathrm{z}^{(h)}_*, \mathrm{y}^{(h)}_*, \mathrm{x}^{(h)}_*) \in \mathfrak{D}^h_{\mathtt{int}} \Big\}$$

where $\mathfrak{D}^h_{\mathtt{int}}$ is defined in (78). Then $\mathfrak{D}^h_{\mathtt{int},\mathtt{aux}}$ is given by

$$\mathfrak{D}^h_{\mathtt{int},\mathtt{aux}} = \mathcal{T}^h_{\hat{\mathfrak{c}}_h \theta} \times B^2_{\hat{\mathfrak{c}}_h \varepsilon_{n-h}} \times B^{*,h-1}_{\hat{\mathfrak{c}}_h \varepsilon} \times (\mathcal{G}_*)_{\overline{\mathfrak{c}}_h \theta, \overline{\mathfrak{c}}_h \bar{r}} \times \mathcal{A}_{\overline{\mathfrak{c}}_h \bar{r}^{(h)}} \times \mathbb{T}^n_{\overline{\mathfrak{c}}_h s} \times \mathbb{T}^n_{\overline{\mathfrak{c}}_h s} \ ,$$

with

$$(\mathcal{G}_*)_{\overline{\mathfrak{c}}_h \theta, \overline{\mathfrak{c}}_h \bar{r}} := (\mathcal{G}_1)_{\hat{\mathfrak{c}}_1 \theta_1} \times \cdots \times (\mathcal{G}_{n-h})_{\hat{\mathfrak{c}}_{n-h} \theta_{n-h}} \times (\mathcal{G}^*_{n-h+1})_{\hat{\mathfrak{c}}_{n-h+1} \overline{r}_{n-h+1}} \times \cdots \times (\mathcal{G}^*_{n-1})_{\hat{\mathfrak{c}}_{n-1} \overline{r}_{n-1}} \ .$$

Next, for $1 \leq h' \leq h$ and any fixed $\bar{\gamma}$, $\bar{K} > 0$ and $\bar{\tau} > 2$, we define

$$\omega^{n-h'}_{\mathtt{sec}}(\hat{\mathrm{y}}^{(h)}_{\mathtt{aux},n-h'}) :=$$

$$\begin{cases} \partial_{\frac{(p^{(1)}_{\mathtt{aux},n-1})^2 + (q^{(1)}_{\mathtt{aux},n-1})^2}{2}, \mathrm{G}^{(1)}_{\mathtt{aux},n-1}, \mathrm{G}^{(1)}_{\mathtt{aux},n}} \mathrm{h}^{n-1}_{\mathtt{sec}}(\hat{\mathrm{y}}^{(h)}_{\mathtt{aux},n-1}) & n \geq 3 \ , \ h' = 1, \ 2 \leq h \leq n-1 \\ \\ \partial_{\frac{(p^{(h')}_{\mathtt{aux},n-h'})^2 + (q^{(h')}_{\mathtt{aux},n-h'})^2}{2}, \mathrm{G}^{(h')}_{\mathtt{aux},n-h'}} \mathrm{h}^{n-h'}_{\mathtt{sec}}(\hat{\mathrm{y}}^{(h)}_{\mathtt{aux},n-h}) & n \geq 3 \ , \quad 2 \leq h' \leq h \leq n-1, \\ & (h', \ h) \neq (n-1, n-1) \\ \\ \partial_{\frac{(p^{(n-1)}_{\mathtt{aux},1})^2 + (q^{(n-1)}_{\mathtt{aux},1})^2}{2}} \mathrm{h}^{n-h}_{\mathtt{sec}}(\hat{\mathrm{y}}^{(n-1)}_{\mathtt{aux},1}) & h' = h = n-1 \ . \end{cases} \tag{105}$$

We then choose the following sub–domain of $\mathfrak{D}^h_{\mathtt{int},\mathtt{aux}}$

$$\begin{aligned}\mathfrak{D}^h_{\mathtt{sec},\mathtt{aux}} \;:=\; &\Big\{(\mathrm{t}^{(h)}_{\mathtt{norm},\mathtt{aux}}, \mathrm{z}^{(h)}_{\mathtt{norm},\mathtt{aux}}, \mathrm{y}^{(h)}_{\mathtt{norm},\mathtt{aux}}, \mathrm{x}^{(h)}_{\mathtt{norm},\mathtt{aux}}) \in \mathfrak{D}^h_{\mathtt{int},\mathtt{aux}} : \\ &|\omega^{n-h'}_{\mathtt{sec}} \cdot \mathrm{k}| \geq \frac{(a^+_{n-h'})^2}{(a^-_{n-h'+1})^3 \theta_{n-h}} \frac{\bar\gamma}{\bar K^{\bar\tau}} \;, \\ &\forall\, \mathrm{k} \in \mathbb{Z}^j \setminus \{0\},\ |\mathrm{k}|_1 \leq \bar K \;, \quad \forall\, 2 \leq h' \leq h \Big\} \;. \end{aligned} \tag{106}$$

Here j is chosen to be 3, 2 or 1 accordingly to the three cases above. The set $\mathfrak{D}^h_{\mathtt{int},\mathtt{aux}}$ is non–empty, if $\bar\gamma$ is chosen suitably small. Indeed, if we put

$$\hat{\mathrm{y}}^{(h)}_{\mathtt{aux},n-h} := \Big(\frac{(p^{(h)}_{\mathtt{aux},n-h})^2 + (q^{(h)}_{\mathtt{aux},n-h})^2}{2}, \mathrm{G}^{(h)}_{\mathtt{aux},n-h}, \Lambda^{(h)}_{\mathtt{aux},n-h},\ \hat{\mathrm{y}}^{(h)}_{\mathtt{aux},n-h+1}\Big)$$

then standard quantitative arguments show that, for any fixed value

$$\big(\bar\Lambda^{(h)}_{\mathtt{aux},n-h},\ \hat{\bar{\mathrm{y}}}^{(h)}_{\mathtt{aux},n-h+1}\big) \in \Pi_{\Lambda^{(h)}_{\mathtt{aux},n-h}, \hat{\mathrm{y}}^{(h)}_{\mathtt{aux},n-h+1}} \mathfrak{D}^h_{\mathtt{int},\mathtt{aux}} \;,$$

the measure of the set $\mathcal{N}_{n-h} \subset B^2_{\bar{\mathfrak{c}}_h \varepsilon_{n-h}} \times \mathcal{G}_{n-h}$ of $(p^{(h)}_{\mathtt{aux},n-h},\ q^{(h)}_{\mathtt{aux},n-h},\ \mathrm{G}^{(h)}_{\mathtt{aux},n-h})$ where the inequality in (106) does not hold may be bounded as

$$\mathrm{meas}\, \mathcal{N}_{n-h} \leq \frac{\bar\gamma}{\mathfrak{c}}\, \mathrm{meas}\, \big(B^2_{\bar{\mathfrak{c}}_h \varepsilon_{n-h}} \times \mathcal{G}_{n-h}\big) \;,$$

(where $\mathfrak{c}$ depends only on the semi–axes ratio and the masses), hence (67) follows. This is because $\omega^{n-h'}_{\mathtt{sec}}(\hat{\mathrm{y}}^{(h)}_{\mathtt{aux},n-h})$ is a diffeomorphism (Compare Appendix C).
Now we inspect the form of $f_{\mathtt{sec},\mathtt{int},\mathtt{aux},h-1}$ in (104). Introducing the following symbols

$$\begin{aligned}
\mathrm{t}^{(h)}_{\mathtt{aux},i} \;&:=\; \begin{cases} (\Theta^{(h)}_{\mathtt{aux},i},\ \cdots,\ \Theta^{(h)}_{\mathtt{aux},n-h-1},\ \vartheta^{(h)}_{\mathtt{aux},i},\ \cdots,\ \vartheta^{(h)}_{\mathtt{aux},n-h-1}) \\ \qquad n \geq 4,\ 2 \leq h \leq n-2,\ 1 \leq i \leq n-h-1 \\ \emptyset \qquad \text{otherwise} \end{cases} \\
\mathrm{y}^{(h)}_{\mathtt{aux},i} \;&:=\; (\mathrm{G}^{(h)}_{\mathtt{aux},i},\ \cdots,\ \mathrm{G}^{(h)}_{\mathtt{aux},n},\ \Lambda^{(h)}_{\mathtt{aux},1},\ \cdots,\ \Lambda^{(h)}_{\mathtt{aux},n}) \\
\mathrm{x}^{(h)}_{\mathtt{aux},i} \;&:=\; (\mathrm{g}^{(h)}_{\mathtt{aux},i},\ \cdots,\ \mathrm{g}^{(h)}_{\mathtt{aux},n},\ \ell^{(h)}_{\mathtt{aux},1},\ \cdots,\ \ell^{(h)}_{\mathtt{aux},n}) \\
\hat{\mathrm{y}}^{(h)}_{\mathtt{aux},i} \;&:=\; (\frac{(p^{(h)}_{\mathtt{aux},i})^2 + (q^{(h)}_{\mathtt{aux},i})^2}{2},\ \cdots,\ \frac{(p^{(h)}_{\mathtt{aux},n-1})^2 + (q^{(h)}_{\mathtt{aux},n-1})^2}{2},\ \mathrm{G}^{(h)}_{\mathtt{aux},i},\ \cdots,\ \mathrm{G}^{(h)}_{\mathtt{aux},n}, \\
&\quad \Lambda^{(h)}_{\mathtt{aux},i},\ \cdots,\ \Lambda^{(h)}_{\mathtt{aux},n}) \\
\hat{\mathrm{x}}^{(h)}_{\mathtt{aux},i} \;&:=\; \begin{cases} (\mathrm{g}^{(h)}_{\mathtt{aux},i+1} - \mathrm{g}^{(h)}_{\mathtt{aux},i},\ \cdots, \mathrm{g}^{(h)}_{\mathtt{aux},n-h} - \mathrm{g}^{(h)}_{\mathtt{aux},n-h-1}) \quad n \geq 4\ \&\ 1 \leq h-1 \leq n-3 \\ \emptyset \qquad \text{otherwise} \end{cases} \\
\hat{\mathrm{X}}^{(h)}_{\mathtt{aux},i} \;&:=\; \begin{cases} (\mathrm{G}^{(h)}_{\mathtt{aux},i},\ \cdots, \mathrm{G}^{(h)}_{\mathtt{aux},n-h}) \quad n \geq 4\ \&\ 1 \leq h-1 \leq n-3 \\ \emptyset \qquad \text{otherwise} \end{cases} \\
\mathrm{z}^{(h)}_{\mathtt{aux},n-h} \;&:=\; (p^{(h)}_{\mathtt{aux},n-h},\ q^{(h)}_{\mathtt{aux},n-h}) \;, \qquad \hat{\mathrm{z}}^{(h)}_{\mathtt{norm},j} := p^{(h)}_{\mathtt{norm},j} + \mathrm{i} q^{(h)}_{\mathtt{norm},j} \\
\tilde{\mathrm{y}}^{(h)}_{\mathtt{aux},i} \;&:=\; (\frac{(p^{(h)}_{\mathtt{aux},n-h+1})^2 + (q^{(h)}_{\mathtt{aux},n-h+1})^2}{2},\ \cdots,\ \frac{(p^{(h)}_{\mathtt{aux},n-1})^2 + (q^{(h)}_{\mathtt{aux},n-1})^2}{2}, \\
&\quad \mathrm{G}^{(h)}_{\mathtt{aux},i},\ \cdots,\ \mathrm{G}^{(h)}_{\mathtt{aux},n}, \Lambda^{(h)}_{\mathtt{aux},i},\ \cdots,\ \Lambda^{(h)}_{\mathtt{aux},n}) \\
\tilde{\mathrm{y}}^{(h)}_{\mathtt{aux}} \;&:=\; \tilde{\mathrm{y}}^{(h)}_{\mathtt{aux},1} \;, \quad \hat{\mathrm{x}}^{(h)}_{\mathtt{aux}} := \hat{\mathrm{x}}^{(h)}_{\mathtt{aux},1} \;, \quad \hat{\mathrm{X}}^{(h)}_{\mathtt{aux}} := \hat{\mathrm{X}}^{(h)}_{\mathtt{aux},1} \;,
\end{aligned}$$

by means of (84), we have

$$\begin{aligned} f_{\mathfrak{sec},\mathrm{int},\mathfrak{aux},h-1}(\mathrm{t}^{(h)}_{\mathfrak{aux}},\mathrm{z}^{(h)}_{\mathfrak{aux}},\mathrm{y}^{(h)}_{\mathfrak{aux}},\mathrm{x}^{(h)}_{\mathfrak{aux}}) &= \mathrm{h}_{\mathfrak{sec},h}(\hat{\mathrm{y}}^{(h)}) + f_{\mathrm{norm},\mathrm{int},\mathfrak{aux},h-1}(\mathrm{t}^{(h)}_{\mathfrak{aux}},\mathrm{z}^{(h)}_{\mathfrak{aux},n-h},\tilde{\mathrm{y}}^{(h)}_{\mathfrak{aux}},\hat{\mathrm{x}}^{(h)}_{\mathfrak{aux}}) \\ &= \sum_{i=n-h}^{n-1} \mathrm{h}^{i}_{\mathfrak{sec}}(\hat{\mathrm{y}}^{(h)}_{i}) \\ &+ \sum_{i=1}^{n-h-1} f^{i}_{\mathrm{norm},\mathrm{int},\mathfrak{aux},h-1}(\mathrm{t}^{(h)}_{\mathfrak{aux},i},\mathrm{z}^{(h)}_{\mathfrak{aux},n-h},\tilde{\mathrm{y}}^{(h)}_{\mathfrak{aux},i},\hat{\mathrm{x}}^{(h)}_{\mathfrak{aux},i}) \end{aligned} \tag{107}$$

where we have let

$$f_{\mathrm{norm},\mathrm{int},\mathfrak{aux},h-1} := f_{\mathrm{norm},\mathrm{int},h-1}\circ\phi^{n-h}_{\mathfrak{aux}}\ , \qquad f^{i}_{\mathrm{norm},\mathrm{int},\mathfrak{aux},h-1} := f^{i}_{\mathrm{norm},\mathrm{int},h-1}\circ\phi^{n-h}_{\mathfrak{aux}}\ . \tag{108}$$

On the domain $\mathfrak{D}^{h}_{\mathfrak{sec},\mathfrak{aux}}$ specified in (106), we aim to construct and real–analytic and canonical transformation

$$\phi^{n-h}_{\mathrm{norm},\mathfrak{aux}}:\ (\mathrm{t}^{(h)}_{\mathrm{norm},\mathfrak{aux}},\mathrm{z}^{(h)}_{\mathrm{norm},\mathfrak{aux}},\mathrm{y}^{(h)}_{\mathrm{norm},\mathfrak{aux}},\mathrm{x}^{(h)}_{\mathrm{norm},\mathfrak{aux}})\in\mathfrak{D}^{h}_{\mathfrak{sec},\mathfrak{aux}}\to(\mathrm{t}^{(h)}_{\mathfrak{aux}},\mathrm{z}^{(h)}_{\mathfrak{aux}}\mathrm{y}^{(h)}_{\mathfrak{aux}},\mathrm{x}^{(h)}_{\mathfrak{aux}})\in\mathfrak{D}^{h}_{\mathrm{int},\mathfrak{aux}} \tag{109}$$

such that the transformed Hamiltonian

$$f_{\mathfrak{sec},\mathfrak{aux},h} := f_{\mathfrak{sec},\mathrm{int},\mathfrak{aux},h-1}\circ\phi^{n-h}_{\mathrm{norm},\mathfrak{aux}}$$

has the form

$$\begin{aligned} f_{\mathfrak{sec},\mathfrak{aux},h} &= \mathrm{h}_{\mathfrak{sec},h}(\hat{\mathrm{y}}^{(h)}_{\mathrm{norm},\mathfrak{aux}}) + f_{\mathrm{norm},\mathfrak{aux},h}(\mathrm{t}^{(h)}_{\mathrm{norm},\mathfrak{aux}},\hat{\mathrm{y}}^{(h)}_{\mathrm{norm},\mathfrak{aux}},\hat{\mathrm{x}}^{(h)}_{\mathrm{norm},\mathfrak{aux}}) \\ &= \sum_{i=n-h}^{n-1}\mathrm{h}^{i}_{\mathfrak{sec}}(\hat{\mathrm{y}}^{(h)}_{\mathrm{norm},\mathfrak{aux},i}) + \sum_{i=1}^{n-h-1} f^{i}_{\mathrm{norm},\mathfrak{aux},h}(\mathrm{t}^{(h)}_{\mathrm{norm},\mathfrak{aux},i},\hat{\mathrm{y}}^{(h)}_{\mathrm{norm},\mathfrak{aux},i},\hat{\mathrm{x}}^{(h)}_{\mathrm{norm},\mathfrak{aux},i}) \\ &+ f_{\mathfrak{exp},\mathfrak{sec},\mathfrak{aux},h}(\mathrm{t}^{(h)}_{\mathrm{norm},\mathfrak{aux}},\mathrm{z}^{(h)}_{\mathrm{norm},\mathfrak{aux}},\mathrm{y}^{(h)}_{\mathrm{norm},\mathfrak{aux}},\mathrm{x}^{(h)}_{\mathrm{norm},\mathfrak{aux}}) \end{aligned}$$

where

$$\hat{\mathrm{x}}^{(h)}_{\mathrm{norm},\mathfrak{aux},i} := \begin{cases} (\mathrm{g}^{(h)}_{\mathrm{norm},\mathfrak{aux},i+1}-\mathrm{g}^{(h)}_{\mathrm{norm},\mathfrak{aux},i},\ \cdots,\mathrm{g}^{(h)}_{\mathrm{norm},\mathfrak{aux},n-h-1}-\mathrm{g}^{(h)}_{\mathrm{norm},\mathfrak{aux},n-h-2}) \\ \qquad \text{if} \qquad n\geq 4\ \&\ 1\leq h-1\leq n-3 \\ \emptyset \qquad\qquad \text{otherwise} \end{cases}$$

$$\begin{aligned} \hat{\mathrm{y}}^{(h)}_{\mathrm{norm},\mathfrak{aux},i} := (&\frac{(p^{(h)}_{\mathrm{norm},\mathfrak{aux},n-h})^2+(q^{(h)}_{\mathrm{norm},\mathfrak{aux},n-h})^2}{2},\ \cdots,\ \frac{(p^{(h)}_{\mathrm{norm},\mathfrak{aux},n-1})^2+(q^{(h)}_{\mathrm{norm},\mathfrak{aux},n-1})^2}{2}, \\ &\mathrm{G}^{(h)}_{\mathrm{norm},\mathfrak{aux},i},\ \cdots,\ \mathrm{G}^{(h)}_{\mathrm{norm},\mathfrak{aux},n},\Lambda^{(h)}_{\mathrm{norm},\mathfrak{aux},i},\ \cdots,\ \Lambda^{(h)}_{\mathrm{norm},\mathfrak{aux},n}) \end{aligned}$$

and $f_{\mathfrak{exp},\mathfrak{sec},\mathfrak{aux},h}$ satisfies the bound for $f_{\mathfrak{exp},\mathfrak{sec},h}$ in (75). This will conclude the proof, up to apply the inverse transformation of (103), with $\mathrm{G}^{(h)}_{\mathfrak{aux},i}$, $\mathrm{g}^{(h)}_{\mathfrak{aux},i}$, $\chi^{(h)}_{*,i}$, $\kappa^{(h)}_{*,i}$ replaced by $\mathrm{G}^{(h)}_{\mathrm{norm},\mathfrak{aux},i}$, $\mathrm{g}^{(h)}_{\mathrm{norm},\mathfrak{aux},i}$, $\chi^{(h)}_{i}$, $\kappa^{(h)}_{i}$, and to take

$$\mathfrak{D}^{h}_{\mathfrak{sec}} := \phi^{n-h}_{\mathfrak{aux}}\big(\mathfrak{D}^{h}_{\mathfrak{sec},\mathfrak{aux}}\big)\ .$$

