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KAM for Reversible Derivative Wave Equations

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Abstract

We prove the existence of Cantor families of small amplitude, analytic, linearly stable quasi-periodic solutions of reversible derivative wave equations.

1. Introduction

An important question in KAM theory for PDEs concerns equations with derivatives in the nonlinearity. Only few results are known, mainly restricted to dispersive equations. For Hamiltonian perturbations of KdV, the existence and stability of quasi-periodic solutions was first proved by Kuksin [18,19] in the late 1990s, see also Kappeler–Pöschel [16]. This approach has been recently extended by Liu–Yuan [15] for Hamiltonian DNLS and by Zhang et al. [28] for the reversible DNLS equation $\mathrm{i}u_t + u_{\mathrm{xx}} + |u_{\mathrm{x}}|^2 u = 0$.

The *derivative* nonlinear wave equation (DNLW), which is *not* dispersive, is *excluded* by these approaches (for semilinear wave equations see [5,7,9,19,21,27]). The existence of periodic solutions (without stability) for the derivative Klein–Gordon equation

$$\mathrm{y}_{tt} - \mathrm{y}_{\mathrm{xx}} + \mathrm{m}\mathrm{y} + \mathrm{y}_t^2 = 0, \quad \mathrm{m} > 0, \quad \mathrm{x} \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}, \tag{1.1}$$

was first proved by Bourgain in [8], extending the approach of Craig–Wayne in [11]. Then Craig [10] focused on the natural question of establishing similar results for more general derivative wave equations

$$\mathrm{y}_{tt} - \mathrm{y}_{\mathrm{xx}} + \mathrm{m}\mathrm{y} = g(\mathrm{x}, \mathrm{y}, \mathrm{y}_{\mathrm{x}}, \mathrm{y}_t), \quad \mathrm{x} \in \mathbb{T}, \tag{1.2}$$

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asking, for example, if $y_{tt} - y_{xx} = y_x^3$ possesses periodic solutions, see [10, section 7.3].

In [3] we recently extended KAM theory for the Hamiltonian model

$$y_{tt} - y_{xx} + my + f(Dy) = 0, \quad m > 0, \quad D := \sqrt{-\partial_{xx} + m}, \quad x \in \mathbb{T}.$$

These kinds of pseudo-differential equations were introduced by Bourgain [7] and Craig [10] as models to study the effect of derivatives versus dispersive phenomena. Clearly [3] does not apply to the derivative wave equations (1.2), which are not Hamiltonian.

In order to prove the existence of periodic/quasi-periodic solutions for (1.2), conditions on the nonlinearity g have to be necessarily imposed. For example, (1.2) with the nonlinear friction term $g = y_t^3$ has no nontrivial smooth periodic/quasi-periodic solutions, see Proposition 1.1. This case may be ruled out by assuming the reversibility condition

$$g(x, y, y_x, -v) = g(x, y, y_x, v) \tag{1.3}$$

satisfied, for example, by (1.1). Under condition (1.3) Equation (1.2) is time-*reversible*, namely the associated first order system

$$y_t = v, \quad v_t = y_{xx} - my + g(x, y, y_x, v) \tag{1.4}$$

is reversible with respect to the involution

$$S(y, v) := (y, -v), \quad S^2 = I. \tag{1.5}$$

For finite-dimensional systems it is known (since Moser [20]) that reversibility may replace the Hamiltonian structure in order to allow the existence of quasi-periodic solutions, see also Arnold [1] and Sevryuk [26]. However, for (1.2) it is *not* sufficient. For example $y_{tt} - y_{xx} = y_x^3$ is time reversible but it has *no* smooth periodic/quasi-periodic solutions except the constants (in Proposition 1.1 we exhibit more general time-reversible nonlinearities for which DNLW has only trivial quasi-periodic solutions). In order to find quasi-periodic solutions we also require the "space-reversibility" assumption

$$g(-x, y, -y_x, v) = g(x, y, y_x, v) \tag{1.6}$$

which rules out nonlinearities like $y_x^3, y_x^5, \ldots$. Actually, condition (1.6) is as natural as (1.3). Indeed, for the wave equation (1.2), the role of time and space variables (t, x) is highly symmetric, and, considering x "as time" (spatial dynamics idea) (1.6) is nothing but the corresponding reversibility condition and terms like $y_x^3, y_x^5, \ldots$ are frictions.

In this paper we prove the *existence* and *stability* of *analytic* quasi-periodic solutions for derivative wave equations (1.2) satisfying (1.3), (1.6), see Theorem 1.1. By the above considerations, this is a very natural class of DNLW equations which may admit quasi-periodic solutions. After Theorem 1.1 we shall further comment on the assumptions. These results were presented in the note [4].

Before describing our main results, we mention the classical bifurcation theorems of Rabinowitz [24] about periodic solutions (with period $T \in \pi\mathbb{Q}$) of dissipative forced derivative wave equations

$$\mathrm{y}_{tt} - \mathrm{y}_{\mathrm{xx}} + \alpha \mathrm{y}_t + \varepsilon F(\mathrm{x}, t, \mathrm{y}, \mathrm{y}_{\mathrm{x}}, \mathrm{y}_t) = 0, \quad \mathrm{x} \in [0, \pi]$$

with Dirichlet boundary conditions, and in [25] with a fully-non-linear forcing term $F = F(\mathrm{x}, t, \mathrm{y}, \mathrm{y}_{\mathrm{x}}, \mathrm{y}_t, \mathrm{y}_{tt}, \mathrm{y}_{t\mathrm{x}}, \mathrm{y}_{\mathrm{xx}})$. Note that for forced PDEs the nonlinearity does not need to be reversible or Hamiltonian.

1.1. Main Results

We consider derivative wave equations (1.2) where $\mathrm{m} > 0$, the nonlinearity $g : \mathbb{T} \times \mathcal{U} \to \mathbb{R}$, $\mathcal{U} \subset \mathbb{R}^3$ open neighborhood of 0, is real analytic and satisfies the assumptions (1.3), (1.6). We require g to vanish at least quadratically at $(\mathrm{y}, \mathrm{y}_{\mathrm{x}}, \mathrm{v}) = (0, 0, 0)$, namely

$$g(\mathrm{x}, 0, 0, 0) = (\partial_{\mathrm{y}} g)(\mathrm{x}, 0, 0, 0) = (\partial_{\mathrm{y}_{\mathrm{x}}} g)(\mathrm{x}, 0, 0, 0) = (\partial_{\mathrm{v}} g)(\mathrm{x}, 0, 0, 0) = 0.$$

Because of (1.3), it is natural to look for "reversible" solutions, namely those such that $\mathrm{y}(t, \mathrm{x})$ is even and $\mathrm{v}(t, \mathrm{x})$ is odd in time, and, because of (1.6), it is natural to restrict things to solutions for which x is even (standing waves). Hence we look for quasi-periodic solutions of (1.2) satisfying

$$\mathrm{y}(t, \mathrm{x}) = \mathrm{y}(t, -\mathrm{x}), \ \forall t, \quad \mathrm{y}(-t, \mathrm{x}) = \mathrm{y}(t, \mathrm{x}), \ \forall \mathrm{x} \in \mathbb{T}. \tag{1.7}$$

For every finite choice of the *tangential* sites $\mathcal{I}^+ \subset \mathbb{N} \backslash \{0\}$, the linear Klein–Gordon equation

$$\mathrm{y}_{tt} - \mathrm{y}_{\mathrm{xx}} + \mathrm{m}\mathrm{y} = 0, \quad \mathrm{x} \in \mathbb{T}, \tag{1.8}$$

possesses the family of quasi-periodic standing wave solutions

$$\mathrm{y} = \sum\nolimits_{j \in \mathcal{I}^+} \sqrt{8\xi_j}\, \lambda_j^{-1} \cos(\lambda_j t) \cos(j\mathrm{x}), \quad \lambda_j := \sqrt{j^2 + \mathrm{m}}, \tag{1.9}$$

parametrized by the "actions" $\xi_j \in \mathbb{R}_+$ and with linear frequencies of oscillations $\bar{\omega} := (\lambda_j)_{j \in \mathcal{I}^+}$.

In order to continue such solutions for the nonlinear equation (1.2)—as it is well known in KAM theory—the leading term of the nonlinearity g has to satisfy some non-degeneracy condition so that the "action-to-frequency" map "twists". For definiteness, we have focused on nonlinearities

$$g = g^{(=3)}(\mathrm{y}, \mathrm{y}_{\mathrm{x}}, \mathrm{y}_t) + g^{(\geqq 5)}(\mathrm{x}, \mathrm{y}, \mathrm{y}_{\mathrm{x}}, \mathrm{y}_t) \tag{1.10}$$

with the cubic leading term

$$g^{(=3)} = \kappa_1 \mathrm{y}^3 + \kappa_2 \mathrm{y}\mathrm{y}_{\mathrm{x}}^2 + \kappa_3 \mathrm{y}\mathrm{y}_{\mathrm{t}}^2, \qquad \kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}, \tag{1.11}$$

and $g^{(\geqq 5)}$ collects terms of order at least five in $(\mathrm{y}, \mathrm{y}_{\mathrm{x}}, \mathrm{y}_t)$. We assume the *non-degeneracy* condition

$$\kappa_1 + (\kappa_2 + \kappa_3) i^2 + \kappa_3 \mathrm{m} \neq 0, \quad \forall i \in \mathcal{I}^+. \tag{1.12}$$

Note that, for each $\mathrm{m} > 0$, condition (1.12) is verified for all the $(\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3$ outside finitely many hyperplanes, for example for *each* $(\kappa_1, \kappa_2, \kappa_3) \neq 0$ with non negative components $\kappa_j \geqq 0$, $j = 1, 2, 3$.

Fix a compact interval $[\mathrm{m}_1, \mathrm{m}_2] \subset (0, \infty)$ and assume that the mass $\mathrm{m} \in [\mathrm{m}_1, \mathrm{m}_2]$ satisfies the finitely many *non-resonance* conditions

$$(\lambda_i^{-1} \pm \lambda_j^{-1}) 4\bar{\omega}, \ \lambda_j^{-1} 4\bar{\omega} \notin (2n-1)\mathbb{Z}^{n/2} \setminus \{0\}, \ \forall i, j \in \mathbb{N} \setminus \mathcal{I}^+, i, j \leqq C_0, \tag{1.13}$$

where $\bar{\omega} := (\lambda_h)_{h \in \mathcal{I}^+}$, $\lambda_h = \sqrt{h^2 + \mathrm{m}}$, n is twice the cardinality of $\mathcal{I}^+$, and C_0 is a suitably large constant depending on $\mathrm{m}_1, \mathrm{m}_2, \mathcal{I}^+$. Note that, for a given set $\mathcal{I}^+$ of tangential sites, condition (1.13) is verified, by analiticity, for all the masses $\mathrm{m} \in [\mathrm{m}_1, \mathrm{m}_2]$ except *finitely* many (and independently of $\kappa_1, \kappa_2, \kappa_3$).

Theorem 1.1. *Assume that the tangential sites* $\mathcal{I}^+ \subset \mathbb{N} \setminus \{0\}$, *the mass* $\mathrm{m} \in [\mathrm{m}_1, \mathrm{m}_2]$ *and* $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$ *satisfy* (1.12)*,* (1.13)*. Then the DNLW equation* (1.2) *with a real analytic nonlinearity satisfying* (1.3)*,* (1.6)*,* (1.10)*–*(1.11) *admits small-amplitude, analytic (both in* t *and* x*), quasi-periodic solutions*

$$\mathrm{y} = \sum_{j \in \mathcal{I}^+} \sqrt{8\xi_j}\, \lambda_j^{-1} \cos(\omega_j^\infty(\xi)\, t) \cos(j\mathrm{x}) + o(\sqrt{\xi}), \quad \omega_j^\infty(\xi) \overset{\xi \to 0}{\longrightarrow} \sqrt{j^2 + \mathrm{m}} \tag{1.14}$$

satisfying (1.7)*, for a Cantor-like set of parameters with density* 1 *at* $\xi = 0$*. The quasi-periodic solutions have zero Lyapunov exponents and the linearized equations can be reduced to constant coefficients (in a phase space of functions even in* x*). The term* $o(\sqrt{\xi})$ *in* (1.14) *is small in some analytic norm.*

Let us comment on the hypothesis of Theorem 1.1.

1. **Reversibility in time and space.** The assumptions (1.3), (1.6), are natural conditions for the existence of quasi-periodic solutions of (1.2), because they imply the reversibility assumption of Moser [20] on the subspace of functions even in x (which does *not* follow by requiring only one of them), and so they allow solution of the homological equations along the KAM proof. Terms like $\mathrm{y}_\mathrm{x}^p, \mathrm{y}_t^p$ with p odd, destroy the oscillations of the Birkhoff normal form and produce drifts of the actions incompatible with the existence of quasi-periodic solutions. Proposition 1.1 proves rigorously these non existence results using suitable Lyapunov functions, for which terms like y_x^p and y_t^p act as friction terms. This shows the role of condition (1.6). As an example, the nonlinearity $g = \mathrm{y}^3 + \mathrm{y}_\mathrm{x}^5$ satisfies all the conditions (1.3), (1.10), (1.11) (and (1.12) holds for each $\mathcal{I}^+$), but *not* (1.6), and non trivial quasi periodic solutions of (1.2) do not exist.

   Thanks to (1.6) we can restrict things to solutions which are even in x and this simplifies the KAM proof because the normal form (4.1) is diagonal. However, as said above, the main reason to assume (1.3) + (1.6) is that they imply the reversibility with respect to the involution used in Moser [20] (see (1.32), (1.33)). This does not follow, for example, by (1.3) and the condition $g(-\mathrm{x}, -\mathrm{y}, \mathrm{y}_\mathrm{x}, \mathrm{v}) = -g(\mathrm{x}, \mathrm{y}, \mathrm{y}_\mathrm{x}, \mathrm{v})$ for which the subspace of functions $(\mathrm{y}, \mathrm{v})(\mathrm{x})$ odd in x is invariant (Dirichlet boundary conditions). One could possibly also deal with other nonlinearities using the involution $(\mathrm{y}(\mathrm{x}), \mathrm{v}(\mathrm{x})) \mapsto (\mathrm{y}(-\mathrm{x}), -\mathrm{v}(-\mathrm{x}))$, which implies the Moser reversibility as well.

2. **Mass** $\mathtt{m} > 0$**.** Also the assumption on the mass $\mathtt{m} \neq 0$ is natural. When $\mathtt{m} = 0$, Proposition 1.2 proves that (1.1) has no smooth solutions for all times except the constants. In Proposition 1.3 we prove other non-existence results of quasi-periodic solutions for DNLW equations satisfying both (1.3), (1.6), but with mass $\mathtt{m} = 0$.
3. **Twist.** The term $g^{(=3)}(\mathtt{y}, \mathtt{y}_{\mathtt{x}}, \mathtt{v})$ in (1.11) is the most general cubic nonlinearity which satisfies (1.3), (1.6) and which is $\mathtt{x}$-independent. Proposition 1.1 proves that for $\mathtt{y}_{\mathtt{x}}^3$, $\mathtt{y}^2\mathtt{y}_{\mathtt{x}}$, $\mathtt{v}^3$, there exist no non-trivial quasi-periodic solutions of (1.2). In (1.10) the leading term $g^{(=3)}$ could also depend explicitly on $\mathtt{x}$ and the higher order nonlinearities have order four, see Remark 7.1.
4. $\mathtt{x}$**-dependence.** The nonlinearity g in (1.2) may explicitly depend on the space variable $\mathtt{x}$. This is a novelty with respect to [3] which used the conservation of momentum, see comments below.
5. **Derivative vs quasi-linear NLW.** Klainermann–Majda [17] exhibited a class of quasi-linear wave equations which do not have smooth periodic (a fortiori quasi-periodic) solutions except for the constants. In this respect [17] may suggest that Theorem 1.1 is optimal regarding the order of (integer) derivatives in the nonlinearity.

The proof of Theorem 1.1 is based on a KAM theorem (see Theorem 4.1) whose key step is, like in [3], to prove the first order asymptotic expansion of the perturbed normal frequencies of the linearized equations along the iteration, see (4.10). This enables us to verify the well known second order Melnikov conditions which allow us to reduce the KAM normal form to constant coefficients. Unlike the case where g does not depend on the derivatives $\mathtt{y}_t$, $\mathtt{y}_{\mathtt{x}}$, this expansion requires hard work. This is achieved by the notion of a *quasi-Töplitz vector field* introduced in Section 3. This class is closed with respect to Lie brackets and Lie transform (Propositions 3.1–3.2). This concept is clearly modelled on the Hamiltonian case in [3,23], and it is related to the Töplitz–Lipschitz functions in Eliasson–Kuksin [12,13] (see also [14]), but there are differences. Actually this notion appears natural for vector fields. We underline two main novelties:

1. As already said, here we consider the general case of $\mathtt{x}$-dependent nonlinearities which break the translation invariance. In [3], and [22,23], the theory of quasi-Töplitz functions was developed for $\mathtt{x}$-independent nonlinearities; namely it relied on the conservation of momentum. This property was used in essential ways, for example in order to prove that the class of quasi-Töplitz functions is closed under a Poisson-bracket. A point of conceptual interest in this paper is that we also show how to use efficiently the notion of momentum when this is not a conserved quantity. Monomial vector fields with a large momentum should be less and less relevant for dynamics. This is efficiently implemented by the introduction of the $\mathtt{a}$-momentum norm (Definition 2.3) which penalizes the high momentum monomials, see (2.24). This allows us to neglect in Proposition 3.1 the high momentum monomial vector fields, by slightly decreasing the parameter $\mathtt{a}$. With this new idea the theory of quasi-Töplitz vector fields is obtained similarly to [3].

2. Another point of conceptual interest is to use the notion of momentum working in a subspace (here of even functions). Until now it was not clear how to proceed, see the end of Section 1.2. In this paper this is achieved by the symmetrization procedure described in Section 5.1. The key observation is that the quasi-Toplitz norm does not increase under symmetrization, Proposition 5.2.

We will add some more technical comments about the proof in Section 1.2. Now we complement Theorem 1.1 with some *non-existence* results.

Proposition 1.1. *Let $p \in \mathbb{N}$ be odd. The DNLW equations (1.2) with*

$$\text{(i) } g = \mathrm{y}_{\mathrm{x}}^p + f(\mathrm{y}), \qquad \text{(ii) } g = \partial_{\mathrm{x}}(\mathrm{y}^p) + f(\mathrm{y}), \qquad \text{(iii) } g = \mathrm{y}_t^p + f(\mathrm{y}) \tag{1.15}$$

have no smooth quasi-periodic solutions except trivial periodic solutions $\mathrm{y}(t,\mathrm{x}) = c(t)$ for (i), (ii) and $\mathrm{y}(t,\mathrm{x}) = c(\mathrm{x})$ for (iii), respectively. If $f \equiv 0$ then $c(\cdot) \equiv const$.

Proof. The function $M := \int_{\mathbb{T}} \mathrm{y}_{\mathrm{x}}\mathrm{y}_t \, \mathrm{dx}$ is a Lyapunov function of (1.15)-(i) since

$$\frac{\mathrm{d}}{\mathrm{d}t} M = \int_{\mathbb{T}} \mathrm{y}_{\mathrm{x}}^{p+1} \, \mathrm{dx} \geqq 0.$$

Hence M strictly increases along the solutions unless $\mathrm{y}_{\mathrm{x}}(t,\mathrm{x}) = 0, \forall t$, namely $\mathrm{y}(t,\mathrm{x}) = c(t)$. Case (ii) is similar. A Lyapunov function of (1.15)-(iii) is $H := \int_{\mathbb{T}} \frac{\mathrm{v}^2}{2} + \frac{\mathrm{y}_{\mathrm{x}}^2}{2} - F(\mathrm{y}) \, \mathrm{dx}$ where $F' = f$. □

The mass term my could be necessary to have the existence of quasi-periodic solutions.

Proposition 1.2. *The DNLW equation*

$$\mathrm{y}_{tt} - \mathrm{y}_{\mathrm{xx}} = \mathrm{y}_t^2, \quad \mathrm{x} \in \mathbb{T}, \tag{1.16}$$

has no smooth solutions defined for all times except the constants.

Proof. We decompose the solution $\mathrm{y}(t,\mathrm{x}) = \mathrm{y}_0(t) + \tilde{\mathrm{y}}(t,\mathrm{x})$ where $\mathrm{y}_0 := \int_{\mathbb{T}} \mathrm{y}(t,\mathrm{x}) \, \mathrm{dx}$ and $\tilde{\mathrm{y}} := \mathrm{y} - \mathrm{y}_0$ has zero average in x. Then, projecting (1.16) on the constants, we get

$$\begin{aligned} \ddot{\mathrm{y}}_0 = \int_{\mathbb{T}} \mathrm{y}_t^2 \, \mathrm{dx} = \int_{\mathbb{T}} (\dot{\mathrm{y}}_0 + \tilde{\mathrm{y}}_t)^2 \, \mathrm{dx} = \dot{\mathrm{y}}_0^2 + 2\dot{\mathrm{y}}_0 \int_{\mathbb{T}} \tilde{\mathrm{y}}_t \, \mathrm{dx} + \int_{\mathbb{T}} \tilde{\mathrm{y}}_t^2 \, \mathrm{dx} = \dot{\mathrm{y}}_0^2 \\ + \int_{\mathbb{T}} \tilde{\mathrm{y}}_t^2 \, \mathrm{dx} \geqq \dot{\mathrm{y}}_0^2. \end{aligned} \tag{1.17}$$

Hence $\mathrm{v}_0 := \dot{\mathrm{y}}_0$ satisfies $\dot{\mathrm{v}}_0 \geqq \mathrm{v}_0^2$, which blows up unless $\mathrm{v}_0 \equiv 0$. But, in this case, (1.17) implies that $\mathrm{y}_t(t,\mathrm{x}) \equiv 0, \forall \mathrm{x}$. Hence $\mathrm{y}(t,\mathrm{x}) = \mathrm{y}(\mathrm{x})$ and (1.16) (and $\mathrm{x} \in \mathbb{T}$) imply that $\mathrm{y}(t,\mathrm{x}) = \text{const}$. □

The above non-existence result may be generalized as follows:

Proposition 1.3. *Let $p, q \in \mathbb{N}$ be even. Then the derivative NLW equations*

$$\mathtt{y}_{tt} - \mathtt{y}_{\mathtt{xx}} = \mathtt{y}_{\mathtt{x}}^{p}, \qquad \mathtt{y}_{tt} - \mathtt{y}_{\mathtt{xx}} = \mathtt{y}_{t}^{p}, \qquad \mathtt{y}_{tt} - \mathtt{y}_{\mathtt{xx}} = \mathtt{y}_{\mathtt{x}}^{p} + \mathtt{y}_{t}^{q}, \quad \mathtt{x} \in \mathbb{T}, \tag{1.18}$$

have no smooth periodic/quasi-periodic solutions except the constants.

Proof. If there exists a periodic solution $(\mathtt{y}(t,\mathtt{x}), \mathtt{v}(t,\mathtt{x}))$ of the first equation, with period T, then

$$\int_0^T \int_{\mathbb{T}} (\mathtt{y}_{tt} - \mathtt{y}_{\mathtt{xx}})\, \mathrm{d}t\, \mathrm{d}\mathtt{x} = 0 = \int_0^T \int_{\mathbb{T}} \mathtt{y}_{\mathtt{x}}^{p}(t,\mathtt{x})\, \mathrm{d}\mathtt{x}\, \mathrm{d}t.$$

Hence, $\forall t \in [0,T]$, $\mathtt{y}_{\mathtt{x}}(t,\mathtt{x}) = 0$, $\forall \mathtt{x} \in \mathbb{T}$, that is $\mathtt{y}(t,\mathtt{x}) = c(t)$. Inserting in (1.18) we get $c_{tt}(t) = 0$ and its only periodic solutions are $c(t) = const$. For quasi-periodic solutions the argument is the same. The other equations can be treated analogously. $\square$

1.2. About the Proof of Theorem 1.1

Complex formulation. In the unknowns

$$u^+ := \frac{1}{\sqrt{2}}(D\mathtt{y} - \mathrm{i}\mathtt{v}), \quad u^- := \frac{1}{\sqrt{2}}(D\mathtt{y} + \mathrm{i}\mathtt{v}), \quad D := \sqrt{-\partial_{\mathtt{xx}} + \mathtt{m}}, \quad \mathrm{i} := \sqrt{-1},$$

systems (1.4) becomes the first order system

$$u_t^+ = \mathrm{i}Du^+ + \mathrm{i}\mathtt{g}(u^+, u^-), \quad u_t^- = -\mathrm{i}Du^- - \mathrm{i}\mathtt{g}(u^+, u^-) \tag{1.19}$$

where

$$\mathtt{g}(u^+, u^-) = -\frac{1}{\sqrt{2}}\, g\left(\mathtt{x}, D^{-1}\left(\frac{u^+ + u^-}{\sqrt{2}}\right), D^{-1}\left(\frac{u_{\mathtt{x}}^+ + u_{\mathtt{x}}^-}{\sqrt{2}}\right), \frac{u^- - u^+}{\mathrm{i}\sqrt{2}}\right). \tag{1.20}$$

Since g is real on real, the subspace $\mathtt{R} := \{\overline{u^+} = u^-\}$ is invariant under the flow evolution of (1.19). Clearly, this corresponds to real valued solutions $(\mathtt{y}, \mathtt{v})$ of (1.4). By (1.6) the subspace of even functions

$$\mathtt{E} := \left\{u^+(\mathtt{x}) = u^+(-\mathtt{x}),\ u^-(\mathtt{x}) = u^-(-\mathtt{x})\right\} \tag{1.21}$$

is invariant. Moreover (1.19) is reversible with respect to the involution

$$S(u^+, u^-) = (u^-, u^+) \tag{1.22}$$

which is nothing but (1.5) in the variables (u^+, u^-).

Dynamical systems formulation. We introduce coordinates by Fourier transform

$$u^+ = \sum\nolimits_{j\in\mathbb{Z}} u_j^+ \mathrm{e}^{\mathrm{i}j\mathtt{x}}, \quad u^- = \sum\nolimits_{j\in\mathbb{Z}} u_j^- \mathrm{e}^{-\mathrm{i}j\mathtt{x}}. \tag{1.23}$$

Then (1.19) becomes the infinite dimensional dynamical system

$$\dot{u}_j^+ = \mathrm{i}\lambda_j u_j^+ + \mathrm{i}\mathtt{g}_j^+(\ldots, u_h^+, u_h^-, \ldots), \quad \dot{u}_j^- = -\mathrm{i}\lambda_j u_j^- - \mathrm{i}\mathtt{g}_j^-(\ldots, u_h^+, u_h^-, \ldots), \tag{1.24}$$

$\forall j \in \mathbb{Z}$, where $\lambda_j := \sqrt{j^2 + \mathtt{m}}$ are the eigenvalues of D and

$$\mathtt{g}_j^+ = \frac{1}{2\pi} \int_{\mathbb{T}} \mathtt{g}\left(\sum\nolimits_{h\in\mathbb{Z}} u_h^+ \mathrm{e}^{\mathrm{i}h\mathrm{x}}, \sum\nolimits_{h\in\mathbb{Z}} u_h^- \mathrm{e}^{-\mathrm{i}h\mathrm{x}}\right) \mathrm{e}^{-\mathrm{i}j\mathrm{x}}\, d\mathrm{x}, \qquad \mathtt{g}_j^- := \mathtt{g}_{-j}^+. \tag{1.25}$$

By (1.23), the "real" subset $\mathtt{R}$ reads $\overline{u_j^+} = u_j^-$ (this is the motivation for the choice of the signs in (1.23)). The invariant subspace $\mathtt{E}$ of even functions in (1.21) reads, under Fourier transform,

$$E := \left\{ u_j^+ = u_{-j}^+,\ u_j^- = u_{-j}^-,\ \forall j \in \mathbb{Z} \right\}. \tag{1.26}$$

By (1.23) the involution (1.22) reads

$$S : (u_j^+, u_j^-) \to (u_{-j}^-, u_{-j}^+), \quad \forall j \in \mathbb{Z}. \tag{1.27}$$

Finally, since g is real analytic, the assumptions (1.3) and (1.6) imply the important property

$$\mathtt{g}_j^{\pm}(\ldots, u_i^+, u_i^-, \ldots) \textit{ has real Taylor coefficients in } (u_i^+, u_i^-). \tag{1.28}$$

This property is compatible with an oscillatory behavior for (1.24), excluding friction phenomena.