We shall obtain the transformation $\phi^{n-h}_{\mathrm{norm},\mathfrak{aux}}$ in (109) via an application of Proposition D.1. Before doing it, we just remark that, since, in our particular case, $f_{\mathrm{norm},\mathrm{int},\mathfrak{aux},h-1}$ depends on $\mathrm{z}^{(h)}_{\mathfrak{aux}}$, $\mathrm{y}^{(h)}_{\mathfrak{aux}}$,

$\mathrm{x}^{(h)}_{\mathfrak{aux}}$ only via $\mathrm{z}^{(h)}_{\mathfrak{aux},n-h}$, $\tilde{\mathrm{y}}^{(h)}_{\mathfrak{aux}}$, $\hat{\mathrm{x}}^{(h)}_{\mathfrak{aux}}$ and is even in $(\mathrm{t}^{(h)}_{\mathfrak{aux}}, \mathrm{z}^{(h)}_{\mathfrak{aux},n-h})$, the proof of Proposition D.1 can be easily handled to show that $\phi^{n-h}_{\mathfrak{norm},\mathfrak{aux}}$ can be chosen of the form

$$\phi^{n-h}_{\mathfrak{norm},\mathfrak{aux}}: \begin{cases} \Theta^{(h)}_{\mathfrak{aux},j} = & \mathcal{F}^{(h)}_{\mathfrak{norm},\mathfrak{aux},j}(\mathrm{t}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \mathrm{z}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h}, \tilde{\mathrm{y}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \hat{\mathrm{x}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}) \\ \vartheta^{(h)}_{\mathfrak{aux},j} - \pi = & \mathcal{G}^{(h)}_{\mathfrak{norm},\mathfrak{aux},j}(\mathrm{t}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \mathrm{z}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h}, \tilde{\mathrm{y}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \hat{\mathrm{x}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}) \\ & j = 1, \cdots, n-h-1 \\ \mathrm{z}^{(h)}_{\mathfrak{aux},n-h} = & \mathcal{Z}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}(\mathrm{t}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \mathrm{z}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h}, \tilde{\mathrm{y}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \hat{\mathrm{x}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}) \\ (\hat{\mathrm{X}}^{(h)}_{\mathfrak{aux}}, \hat{\mathrm{x}}^{(h)}_{\mathfrak{aux}}) = & \mathcal{X}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}(\mathrm{t}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \mathrm{z}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h}, \tilde{\mathrm{y}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \hat{\mathrm{x}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}) \\ \hat{\mathrm{z}}^{(h)}_{\mathfrak{norm},j} = & \hat{\mathrm{z}}^{(h)}_{\mathfrak{norm},\mathfrak{aux},j} e^{\mathrm{i}\psi^{(h)}_{\mathfrak{norm},\mathfrak{aux},j}(\mathrm{t}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \mathrm{z}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h}, \tilde{\mathrm{y}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \hat{\mathrm{x}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}})} \\ & j = n-h+1. \cdots, n-1 \\ \mathrm{y}^{(h)}_{\mathfrak{aux},n-h+1} = & \mathrm{y}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h+1} \\ \mathrm{x}^{(h)}_{\mathfrak{aux},n-h+1} = & \mathrm{x}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h+1} \\ & + \varphi^{(h)}_{\mathfrak{norm},\mathfrak{aux}}(\mathrm{t}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \mathrm{z}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h}, \tilde{\mathrm{y}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \hat{\mathrm{x}}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}) \end{cases}$$

where $\mathcal{F}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}$, $\mathcal{G}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}$ and $\mathcal{Z}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}$ are odd; $\mathcal{X}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}$, $\psi^{(h)}_{\mathfrak{norm},\mathfrak{aux},j}$ and $\varphi^{(h)}_{\mathfrak{norm},\mathfrak{aux}}$ are even under the change

$$(\mathrm{t}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \mathrm{z}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h}) \to -(\mathrm{t}^{(h)}_{\mathfrak{norm},\mathfrak{aux}}, \mathrm{z}^{(h)}_{\mathfrak{norm},\mathfrak{aux},n-h}) \ .$$

Then (87)–(89) follow.

Now we proceed with proving the existence of $\phi^{n-h}_{\mathfrak{norm},\mathfrak{aux}}$. We can choose, in (132),(134) and (135),

$$\begin{aligned} & \nu_i = 2(h+1) \ , \qquad \ell_i = h \ , \qquad m_i = 3i \ , \quad i = 1, \cdots, n-h-1 = N \\ & \mathrm{h}(p,q,I) = \sum_{i=n-h}^{n-1} \mathrm{h}^i_{\mathfrak{sec}}(\hat{\mathrm{y}}^{(h)}_i) \ , \qquad f(p,q,I,\varphi,\eta,\xi) = \sum_{i=1}^{n-h-1} f^i(u_i,p,q,\varphi) \\ & f^i(u_i,p,q,\varphi) := f^{n-h-i}_{\mathfrak{norm},\mathfrak{int},\mathfrak{aux},h-1}(\mathrm{t}^{(h)}_{\mathfrak{aux},n-h-i}, \tilde{\mathrm{y}}^{(h)}_{\mathfrak{aux},n-h-i}, \hat{\mathrm{x}}^{(h)}_{\mathfrak{aux},n-h-i}) \\ & \mathfrak{Z} := \mathfrak{Z}_i := \Big\{ (k',k'',k''') \in \mathbb{Z}^h \times \mathbb{Z}^{h+1} \times \mathbb{Z}^{h+1} : \quad k'_{n-h+1} = \cdots = k'_{n-1} = 0, \\ & \qquad\qquad k''_{n-h+1} = \cdots = k''_n = 0, \ k'''_{n-h} = \cdots = k'''_n = 0, \ k''_1 + \cdots + k''_{n-h} = 0 \Big\} \\ & \mathfrak{L} := \Big\{ (k',k'',k''') \in \mathfrak{Z} : \qquad k'_{n-h} = k''_{n-h} = 0 \Big\} \end{aligned} \tag{110}$$

where we have re–named

$$\begin{aligned} (p,q) &:= (p^{(h)}_{\mathfrak{aux}}, q^{(h)}_{\mathfrak{aux}}) = (p^{(h)}_{\mathfrak{aux},n-h}, \cdots, p^{(h)}_{\mathfrak{aux},n-1}, \ q^{(h)}_{\mathfrak{aux},n-h}, \cdots, q^{(h)}_{\mathfrak{aux},n-1},) \\ I &:= \big(\mathrm{G}^{(h)}_{\mathfrak{aux},n-h}, \cdots, \mathrm{G}^{(h)}_{\mathfrak{aux},n}, \Lambda^{(h)}_{\mathfrak{aux},n-h}, \cdots, \Lambda^{(h)}_{\mathfrak{aux},n}\big) \\ \varphi &:= \big(\mathrm{g}^{(h)}_{\mathfrak{aux},n-h}, \cdots, \mathrm{g}^{(h)}_{\mathfrak{aux},n}, \ell^{(h)}_{\mathfrak{aux},n-h}, \cdots, \ell^{(h)}_{\mathfrak{aux},n}\big) \\ u_i &:= (I, \ \eta^i, \ \xi^i) \ , \qquad \eta := \eta^1 \ , \qquad \xi := \xi^1 \end{aligned}$$

with

$$\begin{aligned}\eta^i &:= \big(\Theta_{\mathfrak{aux},n-h-i}\ ,\ \cdots\ ,\ \Theta_{\mathfrak{aux},n-1},\ \mathrm{G}_{\mathfrak{aux},n-h-i}\ ,\ \cdots\ ,\ \mathrm{G}_{\mathfrak{aux},n-h-1},\\ &\quad \Lambda_{\mathfrak{aux},n-h-i}\ ,\ \cdots\ ,\ \Lambda_{\mathfrak{aux},n-h-1}\big)\\ \xi^i &:= \big(\vartheta_{\mathfrak{aux},n-h-i}\ ,\ \cdots\ ,\ \vartheta_{\mathfrak{aux},n-1},\ \mathrm{g}_{\mathfrak{aux},n-h-i}\ ,\ \cdots\ ,\ \mathrm{g}_{\mathfrak{aux},n-h-1},\\ &\quad \ell_{\mathfrak{aux},n-h-i}\ ,\ \cdots\ ,\ \ell_{\mathfrak{aux},n-h-1}\big)\ .\end{aligned}$$

In order to verify that Proposition D.1 can be applied, we have to check conditions (133) and (138). Due to the choices of $\mathfrak{Z}$, $\mathfrak{L}$ and to the fact that only the function $\mathrm{h}_{\mathfrak{sec}}^{n-h}$ in the summand for $\mathrm{h}_{\mathfrak{sec}}$ in (107) depends on $(p^{(h)}_{\mathfrak{aux},n-h}, q^{(h)}_{\mathfrak{aux},n-h}, \mathrm{G}^{(h)}_{\mathfrak{aux},n-h})$, it is sufficient to check that condition (133) holds with

$$\omega = \omega_{\mathfrak{sec}}^{n-h}\ ,\quad (k',k)\in\mathbb{Z}^2\setminus\{0\}\ ,\quad K=\bar{K}\ .$$

But due to the choice of $\mathfrak{D}^h_{\mathfrak{int},\mathfrak{aux}}$ in (106), we have that (133) is verified, with

$$\mathfrak{a} = \frac{(a^+_{n-h})^2}{(a^-_{n-h+1})^3\theta_{n-h}}\frac{\bar{\gamma}}{\bar{K}^{\bar{\tau}}}\ ,\quad r=\mathfrak{c}_h\frac{\theta_{n-h}\bar{\gamma}}{\bar{K}^{\bar{\tau}+1}}\ ,\quad \varepsilon=\mathfrak{c}_h\sqrt{\theta_{n-h}}\ .$$

It remains to check the inequalities in (138). In view of the definition of f^i following from the formulae (83), (108) and (110), of the definition of $f^i_{\mathfrak{norm},h-1}$ in (73), the definition of $\overline{f^i_{\mathfrak{norm},h-1}}$, the bound for $\widetilde{f^i_{\mathfrak{norm},h-1}}$ in (75), and first inequality in (53), we see that the former of the inequalities in (138) is satisfied with

$$E_i = \frac{1}{\mathfrak{c}_h}\max\Big\{\frac{(a^+_{n-h-i})^2}{(a^+_{n-h-i+1})^3}\ ,\ \mu\bar{K}\big(\frac{a^+_n}{a^-_1}\big)^{\frac{3}{2}}\frac{1}{a^-_{n-h-i+1}}\Big\}\qquad i=1,\ \cdots,\ n-h-1\ . \tag{111}$$

In order to check that also the second inequality in (138) is satisfied, we previously note that the number d_i in (137) can be taken to be

$$d_i = \mathfrak{c}_h\min\big\{\frac{\theta_{n-h}\bar{\gamma}}{\bar{K}^{\bar{\tau}+1}}\ ,\ \theta_{n-h-i}\big\}\ ,\qquad i=1,\ \cdots,\ n-h-1\ .$$

Inserting then the above values for K, $\mathfrak{a}$, E_i and d_i into the left hand side of the second inequality in (138), we find that this can be bounded by

$$\begin{aligned}\frac{1}{\tilde{\mathfrak{c}}_h}\max\Big\{&\frac{\bar{K}^{2\bar{\tau}+2}}{\bar{\gamma}^2}\frac{(a^+_{n-h-i})^2}{(a^+_{n-h})^2}\frac{(a^-_{n-h+1})^3}{(a^-_{n-h-i+1})^3}\ ,\quad \frac{\bar{K}^{\bar{\tau}+1}}{\bar{\gamma}}\frac{(a^+_{n-h-i})^2}{(a^+_{n-h})^2}\frac{(a^-_{n-h+1})^3}{(a^-_{n-h-i+1})^3}\frac{\theta_{n-h}}{\theta_{n-h-i}}\\ &\frac{\bar{K}^{2\bar{\tau}+2}}{\bar{\gamma}^2}\frac{\mu\bar{K}\big(\frac{a^+_n}{a^-_1}\big)^{\frac{3}{2}}}{(a^+_{n-h})^2}\frac{(a^-_{n-h+1})^3}{a^-_{n-h-i+1}}\ ,\quad \frac{\bar{K}^{\bar{\tau}+1}}{\bar{\gamma}}\frac{\mu\bar{K}\big(\frac{a^+_n}{a^-_1}\big)^{\frac{3}{2}}}{(a^+_{n-h})^2}\frac{(a^-_{n-h+1})^3}{a^-_{n-h-i+1}}\frac{\theta_{n-h}}{\theta_{n-h-i}}\Big\}\end{aligned}$$

Using (49), one easily finds that this quantity does not exceed

$$\frac{1}{\hat{\mathfrak{c}}_h}\max\Big\{\mu\big(\frac{a_n}{a_1}\big)^5\frac{\bar{K}^{2\bar{\tau}+2}}{\bar{\gamma}^2},\quad \frac{\bar{K}^{\bar{\tau}+1}\sqrt{\alpha}}{\bar{\gamma}}\Big\}<1\ . \tag{112}$$

where $\hat{\mathfrak{c}}_h$ depends only on the ratio a^-_n/a^+_n and the masses and the inequality follows from (53). This conclude the proof of this case. ∎

5.2.4 Construction of $\phi^{n-1}_{\mathtt{norm}}$

The arguments we have used in the previous section to construct $\phi^1_{\mathtt{norm}}$, $\cdots$, $\phi^{n-2}_{\mathtt{norm}}$ also fit for the case of $\phi^{n-1}_{\mathtt{norm}}$, therefore we shall not repeat them. We only limit to remark that, for this case, Equations (105), (110), (111) and (112) have to be replaced with

$$\omega^{n-1}_{\mathtt{sec}}(\hat{\mathrm{y}}^{(1)}_{\mathtt{aux},n-1}) := \begin{cases} \partial_{\frac{(p^{(1)}_{\mathtt{aux},n-1})^2+(q^{(1)}_{\mathtt{aux},n-1})^2}{2},\mathrm{G}^{(1)}_{\mathtt{aux},n-1},\mathrm{G}^{(1)}_{\mathtt{aux},n}} \mathrm{h}^{n-1}_{\mathtt{sec}}(\hat{\mathrm{y}}^{(1)}_{\mathtt{aux},n-1}) & n \geq 3 \\ \partial_{\mathrm{G}^{(1)}_{\mathtt{aux},2}} \mathrm{h}^2_{\mathtt{sec}}(\hat{\mathrm{y}}^{(1)}_{\mathtt{aux},1}) & n = 2 \ , \end{cases}$$

$$f^i = f^{n-i}_{\mathtt{norm},\mathtt{int},\mathtt{aux},0}(\mathrm{t}^{(1)}_{\mathtt{aux},i}, \hat{\mathrm{y}}^{(1)}_{\mathtt{aux},i}, \hat{\mathrm{x}}^{(1)}_{\mathtt{aux},i}) \ , \quad d_i = \mathfrak{c}_1 \min\Big\{\frac{\theta_{n-1}\bar{\gamma}}{\bar{K}^{\bar{\tau}+1}} \ , \ \theta_{n-i-1}\Big\}$$

$$i = 1, \ \cdots \ , \ n-1 \ , \quad \theta_0 := \theta_1$$

$$E_i = \frac{1}{\hat{\mathfrak{c}}_1} \begin{cases} \max\Big\{\mu(\frac{a^+_n}{a^-_1})^{\frac{3}{2}}\frac{1}{a^-_n}, \ \frac{(a^+_{n-1})^3}{(a^-_n)^4}\Big\} & i = 1 \\ \max\Big\{\mu\bar{K}(\frac{a^+_n}{a^-_1})^{\frac{3}{2}}\frac{1}{a^-_{n-i+1}}, \ \frac{(a^+_{n-i})^2}{(a^-_{n-i+1})^3}\Big\} & n \geq 3, \ i = 2, \cdots \ , \ n-1 \end{cases}$$

$$\frac{1}{\hat{\mathfrak{c}}_1}\max\Big\{\mu(\frac{a_n}{a_1})^5\frac{\bar{K}^{2\bar{\tau}+2}}{\bar{\gamma}^2} \ , \quad \frac{\bar{K}^{2(\bar{\tau}+1)}\alpha}{\bar{\gamma}^2}\Big\} \ . \quad \blacksquare$$

A Computing the domain of holomorphy

A.1 On the analyticity of the solution of Kepler equation

Here is a refinement of Proposition 4.1.

Proposition A.1 *Let $\widehat{e}$ be as in (46). For any $0 < \overline{e} < \widehat{e}$ there exists $\overline{\eta} = \overline{\eta}(\overline{e})$ such that, for any $\overline{\eta} < \eta < 1$ and any $\mathrm{e} \in \mathbb{C}$ with $|\mathrm{e}| \leq \overline{e}$, there exist two positive numbers $\bar{\zeta} = \bar{\zeta}(\eta, \mathrm{e})$, $\overline{\ell} = \overline{\ell}(\eta, \overline{e})$ such that the map*

$$\zeta \in \overline{\mathbb{T}}_{\bar{\zeta}} \to K(\zeta, \mathrm{e}) := \zeta - \mathrm{e}\sin\zeta \tag{113}$$

is injective, its image verifies

$$K(\overline{\mathbb{T}}_{\bar{\zeta}}, \mathrm{e}) \supset \overline{\mathbb{T}}_{\overline{\ell}} \qquad \forall \ \mathrm{e} \in \mathbb{C} : \ |\mathrm{e}| \leq \overline{e} \ .$$

The inverse function

$$\ell \in \overline{\mathbb{T}}_{\overline{\ell}} \to \zeta(\ell, \mathrm{e}) := K^{-1}(\ell, \mathrm{e}) \in \overline{\mathbb{T}}_{\bar{\zeta}_\eta(\mathrm{e})}$$

verifies

$$|1 - \mathrm{e}\cos\zeta(\ell, \mathrm{e})| \geq 1 - \eta \tag{114}$$

Therefore, $\zeta(\ell, \mathrm{e})$ is real–analytic for $\ell \in \overline{\mathbb{T}}_{\overline{\ell}}$.

The proof of Proposition A.1 is elementary and goes along the same lines of [24]. Therefore, we shall present it skipping some detail.

Lemma A.1 *Let $\widehat{e}$ be as in Proposition 4.1. For any $0 < \overline{e} < \widehat{e}$ there exists a unique $\overline{\eta} = \overline{\eta}(\overline{e}) \in (\overline{e}, 1)$ such that*

$$\forall \ \eta \in [\overline{\eta}, \ 1) : \quad \overline{\ell}_\eta(\overline{e}) := \log\Big[\frac{\eta}{\overline{e}} + \sqrt{1 + \frac{\eta^2}{\overline{e}^2}}\Big] - \sqrt{\eta^2 + \overline{e}^2} \geq 0 \ , \qquad \ell_\eta(\overline{e}) = 0 \quad \Longleftrightarrow \quad \eta = \overline{\eta} \ .$$

Proof By definition of $\widehat{e}$, and since the function $\rho \in [0,1] \to \frac{\rho\, e^{\sqrt{1+\rho^2}}}{1+\sqrt{1+\rho^2}}$ increases with ρ, we have

$$\frac{\overline{e}\, e^{\sqrt{1+\overline{e}^2}}}{1+\sqrt{1+\overline{e}^2}} < 1 \ .$$

Consider now the function

$$\eta \in (0,1] \to g_\rho(\eta) := \frac{\rho\, e^{\sqrt{\eta^2+\rho^2}}}{\eta+\sqrt{\eta^2+\rho^2}} \ .$$

This function decreases with η for any $\rho \in (0,1]$. Since

$$g_{\overline{e}}(0) = e^{\overline{e}} > 1 \ , \qquad g_{\overline{e}}(1) = \frac{\overline{e}\, e^{\sqrt{1+\overline{e}^2}}}{1+\sqrt{1+\overline{e}^2}} < 1$$

we find a unique $\overline{\eta} = \overline{\eta}(\overline{e}) \in [0,1]$ such that

$$g_{\overline{e}}(\eta) < 1 \qquad \forall\ \overline{\eta} < \eta < 1 \ , \quad g_{\overline{e}}(\overline{\eta}(\overline{e})) = 1 \ .$$

Since also

$$g_{\overline{e}}(\overline{e}) = \frac{e^{\overline{e}\sqrt{2}}}{1+\sqrt{2}} \geq \frac{e^{\sqrt{2}}}{1+\sqrt{2}} > 1$$

we actually have

$$\overline{e} < \overline{\eta} < 1 \ . \quad \blacksquare$$

Proof of Proposition A.1 We shall prove Proposition A.1 with

$$\begin{aligned} \bar{\zeta}(\eta, \mathrm{e}) &:= \log \frac{\sqrt{\eta^2+\mathrm{e}_2^2}+\sqrt{\eta^2-\mathrm{e}_1^2}}{\sqrt{\mathrm{e}_1^2+\mathrm{e}_2^2}} \\ \overline{\ell}(\eta, \overline{e}) &:= \log\left[\frac{\eta}{\overline{e}} + \sqrt{1+\frac{\eta^2}{\overline{e}^2}}\right] - \sqrt{\eta^2+\overline{e}^2} \end{aligned} \tag{115}$$

where $\mathrm{e} = \mathrm{e}_1 + \mathrm{i}\mathrm{e}_2$. Observe that $\overline{\ell}(\eta, \overline{e}) > 0$ by Lemma A.1. Moreover, since

$$\mathrm{e}_1 \leq |\mathrm{e}| \leq \overline{e} < \overline{\eta} < \eta$$

we have that $\bar{\zeta}(\eta, \mathrm{e})$ is well defined and positive[5]:

$$\bar{\zeta}(\eta, \mathrm{e}) \geq \log \frac{\eta}{\overline{e}} > 0 \ .$$

We split Equation (113) into its real and imaginary part

$$\begin{cases} K_1(\zeta_1, \zeta_2, \mathrm{e}_1, \mathrm{e}_2) := \zeta_1 - (\mathrm{e}_1 \sin\zeta_1 \cosh\zeta_2 - \mathrm{e}_2 \cos\zeta_1 \sinh\zeta_2) = \ell_1 \\ K_2(\zeta_1, \zeta_2, \mathrm{e}_1, \mathrm{e}_2) := \zeta_2 - (\mathrm{e}_1 \cos\zeta_1 \sinh\zeta_2 + \mathrm{e}_2 \sin\zeta_1 \cosh\zeta_2) = \ell_2 \end{cases}$$

[5] Actually, $\bar{\zeta}(\eta, \mathrm{e})$, as a function of $(\mathrm{e}_1, \mathrm{e}_2)$, reaches its positive minimum

$$\overline{\zeta}_{\min} = \log\left[\frac{\eta}{\overline{e}} + \sqrt{1+\frac{\eta^2}{\overline{e}^2}}\right] > \log(1+\sqrt{2})$$

for $(\mathrm{e}_1, \mathrm{e}_2) = (0, \overline{e})$.

(with $\zeta = \zeta_1 + \mathrm{i}\zeta_2$, $\ell = \ell_1 + \mathrm{i}\ell_2$). The equation for the real part gives a unique solution

$$\zeta_1 = \mathcal{Z}_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2, \ell_1)$$

provided

$$|\mathrm{e}_1| \le \eta \ , \qquad |\zeta_2| \le \overline{\zeta}(\eta, \mathrm{e}) \tag{116}$$

since it reduces to an ordinary real Kepler equation

$$\zeta_1 - E_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2) \sin(\zeta_1 - \phi_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2) = \ell_1 \qquad \text{if} \quad E_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2) \neq 0$$

$$\zeta_1 = \ell_1 \qquad \text{otherwise}$$

with

$$E_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2) := \sqrt{\mathrm{e}_1^2 \cosh^2 \zeta_2 + \mathrm{e}_2^2 \sinh^2 \zeta_2}$$
$$\phi_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2) : \qquad E_1 \cos\phi_1 = \mathrm{e}_1 \cosh\zeta_2 \ , \quad E_1 \sin\phi_1 = \mathrm{e}_2 \sinh\zeta_2 \ .$$

and, under condition (116), one has

$$E_1 \le \eta < 1 \ . \tag{117}$$

Observe that this solution $\mathcal{Z}_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2, \ell_1)$ verifies

$$\mathcal{Z}_1(\mathrm{e}_1, \mathrm{e}_2, -\zeta_2, \ell_1) = -\mathcal{Z}_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2, \ell_1) \qquad \mod 2\pi \ . \tag{118}$$

On the other hand, the function

$$\zeta_2 \to \mathcal{K}_2(\mathrm{e}_1, \mathrm{e}_2, \zeta_2, \ell_1) := K_2(\mathcal{Z}_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2, \ell_1), \zeta_2, \mathrm{e}_1, \mathrm{e}_2)$$

is strictly increasing, therefore, it maps the interval $[-\overline{\zeta}(\eta, \mathrm{e}), \overline{\zeta}(\eta, \mathrm{e})]$, onto the interval $[-\mathcal{L}_2\ (\eta, \mathrm{e}, \ell_1), \mathcal{L}_2\ (\eta, \mathrm{e}, \ell_1)]$, where $\mathcal{L}_2(\eta, \mathrm{e}, \ell_1) := \mathcal{K}_2(\mathrm{e}_1, \mathrm{e}_2, \overline{\zeta}(\eta, \mathrm{e}), \ell_1)$ (note that $\mathcal{K}_2(\mathrm{e}_1, \mathrm{e}_2, -\overline{\zeta}(\eta, \mathrm{e}), \ell_1) = -\mathcal{K}_2(\mathrm{e}_1, \mathrm{e}_2, \overline{\zeta}(\eta, \mathrm{e}), \ell_1)$ because of (118)). We have thus proved that the map (113) maps bijectively the strip $\overline{\mathbb{T}}_{\overline{\zeta}(\eta,\mathrm{e})}$ onto the set

$$\ell = \ell_1 + \mathrm{i}\ell_2 \in \mathbb{C} : \quad \ell_1 \in \mathbb{T} \ , \quad \ell_2 \in [-\mathcal{L}_2(\eta, \mathrm{e}, \ell_1), \mathcal{L}_2(\eta, \mathrm{e}, \ell_1)] \ .$$

But the curve

$$\ell_2 = \mathcal{L}_2(\eta, \mathrm{e}, \ell_1) \qquad \ell_1 \in [0, 2\pi)$$

is concave, its minimum points are cusps, where $\mathcal{L}_2$ attains the value

$$\mathcal{L}_{2,\min}(\eta, \mathrm{e}) = \overline{\zeta}(\eta, \mathrm{e}) - \sqrt{\eta^2 - \mathrm{e}_1^2 + \mathrm{e}_2^2} \ .$$

The minimum of this quantity while $|\mathrm{e}| \le \overline{e}$ is just $\overline{\ell}(\eta, \overline{e})$ in (115). Inequality in (114) follows from

$$|1 - \mathrm{e}\cos\zeta| \ge |\operatorname{Re}(1 - \mathrm{e}\cos\zeta)| \ge 1 - |\operatorname{Re}(\mathrm{e}\cos\zeta)|$$

and (by (117))

$$|\operatorname{Re}(\mathrm{e}\cos\zeta)| = |E_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2)\cos(\zeta_1 - \phi_1(\mathrm{e}_1, \mathrm{e}_2, \zeta_2)| \le E_1 \le \eta \ . \quad \blacksquare$$

A.2 Proof of Proposition 4.2

Define

$$\underline{\delta}_j := \sqrt{1-\overline{e}_j^2}\ , \qquad \overline{\delta}_j := \sqrt{1-\underline{e}_j^2}\ .$$

Assume (47), with

$$\mathcal{A} := (1-\sigma^2)\sqrt{\frac{1}{(1+\sigma)^3(1+\sigma^2)^4}}\ , \quad \mathcal{B} := \sqrt{\frac{1}{(1-\sigma^2)(1+\sigma)^3(1+\sigma^2)}}$$

$$\overline{\mathcal{C}}_i := \begin{cases} \mathcal{C}_1(\sigma)\overline{\delta}_i & i=1,\cdots,n-1 \\ \overline{\delta}_n & i=n \end{cases}\ , \quad \underline{\mathcal{C}}_i := \begin{cases} \mathcal{C}_2(\sigma)\sqrt{\underline{\delta}_i^2+2g(\sigma)^2\overline{\delta}_i^2} & i=1,\cdots,n-1 \\ \sqrt{\underline{\delta}_i^2+2g(\sigma)^2\overline{\delta}_n^2} & i=n \end{cases}$$

$$s = \sigma(1-\sigma)$$

where

$$\mathcal{C}_1(\sigma) := \sqrt{1-\sigma^2}\ , \quad \mathcal{C}_2(\sigma) := \sqrt{\frac{(1+\sigma^2)^3}{(1-\sigma^2)^2}} \tag{119}$$

and σ, g are chosen as follows: $g(\sigma')$ is a suitable positive function, depending at most on the ratios $\frac{\Lambda_j^+}{\Lambda_j^-}$, $\frac{\mathrm{G}_i^+}{\mathrm{G}_i^-}$, such[6] that

$$g(\sigma') \to 0 \qquad \text{as} \quad \sigma' \to 0\ , \quad \text{and} \quad |\sin\arg\frac{\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2}{\Lambda_j^2}| \le g(\sigma')\ , \quad j=1,\cdots,n\ , \tag{120}$$

provided

$$\max\Big\{|\arg(\Lambda_i)|, |\arg(\chi_j)|, |\arg(\Theta_j)|,\ |\arg(\vartheta_j)|\Big\} \le \sigma'$$

while σ is so small that, if $\overline{\ell}_1$, $\cdots$, $\overline{\ell}_n$ are as in Proposition 4.1, with $\overline{e}$ replaced by $\overline{e}_1$, $\cdots$, $\overline{e}_n$, then

$$\sigma \le \min\Big\{\frac{3}{4},\ \overline{\ell}_1,\ \cdots,\ \overline{\ell}_n\Big\}$$

and the following inequality is satisfied

$$\frac{\mathcal{C}_1(\sigma)}{\mathcal{C}_2(\sigma)}\frac{\overline{\delta}_j}{\sqrt{\underline{\delta}_j^2+\sqrt{2}g(\sigma)\overline{\delta}_j}} > 1 \qquad \forall\ i=1,\cdots,n\ .$$

Note that this inequality is satisfied for σ suitably small, since, by definition,

$$\overline{\delta}_j > \underline{\delta}_j\ , \quad \mathcal{C}_1(\sigma') \uparrow 1\ , \quad \mathcal{C}_2(\sigma') \downarrow 1\ , \quad g(\sigma') \downarrow 0 \qquad \text{as} \quad \sigma' \to 0\ .$$

Definitions and assumptions in (47) imply, since $\sigma(1-\sigma) < \sigma$,

$$\begin{aligned} &(1-\sigma)\mathrm{G}_n^- < |\chi_i| < \mathrm{G}_n^+(1+\sigma) \\ &|\tan\arg(\chi_{i-1}-\chi_i)| \le \frac{\max|\operatorname{Im}(\chi_{i-1}-\chi_i)|}{\min|\operatorname{Re}(\chi_{i-1}-\chi_i)|} \le \frac{\theta_i}{\mathrm{G}_i^-} \le \sigma \le 1 \\ &|\arg\chi_i| \le |\arg\chi_{n-1}| + \sum_{j=i+1}^{n-1} |\sin^{-1}\frac{|\chi_{j-1}-\chi_j|}{|\chi_j-\chi_{j+1}|}| \le \sigma \le \frac{\pi}{3} \end{aligned} \tag{121}$$

[6] Since, for $j=1,\cdots,n$, $\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2$ depends only on χ_{j-1}, χ_j, Θ_j and ϑ_j as in (17) and all such coordinates, together also with Λ_j, have their anomalies bounded by σ', we can always find such a function $g(\sigma')$.