Abstract KAM theorem. For every choice of the symmetric *tangential sites*

$$\mathcal{I} = \mathcal{I}^+ \cup (-\mathcal{I}^+) \quad \text{with} \quad \mathcal{I}^+ \subset \mathbb{N} \setminus \{0\},\ \sharp\mathcal{I} = n, \tag{1.29}$$

we introduce (after the Birkhoff normal form of Section 7), action-angle variables

$$\begin{aligned} u_j^+ &= \sqrt{\xi_{|j|} + y_j}\, \mathrm{e}^{\mathrm{i}x_j},\ u_j^- = \sqrt{\xi_{|j|} + y_j}\, \mathrm{e}^{-\mathrm{i}x_j},\ j \in \mathcal{I}, \\ (u_j^+, u_j^-) &= (z_j^+, z_j^-) \equiv (z_j, \bar z_j),\ j \notin \mathcal{I}, \end{aligned} \tag{1.30}$$

where $|y_j| < \xi_{|j|}$. Then (1.24) is conjugated to a parameter dependent family of vector fields (as in Section 4)

$$\mathcal{X} := \mathcal{N} + \mathcal{P} \tag{1.31}$$

with a normal form $\mathcal{N}$ as in (4.1), (4.2), and a perturbation $\mathcal{P}$ as in (4.3) which satisfies (A1)–(A4). In particular the vector field (1.31) is

1. REVERSIBLE (Definition 2.5) with respect to the involution

$$S : (x_j, y_j, z_j, \bar z_j) \mapsto (-x_{-j}, y_{-j}, \bar z_{-j}, z_{-j}),\ \forall j \in \mathbb{Z}, \quad S^2 = I, \tag{1.32}$$

   which is nothing but (1.27) in the variables (1.30).
2. REAL-COEFFICIENTS (Definition 2.6), by (1.28).
3. EVEN. The vector field $\mathcal{P} : E \to E$ and so the subspace

$$E := \left\{ x_j = x_{-j},\ y_j = y_{-j},\ j \in \mathcal{I},\ z_j = z_{-j},\ \bar z_j = \bar z_{-j},\ j \in \mathbb{Z} \setminus \mathcal{I} \right\} \tag{1.33}$$

   is invariant under the flow evolution of (1.31).

4. QUASI-TÖPLITZ. The perturbation $\mathcal{P}$ is a *quasi-Töplitz vector field*, Definition 3.4.

The reversibility property (1.32) on the subspace E in (1.33) implies that the average of the term $\mathcal{P}^{(y)}(x, 0, 0, 0)$ is zero (because $\mathcal{P}^{(y)}(x, 0, 0, 0)$ is an odd function in x) along the whole iteration, otherwise quasi-periodic solutions would not exist. Note that we use both (1.3) and (1.6) for the solvability of the homological equations in Lemma 5.1. Then the "real-coefficients" property implies that the corrections to the normal form are purely imaginary (elliptic).

Quasi-Töplitz property. The second order Melnikov non resonance conditions are verified proving that the elliptic frequencies (after the application of the KAM Theorem 4.1) satisfy an asymptotic expansion like $\Omega_j^\infty(\xi) = |j| + c(\xi) + O(1/|j|)$, see (4.10) and (4.4). Indeed, since $c(\xi)$ is *independent* of j, it cancels in the difference $\Omega_j^\infty(\xi) - \Omega_i^\infty(\xi)$ and the measure estimates follow as in the semilinear case (see [21]), where $c(\xi) \equiv 0$. We only state them in Theorem 4.2, whose proof is like that in [3].

The KAM corrections to the frequencies are the coefficients of the linear monomial vector fields $z_j \partial_{z_j}$, $\bar{z}_j \partial_{\bar{z}_j}$ of the perturbation P, and we want to show that, for $|j| > N$, they assume a constant value up to an error of $O(N^{-1})$. Since we need to work with a class of vector fields fulfilling the Lie algebra property, we cannot clearly impose conditions only on these diagonal terms, but we have to consider a larger set of vector fields which are only *approximately* x-independent, linear and diagonal (and y-independent). The quasi-Töplitz vector fields introduced in Section 3 fulfill quantitatively these requirements, see the comments above Definition 3.2.

The symmetrization procedure. In the subspace of functions even in $\mathtt{x}$ the notion of MOMENTUM of a monomial is not well defined. For example the vector fields $z_{-j}\partial_{z_i}$ and $z_j \partial_{z_i}$, that have DIFFERENT momentum, are identified. In other words we can NOT work directly in the cosine basis $\{\cos(j\mathtt{x})\}_{j \geqq 0}$, which would be natural, looking for solutions even in $\mathtt{x}$ (avoiding the double eigenvalues).

Then we proceed as follows. In system (1.31) we think of $x_j, y_j, z_j^\pm$, as independent variables. In this case, since the linear frequencies $\omega_{-j} = \omega_j$, $\Omega_{-j} = \Omega_j$ are resonant, along the KAM iteration, the monomial vector fields of the perturbation

$$
\begin{aligned}
&e^{ik\cdot x}\partial_{x_j}, e^{ik\cdot x} y^i \partial_{y_j}, k \in \mathbb{Z}^n_{\rm odd}, |i| = 0, 1,\ j \in \mathcal{I},\ e^{ik\cdot x} z_{\pm j}\partial_{z_j}, e^{ik\cdot x}\bar{z}_{\pm j}\partial_{\bar{z}_j},\\
&\forall k \in \mathbb{Z}^n_{\rm odd},\ j \in \mathbb{Z} \setminus \mathcal{I},
\end{aligned}
$$

where

$$
k \in \mathbb{Z}^n_{\rm odd} := \left\{ k \in \mathbb{Z}^n \ :\ k_{-j} = -k_j,\ \forall j \in \mathcal{I} \right\} \tag{1.34}
$$

cannot be averaged out. On the other hand, on the invariant subspace E, where we look for the quasi-periodic solutions, the above terms can be replaced by the constant coefficients monomial vector fields, obtained by setting $x_{-j} = x_j, z^\pm_{-j} = z^\pm_j$. Replacing the vector field $\mathcal{P}$ with its symmetrized $\mathcal{SP}$ (Definition 5.2) the $\mathtt{a}$-momentum and quasi-Töplitz norms do not increase (Proposition 5.2). Both $\mathcal{P}$ and $\mathcal{SP}$ determine the same dynamics on the subspace E (Proposition 5.1). The vector field $\mathcal{SP}$ is symmetric and reversible as well (see (5.22)), and the homological equations (5.21) can be solved, see Lemma 5.1. This procedure allows the KAM iteration to be carried out. Remark 5.1 shows that the symmetrization procedure is required at each KAM step.

In Section 7 we finally apply the abstract KAM Theorem 4.1 to prove Theorem 1.1. The main steps are the proof that the vector field of g is quasi-Töplitz (Lemma 7.1), that the Birkhoff normal form transformation preserves the quasi-Töplitz property (Proposition 7.1) and that the frequency-to-action map is twist, see (7.32), (7.34).

2. Vector Fields Formalism

We introduce the main properties of the vector fields used throughout the paper (commutators, momentum, norms, reversibility, degree,...). We shall refer often to section 2 of [3]. The first difference with respect to [3] is that we have to work at the level of vector fields and not of functions (Hamiltonians).

For a finite set $\mathcal{I} \subset \mathbb{Z}$ (possibly empty) and $a \geqq 0$, $p > 1/2$, we define the Hilbert space

$$\ell_{\mathcal{I}}^{a,p} := \Big\{ z = \{z_j\}_{j \in \mathbb{Z} \setminus \mathcal{I}},\ z_j \in \mathbb{C} \ :\ \|z\|_{a,p}^2 := \sum\nolimits_{j \in \mathbb{Z} \setminus \mathcal{I}} |z_j|^2 e^{2a|j|} \langle j \rangle^{2p} < \infty \Big\} \tag{2.1}$$

that, when $\mathcal{I} = \emptyset$, we denote more simply by $\ell^{a,p}$. Let n be the cardinality of $\mathcal{I}$. We consider $V := \mathbb{C}^n \times \mathbb{C}^n \times \ell_{\mathcal{I}}^{a,p} \times \ell_{\mathcal{I}}^{a,p}$ (denoted by E in [3]) with (s, r)-weighted norm

$$v = (x, y, z, \bar{z}) \in V, \quad \|v\|_{s,r} = \frac{|x|_\infty}{s} + \frac{|y|_1}{r^2} + \frac{\|z\|_{a,p}}{r} + \frac{\|\bar{z}\|_{a,p}}{r} \tag{2.2}$$

where $0 < s, r < 1$, and $|x|_\infty := \max_{h=1,\dots,n} |x_h|$, $|y|_1 := \sum_{h=1}^n |y_h|$.

Note that z and $\bar{z}$ are *independent* variables. We shall also use the notation $z_j^+ = z_j$, $z_j^- = \bar{z}_j$, and

$$\mathtt{V} := \big\{ x_1, \dots, x_n, y_1, \dots, y_n, \dots, z_j, \dots, \bar{z}_j, \dots \big\}, \quad j \in \mathbb{Z} \setminus \mathcal{I}. \tag{2.3}$$

As phase space, we consider the toroidal domain

$$D(s, r) := \mathbb{T}_s^n \times D(r) := \mathbb{T}_s^n \times B_{r^2} \times B_r \times B_r \subset V \tag{2.4}$$

where $\mathbb{T}_s^n := \{x \in \mathbb{C}^n : \mathrm{Re}(x) \in \mathbb{T}^n := 2\pi\mathbb{R}^n/\mathbb{Z}^n, \max_{h=1,\dots,n} |\mathrm{Im}\, x_h| < s\}$, $B_{r^2} := \{y \in \mathbb{C}^n : |y|_1 < r^2\}$ and $B_r \subset \ell_{\mathcal{I}}^{a,p}$ is the open ball of radius r centered at zero. If $n = 0$ then $D(s, r) \equiv B_r \times B_r \subset \ell^{a,p} \times \ell^{a,p}$.

We also introduce the "real" phase space

$$\mathtt{R}(s, r) := \Big\{ v = (x, y, z^+, z^-) \in D(s, r) \ :\ x \in \mathbb{T}^n,\ y \in \mathbb{R}^n,\ \overline{z^+} = z^- \Big\} \tag{2.5}$$

where $\overline{z^+}$ is the complex conjugate of z^+.

We consider vector fields of the form

$$X(v) = (X^{(x)}(v), X^{(y)}(v), X^{(z)}(v), X^{(\bar z)}(v)) \in V \tag{2.6}$$

where $v \in D(s,r)$ and $X^{(x)}(v), X^{(y)}(v) \in \mathbb{C}^n$, $X^{(z)}(v), X^{(\bar z)}(v) \in \ell_{\mathcal{I}}^{a,p}$. We also use the differential geometry notation

$$X(v) = X^{(x)}\partial_x + X^{(y)}\partial_y + X^{(z)}\partial_z + X^{(\bar z)}\partial_{\bar z} = \sum\nolimits_{\mathrm{v}\in\mathbb{V}} X^{(\mathrm{v})}\partial_{\mathrm{v}}, \tag{2.7}$$

recall (2.3). Equivalently we write $X(v) = (X^{(\mathrm{v})}(v))_{\mathrm{v}\in\mathbb{V}}$ where each component is a *formal* scalar power series

$$X^{(\mathrm{v})}(v) = \sum_{(k,i,\alpha,\beta)\in\mathbb{I}} X^{(\mathrm{v})}_{k,i,\alpha,\beta}\, \mathrm{e}^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \tag{2.8}$$

with coefficients $X^{(\mathrm{v})}_{k,i,\alpha,\beta} \in \mathbb{C}$ and multi-indices in

$$\mathbb{I} := \mathbb{Z}^n \times \mathbb{N}^n \times \mathbb{N}^{(\mathbb{Z}\setminus\mathcal{I})} \times \mathbb{N}^{(\mathbb{Z}\setminus\mathcal{I})} \tag{2.9}$$

where $\mathbb{N}^{(\mathbb{Z}\setminus\mathcal{I})} := \{\alpha := (\alpha_j)_{j\in\mathbb{Z}\setminus\mathcal{I}} \in \mathbb{N}^{\mathbb{Z}}$ with $|\alpha| := \sum_{j\in\mathbb{Z}\setminus\mathcal{I}}\alpha_j < +\infty\}$. In (2.8) we use the standard multi-indices notation $z^\alpha \bar z^\beta := \Pi_{j\in\mathbb{Z}\setminus\mathcal{I}} z_j^{\alpha_j} \bar z_j^{\beta_j}$.

The formal vector field X is absolutely convergent in V (with norm (2.2)) at $v \in D(s,r)$ if every component $X^{(\mathrm{v})}(v)$, $\mathrm{v} \in \mathbb{V}$, is absolutely convergent and $\|(X^{(\mathrm{v})}(v))_{\mathrm{v}\in\mathbb{V}}\|_{s,r} < +\infty$.

Definition 2.1. (*Monomial vector field*) A monomial vector field is

$$\mathfrak{m}_{k,i,\alpha,\beta;\mathrm{v}'}(v) = \mathfrak{m}_{k,i,\alpha,\beta}(v)\partial_{\mathrm{v}'} \quad \textit{where} \quad \mathfrak{m}_{k,i,\alpha,\beta}(v) := \mathrm{e}^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \tag{2.10}$$

is a scalar monomial.

A vector field X may be decomposed as a formal series of vector field monomials

$$X(v) = \sum_{\mathrm{v}\in\mathbb{V}} \sum_{(k,i,\alpha,\beta)\in\mathbb{I}} X^{(\mathrm{v})}_{k,i,\alpha,\beta} \mathrm{e}^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \partial_{\mathrm{v}}. \tag{2.11}$$

For a subset of indices $I \subset \mathbb{I} \times \mathbb{V}$ we define the projection

$$(\Pi_I X)(v) := \sum_{(k,i,\alpha,\beta,\mathrm{v})\in I} X^{(\mathrm{v})}_{k,i,\alpha,\beta}\, \mathrm{e}^{\mathrm{i}k\cdot x} y^i z^\alpha \bar z^\beta \partial_{\mathrm{v}}. \tag{2.12}$$

The commutator (or Lie bracket) of two vector fields is $[X, Y](v) := \mathrm{d}X(v)[Y(v)] - \mathrm{d}Y(v)[X(v)]$, namely, its v-component is

$$[X, Y]^{(\mathrm{v})} = \sum\nolimits_{\mathrm{v}'\in\mathbb{V}} \partial_{\mathrm{v}'} X^{(\mathrm{v})} Y^{(\mathrm{v}')} - \partial_{\mathrm{v}'} Y^{(\mathrm{v})} X^{(\mathrm{v}')}. \tag{2.13}$$

Given a vector field X, its transformed field under the time 1 flow generated by Y is

$$\mathrm{e}^{\mathrm{ad}_Y} X = \sum\nolimits_{k\geqq 0} \frac{1}{k!} \mathrm{ad}_Y^k X, \qquad \mathrm{ad}_Y X := [X, Y], \tag{2.14}$$

where $\mathrm{ad}_Y^k := \mathrm{ad}_Y^{k-1}\mathrm{ad}_Y$ and $\mathrm{ad}_Y^0 := \mathrm{Id}$.

2.1. Momentum Majorant Norm

Fix a set of indices

$$\mathcal{I} := \{\mathtt{j}_1, \ldots, \mathtt{j}_n\} \subset \mathbb{Z}. \tag{2.15}$$

Definition 2.2. The MOMENTUM of the vector field monomial $\mathfrak{m}_{k,i,\alpha,\beta;\mathrm{v}}$ is

$$\pi(k,\alpha,\beta;\mathrm{v}) := \begin{cases} \pi(k,\alpha,\beta) & \text{if} \quad \mathrm{v} \in \{x_1,\ldots,x_n,y_1,\ldots,y_n\} \\ \pi(k,\alpha,\beta) - \sigma j & \text{if} \quad \mathrm{v} = z_j^\sigma, \ \sigma = \pm, \end{cases} \tag{2.16}$$

where

$$\pi(k,\alpha,\beta) := \sum\nolimits_{i=1}^n \mathtt{j}_i k_i + \sum\nolimits_{j\in\mathbb{Z}\setminus\mathcal{I}} (\alpha_j - \beta_j) j \tag{2.17}$$

is the momentum of the scalar monomial $\mathfrak{m}_{k,i,\alpha,\beta}(v)$.

We say that a vector field X satisfies momentum conservation if and only if it is a linear combination of monomial vector fields with zero momentum.

Let $\mathtt{a} \geqq 0$. Given a vector field X as in (2.11) we define its "$\mathtt{a}$-momentum majorant" vector field

$$(M_{\mathtt{a}}X)(v) := \sum_{\mathrm{v}\in\mathbb{V}} \sum_{(k,i,\alpha,\beta)\in\mathbb{I}} e^{\mathtt{a}|\pi(k,\alpha,\beta;\mathrm{v})|} |X^{(\mathrm{v})}_{k,i,\alpha,\beta}| e^{ik\cdot x} y^i z^\alpha \bar{z}^\beta \partial_{\mathrm{v}} \tag{2.18}$$

where $\pi(k,\alpha,\beta;\mathrm{v})$ is the momentum of the monomial $\mathfrak{m}_{k,i,\alpha,\beta;\mathrm{v}}$ defined in (2.16). When $\mathtt{a} = 0$ we simply write MX instead of M_0X, which coincides with the majorant vector field in [3, section 2.1.2].

Definition 2.3. ($\mathtt{a}$-*momentum majorant-norm*) The $\mathtt{a}$-momentum majorant norm of a formal vector field X as in (2.11) is

$$\|X\|_{s,r,\mathtt{a}} := \sup_{(y,z,\bar{z})\in D(r)} \left\| \left(\sum_{k,i,\alpha,\beta} e^{\mathtt{a}|\pi(k,\alpha,\beta;\mathrm{v})|} |X^{(\mathrm{v})}_{k,i,\alpha,\beta}| e^{|k|s} |y^i||z^\alpha||\bar{z}^\beta| \right)_{\mathrm{v}\in\mathbb{V}} \right\|_{s,r} \tag{2.19}$$

where $|k| := |k|_1 = |k_1| + \cdots + |k_n|$. For a function $f : D(s,r) \to \mathbb{C}$ it reduces to $\|f\|_{s,r,\mathtt{a}} := \sup_{D(r)} \sum_{k,i,\alpha,\beta} e^{\mathtt{a}|\pi(\alpha,\beta,k)|} |f_{k,i,\alpha,\beta}| e^{s|k|} |y^i||z^\alpha||\bar{z}^\beta|$.

When $\mathtt{a} = 0$ the norm $\|\cdot\|_{s,r,0}$ coincides with the "majorant norm" introduced in [3]-Definition 2.6 (where it was simply denoted by $\|\cdot\|_{s,r}$). By (2.19) and (2.18) we get $\|X\|_{s,r,\mathtt{a}} = \|M_{\mathtt{a}}X\|_{s,r,0}$.

Remark 2.1. By the above relation, the norm $\|\cdot\|_{s,r,\mathtt{a}}$ satisfies the same properties of the majorant norm $\|\cdot\|_{s,r,0}$ and the next lemmas for the norm $\|\cdot\|_{s,r,\mathtt{a}}$ follow by the analogous lemmas in [3] for $\|\cdot\|_{s,r,0}$.

Let $|X|_{s,r} := \sup_{v\in D(s,r)} \|X(v)\|_{s,r}$. Arguing as for Lemma 2.11 in [3] we get

Lemma 2.1. *Assume that for some* $s, r > 0, \mathtt{a} \geqq 0$*, the* $\mathtt{a}$*-momentum majorant-norm* $\|X\|_{s,r,\mathtt{a}} < +\infty$*. Then the series in* (2.11)*, resp.* (2.18)*, absolutely converge to the* analytic *vector field* $X(v)$*, resp.* $M_{\mathtt{a}}X(v)$*, for every* $v \in D(s,r)$*. Moreover* $|X|_{s,r}, |M_{\mathtt{a}}X|_{s,r} \leqq \|X\|_{s,r,\mathtt{a}}$.

For a vector field $X : D(s,r) \times \mathcal{O} \to V$ depending on parameters $\xi \in \mathcal{O} \subset \mathbb{R}^n$, we define the λ-Lipschitz (momentum majorant) norm ($\lambda \geqq 0$)

$$\begin{aligned}\|X\|^{\lambda}_{s,r,\mathtt{a},\mathcal{O}} := \|X\|^{\lambda}_{s,r,\mathtt{a}} &:= \|X\|_{s,r,\mathtt{a},\mathcal{O}} + \lambda \|X\|^{\mathrm{lip}}_{s,r,\mathtt{a},\mathcal{O}} \\ &:= \sup_{\xi \in \mathcal{O}} \|X(\xi)\|_{s,r,\mathtt{a}} + \lambda \sup_{\xi,\eta \in \mathcal{O},\ \xi \neq \eta} \frac{\|X(\xi) - X(\eta)\|_{s,r,\mathtt{a}}}{|\xi - \eta|}\end{aligned} \tag{2.20}$$

and we set

$$\mathcal{V}^{\lambda}_{s,r,\mathtt{a}} := \mathcal{V}^{\lambda}_{s,r,\mathtt{a},\mathcal{O}} := \left\{ X : D(s,r) \times \mathcal{O} \to V \ : \ \|X\|^{\lambda}_{s,r,\mathtt{a}} < \infty \right\}.$$

Similarly, we denote by $\mathcal{V}_{s,r,\mathtt{a}}$ the linear space of vector fields with $\|X\|_{s,r,\mathtt{a}} < \infty$. Note that, if X is independent of ξ, then $\|X\|^{\lambda}_{s,r,\mathtt{a}} = \|X\|_{s,r,\mathtt{a}}, \forall \lambda$.

It is easy to check that the $\|\cdot\|^{\lambda}_{s,r,\mathtt{a}}$ norm behaves well under projections (2.12):

Lemma 2.2. (Projection) $\forall I \subset \mathbb{I} \times \mathbb{V}$ *we have* $\|\Pi_I X\|_{s,r,\mathtt{a}} \leqq \|X\|_{s,r,\mathtt{a}}$ *and* $\|\Pi_I X\|^{\mathrm{lip}}_{s,r,\mathtt{a}} \leqq \|X\|^{\mathrm{lip}}_{s,r,\mathtt{a}}$.

Important particular cases are the "ultraviolet" projection

$$(\Pi_{|k| \geqq K} X)(v) := \sum_{|k| \geqq K, i, \alpha, \beta} X^{(\mathrm{v})}_{k,i,\alpha,\beta}\, e^{\mathrm{i} k \cdot x} y^i z^{\alpha} \bar{z}^{\beta} \partial_{\mathrm{v}}, \qquad \Pi_{|k| < K} := \mathrm{Id} - \Pi_{|k| \geqq K} \tag{2.21}$$

and the "high momentum" projection

$$(\Pi_{|\pi| \geqq K} X)(v) := \sum_{|\pi(k,\alpha,\beta;\mathrm{v})| \geqq K} X^{(\mathrm{v})}_{k,i,\alpha,\beta}\, e^{\mathrm{i} k \cdot x} y^i z^{\alpha} \bar{z}^{\beta} \partial_{\mathrm{v}}, \quad \Pi_{|\pi| < K} := \mathrm{Id} - \Pi_{|\pi| \geqq K}. \tag{2.22}$$

By (2.19) the following smoothing estimates follow:

Lemma 2.3. (Smoothing) $\forall K \geqq 1$ *and* $\lambda \geqq 0$

$$\|\Pi_{|k| \geqq K} X\|^{\lambda}_{s',r,\mathtt{a}} \leqq \frac{s}{s'} e^{-K(s-s')} \|X\|^{\lambda}_{s,r,\mathtt{a}}, \qquad \forall\, 0 < s' < s \tag{2.23}$$

$$\|\Pi_{|\pi| \geqq K} X\|^{\lambda}_{s,r,\mathtt{a}'} \leqq e^{-K(\mathtt{a}-\mathtt{a}')} \|X\|^{\lambda}_{s,r,\mathtt{a}}, \qquad \forall\, 0 \leqq \mathtt{a}' \leqq \mathtt{a}. \tag{2.24}$$

The space of analytic vector fields with finite $\mathtt{a}$-momentum majorant norm form a Lie algebra.

Proposition 2.1. (Commutator) *Let* $X, Y \in \mathcal{V}^{\lambda}_{s,r,\mathtt{a}}$*. Then, for* $\lambda \geqq 0, r/2 \leqq r' < r, s/2 \leqq s' < s$*,*

$$\|[X,Y]\|^{\lambda}_{s',r',\mathtt{a}} \leqq 2^{2n+3} \delta^{-1} \|X\|^{\lambda}_{s,r,\mathtt{a}} \|Y\|^{\lambda}_{s,r,\mathtt{a}} \quad \textit{where } \delta := \min\left\{1 - \frac{s'}{s}, 1 - \frac{r'}{r}\right\}. \tag{2.25}$$

Proof. We say that a vector field X has momentum $\pi(X) = h$ if it is an absolutely convergent series of monomial vector fields of momentum h. It results that, if X, Y have momentum $\pi(X)$, $\pi(Y)$, respectively, then $\pi([X, Y]) = \pi(X) + \pi(Y)$. Then the proof of $\|[X, Y]\|_{s',r',\mathtt{a}} \leqq 2^{2n+3}\delta^{-1}\|X\|_{s,r,\mathtt{a}}\|Y\|_{s,r,\mathtt{a}}$ follows as in [3, Lemma 2.15]. The Lipschitz estimate follows as usual. $\square$

2.2. Degree Decomposition

The degree of the monomial vector field $\mathfrak{m}_{k,i,\alpha,\beta;\mathrm{v}}$ is defined as

$$d(\mathfrak{m}_{k,i,\alpha,\beta;\mathrm{v}}) := |i| + |\alpha| + |\beta| - d(\mathrm{v}) \quad \text{where } d(\mathrm{v}) := \begin{cases} 0 & \text{if} \quad \mathrm{v} \in \{x_1, \dots, x_n\} \\ 1 & \text{otherwise,} \end{cases}$$

in particular $d(\partial_x) = 0, d(\partial_y) = d(\partial_{z_j}) = d(\partial_{\bar{z}_j}) = -1$. This notion naturally extends to any vector field by monomial decomposition: we say that a vector field has degree h if it is an absolutely convergent series of monomial vector fields of degree h.