The previous inequalities imply that, firstly

$$|\frac{\Theta_j}{\chi_{j-1}}| \leq \frac{\sigma(1-\sigma)\mathrm{G}_n^-}{(1-\sigma)\mathrm{G}_n^-} \leq \sigma$$

and, similarly,

$$|\frac{\Theta_j}{\chi_j}| \leq \sigma$$

therefore, the inequality for i_j, ι_i is (48) follows. Secondly, the definitions of Θ_i^+, ϑ_i^+ imply that conditions (126) are met and hence Lemma A.2 applies. By the thesis (127), we have[7], for $j = 1$, $\cdots$, $n-1$,

$$\begin{aligned} \left|\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2\right| &\leq \frac{\left|\chi_{j-1}-\chi_j\right|^2}{1-\sigma^2} + (1+\sigma)(1+\sigma^2)|\chi_{j-1}||\chi_j||\vartheta_j-\pi|^2 \\ &\leq \frac{(\mathrm{G}_i^+)^2}{\overline{\mathcal{C}}_j^2} + \frac{(\mathrm{G}_n^+)^2}{\overline{\mathcal{C}}_j^2\mathcal{B}^2}|\vartheta_j-\pi|^2 \\ &\leq \overline{\delta}_j^2(\Lambda_j^-)^2 \ . \end{aligned} \tag{122}$$

For $j = n$,

$$\left|\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2\right| = \left|\chi_{n-1}\right|^2 \leq (\mathrm{G}_n^+)^2 < \overline{\delta}_n^2(\Lambda_n^-)^2 \ .$$

We suddenly have the left bound in (48):

$$1-|e_{i,\mathcal{P}}^2| \leq |1-e_{i,\mathcal{P}}^2| = |\frac{\left|\|\mathrm{C}_{\mathcal{P}}^{(i)}\|^2\right|}{\Lambda_i^2}| \leq \overline{\delta}_i^2 = 1-\underline{e}_i^2 \ ,$$

for $i = 1$, $\cdots$, n. Now we check the right bound. To this end, previously check the following inequality

$$\left||\chi_{j-1}|-|\chi_j|\right| \geq \frac{1-\sigma^2}{1+\sigma^2}\mathrm{G}_j^- \ . \tag{123}$$

Because of the second inequality in (121),

$$|\arg\left[(\chi_{j-1}-\chi_j)(\overline{\chi}_{m-1}-\overline{\chi}_m)\right]| \leq 2\tan^{-1}\sigma \ .$$

Then we have

$$\mathrm{Re}\left[(\chi_{j-1}-\chi_j)(\overline{\chi}_{m-1}-\overline{\chi}_m)\right] \geq \frac{1-\sigma^2}{1+\sigma^2}|\chi_{j-1}-\chi_j||\overline{\chi}_{m-1}-\overline{\chi}_m| \ .$$

Taking the sum for $m = j+1$, $\cdots$, n, gives

$$\begin{aligned} \mathrm{Re}\,(\chi_{j-1}-\chi_j)\overline{\chi}_j &\geq \frac{1-\sigma^2}{1+\sigma^2}|\chi_{j-1}-\chi_j|\sum_{m=j+1}^{n}|\overline{\chi}_{m-1}-\overline{\chi}_m| \geq \frac{1-s^2}{1+s^2}|\chi_{j-1}-\chi_j||\overline{\chi}_j| \\ &\geq \frac{1-\sigma^2}{1+\sigma^2}\mathrm{G}_j^-|\overline{\chi}_j| \end{aligned}$$

[7]Beware that, if $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, we denote

$$\|z\|^2 := z_1^2 + z_2^2 + z_3^3 \ .$$

For a given $z \in \mathbb{C}$, the symbol $|z|$ denotes the usual modulus of $z \in \mathbb{C}$:

$$|z| := \sqrt{(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2} \ .$$

So, Lemma A.3 with

$$A = \chi_{j-1}\ , \quad B = \chi_j\ , \quad \Delta = \mathrm{G}_j^-\ , \quad a = \frac{1-\sigma^2}{1+\sigma^2}$$

gives (123). Then the thesis (128) of Lemma A.2 and the definition of ϑ_j provide, for $j = 1, \cdots, n-1$,

$$\big|\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2\big| \geq \frac{1}{\mathcal{A}^2\underline{\mathcal{C}}_j^2}\Big[\mathcal{A}^2(\mathrm{G}_j^-)^2 - (\mathrm{G}_n^+)^2|\vartheta_j - \pi|^2 \geq \Big] \geq (\underline{\delta}_j^2 + \sqrt{2}g(\sigma)\overline{\delta}_i)(\Lambda_j^+)^2 \tag{124}$$

where $g(\sigma)$ is as in (120). Again, this inequality is implied by the definition of ϑ_j^+ in (47) and the ones of $\mathcal{A}$ and $\mathcal{C}_2$ in (119). By (120), (122) and (124), for $j = 1, \cdots, n$, we have

$$\begin{aligned}
|e_{j,\mathcal{P}}|^2 &= \sqrt{(1 - \operatorname{Re}\frac{\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2}{\Lambda_j^2})^2 + (\operatorname{Im}\frac{\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2}{\Lambda_j^2})^2} \\
&\leq \sqrt{\Big(1 - |\frac{\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2}{\Lambda_j^2}|\Big)^2 + 2|\operatorname{Im}\frac{\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2}{\Lambda_j^2}|} \\
&\leq \sqrt{\Big(1 - \underline{\delta}_j^2 - \sqrt{2}g(\sigma)\overline{\delta}_j\Big)^2 + 2\overline{\delta}_j^2 g(\sigma)^2} \leq 1 - \underline{\delta}_j^2 = \overline{e}_j^2\ .
\end{aligned} \tag{125}$$

For $j = n$,

$$\big|\|\mathrm{C}_{\mathcal{P}}^{(n)}\|^2\big| = |\chi_{n-1}|^2 \geq (\underline{\delta}_n^2 + \sqrt{2}g(\sigma)\overline{\delta}_n)(\Lambda_n^+)^2$$

again implies (125) with $j = n$.

The proof of the inequality on the right in (48) proceeds in a similar way. Indeed, starting with

$$|d_{i,\mathcal{P}}|^2 = \Big|\|x_{\mathcal{P}}^{(i+1)}\|^2 - 2x_{\mathcal{P}}^{(i)} \cdot x_{\mathcal{P}}^{(i+1)} + \|x_{\mathcal{P}}^{(i)}\|^2\Big| \geq \Big|\|x_{\mathcal{P}}^{(i+1)}\|^2\Big| - 2\Big|x_{\mathcal{P}}^{(i)} \cdot x_{\mathcal{P}}^{(i+1)}\Big| - \Big|\|x_{\mathcal{P}}^{(i)}\|^2\Big|$$

and using (as it follows from Proposition A.1)

$$\Big|\|x_{\mathcal{P}}^{(i+1)}\|^2\Big| = |a_{i+1}^2(1 - e_{i+1,\mathcal{P}}\cos\zeta_{i+1})^2| \geq (1 - \eta_{i+1})^2(a_{i+1}^-)^2$$

and analogue arguments as above to evaluate $\Big|x_{\mathcal{P}}^{(i)} \cdot x_{\mathcal{P}}^{(i+1)}\Big|$ and $\Big|\|x_{\mathcal{P}}^{(i)}\|^2\Big|$, one easily finds the ansatz. ∎

Estimates

Lemma A.2 *Fix a number $\sigma > 0$. Assume that, for $1 \leq j \leq n-1$,*

$$\operatorname{Re}\overline{\chi}_j(\chi_{j-1} - \chi_j) > 0\ , \quad |\Theta_j| \leq \sigma\min\{|\chi_{j-1}|,\ |\chi_j|\}\ , \quad |\operatorname{Im}(\vartheta_j - \pi)| \leq \log(1+\sigma)\ . \tag{126}$$

Then

$$\big|\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2\big| \leq \frac{\big|\chi_{j-1} - \chi_j\big|^2}{1-\sigma^2} + (1+\sigma)(1+\sigma^2)|\chi_{j-1}||\chi_j||\vartheta_j - \pi|^2 \tag{127}$$

$$\big|\|\mathrm{C}_{\mathcal{P}}^{(j)}\|^2\big| \geq \frac{\big||\chi_{j-1}| - |\chi_j|\big|^2}{1+\sigma^2} - (1+\sigma)(1+\sigma^2)|\chi_{j-1}||\chi_j||\vartheta_j - \pi|^2 \tag{128}$$

Proof We use the formula (19). By Taylor's, given a, b, $z \in \mathbb{C}$, with $|z| \leq \sigma \min_{t\in[0,1]} |a+t(b-a)|$

$$\begin{aligned}\Big|\sqrt{b^2-z^2}-\sqrt{a^2-z^2}\Big| &= \Big|\int_0^1 \frac{d}{dt}\sqrt{\big(a+t(b-a)\big)^2-z^2}dt\Big| \\ &= \Big|(b-a)\int_0^1 \frac{a+t(b-a)}{\sqrt{\big(a+t(b-a)\big)^2-z^2}}dt\Big| \\ &\leq |b-a|\int_0^1 \frac{|a+t(b-a)|}{\sqrt{\big|a+t(b-a)\big|^2-|z|^2}}dt \\ &\leq \frac{|b-a|}{\sqrt{1-\sigma^2}}\end{aligned}$$

We use this formula with $b := \chi_{j-1}$, $a := \chi_j$, $z := \Theta_j$, with the observation that, for $\operatorname{Re}\overline{\chi}_j(\chi_{j-1}-\chi_j) > 0$, the function

$$t \in [0,1] \to \big|\chi_j + t(\chi_{j-1}-\chi_j)\big|^2 = |\chi_j|^2 + 2t\operatorname{Re}\overline{\chi}_j(\chi_{j-1}-\chi_j) + t^2|\chi_{j-1}-\chi_j|^2$$

reaches its minimum, given by $\min\{|\chi_{j-1}|^2,\ |\chi_j|^2\}$, for $t=0$ or $t=1$. Developing also the function $w \in \mathbb{C} \to \cos w$ around $w=\pi$, with $\varrho := w-\pi = \varrho_1 + \mathrm{i}\varrho_2$ and $|\rho_2| \leq \log(1+\sigma)$

$$\begin{aligned}\big|\cos w + 1\big| &= \Big|\int_0^1 (1-t)\frac{d^2}{dt^2}\cos(\pi+t(w-\pi))\Big| = \frac{1}{2}|\varrho|^2 \sup_{|\varrho'|\leq\varrho}|\cos(\pi+\varrho')| \\ &\leq \frac{1}{2}|\varrho|^2 e^{|\varrho_2|} \leq \frac{1}{2}|\varrho|^2(1+\sigma)\end{aligned}$$

and using again the second inequality in (126), then inequality in (127) follows. The inequality in (128) is obtained via the second inequality in (126) and

$$\begin{aligned}\big|\sqrt{\chi_j^2-\Theta_j^2}-\sqrt{\chi_{j-1}^2-\Theta_j^2}\big| &= \frac{|\chi_{j-1}^2-\chi_j^2|}{\big|\sqrt{\chi_j^2-\Theta_j^2}+\sqrt{\chi_{j-1}^2-\Theta_j^2}\big|} \\ &\geq \frac{\big||\chi_{j-1}|^2-|\chi_j|^2\big|}{\big|\sqrt{\chi_j^2-\Theta_j^2}+\sqrt{\chi_{j-1}^2-\Theta_j^2}\big|} \\ &\geq \frac{\big||\chi_{j-1}|-|\chi_j|\big|}{\sqrt{1+\sigma^2}}\ . \quad \blacksquare\end{aligned}$$

Lemma A.3 *If A, $B \in \mathbb{C}$ and a, $\Delta \in \mathbb{R}_+$ verify $|A-B| \geq \Delta$ and $\operatorname{Re}\overline{B}(A-B) \geq a|B|\Delta$, where $0<a<1$, then $\big||A|-|B|\big| > a\Delta$.*

Proof Let $D := A-B$. Then $\big||A|-|B|\big| = \big||B+D|-|B|\big| \leq a\Delta$ implies

$$|B|^2+|D|^2+2\operatorname{Re}\overline{B}D = |B+D|^2 \leq (|B|+a\Delta)^2 = |B|^2+a^2(\Delta)^2+2a|B|\Delta\ .$$

This contradicts assumptions $|D| \geq \Delta > a\Delta$ and $\operatorname{Re}\overline{B}D \geq a|B|\Delta$. $\blacksquare$

B Proof of Lemma 3.3

In this section, we prove the formulae (39) and (40) given in Lemma 3.3.
We recall the following result

Proposition B.1 ([28]) *Let* $\mathfrak{X} = \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n \subset \mathbb{R}^5 \times \cdots \times \mathbb{R}^5$ *and let*

$$(\ell_k, \mathrm{X}_k) \in \mathbb{T}^1 \times \mathfrak{X}_k \to (y_\phi^{(k)}(\ell_k, \mathrm{X}_k), x_\phi^{(k)}(\ell_k, \mathrm{X}_k)) \in \mathbb{R}^3 \times \mathbb{R}^3 \qquad k = 1, \cdots, n$$

be mappings such that, for $1 \le i < j \le n$

(A) *the map*

$$\phi_{ij} : \qquad (\ell_i, \ell_j, \mathrm{X}_i, \mathrm{X}_j) \to (y_\phi^{(i)}, y_\phi^{(2)}, x_\phi^{(j)}, x_\phi^{(2)})$$

is symplectomorphism of $\mathbb{T}^2 \times \mathfrak{X}_i \times \mathfrak{X}_j$ *into* $\mathbb{R}^{12}$.

(B) *The map* $(\ell_j, \mathrm{X}_j) \to (y_\phi^{(2)}(\ell_j, \mathrm{X}_j), x_\phi^{(2)}(\ell_j, \mathrm{X}_j))$ *verifies*

$$\frac{\|y_\phi^{(2)}(\ell_j, \mathrm{X}_j)\|^2}{2m_j} - \frac{\mathfrak{m}_j \mathfrak{M}_j}{\|x_\phi^{(2)}(\ell_j, \mathrm{X}_j)\|} = -\frac{\mathfrak{m}_j^3 \mathfrak{M}_j^2}{2\Lambda_j^2} \ ;$$

where Λ_j *is the variable conjugated to* ℓ_j *in this symplectomorphism.*

Then the function

$$\mathrm{P}^{(i)}(\ell_i, \mathrm{X}) := -\frac{1}{2\pi} \int_{\mathbb{T}} d\ell_j \, \frac{3(x_\phi^{(i)}(\ell_i, \mathrm{X}_i) \cdot x_\phi^{(j)}(\ell_j, \mathrm{X}_j))^2 - \|x_\phi^{(i)}(\ell_i, \mathrm{X}_i)\|^2 \|x_\phi^{(j)}(\ell_j, \mathrm{X}_j)\|^2}{2\|x_\phi^{(j)}(\ell_j, \mathrm{X}_j)\|^5}$$

is given by

$$\mathrm{P}^{(i)} = \frac{\mathfrak{M}_j \mathfrak{m}_j^2}{4} \frac{3(x_\phi^{(i)} \cdot \mathrm{C}_\phi^{(j)})^2 - \|x_\phi^{(i)}\|^2 \|\mathrm{C}_\phi^{(j)}\|^2}{\|\mathrm{C}_\phi^{(j)}\|^4} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\ell_j}{\|x_\phi^{(j)}\|^2} \ . \tag{129}$$

with $\mathrm{C}_\phi^{(j)}(\mathrm{X}) := x_\phi^{(j)}(\ell_j, \mathrm{X}) \times y_\phi^{(j)}(\ell_j, \mathrm{X})$.

Even though the (i, j) projections of the $\mathcal{P}$–map do not verify assumption (A), one has

Corollary B.1 *The formula* (129) *applies also to the* $\mathcal{P}$*–map, or, more in general, to any Kepler map* $\mathcal{K}$ *related to the the map* $\mathcal{D}e\ell$ *in Definition* 2.5 *via*

$$\mathrm{X}_{\mathcal{D}e\ell} = \mathfrak{F}(\mathrm{X}) \ .$$

Proof $\mathcal{D}e\ell$ verifies (A) and (B). ∎

In particular, we have an expression for the second–order term of the doubly averaged Newtonian potential

$$\overline{f_{\mathcal{K}}^{ij}}^{(2)} := -\frac{m_i m_j}{(2\pi)^2} \int_{\mathbb{T}^2} d\ell_i d\ell_j \, \frac{3(x_{\mathcal{K}}^{(i)}(\ell_i, \mathrm{X}_{\mathcal{K}}) \cdot x_{\mathcal{K}}^{(j)}(\ell_j, \mathrm{X}_{\mathcal{K}}))^2 - \|x_{\mathcal{K}}^{(i)}(\ell_i, \mathrm{X}_{\mathcal{K}})\|^2 \|x_{\mathcal{K}}^{(j)}(\ell_j, \mathrm{X}_{\mathcal{K}})\|^2}{2\|x_{\mathcal{K}}^{(j)}(\ell_j, \mathrm{X}_{\mathcal{K}})\|^5} \ .$$

Corollary B.2 *For any* $\mathcal{K}$ *as in Corollary* B.1,

$$\begin{aligned} \overline{f_{\mathcal{K}}^{ij}}^{(2)} &= m_i m_j \frac{a_i^2}{4a_j^3} \frac{\Lambda_j^3}{\|\mathrm{C}_{\mathcal{K}}^{(j)}\|^5} \Big[-\big(\frac{5}{2} - \frac{3}{2} \frac{\|\mathrm{C}_{\mathcal{K}}^{(i)}\|^2}{\Lambda_i^2}\big) \|\mathrm{C}_{\mathcal{K}}^{(j)}\|^2 \\ &+ \frac{3}{2}\big(5 - 4\frac{\|\mathrm{C}_{\mathcal{K}}^{(i)}\|^2}{\Lambda_i^2}\big)(P_{\mathcal{K}}^{(i)} \cdot \mathrm{C}_{\mathcal{K}}^{(j)})^2 + \frac{3}{2} \frac{\|\mathrm{C}_{\mathcal{K}}^{(i)}\|^2}{\Lambda_i^2} (Q_{\mathcal{K}}^{(i)} \cdot \mathrm{C}_{\mathcal{K}}^{(j)})^2 \Big] \end{aligned} \tag{130}$$

Proof Lemma B.1 implies that

$$\begin{aligned}\overline{f_{\mathcal{K}}^{ij}}^{(2)} &= m_i m_j \frac{M_j m_j^2}{4} \frac{\frac{1}{2\pi}\int_{\mathbb{T}} \Big(3(x_{\mathcal{K}}^{(i)} \cdot \mathrm{C}_{\mathcal{K}}^{(j)})^2 - \|x_{\mathcal{K}}^{(i)}\|^2 \|\mathrm{C}_{\mathcal{K}}^{(j)}\|^2\Big) d\ell_i}{\|\mathrm{C}_{\mathcal{K}}^{(j)}\|^4} \\ &\quad \times \frac{1}{2\pi}\int_{\mathbb{T}} \frac{d\ell_j}{\|x_{\mathcal{K}}^{(j)}\|^2} \ .\end{aligned}$$

By (1)

$$\begin{aligned} x_{\mathcal{K}}^{(i)} \cdot \mathrm{C}_{\mathcal{K}}^{(j)} &= \big(\mathrm{a}_{i,\mathcal{K}} P_{\mathcal{K}}^{(i)} + \mathrm{b}_{i,\mathcal{K}} Q_{\mathcal{K}}^{(i)}\big) \cdot \mathrm{C}_{\mathcal{K}}^{(j)} \\ &= \mathrm{a}_{i,\mathcal{K}} P_{\mathcal{K}}^{(i)} \cdot \mathrm{C}_{\mathcal{K}}^{(j)} + \mathrm{b}_{i,\mathcal{K}} Q_{\mathcal{K}}^{(i)} \cdot \mathrm{C}_{\mathcal{K}}^{(j)} \end{aligned}$$

Therefore, squaring, ℓ_i–averaging and using

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{T}} (\mathrm{a}_{i,\mathcal{K}})^2 d\ell_i &= \frac{a_i^2}{2}\big(5 - 4\frac{\|\mathrm{C}_{\mathcal{K}}^{(i)}\|^2}{\Lambda_i^2}\big) \\ \frac{1}{2\pi}\int_{\mathbb{T}} (\mathrm{b}_{i,\mathcal{K}})^2 d\ell_i &= \frac{a_i^2}{2}\frac{\|\mathrm{C}_{\mathcal{K}}^{(i)}\|^2}{\Lambda_i^2} \\ \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{a}_{i,\mathcal{K}} \mathrm{b}_{i,\mathcal{K}} d\ell_i &= 0 \end{aligned}$$

we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{T}} (x_{\mathcal{K}}^{(i)} \cdot \mathrm{C}_{\mathcal{K}}^{(j)})^2 d\ell_i &= \frac{a_i^2}{2}(5 - 4\frac{\|\mathrm{C}_{\mathcal{K}}^{(i)}\|^2}{\Lambda_i^2})(P_{\mathcal{K}}^{(i)} \cdot \mathrm{C}_{\mathcal{K}}^{(j)})^2 \\ &+ \frac{a_i^2}{2}\frac{\|\mathrm{C}_{\mathcal{K}}^{(i)}\|^2}{\Lambda_i^2}(Q_{\mathcal{K}}^{(i)} \cdot \mathrm{C}_{\mathcal{K}}^{(j)})^2 \ . \end{aligned}$$

Using finally

$$\frac{1}{2\pi}\int_{\mathbb{T}} \|x_{\mathcal{K}}^{(i)}\|^2 d\ell_i = a_i^2\big(\frac{5}{2} - \frac{3}{2}\frac{\|\mathrm{C}_{\mathcal{K}}^{(i)}\|^2}{\Lambda_i^2}\big) \ , \quad \frac{1}{2\pi}\int_{\mathbb{T}} \frac{d\ell_j}{\|x_{\mathcal{K}}^{(j)}\|^2} = \frac{1}{a_j^2}\frac{\Lambda_j}{\|\mathrm{C}_{\mathcal{K}}^{(j)}\|} = \frac{1}{a_j^2}\frac{\Lambda_j}{\chi_i}$$

we obtain (130). ∎

Now we may proceed with proving the formulae in (39) and (40).

Proof of (39) We apply Corollary B.2 with $\mathcal{K} = \mathcal{P}$, $i = n-1$, $j = n$. Using $\|\mathrm{C}_{\mathcal{P}}^{(n)}\| = \chi_{n-1}$ (see (17)), $\mathrm{C}_{\mathcal{P}}^{(n)} = \mathrm{S}_{\mathcal{P}}^{(n)}$ and Eq. (3), Proposition 2.1, and Remark 2.2, we have

$$\begin{aligned} P_{\mathcal{P}}^{(n-1)} \cdot \mathrm{S}_{\mathcal{P}}^{(n)} &= \Theta_{n-1} \\ Q_{\mathcal{P}}^{(n-1)} \cdot \mathrm{S}_{\mathcal{P}}^{(n)} &= \frac{1}{\|\mathrm{C}_{\mathcal{P}}^{(n-1)}\|}\big((\mathrm{S}_{\mathcal{P}}^{(n-1)} - \mathrm{S}_{\mathcal{P}}^{(n)}) \times P_{\mathcal{P}}^{(n-1)}\big) \cdot \mathrm{S}_{\mathcal{P}}^{(n)} \\ &= \frac{1}{\|\mathrm{C}_{\mathcal{P}}^{(n-1)}\|}\mathrm{S}_{\mathcal{P}}^{(n-1)} \times \mathrm{P}_{\mathcal{P}}^{(n-1)} \cdot \mathrm{S}_{\mathcal{P}}^{(n)} \\ &= \frac{1}{\|\mathrm{C}_{\mathcal{P}}^{(n-1)}\|}\sqrt{(\chi_{n-1}^2 - \Theta_{n-1}^2)(\chi_{n-2}^2 - \Theta_{n-1}^2)}\sin\vartheta_{n-1} \ . \end{aligned}$$

∎

Proof of (40) By Corollary B.2 with $\mathcal{K} = \mathcal{P}$, $j = i+1$, we find, for $\overline{f^{i,i+1}_{\mathcal{P}}}^{(2)}$ an expression as in (130), replacing $(n-1, n)$ with $(i, i+1)$.

$$\begin{aligned} P^{(i)}_{\mathcal{P}} \cdot \mathrm{C}^{(i+1)}_{\mathcal{P}} = P^{(i)}_{\mathcal{P}} \cdot (\mathrm{S}^{(i+1)}_{\mathcal{P}} - \mathrm{S}^{(i+2)}_{\mathcal{P}}) &= \Theta_i - P^{(i)}_{\mathcal{P}} \cdot \mathrm{S}^{(i+2)}_{\mathcal{P}} \\ Q^{(i)}_{\mathcal{P}} \cdot \mathrm{C}^{(i+1)}_{\mathcal{P}} = Q^{(i)}_{\mathcal{P}} \cdot (\mathrm{S}^{(i+1)}_{\mathcal{P}} - \mathrm{S}^{(i+2)}_{\mathcal{P}}) &= \frac{1}{\|\mathrm{C}^{(i)}_{\mathcal{P}}\|} (\sqrt{(\chi_i^2 - \Theta_i^2)(\chi_{i-1}^2 - \Theta_i^2)} \sin\vartheta_i \\ &\quad - \mathrm{S}^{(i)}_{\mathcal{P}} \times \mathrm{P}^{(i)}_{\mathcal{P}} \cdot \mathrm{S}^{(i+2)}_{\mathcal{P}} - \mathrm{P}^{(i)}_{\mathcal{P}} \times \mathrm{S}^{(i+1)}_{\mathcal{P}} \cdot \mathrm{S}^{(i+2)}_{\mathcal{P}}) \end{aligned} \tag{131}$$

Now, when $(\Theta_{i+1}, \vartheta_{i+1}) = (0, \pi)$, $\|\mathrm{C}^{(i+1)}_{\mathcal{P}}\|$ reduces to

$$\|\mathrm{C}^{(i+1)}_{\mathcal{P}}\| = \chi_i - \chi_{i+1} \ ,$$

(provided $\arg(\chi_i - \chi_{i+1}) \in (-\frac{\pi}{2},\ \frac{\pi}{2}] \bmod 2\pi$) and $\mathrm{S}^{(i+2)}_{\mathcal{P}} \parallel \mathrm{S}^{(i+1)}_{\mathcal{P}}$, so

$$\mathrm{S}^{(i+2)}_{\mathcal{P}} = \frac{\chi_{i+1}}{\chi_i} \mathrm{S}^{(i+1)}_{\mathcal{P}}$$

and hence, the extra–terms in (131) reduce to

$$\begin{aligned} P^{i)}_{\mathcal{P}} \cdot \mathrm{S}^{(i+2)}_{\mathcal{P}} &= \Theta_i \frac{\chi_{i+1}}{\chi_i} \\ \mathrm{S}^{(i)}_{\mathcal{P}} \times \mathrm{P}^{(i)}_{\mathcal{P}} \cdot \mathrm{S}^{(i+2)}_{\mathcal{P}} &= \frac{\chi_{i+1}}{\chi_i} \sqrt{\chi_{i-1}^2 - \Theta_i^2} \sqrt{\chi_i^2 - \Theta_i^2} \sin\vartheta_i \\ \mathrm{P}^{(i)}_{\mathcal{P}} \times \mathrm{S}^{(i+1)}_{\mathcal{P}} \cdot \mathrm{S}^{(i+2)}_{\mathcal{P}} &= 0 \ . \end{aligned}$$

Then (40) readily follows. ∎

C Checking the non–degeneracy condition

In this section we prove statement **4** of Proposition 5.2.

Due to the form of $\mathrm{h}_{\mathfrak{sec}}$ in (72)–(73) and to the bound for $\widetilde{\mathrm{h}^i_{\mathfrak{sec},h}}$ in (75), it is sufficient to prove that the maps

$$\zeta^{(h)}_i \to \overline{\omega^i_{\mathfrak{sec}}} := \partial_{\zeta^{(h)}_i} \overline{\mathrm{h}^i_{\mathfrak{sec}}}(\zeta^{(h)}_i, \Lambda^{(h)}_{n-h}, \Lambda^{(h)}_{n-h+1})$$

in (73), where

$$\zeta^{(h)}_i = \begin{cases} (\dfrac{(p^{(h)}_1)^2 + (q^{(h)}_1)^2}{2},\ \chi^{(h)}_1) & i = 1 \qquad \& \ n = 2 \\ (\dfrac{(p^{(h)}_{n-1})^2 + (q^{(h)}_{n-1})^2}{2},\ \chi^{(h)}_{n-2},\ \chi^{(h)}_{n-1}) & i = n-1 \qquad \& \ n \geq 3 \\ (\dfrac{(p^{(h)}_i)^2 + (q^{(h)}_i)^2}{2},\ \chi^{(h)}_{i-1}) & i = 2, \cdots, n-2 \qquad \& \ n \geq 4 \\ \dfrac{(p^{(h)}_1)^2 + (q^{(h)}_1)^2}{2} & i = 1 \qquad \& \ n \geq 3 \end{cases}$$

are diffeomorphisms, with non–vanishing Hessian matrices. We shall do this verifications for just one of the cases above, and we choose the second case in the list, $i = n-1$, for $n \geq 3$. The explicit expression of $\overline{\mathrm{h}^{n-1}_{\mathtt{sec}}}$ is given in (97)–(98). We neglect the coefficient $\mathcal{A}_{n-1}$ (which does not depend on $\zeta^{(h)}_{n-1}$) and we denote

$$\widehat{\mathrm{h}^{n-1}_{\mathtt{sec}}} = \mathrm{E}_{n-1} + \Omega_{n-1}\frac{p^2_{n-1}+q^2_{n-1}}{2} + \tau_{n-1}(\frac{p^2_{n-1}+q^2_{n-1}}{2})^2 + \mathrm{O}(p_{n-1}, q_{n-1})^6\Big]$$

the function $\overline{\mathrm{h}^{n-1}_{\mathtt{sec}}}$ thus rescaled, and $\widehat{\omega^{n-1}_{\mathtt{int}}}$ its gradient with respect to $(\frac{(p^{(h)}_{n-1})^2+(q^{(h)}_{n-1})^2}{2}, \chi_{n-2}, \chi_{n-1})$. A perturbative argument shows that, under the choices of Corollary 4.1, the frequency–map with respect to (χ_{n-2}, χ_{n-1}) associated to

$$\mathrm{E}_{n-1} = -\frac{\Lambda^3_n}{2\chi^3_{n-1}}\Big(5 - 3\frac{(\chi_{n-2}-\chi_{n-1})^2}{\Lambda^2_{n-1}}\Big)$$

is an injection of its domain and hence, by another perturbative argument, so is the gradient of $\widehat{\mathrm{h}^{n-1}_{\mathtt{sec}}}$ with respect to the same coordinates, for any fixed value of $\frac{p^2_{n-1}+q^2_{n-1}}{2}$. On the other hand, since τ_{n-1} does not vanish under the same assumptions of Corollary 4.1, $\widehat{\omega^{n-1}_{\mathtt{int}}}$ is an injection. The computation shows that the Jacobian of $\widehat{\omega^{n-1}_{\mathtt{int}}}$ does not vanish. ∎

D Some results from perturbation theory

D.1 A multi–scale normal form theorem

The purpose of this section is to present a normal form result which takes into account different scale lengths. It is a particularization of [31, Normal Form Lemma, p. 192] and uses the same techniques of that paper.

Following [31], the notations are as follows.

- If $A \subset \mathbb{R}^\nu$ is open and connected, $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ is the usual flat torus, r, s are positive numbers, we denote as $A_r := \bigcup_{x\in A}\Big\{z \in \mathbb{C}^\nu : \quad z \in B^\nu_r(x)\Big\}$ the complex r–neighborhood of A. $\mathbb{T}^\nu_s$ will denote the complex set $\mathbb{T} + \mathrm{i}[-s, s]$. As usual, $B^\nu_r(x)$ denotes the ball in $\mathbb{C}^\nu$ with radius r centered at x, accordingly to a prefixed norm $|\cdot|$ of $\mathbb{C}^\nu$.
- If $f = f(u, p, q, \varphi)$ is real–analytic for $(u, p, q, \varphi) \in W_{v,s,\varepsilon} = U_v \times B^{2\ell}_\varepsilon \times \mathbb{T}^\nu_s$, and affords the Taylor–Fourier expansion

$$f = \sum_{k\in\mathbb{Z}^m} f_{k,\alpha,\beta}(u)e^{ik\cdot\varphi}\prod_{j=1}^{\ell}(\frac{p_j - \mathrm{i}q_j}{\sqrt{2}})^{\alpha_j}(\frac{p_j+\mathrm{i}q_j}{\mathrm{i}\sqrt{2}})^{\beta_j}\ ,$$

  we denote as $\|f\|_{v,s,\varepsilon}$ its "sup–(Taylor, Fourier) norm":

$$\|f\|_{v,s,\varepsilon} := \sum_{\substack{(a,b)\in\mathbb{N}^{2\ell}\\ k\in\mathbb{Z}^\nu}} \sup_{u\in U_v} |f_{\alpha,\beta,k}(u)|e^{|k|s}\varepsilon^{|(\alpha,\beta)|}$$

  with $|k| := |k|_1$, $|(\alpha,\beta)| := |\alpha|_1 + |\beta|_1$.

- If f is as in the previous item, $K > 0$ and $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2$ is a sub-lattice of $\mathbb{Z}^\nu \times \mathbb{Z}^\ell$, $T_K f$ and $\Pi_{\mathfrak{L}} f$ denote, respectively, the K–truncation and the $\mathfrak{L}$–projection of f:

$$T_K f := \sum_{|(\alpha,\beta)| \le K,\ |k| \le K} f_{\alpha,\beta,k}(u) e^{ik\cdot\varphi} \prod_{j=1}^{\ell} (\frac{p_j - \mathrm{i}q_j}{\sqrt{2}})^{\alpha_j} (\frac{p_j + \mathrm{i}q_j}{\mathrm{i}\sqrt{2}})^{\beta_j}$$

$$\Pi_{\mathfrak{L}} f := \sum_{\substack{k \in \mathfrak{L}_1 \\ \alpha - \beta \in \mathfrak{L}_2}} f_{\alpha,\beta,k}(u) e^{ik\cdot\varphi} \prod_{j=1}^{\ell} (\frac{p_j - \mathrm{i}q_j}{\sqrt{2}})^{\alpha_j} (\frac{p_j + \mathrm{i}q_j}{\mathrm{i}\sqrt{2}})^{\beta_j} \ .$$

Proposition D.1 (Multi–scale normal form) *Let*

$$\nu \ , \qquad \ell \ , \qquad 1 \le m_1 < \cdots < m_N = m$$

be natural numbers;

$$A \subset \mathbb{R}^\nu,\ B \subset \mathbb{R}^{2\ell},\ C_1,\ C_1' \subset \mathbb{R}^{m_1},\ C_2,\ C_2' \subset \mathbb{R}^{m_2 - m_1},\ \cdots,\ C_N,\ C_N' \subset \mathbb{R}^{m_N - m_{N-1}} \ ,$$

be open and connected sets;

$$r,\ s,\ \varepsilon,\ \rho_1 \ge \rho_2 \cdots \ge \rho_N,\ \rho_1' \ge \rho_2' \cdots \ge \rho_N'$$

positive numbers. Put

$$v_i := (r, \rho_1, \cdots, \rho_i, \rho_1', \cdots, \rho_i') \ , \qquad v := v_N$$

$$U_{v_i}^{(i)} := A_r \times C_{1\rho_1} \times \cdots \times C_{i\rho_i} \times C'_{1\rho_1'} \times \cdots \times C'_{i\rho_i'} \ , \qquad U_v := U_{v_N}^{(N)}$$

$$W_{v_i,s,\varepsilon}^{(i)} := U_{v_i}^{(i)} \times \mathbb{T}_s^\nu \times B_\varepsilon \ , \qquad W_{v,s,\varepsilon} := W_{v_N,s,\varepsilon}^{(N)} \ ,$$

with $i = 1, \cdots, N$.

Let $\mathfrak{a}$, $K > 0$ with $0 < s < 6 \log 5/6$ and $Ks \ge 12$; let also $\mathfrak{L}$ and $\mathfrak{Z}_1, \cdots, \mathfrak{Z}_N$ be sub-lattices of $\mathbb{Z}^\ell \times \mathbb{Z}^\nu$ and let $\mathfrak{Z} := \mathfrak{Z}_1 \cup \cdots \cup \mathfrak{Z}_N$.