The degree d gives to the vector fields the structure of a graded Lie algebra: given two vector fields X, Y of degree respectively $d(X)$ and $d(Y)$, then

$$d([X, Y]) = d(X) + d(Y). \tag{2.26}$$

For a vector field X as in (2.11) we define the homogeneous component of degree $l \in \mathbb{N}$,

$$X^{(l)} := \Pi^{(l)} X := \sum_{|i|+|\alpha|+|\beta|-d(\mathrm{v})=l} X^{(\mathrm{v})}_{k,i,\alpha,\beta}\, \mathrm{e}^{\mathrm{i}k\cdot x} y^i z^\alpha \bar{z}^\beta \partial_{\mathrm{v}} \tag{2.27}$$

and we set

$$X^{\leqq 0} := X^{(-1)} + X^{(0)}. \tag{2.28}$$

Definition 2.4. We denote by $\mathcal{R}^{\leqq 0}$ the vector fields with degree $\leqq 0$. Using the compact notation $\mathrm{u} := (y, z, \bar{z}) = (y, z^+, z^-)$, a vector field in $\mathcal{R}^{\leqq 0}$ writes

$$R = R^{\leqq 0} = R^{(-1)} + R^{(0)}, \quad R^{(-1)} = R^{\mathrm{u}}(x)\partial_{\mathrm{u}}, \quad R^{(0)} = R^x(x)\partial_x + R^{\mathrm{u},\mathrm{u}}(x)\mathrm{u}\,\partial_{\mathrm{u}}, \tag{2.29}$$

where $R^x(x) \in \mathbb{C}^n$, $R^{\mathrm{u}} \in \mathbb{C}^n \times \ell^{a,p}_{\mathcal{I}} \times \ell^{a,p}_{\mathcal{I}}$, $R^{\mathrm{u},\mathrm{u}}(x) \in \mathcal{L}(\mathbb{C}^n \times \ell^{a,p}_{\mathcal{I}} \times \ell^{a,p}_{\mathcal{I}})$. In more extended notation

$$\begin{aligned} R^{\mathrm{u}}(x)\partial_{\mathrm{u}} &= R^y(x)\partial_y + R^z(x)\partial_z + R^{\bar{z}}(x)\partial_{\bar{z}} \\ R^{\mathrm{u},\mathrm{u}}(x)\mathrm{u}\partial_{\mathrm{u}} &= \left(R^{y,y}(x)y + R^{y,z}(x)z + R^{y,\bar{z}}(x)\bar{z}\right)\partial_y \\ &\quad + \left(R^{z,y}(x)y + R^{z,z}(x)z + R^{z,\bar{z}}(x)\bar{z}\right)\partial_z \\ &\quad + \left(R^{\bar{z},y}(x)y + R^{\bar{z},z}(x)z + R^{\bar{z},\bar{z}}(x)\bar{z}\right)\partial_{\bar{z}}. \end{aligned} \tag{2.30}$$

The terms of the vector field that we want to eliminate (or normalize) along the KAM iteration are those in $\mathcal{R}^{\leqq 0}$. The graded Lie algebra property (2.26) implies that $\mathcal{R}^{\leqq 0}$ is closed by Lie bracket:

Lemma 2.4. *If $X, Y \in \mathcal{R}^{\leqq 0}$ then $[X, Y] \in \mathcal{R}^{\leqq 0}$.*

2.3. Reversible, Real-Coefficients, Real-on-real, Even, Vector Fields

We first define the class of reversible/anti-reversible vector fields (this concept was efficiently used in [6] for finding Birkhoff–Lewis periodic solutions of NLW).

Definition 2.5. (Reversibility) A vector field X as in (2.6) is REVERSIBLE with respect to an involution S (namely $S^2 = I$) if $X \circ S = -S \circ X$. A vector field Y is ANTI-REVERSIBLE if $Y \circ S = S \circ Y$.

When the set $\mathcal{I}$ is symmetric as in (1.29) and S is the involution in (1.32), a vector field X is reversible if its coefficients (see (2.8)) satisfy

$$X^{(\mathtt{v})}_{k,i,\alpha,\beta} = \begin{cases} X^{(\hat{\mathtt{v}})}_{-\hat{k},\hat{\imath},\hat{\beta},\hat{\alpha}} & \text{if} \quad \mathtt{v} = x_j, \quad j \in \mathcal{I}, \\ -X^{(\hat{\mathtt{v}})}_{-\hat{k},\hat{\imath},\hat{\beta},\hat{\alpha}} & \text{if} \quad \mathtt{v} = y_j, \quad j \in \mathcal{I}, \\ -X^{(z^{-\sigma}_{-j})}_{-\hat{k},\hat{\imath},\hat{\beta},\hat{\alpha}} & \text{if} \quad \mathtt{v} = z^\sigma_j, \quad j \in \mathbb{Z} \setminus \mathcal{I} \end{cases} \tag{2.31}$$

where

$$\hat{k} := (k_{-j})_{j \in \mathcal{I}}, \ \hat{\imath} := (i_{-j})_{j \in \mathcal{I}}, \ \hat{\beta} := (\beta_{-j})_{j \in \mathbb{Z} \setminus \mathcal{I}}, \ \hat{\alpha} := (\alpha_{-j})_{j \in \mathbb{Z} \setminus \mathcal{I}}, \\ \hat{\mathtt{v}} := (\mathtt{v}_{-j})_{j \in \mathbb{Z}}. \tag{2.32}$$

Definition 2.6. A vector field $X = X^{(x)}\partial_x + X^{(y)}\partial_y + X^{(z^+)}\partial_{z^+} + X^{(z^-)}\partial_{z^-}$ is

- "REAL-COEFFICIENTS" if the Taylor–Fourier coefficients of $X^{(x)}$, $\mathrm{i}X^{(y)}$, $\mathrm{i}X^{(z^+)}$, $\mathrm{i}X^{(z^-)}$ are *real*,
- "ANTI-REAL-COEFFICIENTS" if $\mathrm{i}X$ is real-coefficients,
- "REAL-ON-REAL" if

  $$X^{(x)}(v) = \overline{X^{(x)}(v)}, \ X^{(y)}(v) = \overline{X^{(y)}(v)}, \ X^{(z^-)}(v) = \overline{X^{(z^+)}(v)}, \quad \forall v \in \mathtt{R}(s,r),$$

  where $\mathtt{R}(s, r)$ is defined in (2.5),
- "EVEN" if $X : E \to E$ (see (1.33)).

On the coefficients in (2.8) the REAL-ON-REAL condition amounts to

$$\overline{X^{(\mathtt{v})}_{k,i,\alpha,\beta}} = \begin{cases} X^{(\mathtt{v})}_{-k,i,\beta,\alpha} & \text{if} \quad \mathtt{v} \in \{x_1, \dots, x_n, y_1, \dots, y_n\} \\ X^{(z^{-\sigma}_j)}_{-k,i,\beta,\alpha} & \text{if} \quad \mathtt{v} = z^\sigma_j, \end{cases} \tag{2.33}$$

and the REVERSIBILITY IN SPACE condition to

$$X^{(\mathtt{v})}_{k,i,\alpha,\beta} = X^{(\hat{\mathtt{v}})}_{\hat{k},\hat{\imath},\hat{\alpha},\hat{\beta}} \quad \text{(see (2.32))}. \tag{2.34}$$

Definition 2.7. We denote by

- $\mathcal{R}_{\mathrm{rev}}$ the vector fields which are reversible, real-coefficients, real-on-real and even.
- $\mathcal{R}_{a-\mathrm{rev}}$ the vector fields which are *anti*-reversible, *anti*-real-coefficients, real-on-real and even.

- $\mathcal{R}^{\leqq 0}_{\mathrm{rev}} := \mathcal{R}_{\mathrm{rev}} \cap \mathcal{R}^{\leqq 0}$ and $\mathcal{R}^{\leqq 0}_{a-\mathrm{rev}} := \mathcal{R}_{a-\mathrm{rev}} \cap \mathcal{R}^{\leqq 0}$.

If the vector field X is reversible and Y is anti-reversible then $[X, Y]$ and $\mathrm{e}^{\mathrm{ad}_Y} X$ (recall (2.14)) are reversible. If X, resp. Y, is real-coefficients, resp. anti-real-coefficients, then $[X, Y]$, $\mathrm{e}^{\mathrm{ad}_Y} X$ are real-coefficients. If X, Y are real-on-real, then $[X, Y]$, $\mathrm{e}^{\mathrm{ad}_Y} X$ are real-on-real. If X, Y are even then $[X, Y]$, $\mathrm{e}^{\mathrm{ad}_Y} X$ are even. Therefore we get

Lemma 2.5. *If $X \in \mathcal{R}_{\mathrm{rev}}$ and $Y \in \mathcal{R}_{a-\mathrm{rev}}$ then $[X, Y]$, $e^{\mathrm{ad}_Y} X \in \mathcal{R}_{\mathrm{rev}}$.*

By (2.27), (2.28) and (2.34) we immediately get (the space E was defined in (1.33))

$$X_{|E} \equiv 0 \qquad \Longrightarrow \qquad (X^{\leqq 0})_{|E} \equiv 0. \tag{2.35}$$

Lemma 2.6. *If $X_{|E} \equiv 0$ and Y is even then $([X, Y])_{|E} \equiv 0$, $(e^{\mathrm{ad}_Y} X)_{|E} \equiv 0$.*

3. Quasi-Töplitz Vector Fields

Let $N_0 \in \mathbb{N}$, $\theta, \mu \in \mathbb{R}$ be parameters such that

$$1 < \theta, \mu < 6, \quad 12 N_0^{L-1} + 2\kappa N_0^{b-1} < 1, \quad \kappa := \max_{1 \leqq l \leqq n} |\mathtt{j}_l| \quad (\kappa := 0 \text{ if } \mathcal{I} := \emptyset), \tag{3.1}$$

where $\mathcal{I} := \{\mathtt{j}_1, \ldots, \mathtt{j}_n\}$, see (2.15), and with the three scales

$$0 < b < L < 1, \tag{3.2}$$

see comments before Definition 3.2. In the following we will always take $N \geqq N_0$.

Definition 3.1. A scalar monomial $\mathfrak{m}(k, i, \alpha, \beta) = \mathrm{e}^{\mathrm{i}k \cdot x} y^i z^\alpha \bar{z}^\beta$ is (N, μ)**-low momentum** if

$$|k| < N^b, \quad \alpha + \beta = \gamma \quad \text{with} \quad \sum\nolimits_{l \in \mathbb{Z} \setminus \mathcal{I}} |l| \gamma_l < \mu N^L. \tag{3.3}$$

An (N, μ)-low momentum scalar monomial is (N, μ, h)**-low** if

$$|\pi(k, \alpha, \beta) - h| < N^b. \tag{3.4}$$

We denote by $\mathcal{A}^L_{s,r,\mathtt{a}}(N, \mu)$, respectively $\mathcal{A}^L_{s,r,\mathtt{a}}(N, \mu, h)$, the closure of the vector space generated by (N, μ)-low, resp. (N, μ, h)-low, scalar monomials in the norm $\| \; \|_{s,r,\mathtt{a}}$ in Definition 2.3.
The projection on $\mathcal{A}^L_{s,r,\mathtt{a}}(N, \mu, h)$ will be denoted by $\Pi^{L,h}_{N,\mu}$. Note that it is a projection (see (2.12)) on the subset of indexes $I \subset \mathbb{I}$ satisfying (3.3) and (3.4).

Clearly, the momentum (2.17) of a scalar monomial $\mathfrak{m}(k, i, \alpha, \beta)$, which is (N, μ)-low momentum, satisfies $|\pi(k, \alpha, \beta)| \leqq \kappa N^b + \mu N^L$, by (3.1), (3.3). Hence a scalar monomial $\mathfrak{m}(k, i, \alpha, \beta)$ may be (N, μ, h)-low only if

$$|h| < |\pi(k, \alpha, \beta)| + N^b < \mu N^L + (\kappa + 1)N^b \overset{(3.1)}{<} N. \tag{3.5}$$

In particular

$$\mathcal{A}^L_{s,r,\mathtt{a}}(N, \mu, h) = \emptyset, \qquad \forall\, |h| \geqq N. \tag{3.6}$$

We now define the class of (N, θ, μ)-linear vector fields. They are linear combinations of monomial vector fields supported only on the *high components* $\partial_{z_m^\pm}$, $|m| > \theta N$, which are linear in the *high variables* z_n, $|n| > \theta N$, and with polynomial coefficients in the *low variables* of degree bounded by μN^L, $L < 1$. We allow a mild dependence of the coefficients on the low variables because it is naturally generated by commutators. Finally the momentum and the frequency of each (N, θ, μ)-linear monomial vector field is bounded by N^b with $b < L$. Since $b < 1$ these vector fields are approximately x-independent ($|k| < N^b$) and diagonal ($|\pi| < N^b$). The three scales $0 < b < L < 1$ are "low-high" frequency decomposition which almost decouples the interaction between the low variables and the high modes, and it is used in an essential way in the commutator Proposition 3.1. We denote by e_n the multi-index with the n-th component equal to 1 and with all the others equal to zero.

Definition 3.2. A vector field monomial $\mathfrak{m}(k, i, \alpha, \beta; \mathtt{v})$ is

- (N, μ)**-low** if

$$|\pi(k, \alpha, \beta; \mathtt{v})|, |k| < N^b, \ \alpha + \beta = \gamma \ \text{ with } \sum\nolimits_{l \in \mathbb{Z}\setminus\mathcal{I}} |l|\gamma_l < \mu N^L. \tag{3.7}$$

- (N, θ, μ)**-linear** if

$$\begin{aligned} &\mathtt{v} = z_m^\sigma, |\pi(k, \alpha, \beta; \mathtt{v})|, |k| < N^b, \alpha + \beta = e_n + \gamma \text{ with } |m|, |n| > \theta N, \\ &\sum_{l \in \mathbb{Z}\setminus\mathcal{I}} |l|\gamma_l < \mu N^L. \end{aligned} \tag{3.8}$$

We denote by $\mathcal{V}^L_{s,r,\mathtt{a}}(N, \mu)$, respectively $\mathcal{L}_{s,r,\mathtt{a}}(N, \theta, \mu)$, the closure in the norm $\| \; \|_{s,r,\mathtt{a}}$ of the vector space generated by the (N, μ)-low, respectively (N, θ, μ)-linear, monomial vector fields. The elements of $\mathcal{V}^L_{s,r,\mathtt{a}}(N, \mu)$, resp. $\mathcal{L}_{s,r,\mathtt{a}}(N, \theta, \mu)$, are called (N, μ)-low, resp. (N, θ, μ)-linear, vector fields.

The projections on $\mathcal{V}^L_{s,r,\mathtt{a}}(N, \mu)$, resp. $\mathcal{L}_{s,r,\mathtt{a}}(N, \theta, \mu)$, are denoted by $\Pi^L_{N,\mu}$, resp. $\Pi_{N,\theta,\mu}$. Explicitly $\Pi^L_{N,\mu}$ and $\Pi_{N,\theta,\mu}$ are the projections (see (2.12)) on the subsets of indexes $I \subset \mathbb{I} \times \mathbb{V}$ satisfying (3.7) and (3.8) respectively.

By (3.8) and (3.3), a (N, θ, μ)-linear vector field X has the form

$$X(v) = \sum_{|m|, |n| > \theta N, \sigma, \sigma' = \pm} X^{\sigma,m}_{\sigma',n}(v) z_n^{\sigma'} \partial_{z_m^\sigma} \quad \text{where} \quad X^{\sigma,m}_{\sigma',n} \in \mathcal{A}^L_{s,r,\mathtt{a}}(N, \mu, \sigma m - \sigma' n). \tag{3.9}$$

By Definition 3.1 and (3.1), the coefficients $X^{\sigma,m}_{\sigma',n}(v)$ in (3.9) do not depend on $z_j, \bar{z}_j$ with $|j| \geqq 6N^L$.

Lemma 3.1. *Let* $X \in \mathcal{L}_{s,r,\mathtt{a}}(N,\theta,\mu)$. *Then the coefficients in* (3.9) *satisfy*

$$X^{\sigma,m}_{\sigma',n} = 0 \quad \text{if} \quad \sigma\, \mathtt{s}(m) = -\sigma' \mathtt{s}(n) \tag{3.10}$$

where $\mathtt{s}(m) := \operatorname{sign}(m)$.

Proof. By (3.6) and $|\sigma m - \sigma' n| \overset{(3.10)}{=} |m| + |n| \overset{(3.8)}{\geqq} 2\theta N \overset{(3.1)}{>} N$ we get $\mathcal{A}^L_{s,r,\mathtt{a}}(N,\mu,\sigma m - \sigma' n) = \emptyset$. □

Lemma 3.2. *Let* $\mathfrak{m}_{k,i,\alpha,\beta}$ *be a scalar monomial* (*see* (2.10)) *such that*

$$\alpha + \beta =: \gamma \qquad \text{with} \quad \sum\nolimits_{l \in \mathbb{Z}\setminus\mathcal{I}} |l|\gamma_l < 12N^L. \tag{3.11}$$

Then

$$\Pi_{N,\theta,\mu}\left(\mathfrak{m}_{k,i,\alpha,\beta}\, z_n^{\sigma'}\, \partial_{z_m^\sigma}\right) = \begin{cases} \left(\Pi^{L,\sigma m - \sigma' n}_{N,\mu}(\mathfrak{m}_{k,i,\alpha,\beta})\right) z_n^{\sigma'}\, \partial_{z_m^\sigma} & \text{if} \quad |m|,|n| > \theta N \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It directly follows by (3.1), (3.4) and (3.8). □

3.1. Töplitz Vector Fields

We define the subclass of (N,θ,μ)-linear vector fields which are *Töplitz*.

Definition 3.3. (*Töplitz vector field*) A (N,θ,μ)-linear vector field $X \in \mathcal{L}_{s,r,\mathtt{a}}(N,\theta,\mu)$ is (N,θ,μ)-*Töplitz* if the coefficients in (3.9) have the form

$$X^{\sigma,m}_{\sigma',n} = X^\sigma_{\sigma'}\left(\mathtt{s}(m), \sigma m - \sigma' n\right) \quad \text{for some} \ \ X^\sigma_{\sigma'}(\varsigma, h) \in \mathcal{A}^L_{s,r,\mathtt{a}}(N,\mu,h) \tag{3.12}$$

and $\varsigma \in \{+,-\}$, $h \in \mathbb{Z}$. We denote by $\mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)$ the space of the (N,θ,μ)-*Töplitz* vector fields.

The next lemma is used in the proof of Proposition 3.1.

Lemma 3.3. *Let* $X, Y \in \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)$ *and* $W \in \mathcal{V}^L_{s,r,\mathtt{a}}(N,\mu_1)$ *with* $1 < \mu, \mu_1 < 6$. *For all* $0 < s' < s$, $0 < r' < r$ *and* $\theta' \geqq \theta$, $\mu' \leqq \mu$ *one has*

$$\Pi_{N,\theta',\mu'}[X, W] \in \mathcal{T}_{s',r',\mathtt{a}}(N,\theta',\mu'). \tag{3.13}$$

If moreover

$$\mu N^L + (\kappa+1)N^b < (\theta' - \theta)N \tag{3.14}$$

then

$$\Pi_{N,\theta',\mu'}[X, Y] \in \mathcal{T}_{s',r',\mathtt{a}}(N,\theta',\mu'). \tag{3.15}$$

Proof of (3.13). By definition (recall (3.8)) we have that $X^{(x)}$, $X^{(y)}$ and $X^{(z_m^\sigma)}$ vanish if $|m| \leqq \theta N$. Arguing as in (3.5) we have that $W^{z_j^\sigma} = 0$ if $|j| \geqq \mu_1 N^L + (\kappa+1)N^b$. Note that only the components $[X, W]^{(\mathrm{v})}$ with $\mathrm{v} = z_m^\sigma$ and $|m| > \theta N$ contribute to $\Pi_{N,\theta',\mu'}[X, W]$. Noting that $\theta N > \mu_1 N^L + (\kappa+1)N^b$ (by (3.1) and $N \geqq N_0$) we have

$$[X, W]^{(z_m^\sigma)} = \partial_x X^{(z_m^\sigma)} W^{(x)} + \partial_y X^{(z_m^\sigma)} W^{(y)} + \sum_{\sigma_1, |j| < \mu_1 N^L + \kappa N^b} \partial_{z_j^{\sigma_1}} X^{(z_m^\sigma)} W^{(z_j^{\sigma_1})}. \tag{3.16}$$

By (3.9) and (3.12) we get $X^{(z_m^\sigma)} = \sum_{\sigma', |n| > \theta N} X_{\sigma'}^\sigma(\mathrm{s}(m), \sigma m - \sigma' n) z_n^{\sigma'}$. Let us consider the first term of the right hand side of (3.16). Since $X_{\sigma'}^\sigma(\mathrm{s}(m), \sigma m - \sigma' n), W^{(x_l)} \in \mathcal{A}_{s,r,\mathrm{a}}^L(N, \mu)$ (recall (3.12)), all the monomials in $\partial_x X_{\sigma'}^\sigma(\mathrm{s}(m), \sigma m - \sigma' n) W^{(x)}$ satisfy (3.11). By Lemma 3.2 we have

$$\Pi_{N,\theta',\mu'}\left(\partial_x X^{(z_m^\sigma)} W^{(x)} \partial_{z_m^\sigma}\right) = \begin{cases} \sum_{\sigma', |n| > \theta' N} U_{\sigma',n}^{\sigma,m} z_n^{\sigma'} \partial_{z_m^\sigma}, & \text{if} \quad |m| > \theta' N \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{where} \qquad U_{\sigma',n}^{\sigma,m} := \Pi_{N,\mu'}^{L,\sigma m - \sigma' n}\left(\partial_x X_{\sigma'}^\sigma(\mathrm{s}(m), \sigma m - \sigma' n) W^{(x)}\right).$$

It is immediate to see that $U_{\sigma',n}^{\sigma,m}$ satisfy (3.12). The other terms in (3.16) are analogous. (3.13) follows.

Proof of (3.15). We have by (2.13)

$$[X, Y] =: Z - Z', \qquad \text{where} \quad Z := \sum_{\sigma, |m| > \theta N} \left(\sum_{\sigma_1, |j| > \theta N} \partial_{z_j^{\sigma_1}} X^{(z_m^\sigma)} Y^{(z_j^{\sigma_1})} \right) \partial_{z_m^\sigma} \tag{3.17}$$

and Z' is analogous exchanging the role of X and Y. We have to prove that $\Pi_{N,\theta',\mu'} Z \in \mathcal{T}_{s',r',\mathrm{a}}(N, \theta', \mu')$. By (3.9) and (3.12) we get

$$Z^{(z_m^\sigma)} = \sum_{\sigma_1, |j| > \theta N} \sum_{\sigma', |n| > \theta N} X_{\sigma_1}^\sigma(\mathrm{s}(m), \sigma m - \sigma_1 j) Y_{\sigma'}^{\sigma_1}(\mathrm{s}(j), \sigma_1 j - \sigma' n) z_n^{\sigma'}.$$

Since both $X_{\sigma_1}^\sigma(\mathrm{s}(m), \sigma m - \sigma_1 j)$ and $Y_{\sigma'}^{\sigma_1}(\mathrm{s}(j), \sigma_1 j - \sigma' n)$ belong to $\mathcal{A}_{s,r,\mathrm{a}}^L(N, \mu)$ (recall (3.12)), all the monomials in their product satisfy (3.11). By Lemma 3.2 we get

$$\Pi_{N,\theta',\mu'} Z = \sum_{\sigma, \sigma', |m|, |n| > \theta' N} Z_{\sigma',n}^{\sigma,m} z_n^{\sigma'} \partial_{z_m^\sigma}$$

where

$$Z_{\sigma',n}^{\sigma,m} := \Pi_{N,\mu'}^{L,\sigma m - \sigma' n}\left(\sum_{\sigma_1, |j| > \theta N} X_{\sigma_1}^\sigma(\mathrm{s}(m), \sigma m - \sigma_1 j) Y_{\sigma'}^{\sigma_1}(\mathrm{s}(j), \sigma_1 j - \sigma' n) \right). \tag{3.18}$$

Note that $X^{\sigma,\sigma_1}(\mathrm{s}(m), \sigma m - \sigma_1 j) \in \mathcal{A}^L(N, \mu, \sigma m - \sigma_1 j)$, formula (3.5) and condition (3.14) imply that if $|m| > \theta' N$ then automatically $|j| > |m| - |\sigma m -$

$\sigma_1 j| > \theta' N - \mu N^L - (\kappa+1)N^b > \theta N$ or $X^{\sigma,\sigma_1}(\mathtt{s}(m), \sigma m - \sigma_1 j) = 0$. Then the summation in (3.18) runs over $j \in \mathbb{Z}$. By (3.10) we have $\mathtt{s}(j) = \sigma\sigma_1 \mathtt{s}(m)$. Therefore

$$Z^{\sigma,m}_{\sigma',n} := \Pi^{L,\sigma m - \sigma' n}_{N,\mu'} \Big(\sum\nolimits_{\sigma_1,h} X^{\sigma}_{\sigma_1}(\mathtt{s}(m), h) Y^{\sigma_1}_{\sigma'}(\sigma\sigma_1 \mathtt{s}(m), \sigma m - \sigma' n - h) \Big)$$

satisfying (3.12). □

3.2. Quasi-Töplitz Vector Fields

Given a vector field X and a Töplitz vector $\tilde{X} \in \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)$ we define

$$\hat{X} := N(\Pi_{N,\theta,\mu} X - \tilde{X}). \tag{3.19}$$

Definition 3.4. (*Quasi-Töplitz*) A vector field $X \in \mathcal{V}_{s,r,\mathtt{a}}$ is called (N_0,θ,μ)-quasi-Töplitz if the quasi-Töplitz norm

$$\begin{aligned} \|X\|^T_{s,r,\mathtt{a}} &:= \|X\|^T_{s,r,\mathtt{a},N_0,\theta,\mu} \\ &:= \sup_{N \geqq N_0} \left[\inf_{\tilde{X} \in \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)} \Big(\max\{ \|X\|_{s,r,\mathtt{a}}, \|\tilde{X}\|_{s,r,\mathtt{a}}, \|\hat{X}\|_{s,r,\mathtt{a}} \} \Big) \right] \end{aligned} \tag{3.20}$$

is finite. We define

$$\mathcal{Q}^T_{s,r,\mathtt{a}}(N_0,\theta,\mu) := \Big\{ X : D(s,r) \to V : \|X\|^T_{s,r,\mathtt{a},N_0,\theta,\mu} < \infty \Big\}.$$

In other words, a vector field X is (N_0,θ,μ)-quasi-Töplitz with norm $\|X\|^T_{s,r,\mathtt{a}}$ if, for all $N \geqq N_0$, $\forall \varepsilon > 0$, there is $\tilde{X} \in \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)$ such that

$$\Pi_{N,\theta,\mu} X = \tilde{X} + N^{-1}\hat{X} \quad \text{and} \quad \|X\|_{s,r,\mathtt{a}}, \ \|\tilde{X}\|_{s,r,\mathtt{a}}, \ \|\hat{X}\|_{s,r,\mathtt{a}} \leqq \|X\|^T_{s,r,\mathtt{a}} + \varepsilon. \tag{3.21}$$

We call $\tilde{X} \in \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)$ a "*Töplitz approximation*" of X and $\hat{X}$ the "*Töplitz-defect*".

If $s' \leqq s$, $r' \leqq r$, $\mathtt{a}' \leqq \mathtt{a}$, $N_0' \geqq N_0$, $\theta' \geqq \theta$, $\mu' \leqq \mu$ then

$$\|\cdot\|^T_{s',r',\mathtt{a}',N_0',\theta',\mu'} \leqq \max\{s/s', (r/r')^2\} \|\cdot\|^T_{s,r,\mathtt{a},N_0,\theta,\mu}. \tag{3.22}$$

Lemma 3.4. (Projections 1) *Consider a subset of indices $I \subset \mathbb{I} \times \mathbb{V}$ (see (2.9), (2.3)) such that the projection (see (2.12))*

$$\Pi_I : \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu) \to \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu), \quad \forall N \geqq N_0. \tag{3.23}$$

Then $\Pi_I : \mathcal{Q}^T_{s,r,\mathtt{a}}(N_0,\theta,\mu) \to \mathcal{Q}^T_{s,r,\mathtt{a}}(N_0,\theta,\mu)$ and

$$\|\Pi_I X\|^T_{s,r,\mathtt{a}} \leqq \|X\|^T_{s,r,\mathtt{a}}. \tag{3.24}$$

Moreover, if $X \in \mathcal{Q}^T_{s,r,\mathtt{a}}(N_0,\theta,\mu)$ satisfies $\Pi_I X = X$, then, $\forall N \geqq N_0$, $\forall \varepsilon > 0$, there exists a decomposition $\Pi_{N,\theta,\mu} X = \tilde{X} + N^{-1}\hat{X}$ with a Töplitz approximation $\tilde{X} \in \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)$ satisfying $\Pi_I \tilde{X} = \tilde{X}$, $\Pi_I \hat{X} = \hat{X}$ and $\|\tilde{X}\|_{s,r,\mathtt{a}}, \|\hat{X}\|_{s,r,\mathtt{a}} < \|X\|^T_{s,r,\mathtt{a}} + \varepsilon$.