Let

$$H(u, \varphi, p, q) = \mathrm{h}(p, q, I) + f(u, \varphi, p, q) \tag{132}$$

be real–analytic for $(u, \varphi, p, q) \in W_{v,s,\varepsilon}$, where $u := (I,\ \eta,\ \xi) = (I_1, \cdots, I_\nu, \eta_1, \cdots, \eta_m, \xi_1, \cdots, \xi_m)$. Suppose that

(i) *h depends on (p, q) only via $\frac{p_i^2 + q_i^2}{2}$, with the frequency map $\omega = (\omega_1, \cdots, \omega_\ell, \omega_{\ell+1}, \cdots, \omega_{\ell+\nu})$ defined via*

$$\omega_i := \begin{cases} \partial_{\frac{p_i^2+q_i^2}{2}} \mathrm{h} & 1 \le i \le \ell \\ \\ \partial_{I_{i-\ell}} \mathrm{h} & \ell + 1 \le i \le \ell + \nu \end{cases}$$

verifying

$$|\omega(p, q, I) \cdot (k', k)| \ge \mathfrak{a} \quad \forall\ (k', k) \in \mathfrak{Z} \setminus \mathfrak{L}\ ,\ |(k', k)| \le K \tag{133}$$

and all $(p, q, I) \in B_\varepsilon^{2\ell} \times A_r$;

(ii) *f is a sum*

$$f = \sum_{i=1}^{N} f_i(u_i, \varphi, p, q) \tag{134}$$

where f_i is real–analytic on $W^{(i)}_{v_i,s,\varepsilon}$ and has the form

$$f_i(u_i,\varphi,p,q)=\sum_{(\alpha^--\alpha^+,k)\in\mathfrak{Z}_i} f^i_{k,\alpha^-,\alpha^+}(u_i)\prod_{j=1}^{\nu}e^{\mathrm{i}k_j\varphi_j}\prod_{k=1}^{\ell}\big(\frac{p_k-\mathrm{i}q_k}{\sqrt{2}}\big)^{\alpha^-_k}\big(\frac{p_k+\mathrm{i}q_k}{\sqrt{2}\mathrm{i}}\big)^{\alpha^+_k} \tag{135}$$

with

$$u_i:=(I,\ \eta^i,\ \xi^i):=(I_1,\ \cdots,\ I_\nu,\ \eta_1,\ \cdots,\ \eta_{m_i},\ \xi_1,\ \cdots,\ \xi_{m_i})\ ; \tag{136}$$

(iii) *the following "smallness" conditions hold. If*

$$c_i:=e(1+\ell_i e+m_i e)/2\ ,\qquad d_i:=\min\{rs,\ \varepsilon^2,\ \rho_i\rho_i'\} \tag{137}$$

with e denoting Neper number, then

$$\|f_i\|_{W^{(i)}_{v_i,s,\varepsilon}}\le E_i\ ,\qquad \sum_{i=1}^{N}\frac{7}{6}\big(\frac{9}{8}\big)^{i-1}\frac{2^7c_iKs}{\mathfrak{a}d_i}E_i<1\ . \tag{138}$$

Then, one can find a real-analytic and symplectic transformation

$$\Phi:\quad W_{v/6^N,s/6^N,\varepsilon/6^N}\to W_{v,\sigma,\varepsilon}$$

which conjugates H to

$$H_*(u,\varphi,p,q):=H\circ\Phi=\mathrm{h}(I,p,q)+\sum_{i=1}^{N}g_i(u_i,\varphi,p,q)+\sum_{i=1}^{N}f_i^*(u,\varphi,p,q)\ ,$$

where g_i, f_i verify

$$\begin{aligned}
g_i &= \Pi_{\mathfrak{Z}_i\cap\mathfrak{L}}T_Kg_i\\
\|g_i-\Pi_{\mathfrak{Z}_i\cap\mathfrak{L}}T_Kf_i\|_{v_i/6^N,\sigma/6^N,\varepsilon/6^N} &\le \big(\frac{9}{8}\big)^{2(i-1)}\frac{2^7c_i\,\|f_i\|^2_{v_i,s,\varepsilon}}{\mathfrak{a}d_i}\\
&+ \frac{7}{6}\big(\frac{9}{8}\big)^{2(i-1)}\sum_{j=1}^{i-1}\frac{2^7c_j\,\|f_j\|_{v_j,s,\varepsilon}}{\mathfrak{a}d_j}\|f_i\|_{v_i,s,\varepsilon}\\
&+ \sum_{k=1}^{i-1}\big(\frac{9}{8}\big)^{i-1-k}\frac{2^4c_k\|f_k\|_{v_k,s,\varepsilon}Ks}{\mathfrak{a}d_k}\|f_i\|_{v_i,s,\varepsilon}\\
\|f_i^*\|_{v_i/6^N,s/6^N,\varepsilon/6^N} &\le \big(\frac{9}{8}\big)^{N-1}e^{-Ks/6^i}\|f_i\|_{v_i,s,\varepsilon}
\end{aligned}$$

Finally, Φ is close to the identity in the following sense. Given F, real–analytic on $W^{(i)}_{v_i/6^N,s/6^N,\varepsilon/6^N}$,

$$\|F\circ\Phi-F\|_{v/6^N,s/6^N,\varepsilon/6^N}\le\sum_{k=1}^{N}\big(\frac{9}{8}\big)^{N-k}\frac{2^4c_k\|f_k\|_{v_k,s,\varepsilon}Ks}{\mathfrak{a}d_{k,i}}\|F\|_{v_i/6^N,s/6^N,\varepsilon/6^N}$$

with $d_{k,i}:=\max\{d_k,d_i\}$.

The proof of Proposition D.1 is based on the following

Lemma D.1 *Let $\bar N \in \mathbb{N}$, ν, ℓ, m_i, A, B, C_i, C_i', r, s, ρ_i, ρ_i', $U_{v_i}^{(i)}$, $W_{v_i,s,\varepsilon}^{(i)}$, c_i, d_i, with $i = 1, \cdots, \bar N + 1$, be as in Proposition D.1; $v := (r, \rho_1, \cdots, \rho_{\bar N+1}, \rho_1', \cdots, \rho_{\bar N+1}')$, $U_v := U_{v_{\bar N+1}}^{(\bar N+1)}$, $W_{v,s,\varepsilon} := W_{v_{\bar N+1},s,\varepsilon}^{(\bar N+1)}$. Let*

$$H(p, q, I, \varphi, \eta, \xi) = \mathrm{h}(p, q, I) + g(p, q, I, \varphi, \eta, \xi) + f(p, q, I, \varphi, \eta, \xi) \tag{139}$$

be real–analytic for $(u, \varphi, p, q) \in W_{v,s,\varepsilon}$. Suppose assumption (i) *of Proposition D.1 and, moreover, the following ones*

(ii) *g is a sum*

$$g = \sum_{i=1}^{\bar N} g_i(u_i, \varphi, p, q) \tag{140}$$

where g_i is real–analytic on $W_{v_i,s,\varepsilon}^{(i)}$ and u_i is as in (136);

(iii) *$g_1, \cdots, g_{\bar N}$ and f satisfy*

$$g_i = \Pi_{\mathfrak{L}} g_i \ , \quad f = \Pi_{\mathfrak{Z}} f$$

and

$$\sum_{i=1}^{\bar N} \frac{2^7 c_i K s}{\mathfrak{a} d_i} \|g_i\|_{v_i,s_i,\varepsilon_i} < 1 \ , \quad \|f\|_{v,s,\varepsilon} < \frac{\mathfrak{a} d_{\bar N+1}}{2^7 c_{\bar N+1} K s} \ . \tag{141}$$

Then, one can find a real-analytic and symplectic transformation

$$\Phi : \quad (u', \varphi', p', q') \in W_{v/6,s/6,\varepsilon/6} \to (u, \varphi, p, q) \in W_{v,\sigma,\varepsilon}$$

such that

$$H_* := H \circ \Phi = h + g + g_* + f_* \ ,$$

where $g_* = \Pi_{\mathfrak{Z}\cap\mathfrak{L}} T_K g_*$ is $\mathfrak{Z} \cap \mathfrak{L}$–resonant and the following bounds hold

$$\begin{aligned} \|g_* - T_K \Pi_{\mathfrak{Z}\cap\mathfrak{L}} f\|_{v/6,\sigma/6,\varepsilon/6} &\le (\frac{2^7 c_{\bar N+1} \|f\|_{v,s,\varepsilon}}{\mathfrak{a} d_{\bar N+1}} + \sum_{i=1}^{n} \frac{2^7 c_i \|g_i\|_{v_i,s,\varepsilon}}{\mathfrak{a} d_i}) \|f\|_{v,s,\varepsilon} \\ &\le \frac{\|f\|_{v,s,\varepsilon}}{6} \\ \|f_*\|_{v/6,\sigma/6,\varepsilon/6} &\le e^{-Ks/6} \|f\|_{v,s,\varepsilon} \ . \end{aligned}$$

Finally, Φ is close to the identity in the following sense: for any F which is real–analytic on $W_{v,s,\varepsilon}^{(i)}$,

$$\|F \circ \Phi - \Phi\|_{v/6,s/6,\varepsilon/6} \le \frac{2^4 c_{\bar N+1} \|f\|_{v,s,\varepsilon} K s}{\mathfrak{a} d_i} \|F\|_{v_i,s,\varepsilon} < \frac{1}{8} \|F\|_{v,s,\varepsilon} \ . \tag{142}$$

The following Lemma is a trivial extension[8] of [31, Iterative Lemma]. Its proof is omitted.

[8] In order to obtain the extension it is sufficient to replace ϕ of [31, Appendix A] with

$$\phi = \sum_{\substack{(\alpha-\beta,k)\in\mathfrak{K}\setminus\mathfrak{L} \\ |(\alpha,\beta)|\le K, \ |k|\le K}} \frac{f_{k,\alpha,\beta}(u)}{\mathrm{i}(\alpha-\beta,k)\cdot\omega} e^{\mathrm{i}k\cdot\varphi} \prod_{j=1}^{\ell} (\frac{p_j - \mathrm{i}q_j}{\sqrt{2}})^{\alpha_j} (\frac{p_j + \mathrm{i}q_j}{\mathrm{i}\sqrt{2}})^{\beta_j}$$

Lemma D.2 *Let* $s = (s_1, \cdots, s_\nu)$, $r = (r_1, \cdots, r_\nu)$, $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_\ell)$, $\rho = (\rho_1, \cdots, \rho_m)$, $\rho' = (\rho'_1, \cdots, \rho'_m)$, $v := (r, \rho, \rho')$, $\hat{v} := (\hat{r}, \hat{\rho}, \hat{\rho}') < v/2$, $\hat{s} < s/2$, $\hat{\varepsilon} < \varepsilon/2$,

$$\delta := \min_{\substack{i=1,\cdots,\nu \\ j=1,\cdots,\ell \\ k=1,\cdots,m}} \{\hat{r}_i \hat{s}_i,\ \hat{\varepsilon}_j^2, \hat{\rho}_k \hat{\rho}'_k\} \ .$$

Let

$$H(u,\varphi,p,q) = \mathrm{h}(I,p,q) + g(u,\varphi,p,q) + f(u,\varphi,p,q) \qquad g(u,\varphi,p,q) = \sum_{i=1}^{m} g_i(u,\varphi,p,q)$$

be real–analytic on $W_{v,s,\varepsilon}$. *Assume that inequality* (133) *and*

$$\|f\|_{v,s,\varepsilon} < \frac{\mathfrak{a}\delta}{c} \tag{143}$$

are satisfied. Then one can find a real–analytic and symplectic transformation

$$\Phi : \ W_{v-2\hat{v},s-2\hat{s},\varepsilon-2\hat{\varepsilon}} \to W_{v,s,\varepsilon}$$

defined by the time–one flow[9] $X^1_\phi f := f \circ \Phi$ *of a suitable* ϕ *verifying*

$$\|\phi\|_{v,s,\varepsilon} \leq \frac{\|f\|_{v,s,\varepsilon}}{\mathfrak{a}}$$

such that

$$H_+ := H \circ \Phi = h + g + \Pi_{\mathfrak{L}\cap\mathfrak{Z}} f + f_+$$

and, moreover, the following bounds hold

$$\begin{aligned} \|f_+\|_{v-2\hat{v},s-2\hat{s},\varepsilon-2\hat{\varepsilon}} \ \leq \ & \big(1 - \frac{c}{\mathfrak{a}\delta}\|f\|_{v,s,\varepsilon}\big)^{-1}\Big[\frac{c}{\mathfrak{a}\delta}\|f\|^2_{v,s,\varepsilon} \\ & + e^{-K\hat{s}}\|f\|_{v,s,\varepsilon} + \big(\frac{\varepsilon-\hat{\varepsilon}}{\varepsilon}\big)^K \|f\|_{v,s,\varepsilon} + \|\{\phi, g\}\|_{v-\hat{v},s-\hat{s},\varepsilon-\hat{\varepsilon}}\Big] \end{aligned}$$

Finally, for any real–analytic function F *on* $W_{v,s,\varepsilon}$,

$$\|F \circ \Phi - F\|_{v-2\hat{v},s-2\hat{s},\varepsilon-2\hat{\varepsilon}} \ \leq \ \frac{\|\{\phi, F\}\|_{v-\hat{v},s-\hat{s},\varepsilon-\hat{\varepsilon}}}{1 - \dfrac{c\|f\|_{v,s,\varepsilon}}{\mathfrak{a}\delta}} \ .$$

Proof of Lemma D.1 Following [31], the proof is obtained via iterate applications of Lemma D.2.

To avoid too many indices, we shall prove this lemma taking, in (140), $\bar{N} = 1$; the extension to $\bar{N} \geq 1$ being straightforward. Namely, we take

$$\begin{aligned} & \rho_1 = \cdots = \rho_{m_1} = \bar{\rho} \ , \quad \rho'_1 = \cdots = \rho'_{m_1} = \bar{\rho}' \\ & \rho_{m_1+1} = \cdots = \rho_m = \rho \ , \quad \rho'_{m_1+1} = \cdots = \rho'_m = \rho \end{aligned} \tag{144}$$

[9] The time–one flow generated by ϕ is defined as the differential operator

$$X^1_\phi := \sum_{k=0}^{\infty} \frac{\mathcal{L}^k_\phi}{k!}$$

where $\mathcal{L}^0_\phi f := f$ and $\mathcal{L}^k_\phi f := \{\phi, \mathcal{L}^{k-1}_\phi f\}$, with $k = 1,\ 2,\ \cdots$.

where $1 \le m_1 < m$. Letting

$$v := (r,\ \rho,\ \rho')\ , \qquad \bar v := (r,\ \bar\rho,\ \bar\rho')\ , \quad E := \|f\|_{v,s,\varepsilon}\ , \quad G := \|g\|_{\bar v,s,\varepsilon}\ , \quad \bar c = c_1\ , \quad c = c_2\ ,$$
$$\bar d := \min\{rs,\ \varepsilon^2, \bar\rho\bar\rho'\}\ , \qquad d := \{rs,\ \varepsilon^2, \rho\rho'\}\ ,$$

we rewrite the assumptions in (141) as

$$\frac{2^7\bar c GKs}{\mathfrak{a}\bar d} < 1\ , \quad \frac{2^7 cEKs}{\mathfrak{a}d} < 1\ . \tag{145}$$

The inequality on the right clearly implies (143). So, we apply Lemma D.2 to the Hamiltonian (139), taking $r_1 = \cdots = r_\nu = r$, $s_1 = \cdots = s_\nu = s$, $\varepsilon_1 = \cdots = \varepsilon_\ell = \varepsilon$, ρ_k, ρ'_k as in (144) and

$$\begin{aligned}
&\hat v = \hat v_0 := v/6\ , \quad \hat s = \hat s_0 := s/6\ , \quad \hat\varepsilon = \hat\varepsilon_0 := \varepsilon/6\\
&\hat{\bar v} = \hat{\bar v}_0 := \bar v/6\ , \quad \hat{\bar s} := \hat{\bar s}_0 := \bar s/6\ , \quad \hat{\bar\varepsilon} := \hat{\bar\varepsilon}_0 := \bar\varepsilon/6\\
&\delta := \{\hat r\hat s,\ \hat\varepsilon^2, \hat\rho\hat\rho\} = \frac{d}{36}\ .
\end{aligned}$$

Letting

$$v_1 := v - 2\hat v_0 = 3/4v\ , \qquad s_1 := s - 2\hat s = 2/3s\ , \quad \varepsilon_1 := \varepsilon - 2\hat\varepsilon = 2/3\varepsilon$$

by Lemma D.2, we find a canonical transformation $\Phi_0 = X_{\phi_0}$ which is real–analytic on $W_{v_1,s_1,\varepsilon_1}$ and conjugates H to $H_1 = h + g + g_1 + f_1$, where $g_1 = \Pi_{\mathfrak{L}\cap\mathfrak{Z}} T_K f$ and

$$\begin{aligned}
\|f_1\|_{v_1,s_1,\varepsilon_1} \quad &\le \quad (1 - \frac{36cE}{\mathfrak{a}d})^{-1}\Big[\frac{36cE}{\mathfrak{a}d} + e^{-Ks/6} + (\frac{5}{6})^K\Big]E\\
&+ \quad (1 - \frac{36cE}{\mathfrak{a}d})^{-1}\frac{36\bar c G}{\mathfrak{a}\bar d}E
\end{aligned}$$

where

$$\bar\delta := \min\{\hat r\hat s,\ \hat\varepsilon^2, \hat{\bar\rho}\hat{\bar\rho}'\} = \frac{\bar d}{36}\ .$$