Proof. By (3.21) (recall that $\Pi_{N,\theta,\mu}$ is a projection on an index subset, see Definition 3.2)

$$\Pi_{N,\theta,\mu}\Pi_I X = \Pi_I \Pi_{N,\theta,\mu} X = \Pi_I \tilde{X} + N^{-1}\Pi_I \hat{X}. \tag{3.25}$$

Assumption (3.23) implies that $\Pi_I \tilde{X} \in \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)$ and so $\Pi_I \tilde{X}$ is a Töplitz approximation for $\Pi_I X$. Hence (3.24) follows by $\|\Pi_I X\|_{s,r,\mathtt{a}}$, $\|\Pi_I \tilde{X}\|_{s,r,\mathtt{a}}$, $\|\Pi_I \hat{X}\|_{s,r,\mathtt{a}} < \|X\|^T_{s,r,\mathtt{a}} + \varepsilon$ using Lemma 2.2 and (3.21). Now, if $\Pi_I X = X$, then (3.25) shows that $\Pi_I \tilde{X}$ (which satisfies $\Pi_I(\Pi_I \tilde{X}) = \Pi_I \tilde{X}$), is a Töplitz approximation for X. $\square$

For a vector field $X : D(s,r) \to V$ depending on parameters $\xi \in \mathcal{O}$, we define the norm

$$\|X\|^T_{\vec{p}} := \max\left\{\sup_{\xi\in\mathcal{O}} \|X(\cdot;\xi)\|^T_{s,r,\mathtt{a},N_0,\theta,\mu},\ \|X\|^\lambda_{s,r,\mathtt{a},\mathcal{O}}\right\} \tag{3.26}$$

where, for brevity,

$$\vec{p} := (s, r, \mathtt{a}, N_0, \theta, \mu, \lambda, \mathcal{O}). \tag{3.27}$$

We denote

$$\mathcal{Q}^T_{\vec{p}} := \left\{X \in \mathcal{V}^\lambda_{s,r,\mathtt{a},\mathcal{O}} : X(\cdot;\xi) \in \mathcal{Q}^T_{s,r,\mathtt{a}}(N_0,\theta,\mu),\ \forall \xi \in \mathcal{O} \text{ and } \|X\|^T_{\vec{p}} < \infty\right\}. \tag{3.28}$$

In view of the KAM step we prove that the quasi-Töplitz norm does not increase under suitable projections and that it satisfies smoothing estimates. We denote by Π_{diag} the projection on the space generated by the monomial vector fields $z_j\partial_{z_j}, \bar{z}_j\partial_{\bar{z}_j}$.

Lemma 3.5. (Projections 2) *For all $l \in \mathbb{N}$, $K \in \mathbb{N}$, $N \geqq N_0$, the projections (see (2.27), (2.21), (2.22)) map*

$$\Pi^{(l)}, \Pi_{|k|<K}, \Pi_{|\pi|<K}, \Pi_{\mathrm{diag}} : \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu) \to \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu). \tag{3.29}$$

If $X \in \mathcal{Q}^T_{\vec{p}}$ then

$$\|\Pi^{(l)}X\|^T_{\vec{p}},\ \|\Pi_{|\pi|<K}X\|^T_{\vec{p}}, \|\Pi_{\mathrm{diag}}X\|^T_{\vec{p}}, \|X^{\leqq 0}\|^T_{\vec{p}},\ \|X - X^{\leqq 0}_{|k|<K}\|^T_{\vec{p}} \leqq \|X\|^T_{\vec{p}}. \tag{3.30}$$

Moreover, $\forall\, 0 < s' < s$ and $\forall 0 < \mathtt{a}' < \mathtt{a}$, setting $\vec{p}' = (s', r, \mathtt{a}', N_0, \theta, \mu, \lambda, \mathcal{O})$:

$$\|\Pi_{|k|\geqq K}X\|^T_{\vec{p}'} \leqq e^{-K(s-s')}(s/s')\|X\|^T_{\vec{p}},\ \ \|\Pi_{|\pi|\geqq K}X\|^T_{\vec{p}'} \leqq e^{-K(\mathtt{a}-\mathtt{a}')}\|X\|^T_{\vec{p}}. \tag{3.31}$$

Proof. We prove (3.29) for $\Pi_{|\pi|<K}$; the others are analogous. Since $\tilde{X} \in \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)$ then $\tilde{X}(v) = \sum_{\sigma,\sigma',|m|,|n|>\theta N} \tilde{X}^{\sigma,m}_{\sigma',n}(v) z_n^{\sigma'} \partial_{z_m^\sigma}$ for some $\tilde{X}^{\sigma,m}_{\sigma',n}$ satisfying (3.12). Then

$$\left(\Pi_{|\pi|<K}\tilde{X}\right)(v) = \sum_{\sigma,\sigma',|m|,|n|>\theta N} Y^{\sigma,m}_{\sigma',n}(v) z_n^{\sigma'}\partial_{z_m^\sigma} \quad \text{where} \quad Y^{\sigma,m}_{\sigma',n} := \Pi_{|\pi+\sigma' n-\sigma m|<K}\tilde{X}^{\sigma,m}_{\sigma',n}$$

(recall Definition 3.1). Therefore $Y^{\sigma,m}_{\sigma',n}$ satisfy (3.12) and $\Pi_{|\pi|<K}\tilde{X} \in \mathcal{T}_{s,r,\mathtt{a}}(N,\theta,\mu)$. The estimates (3.30) follow from (3.29) and Lemma 3.4 (in particular (3.24)). The bounds (3.31) follow by (2.23), (2.24) and similar arguments. $\square$

The following proposition shows that the quasi-Töplitz vector fields satisfy modulating the parameters slightly the Lie algebra property.

Proposition 3.1. (Lie bracket) *Assume that* $X^{(1)}, X^{(2)} \in \mathcal{Q}^T_{\vec{p}}$ *(see* (3.28)*) and assume that* $\vec{p}_1 := (s_1, r_1, \mathtt{a}_1, N_1, \theta_1, \mu_1, \lambda, \mathcal{O})$ *with* $N_1 \geqq N_0, \mu_1 \leqq \mu, \theta_1 \geqq \theta, s/2 \leqq s_1 < s, r/2 \leqq r_1 < r, \mathtt{a}_1 < \mathtt{a}$, *satisfy*

$$(\kappa+1)N_1^{b-L} < \mu - \mu_1, \ \ \mu_1 N_1^{L-1} + (\kappa+1)N_1^{b-1} < \theta_1 - \theta, \tag{3.32}$$

$$2N_1 e^{-N_1^b \min\{\mathtt{a}-\mathtt{a}_1, s-s_1\}/2} < 1, \ \ b\min\{\mathtt{a}-\mathtt{a}_1, s-s_1\}N_1^b > 2. \tag{3.33}$$

Then $[X^{(1)}, X^{(2)}] \in \mathcal{Q}^T_{\vec{p}_1}$ *and, for some* $C(n) \geqq 1$,

$$\|[X^{(1)}, X^{(2)}]\|^T_{\vec{p}_1} \leqq C(n)\delta^{-1}\|X^{(1)}\|^T_{\vec{p}}\|X^{(2)}\|^T_{\vec{p}}, \quad \delta := \min\left\{1-\frac{s_1}{s}, 1-\frac{r_1}{r}\right\}. \tag{3.34}$$

The main point in the proof of the above proposition is the following purely algebraic result.

Lemma 3.6. (Splitting lemma) *Let* $X^{(1)}, X^{(2)} \in \mathcal{V}_{s,r,\mathtt{a}}$ *and* (3.32) *hold. Then, for all* $N \geqq N_1$,

$$\begin{aligned}
&\Pi_{N,\theta_1,\mu_1}[X^{(1)}, X^{(2)}] = \\
&\Pi_{N,\theta_1,\mu_1}\Big(\Big[\Pi_{N,\theta,\mu}X^{(1)}, \Pi_{N,\theta,\mu}X^{(2)}\Big] + \Big[\Pi_{N,\theta,\mu}X^{(1)}, \Pi^L_{N,\mu}X^{(2)}\Big] \\
&\qquad + \Big[\Pi^L_{N,\mu}X^{(1)}, \Pi_{N,\theta,\mu}X^{(2)}\Big] + \Big[\Pi_{|k|\geqq N^b \mathrm{or} |\pi|\geqq N^b}X^{(1)}, X^{(2)}\Big] \qquad (3.35)\\
&\qquad + \Big[\Pi_{|k|,|\pi|<N^b}X^{(1)}, \Pi_{|k|\geqq N^b \mathrm{or}|\pi|\geqq N^b}X^{(2)}\Big]\Big). \qquad (3.36)
\end{aligned}$$

Then the proof of Proposition 3.1 follows as in [3] (see Proposition 3.1). The point is to find a Töplitz approximation and a Töplitz defect of $\Pi_{N,\theta_1,\mu_1}[X^{(1)}, X^{(2)}]$, recall (3.21). A Töplitz approximation is obtained by (3.35) substituting $\Pi_{N,\theta,\mu}X^{(i)}$, $i = 1, 2$, with their Töplitz approximations, thus yielding a vector field which is Töplitz by Lemma 3.3. The remaining terms in (3.35) are Töplitz defects. They are small because they contain commutators with the Töplitz defects of $\Pi_{N,\theta,\mu}X^{(i)}$. The last terms (3.36) are exponentially small by (3.33) and (3.31). The momentum-norms of the commutators are estimated by Proposition 2.1.

PROOF OF LEMMA 3.6. We have

$$\begin{aligned}
[X^{(1)}, X^{(2)}] = {} & [\Pi_{|k|,|\pi|<N^b}X^{(1)}, \Pi_{|k|,|\pi|<N^b}X^{(2)}] + [\Pi_{|k|\geqq N^b \mathrm{or}|\pi|\geqq N^b}X^{(1)}, X^{(2)}] \\
& + [\Pi_{|k|,|\pi|<N^b}X^{(1)}, \Pi_{|k|\geqq N^b \mathrm{or}|\pi|\geqq N^b}X^{(2)}].
\end{aligned} \tag{3.37}$$

The last two terms are (3.36). We now prove that the right hand side of (3.37) gives the three terms in (3.35). It is sufficient to study the case where $X^{(h)}$, $h = 1, 2$, are monomial vector fields

$$\mathfrak{m}_h = m_{k^{(h)},i^{(h)},\alpha^{(h)},\beta^{(h)};\mathtt{v}^{(h)}} \text{ (see (2.10))} \quad \text{with} \quad |k^{(h)}|, |\pi(\mathfrak{m}_h)| < N^b, \ h = 1, 2, \tag{3.38}$$

and analyze the conditions under which the projection $\Pi_{N,\theta_1,\mu_1}[\mathfrak{m}_1, \mathfrak{m}_2]$ is not zero.

By the formula of the commutator (2.13) and the definition of the projection Π_{N,θ_1,μ_1} (see Definition 3.2, in particular (3.8)) we have to compute $(D_{\mathtt{v}}\mathfrak{m}_1^{\mathtt{v}'})[\mathfrak{m}_2^{\mathtt{v}}]$ only for $\mathtt{v}' = z_m^\sigma$ with $|m| > \theta_1 N$ and $\mathtt{v} \in \mathbb{V}$, see (2.3).

• CASE I: $\mathtt{v} = x_i$ or $\mathtt{v} = y_i$. By (3.8), in order to have a non trivial projection $\Pi_{N,\theta_1,\mu_1}(D_{\mathtt{v}}\mathfrak{m}_1^{z_m^\sigma})[\mathfrak{m}_2^{\mathtt{v}}]$ it must be

$$\alpha^{(1)} + \beta^{(1)} + \alpha^{(2)} + \beta^{(2)} = e_n + \gamma, \quad |n| > \theta_1 N, \ \sum\nolimits_{l \in \mathbb{Z}\setminus\mathcal{I}} |l|\gamma_l < \mu_1 N^L. \tag{3.39}$$

We claim that

$$\alpha^{(1)}+\beta^{(1)} = e_n+\gamma^{(1)}, \quad \alpha^{(2)}+\beta^{(2)} = \gamma^{(2)}, \quad \sum\nolimits_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|\gamma_l^{(h)} < \mu_1 N^L, \ h = 1, 2, \tag{3.40}$$

which implies that $\mathfrak{m}_1$ is (N, θ_1, μ_1)-linear (see (3.8)), hence (N, θ, μ)-linear, and $\mathfrak{m}_2$ is (N, μ_1)-low (see (3.7)), hence (N, μ)-low. Thus $\Pi_{N,\theta,\mu}\mathfrak{m}_1 = \mathfrak{m}_1$ and $\Pi^L_{N,\mu}\mathfrak{m}_2 = \mathfrak{m}_2$ and we obtain the second (and third by commuting indices) term in the right hand side of (3.35). By (3.39), the other possibility instead of (3.40) is

$$\alpha^{(1)}+\beta^{(1)} = \tilde{\gamma}^{(1)}, \quad \alpha^{(2)}+\beta^{(2)} = e_n+\tilde{\gamma}^{(2)}, \quad \sum\nolimits_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|\tilde{\gamma}_l^{(h)} < \mu_1 N^L, \ h = 1, 2. \tag{3.41}$$

In such a case, since $|\pi(\mathfrak{m}_2)| < N^b$ we get (recall $\mathfrak{m}_2 = \mathfrak{m}_2^{\mathtt{v}}$ with $\mathtt{v} = x, y$)

$$\begin{aligned} N^b > |\pi(k^{(2)}, \alpha^{(2)}, \beta^{(2)})| &\overset{(2.16),(3.41)}{\geqq} |n| - \sum_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|\tilde{\gamma}_l^{(2)} \\ -\kappa|k^{(2)}| &\overset{(3.39),(3.41),(3.38)}{\geqq} \theta_1 N - \mu_1 N^L - \kappa N^b \end{aligned}$$

which contradicts (3.1).

• CASE 2: $\mathtt{v} = z_j^{\sigma_1}$, $j \in \mathbb{Z}\setminus\mathcal{I}$, only for this case we use (3.32). In order to have a non trivial projection $\Pi_{N,\theta_1,\mu_1}(D_{\mathtt{v}}\mathfrak{m}_1^{z_m^\sigma})[\mathfrak{m}_2^{\mathtt{v}}]$, it must be

$$\alpha^{(1)} + \beta^{(1)} + \alpha^{(2)} + \beta^{(2)} - e_j = e_n + \gamma, \quad |n| > \theta_1 N, \quad \sum\nolimits_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|\gamma_l < \mu_1 N^L. \tag{3.42}$$

We have the two following possible cases:

$$\begin{aligned} &\alpha^{(1)} + \beta^{(1)} = e_j + e_n + \gamma^{(1)}, \quad \alpha^{(2)} + \beta^{(2)} = \gamma^{(2)}, \\ &\quad \sum\nolimits_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|\gamma_l^{(h)} < \mu_1 N^L, \ h = 1, 2 \end{aligned} \tag{3.43}$$

$$\begin{aligned} &\alpha^{(1)} + \beta^{(1)} = e_j + \tilde{\gamma}^{(1)}, \quad \alpha^{(2)} + \beta^{(2)} = e_n + \tilde{\gamma}^{(2)}, \\ &\quad \sum\nolimits_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|\tilde{\gamma}_l^{(h)} < \mu_1 N^L, \ h = 1, 2 \end{aligned} \tag{3.44}$$

where $\gamma^{(1)} + \gamma^{(2)} = \tilde\gamma^{(1)} + \tilde\gamma^{(2)} = \gamma$. Note that, since we differentiate $\mathfrak{m}_1$ with respect to $\mathtt{v} = z_j^{\sigma_1}$, the monomial $\mathfrak{m}_1$ must depend on $z_j^{\sigma_1}$ and so the following case does not arise:

$$\alpha^{(1)} + \beta^{(1)} = \tilde\gamma^{(1)}, \quad \alpha^{(2)} + \beta^{(2)} = e_j + e_n + \tilde\gamma^{(2)},$$
$$\textstyle\sum_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|\tilde\gamma_l^{(h)} < \mu_1 N^L, \ h = 1, 2.$$

In the case (3.43), the monomial $\mathfrak{m}_2$ is (N, μ)-low and we claim that $\mathfrak{m}_1$ is (N, θ, μ)-linear. Indeed, since

$$|\pi(\mathfrak{m}_2)| \overset{(2.16)}{=} |\pi(k^{(2)}, \alpha^{(2)}, \beta^{(2)}) - \sigma_1 j| < N^b \tag{3.45}$$

we get $|j| \leqq |\pi(k^{(2)}, \alpha^{(2)}, \beta^{(2)})| + N^b$. Hence

$$\begin{aligned} |j| + \sum\nolimits_l \gamma_l^{(1)}|l| \quad &\leqq \quad |\pi(k^{(2)}, \alpha^{(2)}, \beta^{(2)})| + N^b + \sum\nolimits_l \gamma_l^{(1)}|l| \leqq \kappa|k^{(2)}| \\ &\qquad + \sum\nolimits_l \gamma_l|l| + N^b \\ &\overset{(3.38),(3.42)}{\leqq} (\kappa+1)N^b + \mu_1 N^L \overset{(3.32)}{\leqq} \mu N^L, \end{aligned}$$

namely $\mathfrak{m}_1$ is (N, θ, μ)-linear (see (3.8) with $\gamma = e_j + \gamma^{(1)}$). Hence $\Pi_{N,\theta,\mu}\mathfrak{m}_1 = \mathfrak{m}_1$ and $\Pi^L_{N,\mu}\mathfrak{m}_2 = \mathfrak{m}_2$ and we obtain the second term (and third by commuting indices) on the right hand side of (3.35).

In the case (3.44) we claim that both $\mathfrak{m}_1, \mathfrak{m}_2$ are (N, θ, μ)-linear so we obtain the first term on the right hand side of (3.35). Since, by (3.42), $|n| > \theta_1 N > \theta N$ we already know that $\mathfrak{m}_2$ is (N, θ, μ)-linear. Finally, $\mathfrak{m}_1$ is (N, θ, μ)-linear because

$$\begin{aligned} |j| \overset{(3.45)}{>} |\pi(k^{(2)}, \alpha^{(2)}, \beta^{(2)})| - N^b &\overset{(2.16),(3.44)}{\geqq} |n| - \sum\nolimits_{l\in\mathbb{Z}\setminus\mathcal{I}} |l|\tilde\gamma_l^{(2)} - \kappa|k^{(2)}| - N^b \\ &\overset{(3.42),(3.44),(3.38)}{>} \theta_1 N - \mu_1 N^L - (\kappa+1)N^b \overset{(3.32)}{>} \theta N \end{aligned}$$

concluding the proof. □

The quasi-Töplitz character of a vector field is preserved under the flow of a quasi-Töplitz vector field. As the corresponding Proposition 3.2 of [3], the proof is an iteration of Proposition 3.1.

Proposition 3.2. (Lie series) *Let* $X, Y \in \mathcal{Q}^T_{\vec p}$ (*see* (3.28)). *Assume* $\vec p\,' := (s', r', \mathtt{a}', N_0', \theta', \mu', \lambda, \mathcal{O})$ *satisfies* $s/2 \leqq s' < s$, $r/2 \leqq r' < r$, $\mathtt{a}' < \mathtt{a}$, $\mu' < \mu$, $\theta' > \theta$, *and*

$$N_0' \geqq \max\{N_0, \bar N\}, \quad \bar N := \exp\left(\max\left\{2/b, (L-b)^{-1}, (1-L)^{-1}, 8\right\}\right), \tag{3.46}$$

$$(\kappa+1)(N_0')^{b-L}\ln N_0' \leqq \mu - \mu', \quad (7+\kappa)(N_0')^{L-1}\ln N_0' \leqq \theta' - \theta, \tag{3.47}$$

$$2(N_0')^{-b}\ln^2 N_0' \leqq b\min\{s - s', \mathtt{a} - \mathtt{a}'\}. \tag{3.48}$$

There is $c(n) > 0$ *such that, if the smallness condition*

$$\|X\|_{\vec{p}}^{T} \leqq c(n)\,\delta \tag{3.49}$$

holds (*with* δ *defined in* (2.25)), *then* $e^{\mathrm{ad}_X} Y \in \mathcal{Q}_{\vec{p}'}^{T}$ *and*

$$\|e^{\mathrm{ad}_X} Y\|_{\vec{p}'}^{T} \leqq 2\|Y\|_{\vec{p}}^{T}. \tag{3.50}$$

Moreover, for $h = 0, 1, 2$, *and coefficients* $0 \leqq b_j \leqq 1/j!$, $j \in \mathbb{N}$,

$$\left\|\sum_{j \geqq h} b_j \, \mathrm{ad}_X^j(Y)\right\|_{\vec{p}'}^{T} \leqq 2(C\delta^{-1}\|X\|_{\vec{p}}^{T})^h \|Y\|_{\vec{p}}^{T}. \tag{3.51}$$

4. An Abstract KAM Theorem

We consider a family of linear integrable vector fields with constant coefficients

$$\mathcal{N}(\xi) := \omega(\xi)\partial_x + \mathrm{i}\Omega(\xi) z\partial_z - \mathrm{i}\Omega(\xi)\bar{z}\partial_{\bar{z}} \tag{4.1}$$

defined on the phase space $\mathbb{T}_s^n \times \mathbb{C}^n \times \ell_{\mathcal{I}}^{a,p} \times \ell_{\mathcal{I}}^{a,p}$, where the tangential sites $\mathcal{I} \subset \mathbb{Z}$ are symmetric as in (1.29), the space $\ell_{\mathcal{I}}^{a,p}$ is defined in (2.1), the tangential frequencies $\omega \in \mathbb{R}^n$ and the normal frequencies $\Omega \in \mathbb{R}^{\mathbb{Z}\setminus\mathcal{I}}$ depend on real parameters $\xi \in \mathcal{O} \subset \mathbb{R}^{n/2}$ (where $n/2 =$ cardinality of $\mathcal{I}^+$, see (1.29)), and satisfy

$$\omega_j(\xi) = \omega_{-j}(\xi), \ \forall j \in \mathcal{I}, \quad \Omega_j(\xi) = \Omega_{-j}(\xi), \ \forall j \in \mathbb{Z} \setminus \mathcal{I}. \tag{4.2}$$

For each ξ there is an invariant n-torus $\mathcal{T}_0 = \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\}$ with frequency $\omega(\xi)$. In its normal space, the origin $(z, \bar{z}) = 0$ is an elliptic fixed point with proper frequencies $\Omega(\xi)$. The aim is to prove the persistence of a large portion of this family of linearly stable tori under small perturbations

$$\mathcal{P}(x, y, z, \bar{z}; \xi) = \mathcal{P}^{(x)}\partial_x + \mathcal{P}^{(y)}\partial_y + \mathcal{P}^{(z)}\partial_z + \mathcal{P}^{(\bar{z})}\partial_{\bar{z}}. \tag{4.3}$$

(**A1**) Parameter dependence. The map $\omega : \mathcal{O} \to \mathbb{R}^n$, $\xi \mapsto \omega(\xi)$, is Lipschitz continuous.

With the application to DNLW in mind we assume

(**A2**) Frequency asymptotics.

$$\Omega_j(\xi) = |j| + a(\xi) + b(\xi)|j|^{-1} + O(j^{-2}) \quad \text{as} \quad |j| \to +\infty. \tag{4.4}$$

Moreover the map $(\Omega_j - |j|)_{j \in \mathbb{Z}\setminus\mathcal{I}} : \mathcal{O} \to \ell_\infty$ is Lipschitz continuous.
By (A1) and (A2), the Lipschitz semi-norms of the frequency maps satisfy, for some $1 \leqq M_0 < \infty$,

$$|\omega|^{\mathrm{lip}} + |\Omega|_\infty^{\mathrm{lip}} \leqq M_0 \qquad \text{where} \qquad |\Omega|_\infty^{\mathrm{lip}} := \sup_{\xi \neq \eta \in \mathcal{O}} \frac{|\Omega(\xi) - \Omega(\eta)|_\infty}{|\eta - \xi|} \tag{4.5}$$

and $|z|_\infty := \sup_{j \in \mathbb{Z}\setminus\mathcal{I}} |z_j| < +\infty$.

(**A3**) REGULARITY. The vector field $\mathcal{P}$ in (4.3) maps $\mathcal{P} : D(s,r) \times \mathcal{O} \to \mathbb{C}^n \times \mathbb{C}^n \times \ell_{\mathcal{I}}^{a,p} \times \ell_{\mathcal{I}}^{a,p}$ for some $s, r > 0$. Moreover $\mathcal{P}$ is REVERSIBLE (Definition 2.5), REAL-COEFFICIENTS, REAL-ON-REAL EVEN (Definition 2.6).

Finally, in order to obtain the asymptotic expansion for the perturbed frequencies we also assume

(**A4**) QUASI-TÖPLITZ. The perturbation vector field $\mathcal{P}$ is quasi-Töplitz, see Definition 3.4.

Recalling (4.3) and the notations in (2.30), (2.27), we define

$$\mathcal{P}^y(x)\partial_y := \Pi^{(-1)}\mathcal{P}^{(y)}\partial_y, \qquad \mathcal{P}_* := \mathcal{P} - \mathcal{P}^y(x)\partial_y \tag{4.6}$$

and we denote by $\mathcal{P}_*^{(-1)}, \mathcal{P}_*^{(0)}$ the terms of degree -1 and 0 respectively of $\mathcal{P}_*$, see (2.27). Let

$$\vec{\omega}(\xi) := (\omega_j(\xi))_{j\in\mathcal{I}^+} \in \mathbb{R}^{n/2}, \qquad \text{then} \quad \omega = (\vec{\omega}, \vec{\omega}) \ \text{ by } \ (4.2). \tag{4.7}$$

Theorem 4.1. (KAM theorem) *Fix* $s, r, \mathtt{a} > 0$, $1 < \theta, \mu < 6$, $N_0 \geqq \bar{N}$ *(defined in* (3.46)*). Let* $\gamma \in (0, \gamma_*)$*, where* $\gamma_* = \gamma_*(n, s, \mathtt{a}) < 1$ *is a (small) constant. Let* $\lambda := \gamma/M_0$ *(see* (4.5)*) and* $\vec{p} := (s, r, \mathtt{a}, N_0, \theta, \mu, \lambda, \mathcal{O})$*. Suppose that the vector field* $\mathcal{X} = \mathcal{N} + \mathcal{P}$ *satisfies* (A1)-(A4)*. If*

$$\begin{aligned}\gamma^{-1}\|\mathcal{P}_*\|_{\vec{p}}^{T} \leqq 1 \quad &\textit{and} \quad \varepsilon \\ &:= \max\left\{\gamma^{-2/3}\|\mathcal{P}^y(x)\partial_y\|_{s,r,\mathtt{a},\mathcal{O}}^{\lambda}, \gamma^{-1}\|\mathcal{P}_*^{(-1)}\|_{\vec{p}}^{T}, \gamma^{-1}\|\mathcal{P}_*^{(0)}\|_{\vec{p}}^{T}\right\}\end{aligned} \tag{4.8}$$

is small enough, then

- **(Frequencies)** *There exist Lipschitz functions* $\omega^\infty : \mathbb{R}^{n/2} \to \mathbb{R}^n$, $\Omega^\infty : \mathbb{R}^{n/2} \to \ell_\infty$, $a^\infty : \mathbb{R}^{n/2} \to \mathbb{R}$ *(recall that* $\mathcal{O} \subset \mathbb{R}^{n/2}$*) such that* $\omega^\infty = (\vec{\omega}^\infty, \vec{\omega}^\infty)$, $\vec{\omega}^\infty := (\omega_j^\infty)_{j\in\mathcal{I}^+} \in \mathbb{R}^{n/2}$*, and*

$$|\omega^\infty - \omega| + \lambda|\omega^\infty - \omega|^{\mathrm{lip}}, \ |\Omega^\infty - \Omega|_\infty + \lambda|\Omega^\infty - \Omega|_\infty^{\mathrm{lip}} \leqq C\gamma\varepsilon, \ |a^\infty| \leqq C\gamma\varepsilon, \tag{4.9}$$

$$\omega_j^\infty(\xi) = \omega_{-j}^\infty(\xi), \ \forall j \in \mathcal{I}, \quad \Omega_j^\infty(\xi) = \Omega_{-j}^\infty(\xi), \ \forall j \in \mathbb{Z}\setminus\mathcal{I},$$

$$\sup_{\xi\in\mathbb{R}^{n/2}} |\Omega_j^\infty(\xi) - \Omega_j(\xi) - a^\infty(\xi)| \leqq \gamma^{2/3}\varepsilon\frac{C}{|j|}, \quad \forall |j| \geqq C_\star\gamma^{-1/3}. \tag{4.10}$$

- **(KAM normal form)** *for every* ξ *belonging to*

$$\begin{aligned}\mathcal{O}_\infty := \Big\{\xi \in \mathcal{O} \ : \ &\forall h \in \mathbb{Z}^{n/2}, \ i, j \in \mathbb{Z}\setminus\mathcal{I}, \ p \in \mathbb{Z}, \\ &|\vec{\omega}^\infty(\xi)\cdot h + \Omega_j^\infty| \geqq 2\gamma\langle h\rangle^{-\tau}, \ |\vec{\omega}^\infty(\xi)\cdot h + \Omega_i^\infty(\xi) + \Omega_j^\infty(\xi)| \geqq 2\gamma\langle h\rangle^{-\tau}, \\ &|\vec{\omega}^\infty(\xi)\cdot h - \Omega_i^\infty(\xi) + \Omega_j^\infty(\xi)| \geqq 2\gamma\langle h\rangle^{-\tau} \quad \text{if} \quad h \neq 0 \text{ or } i \neq \pm j, \\ &|\vec{\omega}^\infty(\xi)\cdot h + p| \geqq 2\gamma^{2/3}\langle h\rangle^{-\tau}, \ \text{if } (h,p) \neq (0,0) \\ &|\vec{\omega}(\xi)\cdot h| \geqq 2\gamma^{2/3}\langle h\rangle^{-n/2}, \ \forall 0 < |h| < \gamma^{-1/(7n)}\Big\}\end{aligned} \tag{4.11}$$

there exists an even, analytic, close to the identity diffeomorphism

$$\Phi(\cdot;\xi) : D(s/4, r/4) \ni (x_\infty, y_\infty, z_\infty, \bar{z}_\infty) \mapsto (x, y, z, \bar{z}) \in D(s, r) , \tag{4.12}$$

(*Lipschitz in* ξ) *such that the transformed vector field*

$$\mathcal{X}_\infty = \mathcal{N}_\infty + \mathcal{P}_\infty := \Phi_\star(\cdot;\xi)\mathcal{X} = (D\Phi(\,;\xi))^{-1}\mathcal{X}\circ\Phi(\,;\xi) \quad \textit{has} \quad \left(\mathcal{P}_\infty^{\leqq 0}\right)_{|E} = 0 , \tag{4.13}$$

see (2.28), (1.33). *Moreover* $\mathcal{N}_\infty$ *is a constant coefficients linear normal form vector field as* (4.1) *with frequencies* $\omega^\infty(\xi)$, $\Omega^\infty(\xi)$, *and* $\mathcal{P}_\infty$ *is reversible, real-coefficients, real-on-real, even. Finally* $(\mathcal{X}_\infty)_{|E} = (\mathcal{S}\mathcal{X}_\infty)_{|E}$.

As a consequence we derive

Corollary 4.1. *For all* $\xi \in \mathcal{O}_\infty$, *the map* $\mathbb{T}^{n/2} \ni \vec{x}_\infty \mapsto \Phi((\vec{x}_\infty, \vec{x}_\infty), 0, 0, 0; \xi) \in E$ *is an* $n/2$*-dimensional analytic invariant torus of the vector field* $\mathcal{X} = \mathcal{N} + \mathcal{P}$. *Such a torus is linearly stable on* E *and, in particular, it has zero Lyapunov exponents on* E.

The set $\mathcal{O}_\infty$ in (4.11) could be empty. In the next theorem we bound its measure.

Theorem 4.2. (Measure estimate) *Let* $\mathcal{O} := \mathcal{O}_\rho := \{\xi := (\xi_j)_{j\in\mathcal{I}^+} \in \mathbb{R}^{n/2} \,:\, 0 < \rho/2 \leqq |\xi_j| \leqq \rho\}$. *Assume that the frequencies are affine functions of* ξ

$$\vec{\omega}(\xi) = \bar{\omega} + A\xi \,,\ \bar{\omega} = (\lambda_j)_{j\in\mathcal{I}^+} \in \mathbb{R}^{n/2} \,, \quad \Omega_j(\xi) = \lambda_j + \lambda_j^{-1}\,\vec{a}\cdot\xi \,,\ \forall j \notin \mathcal{I} \,, \tag{4.14}$$

where $A \in \mathrm{Mat}(n/2 \times n/2)$, $\det A \neq 0$, *and* $\vec{a} \in \mathbb{R}^{n/2}$ *are continuous functions in* m. *Fix a compact interval of masses* $[\mathrm{m}_1, \mathrm{m}_2] \subset (0, \infty)$ *and take* $\mathrm{m} \in [\mathrm{m}_1, \mathrm{m}_2]$ *such that*

$$(\lambda_i^{-1} \pm \lambda_j^{-1})(A^T)^{-1}\vec{a} \,, \quad \lambda_j^{-1}(A^T)^{-1}\vec{a} \notin \mathbb{Z}^{n/2}\setminus\{0\} \,,\ \forall i, j \in \mathbb{Z}\setminus\mathcal{I} \,,\ |i|, |j| \leqq C_0 \,, \tag{4.15}$$

for a suitably large constant $C_0 := C_0(\mathrm{m}_1, \mathrm{m}_2, A, \vec{a}, \bar{\omega})$. *Then the Cantor - like set* $\mathcal{O}_\infty$ *defined in* (4.11), *with exponent* $\tau > \max\{n+3, 1/b\}$ (*b is fixed in* (3.2)), *satisfies, for* $\rho \in (0, \rho_0(\mathrm{m}))$ *small,*

$$|\mathcal{O} \setminus \mathcal{O}_\infty| \leqq C(\tau)\rho^{\frac{n}{2}-1}\gamma^{2/3} . \tag{4.16}$$

The proof of Theorem 4.2 is similar to that of the analogous Theorem 4.2 of [3]. The specific form $\Omega_j(\xi)$ in (4.14) is motivated by application to the DNLW, see (7.21). Clearly (4.14) implies (4.4). The asymptotic estimate (4.10) is the key point in order to prove (4.16) (in particular for the second order Melnikov conditions at the third line of (4.11)). At the end of Section 6 we explain how the finitely many conditions in (4.15) are used to estimate the measure

$$\begin{aligned} &|\{\xi \in \mathcal{O} \,:\, |\vec{\omega}^\infty(\xi)\cdot h + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi)| < \gamma\langle h\rangle^{-\tau}\}| \leqq \gamma\rho^{\frac{n}{2}-1}\langle h\rangle^{-\tau} , \\ &\quad h \neq 0 \,, i, j \in \mathbb{Z}\setminus\mathcal{I}. \end{aligned} \tag{4.17}$$

This is the main difference with respect to [3, Lemma 6.1].

5. Homological Equations

The integers $k \in \mathbb{Z}^n$ have indexes in $\mathcal{I}$ (see (1.29)), namely $k = (k_h)_{h \in \mathcal{I}}$.

In the sequel by $a \lessdot b$ we mean that there exists $c > 0$ depending only on n, m, κ such that $a \leqq cb$.

Definition 5.1. (*Normal form vector fields*) The normal form vector fields are

$$\begin{aligned}\mathcal{N} &:= \partial_\omega + \mathrm{Nu}\partial_{\mathrm{u}} = \partial_\omega + \mathrm{i}\Omega z \partial_z - \mathrm{i}\Omega \bar{z}\partial_{\bar{z}} = \omega(\xi)\cdot\partial_x \\ &\quad + \mathrm{i}\sum_{j\in\mathbb{Z}\setminus\mathcal{I}} \Omega_j(\xi) z_j \partial_{z_j} - \mathrm{i}\sum_{j\in\mathbb{Z}\setminus\mathcal{I}} \Omega_j(\xi)\bar{z}_j\partial_{\bar{z}_j}\end{aligned} \tag{5.1}$$

where the frequencies $\omega_j(\xi), \Omega_j(\xi) \in \mathbb{R}$, $\forall\, \xi \in \mathcal{O} \subseteq \mathbb{R}^{n/2}$, are real and symmetric Lipschitz functions

$$\omega_{-j} = \omega_j\,,\ \forall j \in \mathcal{I}\,, \quad \Omega_{-j} = \Omega_j\,,\ \forall j \in \mathbb{Z}\setminus\mathcal{I}\,, \tag{5.2}$$

the matrix N is diagonal

$$\mathrm{N} = \begin{pmatrix} \mathbf{0}_n & 0 & 0 \\ 0 & \mathrm{i}\Omega & 0 \\ 0 & 0 & -\mathrm{i}\Omega \end{pmatrix}, \quad \Omega := \mathrm{diag}_{j\in\mathbb{Z}\setminus\mathcal{I}}(\Omega_j)\,, \tag{5.3}$$

and there exists $j_* > 0$ such that (recall (4.4))

$$\sup_{\xi\in\mathcal{O}} \left|\Omega_j(\xi) - \Omega_j(\xi) - a(\xi)\right| \lessdot \frac{\gamma}{|j|}\,, \quad \forall\, |j| \geqq j_*\,, \tag{5.4}$$

(see (4.4)) for some Lipschitz function $a : \mathcal{O} \to \mathbb{R}$, independent of j.

Note that $\mathcal{N} \in \mathcal{R}^{\leqq 0}_{\mathrm{rev}}$, see Definition 2.7. The symmetry condition (5.2) implies the resonance relations $\Omega_{-j} - \Omega_j = 0$ and $\omega \cdot k = 0$ for all $k \in \mathbb{Z}^n_{\mathrm{odd}}$ defined in (1.34).

5.1. Symmetrization

For a vector field X, we define its "symmetrized" $\mathcal{S}(X)$ by linearity on the monomial vector fields:

Definition 5.2. The symmetrized monomial vector fields are defined by

$$\mathcal{S}(\mathrm{e}^{\mathrm{i}k\cdot x}\partial_{x_j}) := \partial_{x_j}\,, \quad \mathcal{S}(\mathrm{e}^{\mathrm{i}k\cdot x} y^i \partial_{y_j}) := y^i\partial_{y_j}\,,\ \forall k \in \mathbb{Z}^n_{\mathrm{odd}}\,,\ |i| = 0, 1,\ j \in \mathcal{I}, \tag{5.5}$$

$$\mathcal{S}(\mathrm{e}^{\mathrm{i}k\cdot x} z_{\pm j}\partial_{z_j}) := z_j\partial_{z_j}\,, \quad \mathcal{S}(\mathrm{e}^{\mathrm{i}k\cdot x}\bar{z}_{\pm j}\partial_{\bar{z}_j}) := \bar{z}_j\partial_{\bar{z}_j}\,,\ \forall k \in \mathbb{Z}^n_{\mathrm{odd}}\,,\ j \in \mathbb{Z}\setminus\mathcal{I}, \tag{5.6}$$

and $\mathcal{S}$ is the identity on the other monomial vector fields.

By (5.5)–(5.6) we write $\mathcal{S}X = X + X' + X''$ where

$$X' := \sum_{k\in\mathbb{Z}^n_{\text{odd}},\, j\in\mathbb{Z}\setminus\mathcal{I}} X^{(z_j)}_{k,0,e_j,0}(1-\mathrm{e}^{\mathrm{i}k\cdot x})z_j\partial_{z_j} + X^{(\bar z_j)}_{k,0,0,e_j}(1-\mathrm{e}^{\mathrm{i}k\cdot x})\bar z_j\partial_{\bar z_j} \tag{5.7}$$

and

$$\begin{aligned} X'' := & \sum_{k\in\mathbb{Z}^n_{\text{odd}},\, k\neq 0,\, j\in\mathcal{I}} X^{(x_j)}_{k,0,0,0}(1-\mathrm{e}^{\mathrm{i}k\cdot x})\partial_{x_j} \\ & + \sum_{k\in\mathbb{Z}^n_{\text{odd}},\, k\neq 0,\, j\in\mathcal{I},\, |i|=0,1} X^{(y_j)}_{k,i,0,0}(1-\mathrm{e}^{\mathrm{i}k\cdot x})y^i\partial_{y_j} \\ & + \sum_{k\in\mathbb{Z}^n_{\text{odd}},\, j\in\mathbb{Z}\setminus\mathcal{I}} X^{(z_j)}_{k,0,e_{-j},0}(z_j-\mathrm{e}^{\mathrm{i}k\cdot x}z_{-j})\partial_{z_j} + X^{(\bar z_j)}_{k,0,0,e_{-j}}(\bar z_j-\mathrm{e}^{\mathrm{i}k\cdot x}\bar z_{-j})\partial_{\bar z_j}. \end{aligned} \tag{5.8}$$

The "symmetric" subspace E defined in (1.33) is invariant under the flow evolution generated by the vector field X, because $X : E \to E$. Moreover the vector fields X and $\mathcal{S}(X)$ coincide on E:

Proposition 5.1. $X_{|E} = (\mathcal{S}X)_{|E}$.

As a consequence $v(t) \in E$ is a solution of $\dot v = X(v)$ if and only if it is a solution of $\dot v = (\mathcal{S}X)(v)$, and we may replace the vector field X with its symmetrized $\mathcal{S}(X)$ without changing the dynamics on the invariant subspace E. The following lemma shows that both the $\mathtt{a}$-momentum and Töplitz norms of the symmetrized vector field $\mathcal{S}(X)$ are controlled by those of X.

Proposition 5.2. *For $N_1 \geqq N_0$ (defined in (3.1)) which satisfy*

$$N_1 e^{-N_1^b \min\{s,\mathtt{a}\}} \leqq 1\,, \quad bN_1^b \min\{s,\mathtt{a}\} \geqq 1\,, \tag{5.9}$$

the norms of the symmetrized vector field satisfy

$$\begin{aligned} &\text{(i) } \|\mathcal{S}X\|_{s,r,\mathtt{a}} \leqq \|X\|_{s,r,\mathtt{a}}\,, \quad \text{(ii) } \|\mathcal{S}X\|^{\text{lip}}_{s,r,\mathtt{a}} \leqq \|X\|^{\text{lip}}_{s,r,\mathtt{a}}, \\ &\text{(iii) } \|\mathcal{S}X\|^T_{s,r,\mathtt{a},N_1,\theta,\mu} \leqq 9\|X\|^T_{s,r,\mathtt{a},N_1,\theta,\mu}. \end{aligned} \tag{5.10}$$

Moreover, if X is reversible, or real-coefficients, or real-on-real, or even, the same holds for $\mathcal{S}X$.

Proof. In order to prove (5.10)-(i) we first note that the symmetrized monomial vector fields ∂_{x_h}, $y^i\partial_{x_h}$, $z_j\partial_{z_j}$, $\bar z_j\partial_{\bar z_j}$ in (5.5)–(5.6) have zero momentum and are independent of x. Hence their contribution to the a-momentum norm (2.19) is smaller or equal than the contribution of the (not yet symmetrized) monomials $\mathrm{e}^{\mathrm{i}k\cdot x}\partial_{x_j}$, $\mathrm{e}^{\mathrm{i}k\cdot x}y^i\partial_{x_j}$, $\mathrm{e}^{\mathrm{i}k\cdot x}z_{\pm j}\partial_{z_j}$, $\mathrm{e}^{\mathrm{i}k\cdot x}\bar z_{\pm j}\partial_{\bar z_j}$ of X. This proves (5.10)-(i).
Proof of the (5.10)-(ii). The estimate (5.10) follows by

$$\text{(i) } \|X'\|^T_{s,r,\mathtt{a}} \leqq 6\|X\|^T_{s,r,\mathtt{a}}\,, \qquad \text{(ii) } \|X''\|^T_{s,r,\mathtt{a}} \leqq 2\|X\|^T_{s,r,\mathtt{a}}. \tag{5.11}$$

PROOF OF (5.11)-(I). We claim that, for $N \geqq N_1$, the projection $\Pi_{N,\theta,\mu} X' = \tilde{X}' + N^{-1}\hat{X}'$ with

$$\tilde{X}' \in \mathcal{T}_{s,r,\mathtt{a}}, \quad \|\tilde{X}'\|_{s,r,\mathtt{a}} \leqq 6\|X\|^T_{s,r,\mathtt{a}}, \quad \|\hat{X}'\|_{s,r,\mathtt{a}} \leqq 5\|X\|^T_{s,r,\mathtt{a}}, \tag{5.12}$$

implying (5.11) (also because $\|X'\|_{s,r,\mathtt{a}} \leqq 2\|X\|_{s,r,\mathtt{a}}$). In order to prove (5.12) we write the (N,θ,μ)-projection as

$$\Pi_{N,\theta,\mu} X' = U + U^- + U_\perp + U_\perp^- \tag{5.13}$$

where

$$\begin{aligned}
U &:= \sum_{k\in\mathcal{K}_N, |j|>\theta N} X^{(z_j)}_{k,0,e_j,0}(1-\mathrm{e}^{\mathrm{i}k\cdot x}) z_j \partial_{z_j}, \\
U^- &:= \sum_{k\in\mathcal{K}_N, |j|>\theta N} X^{(\bar z_j)}_{k,0,0,e_j}(1-\mathrm{e}^{\mathrm{i}k\cdot x}) \bar z_j \partial_{\bar z_j}, \\
U_\perp &:= \sum_{|j|>\theta N} \Bigg(\sum_{k\in\mathbb{Z}^n_{\mathrm{odd}}\setminus\mathcal{K}_N} X^{(z_j)}_{k,0,e_j,0} \Bigg) z_j \partial_{z_j}, \\
U_\perp^- &:= \sum_{|j|>\theta N} \Bigg(\sum_{k\in\mathbb{Z}^n_{\mathrm{odd}}\setminus\mathcal{K}_N} X^{(\bar z_j)}_{k,0,0,e_j} \Bigg) \bar z_j \partial_{\bar z_j},
\end{aligned}$$

and $\mathcal{K}_N := \{k \in \mathbb{Z}^n_{\mathrm{odd}}, \ |\pi(k)|, |k| < N^b\}, \pi(k) := \sum_{j\in\mathcal{I}} j k_j$. Then (5.11) follows by Steps (1)–(2) below.

STEP (1) *The projection* $\Pi_{N,\theta,\mu}(U+U^-) = (\tilde U + \tilde U^-) + N^{-1}(\hat U + \hat U^-)$ *with*

$$\begin{aligned}
&\tilde U, \tilde U^- \in \mathcal{T}_{s,r,\mathtt{a}}, \ \|\tilde U\|_{s,r,\mathtt{a}}, \|\tilde U^-\|_{s,r,\mathtt{a}} \leqq 6\|X\|^T_{s,r,\mathtt{a}}, \quad \|\hat U\|_{s,r,\mathtt{a}}, \\
&\|\hat U^-\|_{s,r,\mathtt{a}} \leqq 6\|X\|^T_{s,r,\mathtt{a}}.
\end{aligned} \tag{5.14}$$

Since X is quasi-Töplitz, Lemma 3.5 implies that the projection

$$\begin{aligned}
\Pi_{\mathrm{diag}}\Pi^{(0)} X &= \sum_{k\in\mathbb{Z}^n, j\in\mathbb{Z}\setminus\mathcal{I}} X^{(z_j)}_{k,0,e_j,0}\mathrm{e}^{\mathrm{i}k\cdot x} z_j\partial_{z_j} + \sum_{k\in\mathbb{Z}^n, j\in\mathbb{Z}\setminus\mathcal{I}} X^{(\bar z_j)}_{k,0,0,e_j}\mathrm{e}^{\mathrm{i}k\cdot x} \bar z_j\partial_{\bar z_j} \\
&=: W + W'
\end{aligned} \tag{5.15}$$

is quasi-Töplitz as well and ($\|\cdot\|^T_{s,r,\mathtt{a}}$ is short for $\|\cdot\|^T_{s,r,\mathtt{a},N_1,\theta,\mu}$)

$$\|W\|^T_{s,r,\mathtt{a}}, \|W'\|^T_{s,r,\mathtt{a}} \leqq \|\Pi_{\mathrm{diag}}\Pi^{(0)}X\|^T_{s,r,\mathtt{a}} \overset{(3.30)}{\leqq} \|X\|^T_{s,r,\mathtt{a}}.$$

By (3.29) we have $\Pi_{\mathrm{diag}}\Pi^{(0)}\mathcal{T}_{s,r,\mathtt{a}} \subset \mathcal{T}_{s,r,\mathtt{a}}$, hence Lemma 3.4 applied to W implies that for every $N \geqq N_1$ there exist (N-dependent)

$$\tilde W = \sum_{|\pi(k)|,|k|<N^b, |j|>\theta N} \tilde W_k \mathrm{e}^{\mathrm{i}k\cdot x} z_j\partial_{z_j}, \qquad \hat W = \sum_{|\pi(k)|,|k|<N^b, |j|>\theta N} \hat W_{k,j} \mathrm{e}^{\mathrm{i}k\cdot x} z_j\partial_{z_j} \tag{5.16}$$

(note that $\tilde{W}$ is (N, θ, μ)-linear and Töplitz) with

$$\Pi_{N,\theta,\mu} W = \sum_{|\pi(k)|, |k| < N^b, |j| > \theta N} X^{(z_j)}_{k,0,e_j,0} e^{\mathrm{i} k \cdot x} z_j \partial_{z_j} = \tilde{W} + N^{-1} \hat{W} \tag{5.17}$$

and $\|\tilde{W}\|_{s,r,\mathtt{a}}$, $\|\hat{W}\|_{s,r,\mathtt{a}} \leqq \frac{3}{2} \|W\|^T_{s,r,\mathtt{a}} \leqq \frac{3}{2} \|X\|^T_{s,r,\mathtt{a}}$. By (5.13), (5.15), (5.16) and (5.17) we have

$$\begin{aligned} U &= \sum_{k \in \mathcal{K}_N, |j| > \theta N} \tilde{W}_k (1 - e^{\mathrm{i} k \cdot x}) z_j \partial_{z_j} + N^{-1} \sum_{k \in \mathcal{K}_N, |j| > \theta N} \hat{W}_{k,j} (1 - e^{\mathrm{i} k \cdot x}) z_j \partial_{z_j} \\ &=: \tilde{U} + N^{-1} \hat{U} \,. \end{aligned}$$

Note that $\tilde{U}$ is Töplitz. Moreover

$$\begin{aligned} \|\hat{U}\|_{s,r,\mathtt{a}} &\overset{(2.19)}{\leqq} \sup_{\|z\|_{a,p} < r} \left\| \left(\sum_{k \in \mathcal{K}_N} 2 e^{\mathtt{a} |\pi(k)|} e^{s|k|} |\hat{W}_{k,j}| |z_j| \right)_{|j| > \theta N} \right\|_{s,r} \\ &\overset{(5.16)}{\leqq} 2 \|\hat{W}\|_{s,r,\mathtt{a}} \leqq 3 \|X\|^T_{s,r,\mathtt{a}} \,. \end{aligned}$$

An analogous estimate holds true for $\tilde{U}$. A similar decomposition holds for U^- in (5.13).

STEP (2) $N\|U_\perp\|_{s,r,\mathtt{a}}$, $N\|U^-_\perp\|_{s,r,\mathtt{a}} \leqq \|X\|_{s,r,\mathtt{a}}$.

We have

$$\begin{aligned} \|U_\perp\|_{s,r,\mathtt{a}} &\overset{(2.19)}{=} \sup_{\|z\|_{a,p} < r} \left\| \left(\left| \sum_{k \in \mathbb{Z}^n_{\mathrm{odd}} \setminus \mathcal{K}_N} X^{(z_j)}_{k,0,e_j,0} \right| |z_j| \right)_{|j| > \theta N} \right\|_{s,r} \\ &\leqq \sup_{\|z\|_{a,p} < r} \left\| \left(e^{-N^b \min\{s,\mathtt{a}\}} \sum_{|\pi(k)| \ or \ |k| \geqq N^b} e^{\mathtt{a}|\pi(k)| + s|k|} |X^{(z_j)}_{k,0,e_j,0}| |z_j| \right)_{|j| > \theta N} \right\|_{s,r} \\ &\leqq e^{-N^b \min\{s,\mathtt{a}\}} \|X\|_{s,r,\mathtt{a}} \overset{(5.9)}{\leqq} N^{-1} \|X\|_{s,r,\mathtt{a}} \end{aligned}$$

and similarly for $U_\perp$.

PROOF OF (5.11)-(II). The estimate (5.11)-(ii) follows by

$$\|\Pi_{N,\theta,\mu} X''\|_{s,r,\mathtt{a}} \leqq 2N^{-1} \|X\|_{s,r,\mathtt{a}} \,, \quad \forall\, N \geqq N_0 \,. \tag{5.18}$$

In order to prove (5.18) we note that the momentum of $e^{\mathrm{i} k \cdot x} z_{-j} \partial_{z_j}$ with $|k| < N^b$, $|j| > \theta N$, $N \geqq N_1 \geqq N_0$, satisfies

$$|\pi(k, e_{-j}, 0; z_j)| = \left| \sum\nolimits_{h \in \mathcal{I}} h k_h - 2j \right| \geqq 2|j| - \kappa |k| \geqq 2\theta N - \kappa N^b \overset{(3.1)}{>} N > N^b \tag{5.19}$$

(where $\kappa := \max_{h \in \mathcal{I}} |h|$, recall (3.1)). Then by (5.8) and (3.8) the projection $\Pi_{N,\theta,\mu} X'' = V + V'$ with

$$V := \sum_{|j| > \theta N} \left(\sum_{k \in \mathcal{K}_N} X^{(z_j)}_{k,0,e_{-j},0} \right) z_j \partial_{z_j} \,, \qquad V' := \sum_{|j| > \theta N} \left(\sum_{k \in \mathcal{K}_N} X^{(\bar{z}_j)}_{k,0,0,e_{-j}} \right) \bar{z}_j \partial_{\bar{z}_j} \,.$$

We have

$$\begin{aligned}
\|V\|_{s,r,\mathtt{a}} &\overset{(2.19)}{=} \sup_{\|z\|_{a,p}<r} \left\| \left(\left| \sum_{k\in\mathcal{K}_N} X^{(z_j)}_{k,0,e_{-j},0} \right| |z_j| \right)_{|j|>\theta N} \right\|_{s,r} \\
&\overset{(5.19)}{\leqq} \sup_{\|z\|_{a,p}<r} \left\| \left(\sum_{k\in\mathcal{K}_N} \mathrm{e}^{-\mathtt{a}N} \mathrm{e}^{\mathtt{a}|\pi(k,e_{-j},0;z_j)|} |X^{(z_j)}_{k,0,e_{-j},0}||z_{-j}| \right)_{|j|>\theta N} \right\|_{s,r} \\
&\leqq \mathrm{e}^{-\mathtt{a}N} \|X\|_{s,r,\mathtt{a}} \overset{(5.9)}{\leqq} N^{-1}\|X\|_{s,r,\mathtt{a}} \qquad (5.20)
\end{aligned}$$

where in (5.20) we have used that the domain $\{\|z\|_{a,p} < r\}$ is invariant under the map $z_j \mapsto z_{-j}$. Since a similar estimate holds for V', (5.18) follows.