Here, we have used

$$\begin{aligned}
\|\{\phi,\ g\}_{I,\varphi,\eta,\xi}\|_{v-\hat v.s-\hat s,\varepsilon-\hat\varepsilon} \quad &= \quad \|\{\phi,\ g\}_{I,\varphi,\eta^1,\xi^1}\|_{\bar v-\hat{\bar v},s-\hat s,\varepsilon-\hat\varepsilon}\\
&\le \quad \frac{\bar c G}{\mathfrak{a}\bar\delta} = 36\frac{\bar c G}{\mathfrak{a}\bar d}
\end{aligned} \tag{146}$$

since g depends on η, ξ only via $\eta^1 = (\eta_1, \cdots, \eta_{m_1})$, $\xi^1 = (\xi_1, \cdots, \xi_{m_1})$. It is sufficient to consider the case

$$e^{-Ks/6} + (\frac{5}{6})^K \le \frac{18cE}{\mathfrak{a}d}$$

since otherwise the Lemma is proved. In such case, using (145) we can write

$$\begin{aligned}
E_1 = \|f_1\|_{v_1,s_1,\varepsilon_1} \quad &\le \quad \frac{32}{23}\Big(\frac{9}{32}\frac{2^7cEKs}{\mathfrak{a}d} + \frac{9}{64}\frac{2^7cEKs}{\mathfrak{a}d} + \frac{9}{32}\frac{2^7\bar cGKs}{\mathfrak{a}\bar d}\Big)\frac{E}{Ks}\\
&< \quad \frac{E}{Ks}\max\Big\{\frac{2^7cEKs}{\mathfrak{a}d},\ \frac{2^7\bar cGKs}{\mathfrak{a}\bar d}\Big\} < \frac{E}{4}
\end{aligned} \tag{147}$$

Let

$$L := \big[\frac{Ks}{12\log 2}\big]\ .$$

Note that

$$L \geq 1 \ , \qquad Ks > 8L \ , \tag{148}$$

since we have assumed $Ks \geq 12$. We want to prove that Lemma D.2 can be applied L times with parameters

$$\hat{v}_i = \frac{v}{4L} \ , \quad \hat{\varepsilon}_i = \frac{\varepsilon}{4L} \ , \quad \hat{s}_i = \frac{s}{4L} \ , \quad \delta_i = \frac{d}{16L^2} \ , \quad i = 1, \cdots, L \ . \tag{149}$$

For $L = 1$, this follows from (147):

$$E_1 := \|f_1\|_{v_1, s_1, \varepsilon_1} \quad \leq \quad \frac{E}{Ks} \leq 2^{-7} \frac{\mathfrak{a} d}{c(Ks)^2} < 2^{-13} \frac{\mathfrak{a}\delta_1}{c}$$

which is implied by the inequality in (147) and assumption (141). We then assume $L \geq 2$. Suppose, by induction, that, for a certain $1 \leq i \leq L-1$, and any $1 \leq j \leq i$, we have conjugated H to

$$H_j = \mathrm{h} + g + \bar{g}_j + f_j$$

where $\bar{g}_j = \sum_{k=0}^{j-1} \Pi_{\mathfrak{L} \cap \mathfrak{Z}} T_K f_k$

$$E_j := \|f_j\|_{v_j, s_j, \varepsilon_j} \leq \min\Big\{ \frac{E}{4^j} \ , \ 2^{-6} \frac{\mathfrak{a}\delta_j}{c} \Big\} \tag{150}$$

where $\hat{v}_0$, $\hat{s}_0$, $\hat{\varepsilon}_0$ are as above, $v_0 := v$, $s_0 := s$, $\varepsilon_0 := \varepsilon$ and $v_j = v_{j-1} - 2\hat{v}_{j-1}$. Then by Lemma D.2, on the domain $W_{v_{j+1}, s_{j+1}, \varepsilon_{j+1}}$, we fined a real–analytic transformation $\Phi_i = X_{\phi_i}$, which conjugates H_i to

$$H_{i+1} = \mathrm{h} + g + \bar{g}_{i+1} + f_{i+1}$$

where $\bar{g}_{i+1} = \bar{g}_i + \Pi_{\mathfrak{L} \cap \mathfrak{K}} f_i = \sum_{k=0}^{i} \Pi_{\mathfrak{L} \cap \mathfrak{Z}} T_K f_k$. We prove that (150) is satisfied for $j = i+1$. Using[10] the assumption on the right in (145), (147), the inequality for Ks in (148) and the definition of δ_i in (149), we have

$$\|\{\bar{g}_i, \phi_i\}\|_{v_i - \hat{v}_i, s_i - \hat{s}_i, \varepsilon_i - \hat{\varepsilon}_i} \quad \leq \quad \Big[\frac{c}{\mathfrak{a}\delta_i}\big(E_1 + \frac{E}{L}\big)\Big] E_i \leq \Big[\frac{c}{\mathfrak{a}\delta_i}\frac{E}{Ks} + \frac{c}{\mathfrak{a}\delta_i}\frac{E}{L}\Big] E_i < \frac{E_i}{32} \ .$$

Moreover, by a similar argument as in (146) and since g is actually real–analytic in the larger domain

$$W_{\bar{v}, s, \varepsilon} \supset W_{\bar{v}_i - \hat{v}_i + \bar{v}, s_i - \hat{s}_i + \hat{s}, \varepsilon_i - \hat{\varepsilon}_i + \hat{\varepsilon}} \ ,$$

we have

$$\|\{g, \phi_i\}\|_{v_i - \hat{v}_i, s_i - \hat{s}_i, \varepsilon_i - \hat{\varepsilon}_i} \quad = \quad \|\{g, \phi_i\}\|_{\bar{v}_i - \hat{\bar{v}}_i, s_i - \hat{s}_i, \varepsilon_i - \hat{\varepsilon}_i} \leq \frac{\bar{c} E_i}{\mathfrak{a}\bar{\delta}_i} \frac{G}{L} < \frac{E_i}{64} \ ,$$

where

$$\bar{\delta}_i := \min\{\hat{r}_i \hat{s}_i, \bar{\rho}_i \hat{\bar{\rho}}'_i\} = \frac{\bar{d}}{16L^2} \ , \quad i = 1, \cdots, L \ .$$

[10] For the proof of inequality $\|\{g_i, \phi_i\}\|_{v_i - \hat{v}_i, s_i - \hat{s}_i, \varepsilon_i - \hat{\varepsilon}_i} \leq \frac{c E_i}{\mathfrak{a}\delta_1}(E_1 + \frac{E}{L})$, compare [31, Proof of the Normal Form Lemma].

Then we find[11]

$$\begin{aligned} E_{i+1} = \|f_{i+1}\|_{v_{i+1},s_{i+1},\varepsilon_{i+1}} &\leq (1-\frac{cE_i}{\mathfrak{a}\delta_1})^{-1}\Big[\frac{cE_i}{\mathfrak{a}\delta_1} + e^{-K\hat{s}_i} + \big(\frac{\varepsilon_i-\hat{\varepsilon}_i}{\varepsilon_i}\big)^K\Big]E_i \\ &+ (1-\frac{cE_i}{\mathfrak{a}\delta_1})^{-1}\|\{\bar{g}_i,\phi_i\}\|_{v_i-\hat{v}_i,s_i-\hat{s}_i,\varepsilon_i-\hat{\varepsilon}_i} \\ &+ (1-\frac{cE_i}{\mathfrak{a}\delta_1})^{-1}\|\{g,\phi_i\}\|_{\bar{v}_i-\bar{\hat{v}}_i,s_i-\hat{s}_i,\varepsilon_i-\hat{\varepsilon}_i} \\ &\leq \frac{64}{63}\Big[\frac{1}{64}+\frac{1}{8}+(\frac{4}{7})^{16}+\frac{1}{32}+\frac{1}{64}\Big]E_i \\ &< \frac{E_i}{4} < E_1 < 2^{-6}\frac{\mathfrak{a}\delta_1}{c}\ . \end{aligned}$$

since $i \geq 1$. Then we let $\Phi := \Phi_0 \circ \cdots \circ \Phi_L$, $H_* := H \circ \Phi = h + g + \bar{g}_{L+1} + f_{L+1}$, $g_* := g_{L+1}$, $f_* := f_{L+1}$ and we have, by telescopic inequalities and (147),

$$\begin{aligned} \|g_* - \Pi_{\mathfrak{L}\cap\mathfrak{K}} T_K f\|_{v/6,s/6,\varepsilon/6} &= \sum_{i=1}^{L}\|\Pi_{\mathfrak{L}\cap\mathfrak{K}} T_K f_i\| \leq \sum_{i=1}^{L} E_i \leq E_1 \sum_{i=1}^{L}\frac{1}{4^{i-1}} \\ &= \frac{4}{3}E_1 \leq (\frac{2^7 cE}{\mathfrak{a}d} + \frac{2^7\bar{c}G}{\mathfrak{a}\bar{d}})E \end{aligned}$$

Now we prove (142). Let $F \in W_{\bar{v},s,\varepsilon}$, $F_{-1} := F$, $F_i := F \circ \Phi_0 \circ \cdots \circ \Phi_i$, $i = 0, \cdots, L$. Then

$$\begin{aligned} \|F\circ\Phi - F\|_{\bar{v}/6,s/6,\varepsilon/6} &= \|F_L - F\|_{\bar{v}_{L+1},s_{L+1},\varepsilon_{L+1}} \leq \sum_{i=0}^{L}\|F_{i-1}\circ\Phi_i - F_{i-1}\|_{\bar{v}_{i+1},s_{i+1},\varepsilon_{i+1}} \\ &\leq \sum_{i=0}^{L}\frac{\frac{\bar{c}E_i}{\mathfrak{a}\bar{\delta}_i}}{(1-\frac{\bar{c}E_i}{\mathfrak{a}\bar{\delta}_i})}\|F\|_{\bar{v}_i,s_i,\varepsilon_i} \leq \frac{\sum_{i=0}^{L}\frac{\bar{c}E_i}{\mathfrak{a}\bar{\delta}_i}}{\prod_{i=0}^{L}(1-\frac{\bar{c}E_i}{\mathfrak{a}\bar{\delta}_i})}\|F\|_{\bar{v},s,\varepsilon} \\ &\leq \sum_{i=0}^{L}\frac{\bar{c}E_i}{\mathfrak{a}\bar{\delta}_i}e^{\frac{5}{4}\sum_{i=0}^{L}\frac{\bar{c}E_i}{\mathfrak{a}\bar{\delta}_i}}\|F\|_{\bar{v},s,\varepsilon} \leq \frac{2^5\bar{c}E_0 Ks}{\mathfrak{a}d}\|F\|_{\bar{v},s,\varepsilon} \end{aligned}$$

where we have used $\frac{\bar{c}E_i}{\mathfrak{a}\bar{\delta}_i} < 1/24$ that, for $0 \leq x \leq 1/24$, $\log(1-x)^{-1} < \frac{5}{4}x$ and

$$\begin{aligned} \sum_{i=0}^{L}\frac{\bar{c}E_i}{\mathfrak{a}\bar{\delta}_i} &= \frac{\bar{c}E_0}{\mathfrak{a}\bar{\delta}_0} + \sum_{i=1}^{L}\frac{\bar{c}E_i}{\mathfrak{a}\bar{\delta}_i} \leq \frac{2^6\bar{c}E_0}{\mathfrak{a}d} + \frac{\bar{c}E_1}{\mathfrak{a}\bar{\delta}_1}\sum_{i=1}^{L}\frac{1}{4^{i-1}} \\ &\leq \frac{2^6\bar{c}E_0}{\mathfrak{a}d} + \frac{4}{3}\frac{\bar{c}E_1}{\mathfrak{a}\bar{\delta}_1} < \frac{2^4\bar{c}E_0 Ks}{\mathfrak{a}d}\ . \end{aligned}$$

The proof for $F \in W_{v,s,\varepsilon}$ is similar. ∎

Proof of Proposition D.1 For simplicity of notations, we prove Proposition D.1 in the case $\nu = \ell = 1$; the generalization to any ν, ℓ being straightforward. Consider the Hamiltonian

$$H_0(u_1,\varphi,p,q) := \mathrm{h}(I,p,q) + f_1(u_1,\varphi,p,q)\ , \qquad (u_1,\varphi,p,q) \in W^{(1)}_{v_1,s,\varepsilon}\ .$$

[11] Since $K > 8L$ and $L \geq 2$, one has $(1-\frac{3}{2L})^K \leq \frac{1}{(1+\frac{3}{2L})^{8L}}$ with the r.h.s bounded above by $(4/7)^{16}$ (it decreases to e^{-12} as $L \to +\infty$).

To this Hamiltonian let us apply Lemma D.1, with $g \equiv 0$, so as to conjugate it to

$$H_1 := H_0 \circ \Phi_1 = h + g_1 + f_{*1}^{(1)} \ , \qquad (u_1, \varphi, p, q) \in W^{(1)}_{v_1/6, s/6, \varepsilon/6}$$

where g_1, $f_{*1}^{(1)}$ correspond to g_*, f_*, hence satisfy

$$\begin{aligned} \|f_{*1}^{(1)}\|_{v_1/6,s/6,\varepsilon/6} &\leq e^{-Ks/6}\|f_1^{(i)}\|_{v_1,s,\varepsilon} \\ \|g_1\|_{v_1/6,s/6,\varepsilon/6} &\leq \frac{7}{6}\|f_1\|_{v_1,s,\varepsilon} \\ \|g_1 - \Pi_{\mathfrak{L}\cap\mathfrak{Z}} T_K f_1\|_{v_1/6,s/6,\varepsilon/6} &\leq \frac{2^7 c_1 \, \|f_1\|^2_{v_1,s,\varepsilon}}{\mathfrak{a} d_1} \end{aligned}$$

Then we have

$$H^{(1)}(u, \varphi, p, q) := H \circ \Phi_1 = H_0 \circ \Phi_1 + \sum_{j=2}^{N} f_j \circ \Phi_1 = \mathrm{h} + g_1 + f_{1*}^{(1)} + \sum_{j=2}^{N} f_j^{(1)}$$

where $f_j^{(1)} := f_j \circ \Phi_1$. Assume, inductively, that, for some $1 \leq i \leq N-1$ and any $1 \leq j \leq i$ we have conjugated H to

$$H^{(j)}(u, \varphi, p, q) = H \circ \Phi_1 \circ \cdots \circ \Phi_j = \mathrm{h} + \sum_{k=1}^{j} g_k + \sum_{k=1}^{j} f_{k*}^{(j)} + \sum_{k=j+1}^{N} f_k^{(j)}$$

where

$$\Phi_j : \ W^{(j)}_{v/6^j, s/6^j, \varepsilon/6^j} \to W^{(j-1)}_{v/6^{j-1}, s/6^{j-1}, \varepsilon/6^{j-1}}$$

transforms

$$H_{j-1} := \mathrm{h} + \sum_{k=1}^{j-1} g_k + f_j^{(j-1)}$$

into

$$H_{j-1} \circ \Phi_j = \mathrm{h} + \sum_{k=1}^{j} g_k + f_{*j}^{(j)} \ .$$

The Hamiltonian

$$H_i(u_{i+1}, \varphi, p, q) := \mathrm{h} + \sum_{k=1}^{i} g_k(u_k, \varphi, p, q) + f_{i+1}^{(i)}(u_{i+1}, \varphi, p, q)$$

is real–analytic for $(u_{i+1}, \varphi, p, q) \in W^{(i+1)}_{v_{i+1}/6^i, s/6^i, \varepsilon/6^i}$ and satisfies the assumptions of Lemma D.1, with $\bar{N} = i$. Then one can find $\Phi_{i+1} : \ W^{(i+1)}_{v_{i+1}/6^{i+1}, s/6^{i+1}, \varepsilon/6^{i+1}} \to W^{(i+1)}_{v_{i+1}/6^i, s/6^i, \varepsilon/6^i}$ such that

$H_i \circ \Phi_{i+1} = \mathrm{h} + \sum_{k=1}^{i+1} g_k + f_{*i+1}^{(i+1)}$, where

$$\begin{aligned}
\|f_{*i+1}^{(i+1)}\|_{v_{i+1}/6^{i+1},s/6^{i+1},\varepsilon/6^{i+1}} &\leq e^{-Ks/6^{i+1}} \|f_{i+1}^{(i)}\|_{v_{i+1}/6^i,s/6^i,\varepsilon/6^i} \\
&\leq (\frac{9}{8})^i e^{-Ks/6^{i+1}} \|f_{i+1}\|_{v_{i+1},s,\varepsilon} \\
\|g_{i+1}\|_{v_{i+1}/6^{i+1},s/6^{i+1},\varepsilon/6^{i+1}} &\leq \frac{7}{6}\|f_{i+1}^{(i)}\|_{v_{i+1}/6^i,s/6^i,\varepsilon/6^i} \leq \frac{7}{6}(\frac{9}{8})^i \|f_{i+1}\|_{v_{i+1},s,\varepsilon} \\
\|g_{i+1} - \Pi_{\mathfrak{L}\cap\mathfrak{Z}} T_K f_{i+1}\|_{v_{i+1}/6^{i+1},s/6^{i+1},\varepsilon/6^{i+1}} &\leq \|g_{i+1} - \Pi_{\mathfrak{L}\cap\mathfrak{Z}} T_K f_{i+1}^{(i)}\|_{v_{i+1}/6^{i+1},s/6^{i+1},\varepsilon/6^{i+1}} \\
&+ \|\Pi_{\mathfrak{L}\cap\mathfrak{Z}} T_K f_{i+1}^{(i)} - \Pi_{\mathfrak{L}\cap\mathfrak{Z}} T_K f_{i+1}\|_{v_{i+1}/6^{i+1},s/6^{i+1},\varepsilon/6^{i+1}} \\
&\leq \|g_{i+1} - \Pi_{\mathfrak{L}\cap\mathfrak{Z}} T_K f_{i+1}^{(i)}\|_{v_{i+1}/6^{i+1},s/6^{i+1},\varepsilon/6^{i+1}} \\
&+ \|f_{i+1}^{(i)} - f_{i+1}\|_{v_{i+1}/6^{i+1},s/6^{i+1},\varepsilon/6^{i+1}} \\
&\leq (\frac{9}{8})^{2i} \frac{2^7 c_{i+1} \|f_{i+1}\|_{v_{i+1},s,\varepsilon}^2}{\mathfrak{a} d_{i+1}} \\
&+ \frac{7}{6}(\frac{9}{8})^{2i} \sum_{j=1}^{i} \frac{2^7 c_j \|f_j\|_{v_j,s,\varepsilon}}{\mathfrak{a} d_j} \|f_{i+1}\|_{v_{i+1},s,\varepsilon} \\
&+ \sum_{k=1}^{i} (\frac{9}{8})^{i-k} \frac{2^4 c_k \|f_k\|_{v_k,s,\varepsilon} Ks}{\mathfrak{a} d_k} \|f_{i+1}\|_{v_{i+1},s,\varepsilon}
\end{aligned}$$

with $f_{k*}^{(i+1)} := f_{k*}^{(i)} \circ \Phi_{i+1}$ for $1 \leq k \leq i+1$ and $f_k^{(i+1)} := f_k^{(i)} \circ \Phi_{i+1}$ for $i+2 \leq k \leq N$. Then we find

$$\begin{aligned}
H^{(i+1)} &:= H^{(i)} \circ \Phi_{i+1} = (\mathrm{h} + \sum_{k=1}^{i} g_k + \sum_{k=1}^{i} f_{k*}^{(i)} + \sum_{k=i+1}^{N} f_k^{(i)}) \circ \Phi_{i+1} \\
&= H_i \circ \Phi_{i+1} + (\sum_{k=1}^{i} f_{k*}^{(i)} + \sum_{k=i+2}^{N} f_k^{(i)}) \circ \Phi_{i+1} \\
&= \mathrm{h} + \sum_{k=1}^{i+1} g_k + \sum_{k=1}^{i+1} f_{k*}^{(i+1)} + \sum_{k=i+2}^{N} f_k^{(i+1)}
\end{aligned}$$

and hence, after N steps,

$$H^{(N)} := H \circ \Phi_1 \cdots \circ \Phi_N = \mathrm{h} + \sum_{k=1}^{N} g_k + \sum_{k=1}^{i+1} f_{k*}^{(N)}$$

satisfies the thesis of Proposition D.1. ∎

D.2 A slightly–perturbed integrable system

The following result is well known in the literature of close–to be integrable systems, hence its proof is omitted. Note that it deals with an integrable system, close to another integrable one.