Finally, the vector field $\mathcal{S}X$ is even because $\mathcal{S}X_{|E} = X_{|E}$ (Proposition 5.1) and X is even. Since X is real-coefficients, Definition 5.2 immediately implies that $\mathcal{S}X$ is real-coefficients. Since X is reversible and real-on-real, (2.31) and (2.33) enable to check that X', X'' in (5.7)–(5.8) are reversible and real-on-real, and so $\mathcal{S}X$. □

Remark 5.1. The assumptions $X \in \mathcal{R}_{\mathrm{rev}}$, $Y \in \mathcal{R}_{a-\mathrm{rev}}$, $X = \mathcal{S}X$, $Y = \mathcal{S}Y$ are not sufficient to imply $[X, Y] = \mathcal{S}[X, Y]$, as the example $X = \mathrm{i}(z_{-1}\partial_{z_2} + z_1\partial_{z_{-2}} - \bar z_1\partial_{\bar z_{-2}} - \bar z_{-1}\partial_{\bar z_2})$, $Y = z_2\partial_{z_1} + z_{-2}\partial_{z_{-1}} + \bar z_{-2}\partial_{\bar z_{-1}} + \bar z_2\partial_{\bar z_1}$ shows.

5.2. Homological Equations and Quasi-Töplitz Property

We consider the homological equation

$$\mathrm{ad}_{\mathcal{N}} F = R - [R] \qquad (5.21)$$

where

$$R \in \mathcal{R}^{\leqq 0}_{\mathrm{rev}} \ \text{(see Definition 2.7)}, \qquad R = \mathcal{S}R \ \text{(see Definition 5.2)} \qquad (5.22)$$

and

$$[R] := \langle R^x\rangle\partial_x + \sum\nolimits_{j\in\mathbb{Z}\setminus\mathcal{I}} \langle R^{z_j z_j}\rangle z_j\partial_{z_j} + \langle R^{\bar z_j \bar z_j}\rangle \bar z_j \partial_{\bar z_j}\,, \qquad (5.23)$$

where $\langle\cdot\rangle$ denotes the average with respect to the angles x. By Lemmata 2.4 and 2.5 and since $\mathcal{N} \in \mathcal{R}^{\leqq 0}_{\mathrm{rev}}$ (see Definition 5.1), the action $\mathrm{ad}_{\mathcal{N}} : \mathcal{R}^{\leqq 0}_{a-\mathrm{rev}} \to \mathcal{R}^{\leqq 0}_{\mathrm{rev}}$. The commutator

$$\mathrm{ad}_{\mathcal{N}} F = [F, \mathcal{N}] = \begin{cases} (\partial_\omega F^{\mathrm{u}} - \mathrm{N}F^{\mathrm{u}})\,\partial_{\mathrm{u}} & \text{if } F = F^{(-1)} \\ \partial_\omega F^x\partial_x + (\partial_\omega F^{\mathrm{u},\mathrm{u}} + [F^{\mathrm{u},\mathrm{u}}, \mathrm{N}])\,\mathrm{u}\partial_{\mathrm{u}} & \text{if } F = F^{(0)} \end{cases} \qquad (5.24)$$

(recall the notations in (2.29)–(2.30)) where $[F^{\mathrm{u},\mathrm{u}}, \mathrm{N}] = F^{\mathrm{u},\mathrm{u}}\mathrm{N} - \mathrm{N}F^{\mathrm{u},\mathrm{u}}$ is the usual commutator between matrices (and N is defined in (5.3)). We solve (5.21) when

$$R = R^{(h)}_K := \Pi_{|k|<K}\Pi_{|\pi|<K} R^{(h)}\,, \quad h = 0, -1\,, \quad K \in \mathbb{N} \qquad (5.25)$$

(recall the projections (2.21), (2.22) and (2.27)).

Definition 5.3. (*Melnikov conditions*) Let $\gamma > 0$. The frequencies $\omega(\xi) = (\vec{\omega}(\xi), \vec{\omega}(\xi))$, $\vec{\omega} \in \mathbb{R}^{n/2}$, $\Omega(\xi)$ satisfy the Melnikov conditions (up to $K > 0$) at $\xi \in \mathbb{R}^{n/2}$, if: $\forall\, h \in \mathbb{Z}^{n/2}$, $|h| < K, i, j \in \mathbb{Z} \setminus \mathcal{I}$,

$$|\vec{\omega}(\xi) \cdot h| \geqq \gamma \langle h \rangle^{-\tau} \quad \text{if} \quad h \neq 0\,, \tag{5.26}$$

$$|\vec{\omega}(\xi) \cdot h + \Omega_j| \geqq \gamma \langle h \rangle^{-\tau}\,, \tag{5.27}$$

$$|\vec{\omega}(\xi) \cdot h + \Omega_i(\xi) + \Omega_j(\xi)| \geqq \gamma \langle h \rangle^{-\tau}\,, \tag{5.28}$$

$$|\vec{\omega}(\xi) \cdot h - \Omega_i(\xi) + \Omega_j(\xi)| \geqq \gamma \langle h \rangle^{-\tau} \quad \text{if} \quad h \neq 0 \quad \text{or} \quad i \neq \pm j\,, \tag{5.29}$$

where $\langle h \rangle := \max\{|h|, 1\}$ and $\tau > 1/b$.

For $k \in \mathbb{Z}^n$ we set $k_\pm := (k_j)_{j \in \mathcal{I}^\pm} \in \mathbb{Z}^{n/2}$, namely $k = (k_+, k_-)$. Then

$$\omega \cdot k = \vec{\omega} \cdot h\,, \quad \text{with} \quad h := k_+ + k_- \in \mathbb{Z}^{n/2} \quad \text{and} \quad k \notin \mathbb{Z}^n_{\text{odd}} \overset{(1.34)}{\Longrightarrow} h \neq 0\,. \tag{5.30}$$

Note that $|h| \leqq |k_+| + |k_-| = |k|$.

Lemma 5.1. (Solution of homological equations) *Let* $s, r, \mathtt{a} > 0$, $K > 0$. *Let* $\mathcal{O} \subset \mathbb{R}^{n/2}$ *and assume that the Melnikov conditions* (5.26)–(5.29) *are satisfied* $\forall\, \xi \in \mathcal{O}$. *Then,* $\forall\, \xi \in \mathcal{O}$, *the homological equation* (5.21) *with* $R = R(\cdot; \xi)$ *as in* (5.22)*,*(5.25) *has a unique solution* $F = F(\cdot; \xi)$

$$F \in \mathcal{R}^{\leqq 0}_{a-\text{rev}}\,, \qquad F = \mathcal{S} F\,, \qquad F = \Pi_{|k|<K} \Pi_{|\pi|<K} F$$

with $\langle F^y \rangle = 0$, $\langle F^{y,y} \rangle = 0$, $\langle F^{z_i^\pm, z_i^\pm} \rangle = 0$. *It satisfies*

$$\|F\|_{s,r,\mathtt{a},\mathcal{O}} \leqq \gamma^{-1} K^\tau \|R\|_{s,r,\mathtt{a},\mathcal{O}} \tag{5.31}$$

$$\|F\|^{\text{lip}}_{s,r,\mathtt{a},\mathcal{O}} \lessdot \gamma^{-1} K^\tau \|R\|^{\text{lip}}_{s,r,\mathtt{a},\mathcal{O}} + \gamma^{-2} K^{2\tau+1} \left(|\omega|^{\text{lip}}_{\mathcal{O}} + |\Omega|^{\text{lip}}_{\mathcal{O}} \right) \|R\|_{s,r,\mathtt{a},\mathcal{O}}\,. \tag{5.32}$$

Proof. By (5.24) the homological equation (5.21) splits into

$$\partial_\omega F^{\mathrm{u}} - \mathrm{N} F^{\mathrm{u}} = R^{\mathrm{u}}\,, \quad \partial_\omega F^x = R^x - \langle R^x \rangle\,, \quad \partial_\omega F^{\mathrm{u},\mathrm{u}} + [F^{\mathrm{u},\mathrm{u}}, \mathrm{N}] = R^{\mathrm{u},\mathrm{u}} - [R]^{\mathrm{u},\mathrm{u}}\,. \tag{5.33}$$

Since $R = \mathcal{S} R$ (recall (5.22)), by (5.5) we get

$$R^x(x) = \langle R^x \rangle + \sum\nolimits_{k \notin \mathbb{Z}^n_{\text{odd}}} R^x_k e^{\mathrm{i} k \cdot x}\,, \quad \text{similarly for} \quad R^y(x)\,,\ R^{y,y}(x)\,. \tag{5.34}$$

Since R is reversible and even the average

$$\langle R^y \rangle = 0\,, \quad \langle R^{y,y} \rangle = 0 \tag{5.35}$$

By (5.3), the first equation in (5.33) amounts to $\partial_\omega F^y = R^y$, $\partial_\omega F^z - \mathrm{i}\Omega F^z = R^z$, $\partial_\omega F^{\bar z} + \mathrm{i}\Omega F^{\bar z} = R^{\bar z}$. By (5.3), the third equation in (5.33) splits into $\partial_\omega F^{y,y} = R^{y,y}$, $\partial_\omega F^{y,z} + \mathrm{i} F^{y,z}\Omega = R^{y,z}$ (and the analogous equations for $F^{y,\bar z}$, $F^{z,y}$, $F^{\bar z,y}$), $\partial_\omega F^{z,\bar z} - \mathrm{i} F^{z,\bar z}\Omega - \mathrm{i}\Omega F^{z,\bar z} = R^{z,\bar z}$ (analogously for $F^{\bar z,z}$),

$$\partial_\omega F^{z,z} + \mathrm{i} F^{z,z}\Omega - \mathrm{i}\Omega F^{z,z} = R^{z,z} - [R]^{z,z} \tag{5.36}$$

(analogously for $F^{\bar{z},\bar{z}}$). By (5.26), (5.34), (5.35) and (5.30) the equations for F^x, F^y, F^{yy} are uniquely (having zero average) solved, that is, $F^x(x) = \sum_{k \notin \mathbb{Z}^n_{\rm odd}} F^x_k e^{ik\cdot x}$ with $F^x_k := -iR^x_k/\omega \cdot k$. Similarly the equations for F^{z^σ}, F^{y,z^σ}, $F^{z^\sigma,y}$, $\sigma = \pm$ and $F^{z,\bar{z}}$, $F^{\bar{z},z}$ are solved by (5.27) and (5.28) respectively.

For $i, j \in \mathbb{Z} \setminus \mathcal{I}$, developing in Fourier series $F^{z_i z_j}(x) = \sum_{k \in \mathbb{Z}^n} F^{z_i z_j}_k e^{ik\cdot x}$, Equation (5.36) becomes

$$\mathrm{i}(\omega \cdot k + \Omega_j - \Omega_i) F^{z_i z_j}_k = R^{z_i z_j}_k - [R]^{z_i z_j}_k \,. \tag{5.37}$$

If $i \neq \pm j$ then (5.37) is easily solved by (5.29). Otherwise, since $R = \mathcal{S}R$ and by (5.6),

$$\begin{aligned} &\text{if } i = j \implies R^{z_i z_i}_k = 0\,, \ \forall k \in \mathbb{Z}^n_{\rm odd} \setminus \{0\}; \\ &\text{if } i = -j\,,\ (i \neq 0) \implies R^{z_i z_{-i}}_k = 0\,,\ \forall k \in \mathbb{Z}^n_{\rm odd}\,. \end{aligned} \tag{5.38}$$

Then (5.37) is solved by (5.29) and (5.30).

The properties of anti-reversibility, anti-real-coefficients, real-on-real, and parity for the vector field solution F are easily verified. The estimates (5.31)–(5.32) directly follow by bounds on the small divisors in the Melnikov conditions (5.26)–(5.29) (and (5.30)) and the expression of F. □

The solution of the homological equation is quasi-Töplitz.

Proposition 5.3. (Quasi-Töplitz) *Let the normal form $\mathcal{N}$ be as in Definition 5.1 and assume that $R \in \mathcal{Q}^T_{s,r,\mathtt{a}}(N_0, \theta, \mu)$. Let F be the (unique) solution of the homological equation (5.21) found in Lemma 5.1, for all $\xi \in \mathcal{O}$ satisfying the Melnikov conditions (5.26)–(5.29). If, in addition,*

$$|\vec{\omega}(\xi)\cdot h + p| \geqq \gamma^{2/3}\langle h\rangle^{-\tau}\,, \quad \forall |h| \leqq K,\ p \in \mathbb{Z},\ (h,p) \neq (0,0)\,, \tag{5.39}$$

then $F = F(\cdot;\xi) \in \mathcal{Q}^T_{s,r,\mathtt{a}}(N_0^*, \theta, \mu)$ with

$$N_0^* := \max\left\{N_0\,,\, j_*,\ \hat{c}\gamma^{-1/3}K^{\tau+1}\right\} \tag{5.40}$$

for a (suitably large) constant $\hat{c} := \hat{c}(\mathtt{m}, \kappa) \geqq 1$. Moreover

$$\|F(\cdot;\xi)\|^T_{s,r,\mathtt{a},N_0^*,\theta,\mu} \leqq 4\hat{c}\gamma^{-1}K^{2\tau}\|R(\cdot;\xi)\|^T_{s,r,\mathtt{a},N_0,\theta,\mu}\,. \tag{5.41}$$

Proof. The proof follows step by step the one of the analogous Proposition 5.1 of [3]. □

6. Proof of Theorem 4.1

6.1. First Step

We perform a preliminary change of variables in order to improve the smallness conditions of the perturbation. In particular we want to average out the term $\mathcal{P}^y(x)\partial_y$ defined in (4.6). We introduce the symmetrized vector fields (see Definition 5.2)

$$R^y(x)\partial_y := \mathcal{S}\mathcal{P}^y(x)\partial_y\,, \qquad R := \mathcal{S}\mathcal{P}\,, \qquad X := \mathcal{S}\mathcal{X} = \mathcal{N} + R \tag{6.1}$$

(since $\mathcal{S}\mathcal{N} = \mathcal{N}$). By assumption (A3) and the last statement of Proposition 5.2, $R \in \mathcal{R}_{\mathrm{rev}}$ (see Definition 2.7). Moreover Proposition 5.1 implies that $X_{|E} = \mathcal{X}_{|E}$.

Next we study the homological equation

$$-\mathrm{ad}_{\mathcal{N}} F + \Pi_{|k|<\gamma^{-1/(7n)}} R^y\partial_y = \langle R^y\rangle\partial_y \overset{(5.35)}{=} 0 \tag{6.2}$$

because R is reversible and even.

Lemma 6.1. *For all ξ in $\mathcal{O}_* := \big\{\xi \in \mathcal{O} : |\vec{\omega}(\xi)\cdot h| \geqq \gamma^{2/3}\langle h\rangle^{-n/2}, \forall 0 < |h| < \gamma^{-1/(7n)}\big\}$ the homological equation (6.2) admits a unique solution with $\langle F\rangle = 0$ which satisfies*

$$\|F\|^T_{3s/4,r,\mathrm{a},N_0,\theta,\mu,\lambda,\mathcal{O}_*} = \|F\|^\lambda_{3s/4,r,\mathrm{a},\mathcal{O}_*} \leqq C(s)\varepsilon\,. \tag{6.3}$$

Moreover $F \in \mathcal{R}^{\leqq 0}_{a-\mathrm{rev}}$ and $\mathcal{S}F = F$.

We now apply Proposition 3.2 with $\vec{p} \rightsquigarrow (3s/4, r, \mathrm{a}, N_0, \theta, \mu, \lambda, \mathcal{O}_*)$ and $\vec{p}\,' \rightsquigarrow \vec{p}_0$ with $\vec{p}_0 := (s/2, r/2, \mathrm{a}/2, N_0^{(0)}, 4\theta/3, 3\mu/4, \lambda, \mathcal{O}_*)$ where $N_0^{(0)} \geqq \max\{N_0, \bar{N}\}$ (recall (3.46)) is chosen large enough so that (3.46), (3.47), (3.48) are satisfied and (6.3) imply condition (3.49) for ε sufficiently small. Let $\bar{\Phi}$ be the time 1-flow of F (so that $\mathrm{e}^{\mathrm{ad}_F} = \bar{\Phi}_\star$). Since the quasi-Töplitz norm is non-increasing with N_0 (see (3.22)) we may also take $N_0 \geqq \bar{N}$ large enough so that (5.9) (with $N_0 \rightsquigarrow N_1$) holds. Hence

$$\begin{aligned}\|\mathrm{e}^{\mathrm{ad}_F}(R - R^y\partial_y)\|^T_{\vec{p}_0} &\overset{(3.50)}{\leqq} 2\|R - R^y\partial_y\|^T_{s,r,\mathrm{a},N_0,\theta,\mu,\lambda,\mathcal{O}_*}\\ &\overset{(6.1),(5.10)}{\leqq} 18\|\mathcal{P} - \mathcal{P}^y(x)\partial_y\|^T_{s,r,\mathrm{a},N_0,\theta,\mu,\lambda,\mathcal{O}_*} \overset{(4.6),(4.8)}{<} 18\gamma\,.\end{aligned} \tag{6.4}$$

Similarly (3.51) (with $h \rightsquigarrow 1$, $b_j \rightsquigarrow 1/j!$) implies, for $h = -1, 0$,

$$\begin{aligned}&\Big\|\Big(\mathrm{e}^{\mathrm{ad}_F}(R - R^y\partial_y) - (R - R^y\partial_y)\Big)^{(h)}\Big\|^T_{\vec{p}_0}\\ &\lessdot \|\mathcal{P}_*\|^T_{s,r,\mathrm{a},N_0,\theta,\mu,\lambda,\mathcal{O}_*}\|F\|^T_{3s/4,r,\mathrm{a},N_0,\theta,\mu,\lambda,\mathcal{O}_*}\\ &\overset{(6.3),(4.8)}{\leqq} C(s)\gamma\varepsilon\,.\end{aligned} \tag{6.5}$$

Since the commutator $[F, R^y(x)\partial_y] = [F^y(x)\partial_y, R^y(x)\partial_y] = 0$ we deduce $e^{\mathrm{ad}_F}(R^y\partial_y) = R^y\partial_y$, and, using also (6.2), we get $e^{\mathrm{ad}_F}\mathcal{N} = \mathcal{N} + \mathrm{ad}_F\mathcal{N}$. Hence, using (6.2),

$$e^{\mathrm{ad}_F} X = \mathcal{N} + \Pi_{|k|\geqq\gamma^{-1/(7n)}} R^y\partial_y + e^{\mathrm{ad}_F}(R - R^y\partial_y) =: \mathcal{N}_0 + P_0 \tag{6.6}$$

where $\mathcal{N}_0 := \mathcal{N}$. Then we consider the symmetrized vector field

$$X_0 := \mathcal{S}(e^{\mathrm{ad}_F} X) = \mathcal{N}_0 + R_0\,, \qquad R_0 := \mathcal{S}P_0\,. \tag{6.7}$$

Since $R^y(x)\partial_y$ depends on the variable x only we have

$$\|\mathcal{S}\Pi_{|k|\geqq\gamma^{-1/(7n)}} R^y(x)\partial_y\|^T_{\vec{p}_0} = \|\mathcal{S}\Pi_{|k|\geqq\gamma^{-1/(7n)}} R^y(x)\partial_y\|^\lambda_{s/2,r,\mathtt{a},\mathcal{O}} \lessdot \gamma\varepsilon\,, \tag{6.8}$$

arguing as for (6.3), using (5.10), (2.23), and for $\gamma < \gamma_*$ small (depending on s and n). Recollecting (6.7), (6.6), (6.4), (6.8) and (6.5) we get

Lemma 6.2. *The constants $\bar{\varepsilon}_0 := \varepsilon_0^{(-1)} + \varepsilon_0^{(0)}$, $\varepsilon_0^{(h)} := \gamma^{-1}\|R_0^{(h)}\|^T_{\vec{p}_0}$, $h = -1, 0$, $\Theta_0 := \gamma^{-1}\|R_0\|^T_{\vec{p}_0}$ satisfy $\varepsilon_0^{(h)} \leqq C(s,n)\varepsilon$, $h = -1, 0$, $\Theta_0 \leqq 2^8$, where ε is defined in* (4.8).

The vector fields $P_0, R_0 \in \mathcal{R}_{\mathrm{rev}}$ because $F \in \mathcal{R}_{a-\mathrm{rev}}$ (Lemma 6.1), $R \in \mathcal{R}_{\mathrm{rev}}$, and using Proposition 5.2. Similarly, since $\mathcal{X} \in \mathcal{R}_{\mathrm{rev}}$ (by the hypothesis of Theorem 4.1) the vector field

$$\mathcal{X}_0 := e^{\mathrm{ad}_F}\mathcal{X} = \bar{\Phi}_\star\mathcal{X} \in \mathcal{R}_{\mathrm{rev}}\,. \tag{6.9}$$

Proposition 5.1 implies that $X_{|E} = (\mathcal{S}\mathcal{X})_{|E} = \mathcal{X}_{|E}$ (see (6.1)) and $X_{0|E} = (e^{\mathrm{ad}_F} X)_{|E}$ (see (6.7)). Moreover, since F is even, Lemma 2.6 (applied with $Y \rightsquigarrow F$) and (6.9) imply

$$\mathcal{X}_{0|E} = X_{0|E}\,. \tag{6.10}$$

6.2. The KAM Step

We now describe the iterative scheme which produces a sequence of quasi-Töplitz vector fields X_ν with parameters $\vec{p}_\nu = (s_\nu, r_\nu, \mathtt{a}_\nu, N_0^{(\nu)}, \theta_\nu, \mu_\nu, \lambda, \mathcal{O}_\nu)$, $\lambda = \gamma/M_0$, and such that $X_\nu^{\leqq 0}{}_{|E}$ tends to zero as $\nu \to +\infty$. For compactness of notation we drop the index ν and write "+" for $\nu + 1$.

Iterative hypotheses. Suppose $1 < \theta, \mu < 6$, $N_0 \geqq \bar{N}$ (defined in (3.46)), $\mathcal{O} \subseteq \mathbb{R}^{n/2}$. Let $X = \mathcal{N} + R$, where $\mathcal{N}$ is a normal form vector field (see Definition 5.1) with Lipschitz frequencies $\omega(\xi), \Omega(\xi)$, $\xi \in \mathbb{R}^{n/2}$ and (5.4) holds with some $a(\xi)$, $\forall |j| \geqq 6N_0$ (namely $j_* = 6N_0$). Moreover $|\omega|^{\mathrm{lip}}_{\mathbb{R}^{n/2}}, |\Omega|^{\mathrm{lip}}_{\mathbb{R}^{n/2}} \leqq M \leqq 2M_0$. The perturbation R satisfies $\|R\|^T_{\vec{p}} < \infty$, $R \in \mathcal{R}_{\mathrm{rev}}$, $\mathcal{S}R = R$. We finally fix some K and we assume that $6N_0 \geqq \hat{c}\gamma^{-1/3}K^{\tau+1}$ (where $\hat{c}$ is the constant introduced in (5.40)).

We now describe a *KAM step*, namely a change of variables generated by the time-1 flow of a vector field F and such that

$$X_+ := \mathcal{S} e^{\mathrm{ad}_F} X =: \mathcal{S}\Phi_\star X = \mathcal{N}_+ + R_+ \tag{6.11}$$

still satisfies the iterative hypotheses, with slightly different parameters, and a much smaller new perturbation R_+, see (6.27).

The new normal form $\mathcal{N}_+$. Set (recall (2.29))

$$\begin{aligned} R_K^{\leqq 0} &:= \Pi_{|k|<K}\Pi_{|\pi|<K} R^{\leqq 0} = \Pi_{|k|<K}\Pi_{|\pi|<K} R^{(-1)} + \Pi_{|k|<K}\Pi_{|\pi|<K} R^{(0)} \\ &=: R_K^{(-1)} + R_K^{(0)} . \end{aligned} \tag{6.12}$$

Since $R \in \mathcal{R}_{\mathrm{rev}}$ then $R_K^{\leqq 0} \in \mathcal{R}_{\mathrm{rev}}^{\leqq 0}$ and $\mathcal{S} R_K^{\leqq 0} = R_K^{\leqq 0}$. The new normal form is defined for $\xi \in \mathcal{O}$ as

$$\mathcal{N}^+ := \mathcal{N} + \hat{\mathcal{N}}, \tag{6.13}$$

$$\begin{aligned} \hat{\mathcal{N}} &\overset{(5.23)}{:=} [R_K^{\leqq 0}] = \langle R^x \rangle \partial_x + \sum_{j \in \mathbb{Z}\setminus\mathcal{I}} \langle R^{z_j z_j} \rangle z_j \partial_{z_j} + \langle R^{\bar z_j \bar z_j} \rangle \bar z_j \partial_{\bar z_j} = \hat\omega \cdot \partial_x \\ &\quad + \mathrm{i} \sum_{j \in \mathbb{Z}\setminus\mathcal{I}} \hat\Omega_j z_j (\partial_{z_j} - \bar z_j \partial_{\bar z_j}) \end{aligned} \tag{6.14}$$

because, since $R_K^{\leqq 0}$ is real-coefficients and real-on-real (Definition 2.6)

$$\begin{aligned} &\langle R^{z_j z_j} \rangle = \mathrm{i}\hat\Omega_j\,,\ \hat\Omega_j \in \mathbb{R}\,, \quad \langle R^{\bar z_j \bar z_j} \rangle \overset{(2.33)}{=} -\mathrm{i}\hat\Omega_j\,,\ \forall\, j \in \mathbb{Z}\setminus\mathcal{I}\,, \\ &\hat\omega_j := \langle R^{x_j} \rangle \in \mathbb{R}\,, \forall\, j \in \mathcal{I}\,. \end{aligned} \tag{6.15}$$

Moreover, since R is even, $\hat\omega$, $\hat\Omega$ satisfy (5.2), namely $\hat\omega_j \overset{(2.34)}{=} \hat\omega_{-j}$, $\hat\Omega_j \overset{(2.34)}{=} \hat\Omega_{-j}$. Note that $\hat{\mathcal{N}}$ only depends on $R^{(0)}$ and that $\hat{\mathcal{N}} - \langle R^x \rangle \partial_x = \Pi_{\mathrm{diag}} R$.

The following lemma on the asymptotic of the frequencies is based on the projection Lemma 3.5 for Π_{diag} similarly to Lemma 5.2 of [3].