Theorem D.1 *One can find a number* $\mathfrak{c}_0$ *such that, for any real–analytic, one–dimensional, system*

$$\mathrm{H}(P,Q) = \mathrm{h}(\frac{P^2+Q^2}{2}) + f(P,Q) \qquad (P,Q) \in \mathfrak{B} = B_\varepsilon^2(0) \subset \mathbb{C}^2$$

and any $0 < \bar\varepsilon < \varepsilon$, *such that*

$$\inf_{B^2_{\bar\varepsilon}} |\partial \mathrm{h}| \geq \mathfrak{a} \ , \qquad \sup_{B^2_{\bar\varepsilon}} |f| \leq \mathfrak{e} \ , \qquad \frac{1}{\mathfrak{c}_0} \frac{\mathfrak{e}}{\mathfrak{a} \bar\varepsilon^2} < 1 \ , \tag{151}$$

one can find a real–analytic transformation

$$\phi_* : \quad (P_*, Q_*) \in B^2_{\varepsilon - \bar\varepsilon} \to (P, Q) \in B^2_\varepsilon$$

which conjugates H *to a function* $\mathrm{H}_* = \mathrm{H} \circ \phi_*$ *depending only on* $\frac{P_*^2 + Q_*^2}{2}$. *The assertion can be extended to the case that* h, f *are functions of other canonical coordinates* $(P', Q', \mathrm{y}, \mathrm{x})$, *depending on them only via* $\mathrm{Y} = (\mathrm{y}, \frac{P_1'^2 + Q_1'^2}{2}, \cdots, \frac{P_m'^2 + Q_m'^2}{2})$, *with* $\mathrm{y} \in \mathcal{Y}_\rho$, $(P_j', Q_j') \in B^2_{\varepsilon_j'}$. *In this case, letting* $(P_*, Q_*) \to \phi_*(P_*, Q_*; \mathrm{Y})$ *the transformation obtained for any fixed value of* Y, *there exists a canonical, real–analytic, transformation* Φ_* *of the form*

$$\Phi_* : \qquad (P, Q) = \phi_*(P_*, Q_*; \mathrm{Y}_*) \quad \mathrm{y} = \mathrm{y}_* \ , \quad \mathrm{x} = \mathrm{x}_* + \varphi(\mathrm{Y}_*) \ , \quad P_j' + \mathrm{i} Q_j' = e^{\mathrm{i}\psi_j(\mathrm{Y}_*)} (P_{*j}' + \mathrm{i} Q_{*j}')$$

which conjugates H *to a function* $\mathrm{H}_* = \mathrm{H} \circ \Phi_*$ *depending only on* $\frac{P_*^2 + Q_*^2}{2}$ *and* Y_*. *In this case, the functions* φ_j, ψ_j *verify*

$$|\varphi_j| \leq \frac{1}{\mathfrak{c}_0} \frac{\mathfrak{e}}{\mathfrak{a} \rho_j} \ , \qquad |\psi_j| \leq \frac{1}{\mathfrak{c}_0} \frac{\mathfrak{e}}{\mathfrak{a} \varepsilon_j'^2} \ .$$

E More on the geometrical structure of the $\mathcal{P}$–coordinates, compared to Deprit's coordinates

In this section we aim to point out differences and similarities between the $\mathcal{P}$–coordinates and the coordinates denoted as $(\Psi, \Gamma, \Lambda, \psi, \gamma, \ell)$ in [7, 27, 9].

We recall that the "planetary" coordinates $(\Psi, \Gamma, \Lambda, \psi, \gamma, \ell)$ may be derived (after a canonical transformation) from a more general set of canonical coordinates studied by A. Deprit. In their planetary form, the coordinates $(\Psi, \Gamma, \Lambda, \psi, \gamma, \ell)$ have been rediscovered[12] by the author during her PhD, under the strong motivation of their application to the planetary problem [27, 9]. Let us recall their definition[13], in the spirit of *Kepler maps* (Definition 2.2).

Let $\mathrm{C}^{(i)}_{\mathcal{E}}$, $\mathrm{S}^{(i)}_{\mathcal{E}}$ be as in (8) of Section 2 and define the $\mathcal{D}ep$–nodes

$$n_i := \begin{cases} k^{(3)} \times \mathrm{S}^{(1)}_{\mathcal{E}} & i = 0 \\ \mathrm{S}^{(i)}_{\mathcal{E}} \times \mathrm{S}^{(i+1)}_{\mathcal{E}} = -\mathrm{S}^{(i)}_{\mathcal{E}} \times \mathrm{C}^{(i)}_{\mathcal{E}} & i = 1, \cdots, n-1 \ . \\ -n_{n-1} & i = n \end{cases} \tag{152}$$

Then let

$$\mathcal{E}_{\mathcal{D}ep} := \left\{ ((\mathfrak{E}_1, \cdots, \mathfrak{E}_n) \subset E^3 \times \cdots \times E^3) : \quad 0 < e_i < 1 \ , \quad n_{i-1} \neq 0 \quad \forall\, i = 1, \cdots, n \right\} .$$

On $\mathcal{E}_{\mathcal{D}ep}$, define the map

$$\tau^{-1}_{\mathcal{D}ep} : \quad (\mathfrak{E}_1, \cdots, \mathfrak{E}_n) \in \mathcal{E}_{\mathcal{D}ep} \to \mathrm{X}_{\mathcal{D}ep} \in \mathfrak{X}_{\mathcal{D}ep} = \tau^{-1}_{\mathcal{D}ep}(\mathcal{E}_{\mathcal{D}ep})$$

[12] The proof of their symplectic character found in [27] has been published in [7]. Another proof has been given in [34].

[13] For sake of uniformity, we use slightly different notations with respect to the ones in [7], actually closer to the ones of the paper [12]).

where

$$\mathrm{X}_{\mathcal{D}ep} = (\Psi, \Gamma, \Lambda, \psi, \gamma) \in \mathbb{R}^n \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{T}^n \times \mathbb{T}^n$$

where

$$\Psi = (\Psi_{-1}, \Psi_0, \bar{\Psi}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n-2}_+ \quad \psi = (\psi_{-1}, \psi_0, \bar{\Psi}) \in \mathbb{T} \times \mathbb{T} \times \mathbb{T}^{n-2}$$

$$\Gamma = (\Gamma_1, \cdots, \Gamma_n) \in \mathbb{R}^n_+ \qquad \gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{T}^n$$

$$\Lambda = (\Lambda_1, \cdots, \Lambda_n) \in \mathbb{R}^n_+$$

with

$$\bar{\Psi} = (\Psi_1, \cdots, \Psi_{n-2}) \quad \bar{\psi} = (\psi_1, \cdots, \psi_{n-2})$$

are defined as follows. The coordinates Λ_j are as in (11), while $(\Psi, \Gamma, \psi, \gamma)$ are defined as

$$\Psi_{i-2} = \begin{cases} Z := \mathrm{S}^{(1)}_{\mathcal{E}} \cdot k^{(3)} \\ \\ |\mathrm{S}^{(i)}_{\mathcal{E}}| \end{cases} \qquad \psi_{i-2} = \begin{cases} \zeta := \alpha_{k^{(3)}}(k^{(1)}, n_0) & i = 1 \\ \\ \alpha_{\mathrm{S}^{(i-1)}_{\mathcal{E}}}(n_{i-2}, n_{i-1}) & 2 \leq i \leq n \end{cases} \tag{153}$$

$$\Gamma_i := |\mathrm{C}^{(i)}_{\mathcal{E}}| \qquad \gamma_i := \alpha_{\mathrm{C}^{(i)}_{\mathcal{E}}}(n_i, P^{(i)}) \qquad 1 \leq i \leq n$$

Then $\tau^{-1}_{\mathcal{D}ep}$ is a bijection [12, 27, 7, 34].

Definition E.1 We call *Deprit's map*, or $\mathcal{D}ep$ map, the Kepler map

$$\mathcal{D}ep: \qquad \mathrm{Dep} = (\mathrm{X}_{\mathcal{D}ep}, \ell) \in \mathcal{D}_{\mathcal{D}ep} = \mathfrak{X}_{\mathcal{D}ep} \times \mathbb{T}^n \to (y, x) \in \mathbb{R}^{3n} \times \mathbb{R}^{3n}$$

associated to $\tau_{\mathcal{D}ep}$.

Comparing $\mathcal{P}$ and $\mathcal{D}ep$

a) Both the $\mathcal{P}$ and $\mathcal{D}ep$–coordinates reduce the system to $(3n-2)$ degrees of freedom. They share the following three coordinates (two actions and an angle)

$$\Psi_{-1} = Z = \Theta_0\ , \quad \psi_{-1} = \zeta = \vartheta_0\ , \quad \Psi_0 = \mathrm{G} = \chi_0$$

which are integrals of the system. As a consequence, the coordinates (Z, ζ) and, respectively,

$$\mathrm{g} := \psi_0\ , \qquad \mathfrak{g} := \kappa_0$$

do not appear into the Hamiltonian. Note that $\mathcal{D}ep$ and $\mathcal{P}$ share also the fixed node $n_0 = \nu_1$.

b) The angle g for the set $\mathcal{D}ep$ describes the motion of the node n_1 in (152) and, by the cyclic character of g, this motion is negligible. Its counterpart in the set $\mathcal{P}$ is the the node n_1 in (10), the negligible motion of which is governed by $\mathfrak{g}$.

c) Compare the diagrams in (20) and (21) with the two ones associated to the $\mathcal{D}ep$–map, respectively:

$$
\begin{array}{ccccccccccc}
 & & n_0 & & n_1 & & \vdots & & n_{n-2} & & n_{n-1} \\
 & & \Uparrow & & \Uparrow & & \vdots & & \Uparrow & & \Uparrow \\
k^{(3)} & \to & \mathrm{S}_{\mathcal{E}}^{(1)} & \to & \mathrm{S}_{\mathcal{E}}^{(2)} & \to & \cdots & \to & \mathrm{S}_{\mathcal{E}}^{(n-1)} & \to & \mathrm{S}_{\mathcal{E}}^{(n)} = \mathrm{C}_{\mathcal{E}}^{(n)} \\
 & & \downarrow & & \downarrow & & \vdots & & \downarrow & & \\
 & & \mathrm{C}_{\mathcal{E}}^{(1)} & & \mathrm{C}_{\mathcal{E}}^{(2)} & & \vdots & & \mathrm{C}_{\mathcal{E}}^{(n-1)} & & \\
 & & \Downarrow & & \Downarrow & & \vdots & & \Downarrow & & \\
 & & -n_1 & & -n_2 & & \vdots & & -n_{n-1} & &
\end{array}
$$

and

$$
\begin{array}{ccccccccccc}
\mathrm{F}_0 & \to & \mathrm{F}_1^* & \to & \cdots & \to & \mathrm{F}_i^* & \to & \cdots & \to & \mathrm{F}_n^* = \mathrm{G}_n^* \\
 & & \downarrow & & \vdots & & \downarrow & & \vdots & & \downarrow \\
 & & \mathrm{G}_1^* & & & & \mathrm{G}_i^* & & & & \mathrm{G}_n^*
\end{array}
$$

where

$$
\mathrm{F}_i^* = (n_{i-1},\ \cdot\ , \mathrm{S}_{\mathcal{E}}^{(i)}) \quad \mathrm{G}_i^* = (-n_i,\ \cdot\ , \mathrm{C}_{\mathcal{E}}^{(i)}) \qquad i = 1, \cdots, n\ .
$$

Note that, analogously to (20), n_i in (152) is the skew-product of its two previous vectors in the tree (20).

d) While $\mathcal{D}ep$ is not defined for the planar problem, $\mathcal{P}$ is, and, in that case, the coordinates $(\Theta, \chi, \vartheta, \kappa)$ in (11) reduce to[14]

$$
\Theta_i = \begin{cases} \chi_0 \\ \\ 0 \end{cases} \qquad \vartheta_i = \begin{cases} 0 \\ \\ \pi \end{cases} \qquad \kappa_i = \begin{cases} \arg P^{(1)} - \dfrac{\pi}{2} & i = 0 \\ \\ \widehat{P^{(i)} P^{(i+1)}} + \pi & i = 1, \cdots, n-1 \end{cases}
$$

$$
\chi_i = \sum_{j=i+1}^{n} \|\mathrm{C}_{\mathcal{E}}^{(j)}\|
$$

while the (Λ, ℓ) remain unchanged.

e) The $\mathcal{P}$–map is singular when some eccentricity e_i vanishes or some of the following relations hold

$$
\mathrm{S}_{\mathcal{E}}^{(1)} \parallel k^{(3)} \qquad P^{(i)} \parallel \mathrm{S}_{\mathcal{E}}^{(i)} \qquad \mathrm{S}_{\mathcal{E}}^{(i+1)} \parallel P^{(i)}\ .
$$

[14] Here by "planar case" we mean $\mathrm{C}_{\mathcal{E}}^{(1)} \parallel \cdots \parallel \mathrm{C}_{\mathcal{E}}^{(n)} \parallel k^{(3)}$. Note that, to be more precise, ϑ_0 and κ_0 would not exist in that case (since $\nu_1 = 0$). However, since they are both cyclic angles, we can fix them to an arbitrary value. The choice above corresponds to replace ν_1 with $k^{(1)}$.

The former of such relations is negligible, while the other ones have no physical meaning. Therefore, the only physically relevant singularities of $\mathcal{P}$ are for zero–eccentric motions.
The $\mathcal{D}ep$–map is singular when some eccentricity e_i vanishes or some of the following relations hold

$$\mathrm{S}_{\mathcal{E}}^{(1)} \parallel k^{(3)} \qquad \mathrm{S}_{\mathcal{E}}^{(i+1)} \parallel \mathrm{S}_{\mathcal{E}}^{(i)} \qquad i = 1, \cdots, n-1 \ .$$

The configurations $\mathrm{S}_{\mathcal{E}}^{(i)} \parallel \mathrm{S}_{\mathcal{E}}^{(i+1)}$ have a relevant physical meaning, since the planar case corresponds to the intersection of all such configurations. A complete regularization of *all* the singularities of the $\mathcal{D}ep$–map has been obtained in [27, 9], which allowed to overcome the problem of the *rotational degeneracy* (see [8] for information) of the planetary problem and to construct the Brkhoff normal form of it. It works at expenses of one extra–degree of freedom.

f) The Euclidean lengths $\|\mathrm{C}_{\mathcal{E}}^{(i)}\|$ of the planets' angular momenta are the actions Γ_i among $\mathcal{D}ep$–coordinates: see (153). In terms of the $\mathcal{P}$–coordinates they have more involved expressions in (17). As mentioned in the previous item, this makes more difficult regularizing singular configurations with zero eccentricity. The formula simplifies in the planar case:

$$\|\mathrm{C}_{\mathcal{E}}^{(i)}\| = \begin{cases} |\chi_{i-1} - \chi_i| & i = 1, \cdots, n-1 \\ \\ \chi_{n-1} & i = n \end{cases}$$

where $|w| := \sqrt{w^2}$, for a given $w \in \mathbb{C}$.

g) Reflections are not well described in the framework of the $\mathcal{D}ep$–reduction: Compare, *e.g.*, [29, Section 4.4]. Instead, in the framework of the $\mathcal{P}$–reduction, the transformation

$$(\bar{\Theta}, \bar{\vartheta}) \to (-\bar{\Theta}, 2k\pi - \bar{\vartheta}) \qquad k \in \mathbb{Z}^{n-1}$$

corresponds to changing the sign of the second component of any $y^{(i)}$ and any $x^{(i)}$. Therefore, any of the points

$$(\bar{\Theta}, \bar{\vartheta}) = (0, k\pi) \qquad k \in \mathbb{Z}^{n-1}$$

is an equilibrium point for the Hamiltonian, corresponding to a co–planar configuration. Compare Proposition 2.2.

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