Lemma 6.3. *It results that* $\sup_{\xi\in\mathcal{O}} |\hat\omega|, |\hat\Omega|_\infty \leqq 2\|R^{(0)}\|_{s,r,\mathtt{a}}$, $|\hat\omega|^{\mathrm{lip}}_{\mathcal{O}}, |\hat\Omega|^{\mathrm{lip}}_{\infty,\mathcal{O}} \leqq 2\|R^{(0)}\|^{\mathrm{lip}}_{s,r,\mathtt{a}}$ *and there exist* $\hat a : \mathcal{O} \to \mathbb{R}$ *satisfying* $\sup_{\xi\in\mathcal{O}} |\hat a(\xi)| \leqq 2\|R^{(0)}\|^T_{s,r,\mathtt{a},N_0,\theta,\mu}$ *such that*

$$\sup_{\xi\in\mathcal{O}} |\hat\Omega_j(\xi) - \hat a(\xi)| \leqq \frac{40}{|j|} \|R^{(0)}\|^T_{s,r,\mathtt{a},N_0,\theta,\mu}\,, \quad \forall\, |j| \geqq 6(N_0+1)\,.$$

The new vector field X_+. We decompose

$$X = \mathcal{N} + R = \mathcal{N} + R_K^{\leqq 0} + (R - R_K^{\leqq 0})$$

where $R_K^{\leqq 0}$ is defined in (6.12). We apply Lemma 5.1 and Proposition 5.3 with $\mathcal{O} \rightsquigarrow \mathcal{O}_+ := \{\xi \in \mathcal{O} \mid (5.26) - (5.29) \text{ and } (5.39) \text{ hold}\}$. Let $F = F_K^{\leqq 0} = F_K^{(-1)} + F_K^{(0)} \in \mathcal{R}_{a-\mathrm{rev}}^{\leqq 0}$ be the unique solution of the homological equation

$$\mathrm{ad}_{\mathcal{N}} F = R_K^{\leqq 0} - [R_K^{\leqq 0}]\,. \tag{6.16}$$

The bounds (5.32), $|\omega|^{\mathrm{lip}}, |\Omega|^{\mathrm{lip}} \leqq M \leqq 2M_0$, and (5.41) (with $R \rightsquigarrow R_K^{(h)}, h = -1, 0$) imply

$$\|F^{(h)}\|_{\vec{p}_\star}^T \lessdot \gamma^{-1} K^{2\tau+1} \|R^{(h)}\|_{\vec{p}}^T, \quad h=-1,0, \quad \text{where} \quad \vec{p}_\star := (s, r, \mathtt{a}, 6N_0, \theta, \mu, \lambda, \mathcal{O}_+)\,. \tag{6.17}$$

Note that in (5.40)–(5.41) $N_0^* = 6N_0$ because, by the iterative hypothesis, $j_* = 6N_0 \geqq \hat{c}\gamma^{-1/3} K^{\tau+1}$.

We introduce the new parameters

$$\vec{p}_+ := (s_+, r_+, \mathtt{a}_+, N_0^+, \theta_+, \mu_+, \lambda, \mathcal{O}_+)\,, \tag{6.18}$$

where $s/2 \leqq s_+ < s$, $r/2 \leqq r_+ < r$, $0 < \mathtt{a}_+ < \mathtt{a}$, $N_0^+ \geqq 7N_0$, $\theta_+ > \theta$, $\mu_+ < \mu$, such that

$$(\kappa + 1)(N_0^+)^{b-L} \ln N_0^+ \leqq \mu - \mu_+, \quad (7 + \kappa)(N_0^+)^{L-1} \ln N_0^+ \leqq \theta_+ - \theta\,, \tag{6.19}$$

$$2(N_0^+)^{-b} \ln^2 N_0^+ \leqq b \min\{s - s_+, \mathtt{a} - \mathtt{a}_+\}\,, \tag{6.20}$$

and note that $N_0^+ \geqq \bar{N}$ defined in (3.46) (by the iterative hypothesis $N_0 \geqq \bar{N}$). If, moreover, the smallness condition

$$\|F\|_{\vec{p}_\star}^T \leqq c(n)\,\delta_+\,, \quad \delta_+ := \min\left\{1 - \frac{s_+}{s}, 1 - \frac{r_+}{r}\right\} \tag{6.21}$$

holds (see (3.49)), then Proposition 3.2 (with $\vec{p} \rightsquigarrow \vec{p}_\star$, $\vec{p}\,' \rightsquigarrow \vec{p}_+$, $\delta \rightsquigarrow \delta_+$) implies that the time 1-flow generated by F maps $D(s_+, r_+)$ into $D(s, r)$. The transformed and symmetrized vector field is

$$X^+ := \mathcal{S} e^{\mathrm{ad}_F} X \overset{(2.14)}{=} \mathcal{S}\left(X + \mathrm{ad}_F(X) + \sum\nolimits_{j \geqq 2} \frac{1}{j!} \mathrm{ad}_F^j(X)\right) = \mathcal{N}^+ + R^+ \tag{6.22}$$

with the new normal form $\mathcal{N}^+$ defined in (6.14) and, by (6.16), the new perturbation

$$R^+ := \mathcal{S}\left(R - R_K^{\leqq 0} + \mathrm{ad}_F(R^{\leqq 0}) + \mathrm{ad}_F(R^{\geqq 1}) + \sum\nolimits_{j \geqq 2} \frac{1}{j!} \mathrm{ad}_F^j(X)\right) \tag{6.23}$$

where $R^{\geqq 1} := \sum_{j \geqq 1} R^{(j)}$, see (2.27), so that $R = R^{\leqq 0} + R^{\geqq 1}$.

We set

$$\varepsilon^{(h)} := \gamma^{-1} \|R^{(h)}\|_{\vec{p}}^T\,, \ h = -1, 0\,, \quad \bar{\varepsilon} := \varepsilon^{(-1)} + \varepsilon^{(0)}\,, \quad \Theta := \gamma^{-1} \|R\|_{\vec{p}}^T \tag{6.24}$$

and the corresponding quantities $\varepsilon_+^{(h)}, \bar{\varepsilon}_+, \Theta_+$ for R^+ with parameters $\vec{p}_+$ defined in (6.18).

Proposition 6.1. (KAM step) *Assume that the parameters $\vec{p}$, $\vec{p}_+$ (see* (6.18)*) satisfy* (6.19)*,* (6.20)*, and that*

$$\delta_+^{-1} K^{2\tau+1}\bar{\varepsilon} \ \ \text{is small enough}\,, \quad \Theta \leqq 2^9\,, \tag{6.25}$$

where δ_+ is defined in (6.21)*. Then, by* (6.17)*, the solution $F \in \mathcal{R}^{\leqq 0}_{\mathrm{rev}}$ of the homological equation* (6.16) *satisfies* (6.21) *and the transformed vector field X^+ in* (6.22) *is well defined. The new normal form is* (6.13)*-*(6.14) *with frequencies satisfying Lemma 6.3. The new perturbation $R^+ \in \mathcal{R}_{\mathrm{rev}}$ in* (6.23) *satisfies $R^+ = \mathcal{S}R^+$ and (see* (6.24)*)*

$$\begin{aligned} \varepsilon_+^{(-1)} &\lessdot \delta_+^{-2} K^{4\tau+2}\bar{\varepsilon}^2 + \varepsilon^{(-1)}\, e^{-K\min\{s-s_+,\, \mathtt{a}-\mathtt{a}_+\}} \\ \varepsilon_+^{(0)} &\lessdot \delta_+^{-2} K^{4\tau+2}\left(\varepsilon^{(-1)} + \bar{\varepsilon}^2\right) + \varepsilon^{(0)}\, e^{-K\min\{s-s_+,\, \mathtt{a}-\mathtt{a}_+\}} \end{aligned} \tag{6.26}$$

$$\Theta_+ \leqq \Theta(1 + C\delta_+^{-2}K^{4\tau+2}\bar{\varepsilon})\,. \tag{6.27}$$

Proof. We analyze each term of R^+ in (6.23). We first claim that

$$\left\| \mathrm{ad}_F(R^{\leqq 0})\right\|^T_{\vec{p}_+} + \left\|\sum_{j\geqq 2}\frac{1}{j!}\mathrm{ad}_F^j(X)\right\|^T_{\vec{p}_+} \lessdot \delta_+^{-2}\gamma K^{2(2\tau+1)}\bar{\varepsilon}^2\,. \tag{6.28}$$

We have

$$\begin{aligned} \sum_{j\geqq 2}\frac{1}{j!}\mathrm{ad}_F^j(X) &= \sum_{j\geqq 2}\frac{1}{j!}\mathrm{ad}_F^j(\mathcal{N}+R) = \sum_{j\geqq 2}\frac{1}{j!}\mathrm{ad}_F^{j-1}(\mathrm{ad}_F\mathcal{N}) \\ &\quad + \sum_{j\geqq 2}\frac{1}{j!}\mathrm{ad}_F^j(R) \\ &\overset{(6.16)}{=} \sum_{j\geqq 2}\frac{1}{j!}\mathrm{ad}_F^{j-1}([R_K^{\leqq 0}] - R_K^{\leqq 0}) + \sum_{j\geqq 2}\frac{1}{j!}\mathrm{ad}_F^j(R)\,. \end{aligned}$$

As we have already noticed, by (6.19), (6.20), (6.21) we can apply Proposition 3.2 (with $\vec{p} \rightsquigarrow \vec{p}_\star$, $\vec{p}\,' \rightsquigarrow \vec{p}_+$, $\delta \rightsquigarrow \delta_+$, $h \rightsquigarrow 2$) obtaining

$$\left\|\sum_{j\geqq 2}\frac{1}{j!}\mathrm{ad}_F^j(R)\right\|^T_{\vec{p}_+} \overset{(3.51)}{\lessdot} \left(\delta_+^{-1}\|F\|^T_{\vec{p}_\star}\right)^2 \|R\|^T_{\vec{p}_\star} \overset{(6.17),(6.24)}{\lessdot} \delta_+^{-2}K^{2(2\tau+1)}\bar{\varepsilon}^2\gamma\,\Theta\,. \tag{6.29}$$

In the same way we get (with $h \rightsquigarrow 1$)

$$\begin{aligned} &\left\|\sum_{j\geqq 2}\frac{1}{j!}\mathrm{ad}_F^{j-1}\left([R_K^{\leqq 0}]-R_K^{\leqq 0}\right)\right\|^T_{\vec{p}_+} = \left\|\sum_{j\geqq 1}\frac{1}{(j+1)!}\mathrm{ad}_F^{j}\left([R_K^{\leqq 0}]-R_K^{\leqq 0}\right)\right\|^T_{\vec{p}_+} \\ &\overset{(3.51)}{\lessdot} \delta_+^{-1}\|F\|^T_{\vec{p}_\star}\|[R_K^{\leqq 0}]-R_K^{\leqq 0}\|^T_{\vec{p}_\star} \leqq \delta_+^{-1}\|F\|^T_{\vec{p}_\star}\|R_K^{\leqq 0}\|^T_{\vec{p}_\star} \overset{(6.17),(6.24)}{\lessdot} \delta_+^{-1}K^{2\tau+1}\gamma\bar{\varepsilon}^2. \end{aligned} \tag{6.30}$$

Finally, by Proposition 3.1, applied with $\vec{p} \rightsquigarrow \vec{p}_\star$, $\vec{p}_1 \rightsquigarrow \vec{p}_+$, $\delta \rightsquigarrow \delta_+$ (note that conditions (3.32)–(3.33) follow by (6.19)–(6.20)), we get

$$\left\|\mathrm{ad}_F(R^{\leqq 0})\right\|^T_{\vec{p}_+} \overset{(3.34)}{\lessdot} \delta_+^{-1}\|F\|^T_{\vec{p}_\star}\|R^{\leqq 0}\|^T_{\vec{p}_\star} \overset{(6.17),(6.24)}{\lessdot} \delta_+^{-1}K^{2\tau+1}\gamma\,\bar{\varepsilon}^2\,. \tag{6.31}$$

The bounds (6.29), (6.30), (6.31), and $\Theta \leqq 2^9$ (see (6.25)), prove (6.28).

We now prove (6.27). Again by Proposition 3.1 we get

$$\left\| \mathrm{ad}_F(R^{\geqq 1}) \right\|^T_{\vec{p}_+} \lessdot \delta_+^{-1} \| F \|^T_{\vec{p}_\star} \| R^{\geqq 1} \|^T_{\vec{p}} \overset{(6.17),(6.24)}{\lessdot} \delta_+^{-1} K^{2\tau+1} \gamma \, \bar{\varepsilon} \, \Theta \tag{6.32}$$

and (6.27) follows by (6.23), (5.10), (3.30), (6.24) (6.32), (6.28) and $\bar{\varepsilon} \leqq 3\Theta$ (which follows by (6.24) and (3.30)).

We now consider $R_+^{(h)}$, $h = 0, -1$. Recalling the degree decomposition $F = F^{(-1)} + F^{(0)}$, formula (2.26) implies that the term $\mathrm{ad}_F R^{\geqq 1}$ in (6.23) does not contribute to $R_+^{(-1)}$. On the other hand, its contribution to $R_+^{(0)}$ is $[R^{(1)}, F^{(-1)}]$. Again by (3.34), (6.17), (6.24) and (3.30), we get

$$\| [R^{(1)}, F^{(-1)}] \|^T_{\vec{p}_+} \lessdot \delta_+^{-1} \gamma K^{2\tau+1} \varepsilon^{(-1)} \Theta \, . \tag{6.33}$$

The contribution of $R - R_K^{\leqq 0}$ in (6.23) to $R_+^{(h)}$, $h = 0, -1$, is $\Pi_{|k|<K} \Pi_{|\pi| \geqq K} R^{(h)} + \Pi_{|k| \geqq K} R^{(h)}$. By (3.31) (recall $ss_+^{-1} < 2$), (3.30), and (6.24), we get

$$\left\| \Pi_{|k|<K} \Pi_{|\pi| \geqq K} R^{(h)} + \Pi_{|k| \geqq K} R^{(h)} \right\|^T_{\vec{p}_+} \leqq 3 \mathrm{e}^{-K \min\{s - s_+, \, \mathtt{a} - \mathtt{a}_+\}} \gamma \varepsilon^{(h)} \, . \tag{6.34}$$

In conclusion, (6.26) follows by (6.23), (5.10), (6.28), (6.33), (6.34) and $\Theta \leqq 2^9$. □

KAM iteration. Once the KAM step has been proved, the proof of Theorem 4.1 is concluded by an usual KAM iteration. The scheme is very similar to that in [3] (and [2]) and we skip it. We only focus on the main difference, which is the symmetrization procedure.

For every $i \in \mathbb{N}$ we construct a close-to-the-identity, analytic, even (Definition 2.6) change of variables Φ^i (obtained as the time-1 flow of the solution F_i of the homological equation (6.16) at the i^{th} step) such that (recall (6.11) and (6.22))

$$X_i := \mathcal{S} \Phi^i_\star X_{i-1} =: \mathcal{N}_i + R_i \, , \quad R_i \in \mathcal{R}_{\mathrm{rev}} \, , \;\; R_i = \mathcal{S} R_i \tag{6.35}$$

($\Phi^i_\star$ is the lift to the tangent space (recall (4.13)). Since the algorithm is "quadratic" (recall (6.26)), the quasi-Töplitz (with suitable i-dependent parameters) norm of the -1 and 0 degree terms of R_i converges *super-exponentially* to zero. Let

$$\begin{aligned} X_\infty := \lim_{i \to \infty} X_i = \lim_{i \to \infty} \mathcal{S} \Phi^i_\star X_{i-1} = \mathcal{N}_\infty + R_\infty \quad \text{where} \\ \mathcal{N}_\infty := \lim_{i \to \infty} \mathcal{N}_i , \, R_\infty := \lim_{i \to \infty} R_i \, . \end{aligned} \tag{6.36}$$

By (6.35) and the convergence of the -1 and 0 degree terms of R_i we get

$$R_\infty = \mathcal{S} R_\infty \, , \qquad R_\infty^{\leqq 0} = 0 \, . \tag{6.37}$$

The transformation Φ in (4.12) is defined by $\Phi := \lim_{\nu \to \infty} \bar{\Phi} \circ \Phi^0 \circ \Phi^1 \circ \cdots \circ \Phi^\nu$ where $\bar{\Phi}$ is defined in Section 6.1 as the time 1-flow of F defined in Lemma 6.1.

The map Φ is even because Φ^i, $i \geqq 0$, and $\bar{\Phi}$ are even. Let us show the proof of (4.13). We have that

$$\mathcal{X}_\infty = \Phi_\star \mathcal{X} = \lim_{i\to\infty} \mathcal{X}_i \quad \text{where} \quad \mathcal{X}_i := \Phi^i_\star \mathcal{X}_{i-1}\,,\ i \geqq 1\,,\ \mathcal{X}_0 \text{ defined in (6.9)}\,. \tag{6.38}$$

The vector field $\mathcal{X}_\infty \in \mathcal{R}_{\rm rev}$ because $\mathcal{X}_0 \in \mathcal{R}_{\rm rev}$ (see (6.9)) and each $\mathcal{X}_i \in \mathcal{R}_{\rm rev}$ because $\Phi^i_\star = e^{\mathrm{ad}_{F_i}}$ with $F_i \in \mathcal{R}_{a-\rm rev}$ (then use Lemma 2.5). The relation between the "auxiliary" vector field X_∞ and the "true" vector field $\mathcal{X}_\infty$ is given by the following

Lemma 6.4. $(\mathcal{X}_\infty)_{|E} = (X_\infty)_{|E}$.

Proof. The lemma follows by proving $(\mathcal{X}_i)_{|E} = (X_i)_{|E}, \forall\, i \geqq 0$. The inductive basis for $i = 0$ is (6.10). Let us assume that $(\mathcal{X}_{i-1})_{|E} = (X_{i-1})_{|E}$. Then

$$\begin{aligned}(\mathcal{X}_i)_{|E} - (X_i)_{|E} &\overset{(6.35),(6.38)}{=} (\Phi^i_\star \mathcal{X}_{i-1})_{|E} - (\mathcal{S}\Phi^i_\star X_{i-1})_{|E}\\ &= \Big(\Phi^i_\star(\mathcal{X}_{i-1} - X_{i-1})\Big)_{|E} \equiv 0\end{aligned}$$

by Proposition 5.1 and Lemma 2.6 (used with $X \rightsquigarrow \mathcal{X}_{i-1} - X_{i-1}, Y \rightsquigarrow F_i$ with $e^{\mathrm{ad}_{F_i}} = \Phi^i_\star$). $\square$

We have already chosen $\mathcal{N}_\infty$ in (6.36), then $\mathcal{P}_\infty$ in (4.13) is $\mathcal{P}_\infty = \mathcal{X}_\infty - \mathcal{N}_\infty$. It is now simple to show that $(\mathcal{P}^{\leqq 0}_\infty)_{|E} = 0$. Indeed

$$\begin{aligned}\left(\mathcal{P}^{\leqq 0}_\infty\right)_{|E} &= \Big((\mathcal{X}_\infty - \mathcal{N}_\infty)^{\leqq 0}\Big)_{|E} \overset{(6.36)}{=} \Big((\mathcal{X}_\infty - X_\infty + R_\infty)^{\leqq 0}\Big)_{|E}\\ &\overset{(6.37)}{=} \Big((\mathcal{X}_\infty - X_\infty)^{\leqq 0}\Big)_{|E} \overset{(2.35)}{\equiv} 0\,.\end{aligned}$$

by Lemma 6.4. Finally $\mathcal{P}_\infty \in \mathcal{R}_{\rm rev}$ because $\mathcal{N}_\infty$ and $\mathcal{X}_\infty \in \mathcal{R}_{\rm rev}$. This concludes the proof of (4.13).

Proof of (4.17). By $\det A \neq 0$ and (4.9) the action-to-frequency map $\vec{\omega}^\infty$ is invertible. Introducing $\zeta = \vec{\omega}^\infty(\xi)$ as parameters, we obtain $\xi = (\vec{\omega}^\infty)^{-1}(\zeta) = A^{-1}(\zeta - \bar{\omega}) + O(\varepsilon\gamma)$ and, using also (4.14),

$$\begin{gathered}\vec{\omega}^\infty(\xi)\cdot h + \Omega^\infty_i(\xi) - \Omega^\infty_j(\xi) = f_{h,i,j}(\zeta) + r_{i,j}(\zeta),\\ |r_{i,j}| = O(\gamma\varepsilon), \quad |r_{i,j}|^{\rm lip} = O(\varepsilon)\,,\\ f_{h,i,j}(\zeta) := c_{h,i,j}\cdot\zeta + d_{i,j}\,,\ c_{h,i,j} := h + (\lambda_i^{-1} - \lambda_j^{-1})A^{-T}\vec{a},\\ d_{i,j} := \lambda_i - \lambda_j - (\lambda_i^{-1} - \lambda_j^{-1})\bar{\omega}\cdot A^{-T}\vec{a}.\end{gathered}$$

Then (4.17) follows immediately if $|c_{h,i,j}| > \bar{c} > 0$ because $|c_{h,i,j}\cdot\partial_\zeta f_{h,i,j}| \geqq \bar{c}^2 > 0$ and $|r_{i,j}|^{\rm lip} = O(\varepsilon)$. Now, since $h \in \mathbb{Z}^{n/2}\setminus\{0\}$ and $|(\lambda_i^{-1} - \lambda_j^{-1})A^{-T}\vec{a}| = O(\lambda_i^{-1} + \lambda_j^{-1})$, the coefficient $|c_{h,i,j}| > 1/2$, for $\min\{|i|,|j|\} \geqq C$ large. On the other hand, if $|i| \leqq C$ and $|j| \geqq C_0$ with C_0 large enough (or permuting the role of i and j) the coefficient $|d_{i,j}| \geqq 1$. In this case $|f_{h,i,j} + r_{i,j}| > 1/8$ for all $\zeta \in \vec{\omega}^\infty(\mathcal{O})$ unless $|c_{h,i,j}\cdot\bar{\omega}| \geqq 1/4$ (for ε, ρ small). Hence $|\bar{\omega}\cdot\partial_\zeta f_{h,i,j}| = |\bar{\omega}\cdot c_{h,i,j}| > 1/4$ and, again, (4.17) follows. Finally, the first condition in (4.15) and $h \neq 0$ imply $\min\{|c_{h,i,j}| \text{ for } |i|,|j| \leqq C_0\} > 0$ and so (4.17).

7. Proof of Theorem 1.1

By hypothesis, the analytic nonlinearity g has the convergent Taylor expansion

$$g(\mathrm{x},\mathrm{y},\mathrm{y}_{\mathrm{x}},\mathrm{v}) = \kappa_1 \mathrm{y}^3 + \kappa_2 \mathrm{y}\mathrm{y}_{\mathrm{x}}^2 + \kappa_3 \mathrm{y}\mathrm{y}_t^2 + \sum\nolimits_{k+h+l\geqq 5} g^{(k,h,l)}(\mathrm{x})\mathrm{y}^k \mathrm{y}_{\mathrm{x}}^h \mathrm{v}^l$$

where $k, h, l \in \mathbb{N}$ and

$$\|g^{(k,h,l)}\|_{a_0,p} < C^{k+h+l} \quad \text{for some} \quad a_0 > 0\,, \ p > 1/2\,, \ C > 0\,, \tag{7.1}$$

having identified each function $g^{(k,h,l)}(\mathrm{x})$ with the Fourier series $\{g^{(k,h,l)}_{j_0}\}_{j_0\in\mathbb{Z}} \in \ell^{a_0,p}$, recall (2.1). As phase space we consider $u, \bar{u} \in \ell^{a,p}$ with $a := a_0/2$. The coefficients $\mathtt{g}_j^+$ in (1.25) are

$$\mathtt{g}_j^+ := \mathtt{g}_j = -\sum_{d=3, d\geqq 5} \ \sum_{j_0+\sum_{i=1}^d \sigma_i j_i = j} (\sqrt{2})^{-d-1} \mathtt{g}_{\vec{\sigma},\vec{\jmath},j_0} u_{\vec{\jmath}}^{\vec{\sigma}} =: \mathtt{g}_j^{(=3)} + \mathtt{g}_j^{(\geqq 5)} \tag{7.2}$$

where $\vec{\jmath} = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, $\vec{\sigma} = (\sigma_1, \ldots, \sigma_d) \in \{\pm\}^d$ and $u_{\vec{\jmath}}^{\vec{\sigma}} = \prod_{i=1}^d u_{j_i}^{\sigma_i}$. The coefficients $g_{\vec{\sigma},\vec{\jmath},j_0}$ are

$$g_{\vec{\sigma},\vec{\jmath},j_0} = \sum_{h+k+l=d} (-1)^l\, \mathrm{i}^{h+l} \sigma_{k+1}\cdots\sigma_{k+h+l} \frac{j_{k+1}\cdots j_{k+h}}{\lambda_{j_1}\cdots\lambda_{j_{k+h}}} g_{j_0}^{(k,h,l)}\,.$$

We consider (1.24) as the equations of motion of the vector field $\mathcal{N}_0 + G$ where (recall (1.25))

$$\begin{aligned} \mathcal{N}_0 &:= \sum_{\sigma=\pm,\, j\in\mathbb{Z}} \sigma \mathrm{i}\lambda_j u_j^\sigma \partial_{u_j^\sigma}\,, \quad G = \sum_{\sigma=\pm,\, j\in\mathbb{Z}} G^{(u_j^\sigma)} \partial_{u_j^\sigma}\,, \\ G^{(u_j^\sigma)} &:= \mathrm{i}\sigma \mathtt{g}_{\sigma j}\,, \quad G = G^{(=3)} + G^{(\geqq 5)}. \end{aligned} \tag{7.3}$$

Note that $G^{(u_{-j}^+)} = -G^{(u_j^-)}$ and that $G^{(=3)}$ has zero momentum by (7.2), (7.3). Moreover G is reversible (with respect to the involution S in (1.27)), real-coefficients, real-on-real, even, namely $G \in \mathcal{R}_{\mathrm{rev}}$ (Definition 2.7 in absence of x, y-variables).

Lemma 7.1. *Set* $\mathtt{a} := a_0/2$ (*where* a_0 *is defined in* (7.1)). *Then, for* $R := R(C) > 0$ *small enough* (*where* C *is defined in* (7.1)), *it results*

$$\|G\|_{R,\mathtt{a}}\,, \ \|G^{(=3)}\|_{R,\mathtt{a}} \lessdot R^2\,, \quad \|G^{(\geqq 5)}\|_{R,\mathtt{a}} \lessdot R^4\,. \tag{7.4}$$

Moreover $G, G^{(=3)}, G^{(\geqq 5)} \in \mathcal{Q}^T_{R,\mathtt{a}}(N_0, 3/2, 4)$, *for* N_0 *satisfying* (3.1), *and*

$$\|G\|^T_{R,\mathtt{a},N_0,3/2,4}\,, \ \|G^{(=3)}\|^T_{R,\mathtt{a},N_0,3/2,4} \lessdot R^2\,, \quad \|G^{(\geqq 5)}\|^T_{R,\mathtt{a},N_0,3/2,4} \lessdot R^4\,. \tag{7.5}$$

Proof. We first note that (recall also that $a := a_0/2$)

$$\left\| \left(\sum_{j_0+\sum_{i=1}^d \sigma_i j_i = j} \mathrm{e}^{\mathtt{a}|j_0|} |g_{j_0}^{(k,h,l)}||u_{\vec{j}}^{\vec{\sigma}}| \right)_{j\in\mathbb{Z}} \right\|_{a,p} \lessdot \|g^{(k,h,l)}\|_{a_0,p} (\|u\|_{a,p} + \|\bar{u}\|_{a,p})^d . \tag{7.6}$$

Indeed

$$\sum_{j_0+\sum_{i=1}^d \sigma_i j_i = j} \mathrm{e}^{\mathtt{a}|j_0|} |g_{j_0}^{(k,h,l)}||u_{\vec{j}}^{\vec{\sigma}}| \leqq \left(f^{(k,h,l)} * \tilde{u} * \tilde{u} * \cdots * \tilde{u} \right)_j , \qquad \forall\, j \in \mathbb{Z} ,$$

where $f^{(k,h,l)} := (\mathrm{e}^{\mathtt{a}|j_0|} g_{j_0}^{(k,h,l)})_{j_0\in\mathbb{Z}}$, $\tilde{u} := (\tilde{u}_n)_{n\in\mathbb{Z}}$, $\tilde{u}_n := |u_n| + |\bar{u}_n|$, $*$ denotes the convolution of sequences and

$$\begin{aligned} \| f^{(k,h,l)} * \tilde{u} * \tilde{u} * \cdots * \tilde{u} \|_{a,p} &\lessdot \| f^{(k,h,l)} \|_{a,p} \|\tilde{u}\|_{a,p}^d \\ &\lessdot \|g^{(k,h,l)}\|_{a_0,p} (\|u\|_{a,p} + \|\bar{u}\|_{a,p})^d \end{aligned}$$

by the Hilbert algebra property of $\ell^{a,p}$ and since $\mathtt{a} = a_0 - a$.

Now we rewrite the sum in (7.2) as $\mathtt{g}_j = \sum_{|\alpha|+|\beta|\geqq 3} (\mathtt{g}_j)_{\alpha,\beta} u^\alpha \bar{u}^\beta$ where $(\mathtt{g}_j)_{\alpha,\beta}$ can be explicitly computed from (7.2) but has a complicated combinatorics. In order to compute the norm $\|G\|_{R,\mathtt{a}}$ we note that $1/|\lambda_l| \lessdot 1$, $|l|/|\lambda_l| \leqq 1$ and

$$u_{\vec{j}}^{\vec{\sigma}} = u^\alpha \bar{u}^\beta \quad \Longrightarrow \quad \pi(\alpha, \beta; u_j^\sigma) = \sum\nolimits_{1\leqq i\leqq d} \sigma_i j_i - \sigma j . \tag{7.7}$$

We have (recall (7.3))

$$\begin{aligned} \|G\|_{R,\mathtt{a}} &\overset{(2.2)}{=} \sup_{\|u\|_{a,p},\|\bar{u}\|_{a,p}<R} R^{-1} \sum_{\sigma=\pm} \left\| \left(\sum_{|\alpha|+|\beta|\geqq 3} \mathrm{e}^{\mathtt{a}|\pi(\alpha,\beta;u_j^\sigma)|} |(g_{\sigma j})_{\alpha,\beta}||u^\alpha||\bar{u}^\beta| \right)_{j\in\mathbb{Z}} \right\|_{a,p} \\ &\overset{(7.7)}{\lessdot} R^{-1} \sup_{\|u\|_{a,p},\|\bar{u}\|_{a,p}<R} \left\| \left(\sum_{d\geqq 3} (\sqrt{2})^{-d-1} \sum_{j_0+\sum_{i=1}^d \sigma_i j_i = j} \sum_{h+k+l=d} \mathrm{e}^{\mathtt{a}|j_0|} |g_{j_0}^{(k,h,l)}||u_{\vec{j}}^{\vec{\sigma}}| \right)_{j\in\mathbb{Z}} \right\|_{a,p} \\ &\overset{(7.6)}{\lessdot} R^{-1} \sup_{\|u\|_{a,p},\|\bar{u}\|_{a,p}<R} \sum_{d\geqq 3} \sum_{h+k+l=d} (\sqrt{2})^{-d-1} \|g^{(k,h,l)}\|_{a_0,p} (\|u\|_{a,p} + \|\bar{u}\|_{a,p})^d \overset{(7.1)}{\lessdot} R^2 \end{aligned}$$

proving (7.4) for R small enough with respect to the constant C in (7.1).

Let us now prove the estimate (7.5) for the quasi-Töplitz norm of G (the estimates for $G^{(=3)}$ and $G^{(\geqq 5)}$ are analogous). For $N \geqq N_0$, by (7.2) and (7.3) we deduce that

$$\Pi_{N,3/2,4} G = \sum_{|m|,|n|>(3/2)N} G_{\sigma',n}^{\sigma,m} u_n^{\sigma'} \partial_{u_m^\sigma} = \tilde{G} + N^{-1} \hat{G}$$

where (recall (3.8), (3.9), (3.10))

$$G_{\sigma',n}^{\sigma,m} := -\mathrm{i}\sigma \sum_{d\geqq 2} \sum_{\substack{\sum_{i=1}^d |j_i|<4N^L, |j_0|<N^b \\ j_0+\sum_{i=1}^d \sigma_i j_i = \sigma m - \sigma' n}} (\sqrt{2})^{-d-2} \sum_{h+k+l=d+1} (\mathrm{i})^{h+l} (-1)^l g_{j_0}^{(k,h,l)} c_{\vec{\sigma},\vec{j},\sigma',n}^{(k,h,l)} u_{\vec{j}}^{\vec{\sigma}}$$

$$c_{\vec{\sigma},\vec{j},\sigma',n}^{(k,h,l)} := \frac{j_{k+1} \ldots j_{k+h-1} \sigma_{k+1} \ldots \sigma_{k+h+l-1}}{\lambda_{j_1} \ldots \lambda_{j_{k+h-1}}} \left(k \frac{\sigma_k j_k}{\lambda_n} + h \frac{\sigma' n}{\lambda_n} + l \frac{\sigma'}{\lambda_{j_{k+h}}} \right).$$

The Töplitz approximation $\tilde{G}$ is obtained by substituting the coefficients $c^{(k,h,l)}_{\vec{\sigma},\vec{j},\sigma',n}$ with their Töplitz approximation $\tilde{c}^{(k,h,l)}_{\vec{\sigma},\vec{j},\sigma',n}$ defined by replacing $1/\lambda_n$ by 0 and n/λ_n by the sign $\mathtt{s}(n)$.

Since $0 \leqq \lambda_n - |n| \leqq \sqrt{\mathtt{m}}$, $\forall n \in \mathbb{Z}$, and $\lambda_n \geqq |n| > (3/2)N$, the Taylor coefficients of $\tilde{G}$ and of the corresponding defect $\hat{G}$ are uniformly bounded. Then, arguing as in the proof of (7.4), we deduce that $\|\tilde{G}\|_{R,\mathtt{a}}, \|\hat{G}\|_{R,\mathtt{a}} \lessdot R^2$. Note that $\tilde{c}^{(k,h,l)}_{\vec{\sigma},\vec{j},\sigma',n}$ depends on n only through $\mathtt{s}(n)$. Since by (3.10) $\mathtt{s}(n) = \sigma\sigma'\mathtt{s}(m)$ we have that $\tilde{G} \in \mathcal{T}_{R,\mathtt{a}}(N, 3/2, 4)$ (Definition 3.3). By Definition 3.4 we get (7.5). $\square$

For the Birkhoff normal form step we need the following lemma proved in [3] (Lemma 7.2 and formula (7.21), see also [21]).

Lemma 7.2. *There exists an absolute constant $c_* > 0$, such that, for every $\mathtt{m} \in (0,\infty)$ and $j_i \in \mathbb{Z}$, $\sigma_i = \pm$, $i = 1,2,3,4$ satisfying $\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$ but not satisfying*

$$j_1 = j_2\,,\ j_3 = j_4\,,\ \sigma_1 = -\sigma_2\,,\ \sigma_3 = -\sigma_4 \ \text{(or permutations of the indexes)}\,, \tag{7.8}$$

we have

$$\begin{aligned} &|\sigma_1\lambda_{j_1} + \sigma_2\lambda_{j_2} + \sigma_3\lambda_{j_3} + \sigma_4\lambda_{j_4}| \geqq \frac{c_*\mathtt{m}}{(n_0^2+\mathtt{m})^{3/2}} > 0 \quad \textit{where} \\ &n_0 := \min\{\langle j_1\rangle, \langle j_2\rangle, \langle j_3\rangle, \langle j_4\rangle\}\,. \end{aligned} \tag{7.9}$$

Then we define the projections G_1 and G_2 of $G^{(=3)}$ as follows: the vector field $-\mathrm{i}G_1^{(u_j^+)}$ is the projection of $\mathtt{g}_j^{(=3)}$ (recall (7.2)) onto the indexes $(\sigma_1,\sigma_2,\sigma_3,+)$, (j_1,j_2,j_3,j) which satisfy (7.8) with $j_1 \in \mathcal{I}$. Let $-\mathrm{i}G_2^{(u_j^+)}$ be the projection of $\mathtt{g}_j^{(=3)}$ onto the indexes $j_1, j_2, j_3 \notin \mathcal{I}$ if $j \notin \mathcal{I}$ and zero otherwise. We have that

$$\|G_1\|_{R,\mathtt{a}} = \|G_1\|_{R,0}\,,\ \|G_2\|_{R,\mathtt{a}} = \|G_2\|_{R,0} \lessdot R^2\,, \tag{7.10}$$

and, for N_0' large enough,

$$\|G_1\|^T_{R,\mathtt{a},N_0,3/2,4} \lessdot R^2\,, \qquad \|G_2\|^T_{R,\mathtt{a},N_0,3/2,4} \lessdot R^2\,. \tag{7.11}$$

The estimates (7.10) and (7.11) follows by (3.24) and the analogous estimates (7.4) and (7.5) for G, since G_1, G_2 are projections (recall (2.12)) of G, satisfying (3.23).

Proposition 7.1. (Birkhoff normal form) *For any $\mathcal{I}$ as in (1.29), and $\mathtt{m} > 0$, there exists $R_0 > 0$ and a real analytic change of variables $\Gamma : B_{R/2} \times B_{R/2} \subset \ell^{a,p} \times \ell^{a,p} \to B_R \times B_R \subset \ell^{a,p} \times \ell^{a,p}$, $0 < R < R_0$, that takes the vector field $\mathcal{N}_0 + G$ into*

$$\Big(D\Gamma^{-1}[\mathcal{N}_0 + G]\Big) \circ \Gamma = \mathcal{N}_0 + G_1 + G_2 + G_3 \tag{7.12}$$

where G_1, G_2 satisfy (7.11), G_3 satisfy $G_3^{(u^+_{-j})} = -G_3^{(u_j^-)}$ and for N_0' large enough

$$\|G_3\|^T_{R/2,\mathtt{a}/2,N_0',7/4,3} \lessdot R^4\,. \tag{7.13}$$

Finally $\mathcal{N}_0 + G_1 + G_2 + G_3 \in \mathcal{R}_{\mathrm{rev}}$ (recall Definition 2.7 in absence of x, y-variables).

Proof. Let us define the generating function $F := \sum_{j\in\mathbb{Z},\sigma=\pm} F^{(u_j^\sigma)}\partial_{u_j^\sigma}$ with

$$F^{(u_j^\sigma)} := \sum_{\substack{\sigma_1\lambda_{j_1}+\sigma_2\lambda_{j_2}+\sigma_3\lambda_{j_3}-\sigma\lambda_j\neq 0,\\ (j_1,j_2,j_3,j)\notin(\mathcal{I}^c)^4,\vec{\sigma}\cdot\vec{j}=\sigma j}} -\frac{1}{4}\frac{\sigma}{\sigma_1\lambda_{j_1}+\sigma_2\lambda_{j_2}+\sigma_3\lambda_{j_3}-\sigma\lambda_j}\,\mathtt{g}_{\vec{\sigma},\vec{j},0}\,u_{\vec{j}}^{\vec{\sigma}}\,. \tag{7.14}$$

By Lemma 7.2 and arguing as in Lemma 7.1 we get $\|F\|_{R,\mathtt{a}} = \|F\|_{R,0} \lessdot R^2$. Moreover we claim that

$$\|F\|^T_{R,\mathtt{a},N_0,3/2,4} \lessdot R^2\,. \tag{7.15}$$

For $N \geqq N_0$, by (7.14) we wish to write $\Pi_{N,3/2,4}F = \tilde{F} + N^{-1}\hat{F}$, where (recall (3.8), (3.9) and (3.10))

$$\tilde{F} = \sum_{|m|,|n|>(3/2)N} \tilde{F}^{\sigma,m}_{\sigma',n}\, z_n^{\sigma'}\,\partial_{z_m^\sigma}$$

is Töplitz. We define $\tilde{F}$ by using (7.14) with $j, j_3, \sigma_3 \rightsquigarrow m, n, \sigma'$ and substituting as follows: $\mathtt{g}_{\vec{\sigma},\vec{j},0}$ by its Töplitz approximation (given in Lemma 7.1) and $d := \sigma_1\lambda_{j_1}+\sigma_2\lambda_{j_2}+\sigma'\lambda_n-\sigma\lambda_m$ by $\tilde{d} := \sigma_1\lambda_{j_1}+\sigma_2\lambda_{j_2}+\sigma'|n|-\sigma|m|$. To estimate the Töplitz defect $\hat{F}$ we consider first the case $\sigma = \sigma'$. We have

$$|d-\tilde{d}| = |\lambda_n-\lambda_m-|n|+|m|| \lessdot (|n|^{-1}+|m|^{-1}) \lessdot |n|^{-1}\,,$$

noting that $1/2 \leqq |n|/|m| \leqq 2$ by $\sigma_1 j_1+\sigma_2 j_2 = \sigma m-\sigma' n$ and $|j_1|+|j_2| < 4N^L$, for $N \geqq N_0$ large enough. Then, since by (7.9), $1 \lessdot |d|$, for $|n| \geqq (3/2)N$ and N_0 large enough, $1 \lessdot |d| - |\tilde{d}-d| \leqq |\tilde{d}|$. In particular $|\tilde{d}| \geqq const. > 0 > 0$ and $\tilde{F}$, $\hat{F}$ are well defined. Moreover

$$\left|\frac{d}{\tilde{d}}-1\right| = \left|\frac{1}{\tilde{d}}(d-\tilde{d})\right| \lessdot \frac{1}{|n|} \qquad \text{and} \qquad |\lambda_n-|n|| \lessdot \frac{1}{|n|}$$

and, therefore, $||n|-d\tilde{d}^{-1}\lambda_n| \lessdot 1$. In the case $\sigma = -\sigma'$, since $|j_1|+|j_2| < 4N^L$ and $\lambda_m \geqq |m| \geqq N$, we get $|d| \geqq |n|$. Recalling that we have, both in the case $\sigma = \sigma'$ and $\sigma = -\sigma'$, that the Taylor coefficients of $\tilde{F}$, $\hat{F}$ are uniformly bounded, arguing as in the proof of Lemma 7.1, we get $\|\tilde{F}\|_{R,\mathtt{a}}$, $\|\hat{F}\|_{R,\mathtt{a}} \lessdot R^2$. We note that $\tilde{F} \in \mathcal{T}_{R,\mathtt{a}}(N,3/2,4)$; indeed $\sigma = \sigma'$ and by (3.10) $\mathtt{s}(m) = \mathtt{s}(n)$, so that $\tilde{d} := \sigma_1\lambda_{j_1}+\sigma_2\lambda_{j_2}+\mathtt{s}(m)(\sigma' n-\sigma m)$. Then by Definition 3.4 we deduce (7.15).

With $\mathcal{N}_0$ defined in (7.3) we have

$$\left[\mathcal{N}_0, u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\,\partial_{u_j^\sigma}\right] = \mathrm{i}(\sigma_1\lambda_{j_1}+\sigma_2\lambda_{j_2}+\sigma_3\lambda_{j_3}-\sigma\lambda_j)u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\,\partial_{u_j^\sigma}\,.$$

Then F in (7.14) solves the homological equation $[\mathcal{N}_0, F] + G^{(=3)} + G^{(=3)} = G_1 + G_2$. Then we define Γ as the time-1 flow generated by the vector field F and (7.13) follows by Proposition 3.2 taking $R < R_0$ small enough and N_0' large enough.

We claim that $F \in \mathcal{R}_{a-\mathrm{rev}}$. Indeed F is real-on-real (recall Definition 2.6) by (7.14). F is anti-real-coefficients since the Taylor coefficients in (7.14) are real. F is anti-reversible (recall Definition 2.5) with respect to the involution S in (1.27) since by (7.14) we have $F^{(u_j^\sigma)} \circ S = F^{(u_{-j}^{-\sigma})}$. Finally F is even (recall Definition 2.6) since, again by (7.14) $F^{(u_j^\sigma)}_{|E} = F^{(u_{-j}^\sigma)}_{|E}$ (with E defined in (1.26)).

Then $\mathcal{N}_0 + G_1 + G_2 + G_3 = \mathrm{e}^{\mathrm{ad} F}(\mathcal{N}_0 + G) \in \mathcal{R}_{\mathrm{rev}}$ by Lemma 2.5. □

7.1. Action-Angle Variables and Conclusion of Proof of Theorem 1.1

Let us denote by $(u^+, u^-) = \Phi(x, y, z^+, z^-; \xi)$ the change of variable introduced in (1.30). For $\rho > 0$, let (recall (1.29))

$$\mathcal{O}_\rho := \left\{ \xi \in \mathbb{R}^{n/2} \,:\, \rho/2 \leqq \xi_j \leqq \rho\,,\ \ j \in \mathcal{I}^+ \right\}. \tag{7.16}$$

A vector field $X = (X^{(u^+)}, X^{(u^-)})$ is transformed by the change of variable Φ in

$$Y := \Phi_\star X = \big(D\Phi^{-1}[X]\big) \circ \Phi, \quad \text{with} \quad Y^{(z_j^\sigma)} = X^{(u_j^\sigma)} \circ \Phi, \quad \sigma = \pm,\ j \in \mathbb{Z}\backslash\mathcal{I},$$

$$Y^{(x_j)} = -\frac{\mathrm{i}}{2}\left(\frac{1}{u_j^+} X^{(u_j^+)} - \frac{1}{u_j^-} X^{(u_j^-)}\right) \circ \Phi, \quad Y^{(y_j)} = \left(u_j^- X^{(u_j^+)} + u_j^+ X^{(u_j^-)}\right) \circ \Phi,\ j \in \mathcal{I}.$$

Lemma 7.3. (Lemma 7.6 of [3]) *Let us take*

$$0 < 16 r^2 < \rho\,, \qquad \rho = C_* R^2 \quad \text{with} \quad C_*^{-1} := 48 n \kappa^{2p} e^{2(s + a\kappa)}\,. \tag{7.17}$$

where $a = a_0/2$, $p > 1/2$ and κ is defined in (3.1). Then, for all $\xi \in \mathcal{O}_\rho \cup \mathcal{O}_{2\rho}$, the map $\Phi(\,\cdot\,; \xi) : D(s, 2r) \to B_{R/2} \times B_{R/2} \subset \ell^{a,p} \times \ell^{a,p}$ is well defined and analytic ($D(s, 2r)$ is defined in (2.4)).

Given a vector field $X : B_{R/2} \times B_{R/2} \to \ell^{a,p} \times \ell^{a,p}$, the previous Lemma and (7.17) show that the transformed vector field $Y := \Phi_\star X : D(s, 2r) \to \ell^{a,p} \times \ell^{a,p}$. It results that, if X is quasi-Töplitz in the variables $(u, \bar u)$ then Y is quasi-Töplitz in the variables $(x, y, z, \bar z)$ (see Definition 3.4). We define the space of vector fields

$$\mathcal{V}^d_{R,\mathtt{a}} := \left\{ X := X(u, \bar u) : \|X\|_{R,\mathtt{a}} < \infty \text{ and } X^{(u_j^\sigma)} = \sum_{|\alpha^{(2)} + \beta^{(2)}| \geqq d} X^{(u_j^\sigma)}_{\alpha,\beta} u^\alpha \bar u^\beta \right\}.$$

Proposition 7.2. (Quasi–Töplitz) *Let N_0, θ, μ, μ' satisfy (3.1) and*

$$(\mu' - \mu) N_0^L > N_0^b\,, \qquad N_0 2^{-\frac{N_0^b}{2\kappa} + 1} < 1\,. \tag{7.18}$$

If $X \in \mathcal{Q}^T_{R/2,\mathtt{a}}(N_0, \theta, \mu') \cap \mathcal{V}^d_{R/2,\mathtt{a}}$ with $d = 0, 1$, then $Y := \Phi_\star X \in \mathcal{Q}^T_{s,r,\mathtt{a}}(N_0, \theta, \mu)$ and

$$\|Y\|^T_{s,r,\mathtt{a},N_0,\theta,\mu,\mathcal{O}_\rho} \lessdot (8r/R)^{d-2} \|X\|^T_{R/2,\mathtt{a},N_0,\theta,\mu'}\,. \tag{7.19}$$

The proof of Proposition 7.2 follows closely the analogous Proposition 7.2 in [3] (replacing the Hamiltonians with the vector fields). The following lemma holds (see Lemma 7.11 in [3]):

Lemma 7.4. *Let* $X \in \mathcal{V}_{R/2,\mathtt{a}}$, $Y := \Phi_\star X$ *and* $Y_0(x, y) := Y(x, y, 0, 0) - Y^{(y)}(x, 0, 0, 0)\partial_y$. *Then, assuming* (7.17), $\|Y_0\|_{s,2r,\mathtt{a},\mathcal{O}_\rho\cup\mathcal{O}_{2\rho}} \lessdot (R/r)\|X\|_{R/2,\mathtt{a}}$.

Recalling (7.17), the vector field $\mathcal{N}_0 + G_1 + G_2 + G_3$ in (7.12) is transformed by the change of variable (1.30) into

$$\Phi_\star(\mathcal{N}_0 + G_1 + G_2 + G_3) = \mathcal{N} + \mathcal{P} = \mathcal{N} + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 \tag{7.20}$$

where the normal form $\mathcal{N}$ is as in (4.1) with frequencies (satisfying (4.2)) as in (4.14)

$$\begin{aligned} \omega_j(\xi) &= \lambda_j + \lambda_j^{-1}\left(-\frac{1}{4}\vec{a}_{|j|}\xi_{|j|} + \vec{a}\cdot\xi\right), \forall j \in \mathcal{I}, \\ \Omega_j(\xi) &= \lambda_j + \lambda_j^{-1}\,\vec{a}\cdot\xi\,, \forall j \notin \mathcal{I}, \end{aligned} \tag{7.21}$$

$$\vec{a} := \sum\nolimits_{1\leqq l\leqq 3}\kappa_l\vec{a}^{(l)} \in \mathbb{R}^{n/2},\ \vec{a}_i^{(1)} := -\lambda_i^{-2},$$
$$\vec{a}_i^{(2)} := -i^2\lambda_i^{-2},\ \vec{a}_i^{(3)} := -1,\ \forall i \in \mathcal{I}^+. \tag{7.22}$$

Moreover the three terms of the perturbation are

$$\begin{aligned} \mathcal{P}_1^{(x_j)} &:= \frac{1}{\lambda_j}\left(-\frac{1}{4}\vec{a}_{|j|}\cdot y_j + \frac{1}{2}(\vec{a},\vec{a})y\right),\ \mathcal{P}_1^{(y_j)} = 0,\ j \in \mathcal{I}, \\ \mathcal{P}_1^{(z_j^\sigma)} &:= -\frac{\sigma\mathrm{i}}{2\lambda_j}(\vec{a},\vec{a})\cdot y\, z_j^\sigma\,,\ \sigma = \pm,\ j \notin \mathcal{I}, \\ \mathcal{P}_2 &:= \Phi_\star G_2\ \ \text{(note that } \mathcal{P}_2^{(x)} = \mathcal{P}_2^{(y)} = 0\,,\ \mathcal{P}_2^{(z_j^\pm)} = G_2^{(u_j^\pm)},\ j \notin \mathcal{I}\text{)}, \\ \mathcal{P}_3 &:= \Phi_\star G_3\,. \end{aligned} \tag{7.23}$$

As in (4.6) we decompose the perturbation

$$\begin{aligned} \mathcal{P} &= \mathcal{P}^y(x;\xi)\partial_y + \mathcal{P}_*\,,\quad \mathcal{P}^y(x;\xi)\partial_y := \Pi^{(-1)}\mathcal{P}^{(y)}\partial_y = \Pi^{(-1)}\mathcal{P}_3^{(y)}\partial_y \\ &= \mathcal{P}_3^{(y)}(x,0,0,0;\xi)\partial_y. \end{aligned} \tag{7.24}$$

Lemma 7.5. *Let* $s, r > 0$ *as in* (7.17) *and* N *large enough (with respect to* $\mathtt{m}, \mathcal{I}, L, b$*). Then*

$$\|\mathcal{P}^y\partial_y\|_{s,r,\mathtt{a}/2,\mathcal{O}}^\lambda \lessdot (1+\lambda/\rho)R^6r^{-2}\,,\qquad \|\mathcal{P}_*\|_{\vec{p}}^T \lessdot (1+\lambda/\rho)(r^2+R^5r^{-1})\,, \tag{7.25}$$

where

$$\mathcal{O} = \mathcal{O}(\rho) := \left\{\xi \in \mathbb{R}^n\ :\ 2\rho/3 \leqq \xi_l \leqq 3\rho/4\,,\ \ l = 1,\ldots,n\right\} \subset \mathcal{O}_\rho \tag{7.26}$$

(*the set* $\mathcal{O}_\rho$ *was defined in* (7.16)) *and* $\vec{p} := (s, r, \mathtt{a}/2, N, 2, 2, \lambda, \mathcal{O})$.

Proof. By the definition (7.24) we have

$$\|\mathcal{P}^y\partial_y\|_{s,r,\mathtt{a}/2,\mathcal{O}_\rho} = \|\Pi^{(-1)}\mathcal{P}_3^{(y)}\partial_y\|_{s,r,\mathtt{a}/2,\mathcal{O}_\rho} \overset{\text{Lemma 2.2}}{\leqq} \|\mathcal{P}_3^{(y)}\partial_y\|_{s,r,\mathtt{a}/2,\mathcal{O}_\rho} \overset{(7.19),(7.23)}{\lessdot} \left(\frac{r}{R}\right)^{-2}\|G_3\|^T_{R/2,\mathtt{a}/2,N,7/4,3} \overset{(7.13)}{\lessdot} \frac{R^6}{r^2} \tag{7.27}$$

(applying (7.19) with $d \rightsquigarrow 0$, $N_0 \rightsquigarrow N$, $\theta \rightsquigarrow 7/4$, $\mu \rightsquigarrow 2$, $\mu' \rightsquigarrow 3$) and taking N large enough so that (7.18) holds and $N \geqq N_0'$ defined in Proposition 7.1. By (7.20), (7.23) and (7.24) we write

$$\mathcal{P}_* = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_4 + \mathcal{P}_5 \qquad \text{where} \tag{7.28}$$

$$\begin{aligned}\mathcal{P}_4 &:= \mathcal{P}_3(x,y,z,\bar z;\xi) - \mathcal{P}_3(x,y,0,0;\xi),\\ \mathcal{P}_5 &:= \mathcal{P}_3(x,y,0,0;\xi) - \mathcal{P}_3^{(y)}(x,0,0,0;\xi)\partial_y\,.\end{aligned}$$

We claim that

$$\|\mathcal{P}_1\|^T_{s,r,\mathtt{a}/2,N,2,2,\mathcal{O}_\rho},\quad \|\mathcal{P}_2\|^T_{s,r,\mathtt{a}/2,N,2,2,\mathcal{O}_\rho} \lessdot r^2\,. \tag{7.29}$$

Indeed the estimate on $\mathcal{P}_1$ follows since $\mathcal{P}_1$ is Töplitz and $\|\mathcal{P}_1\|_{s,r,\mathtt{a}/2,\mathcal{O}_\rho} \lessdot r^2$ by (7.23). On the other hand the estimate on $\mathcal{P}_2$ follows by (7.23) and (7.11) with $N \geqq N_0$ large enough to fulfill (3.1).

By (7.23) and (7.19) (with $d \rightsquigarrow 1$, $N_0 \rightsquigarrow N$, $\mu \rightsquigarrow 2$, $\mu' \rightsquigarrow 3$), for N large enough, we get

$$\|\mathcal{P}_4\|^T_{s,r,\mathtt{a}/2,N,2,2,\mathcal{O}_\rho} \lessdot \left(\frac{r}{R}\right)^{-1}\|G_3\|^T_{R/2,\mathtt{a}/2,N_0',7/4,3} \overset{(7.13)}{\lessdot} \left(\frac{r}{R}\right)^{-1} R^4 = \frac{R^5}{r}\,. \tag{7.30}$$

Since $\mathcal{P}_5$ does not depend on the variables $z^\pm$ we get

$$\begin{aligned}\|\mathcal{P}_5\|^T_{s,r,\mathtt{a}/2,N,2,2,\mathcal{O}_\rho} &= \|\mathcal{P}_5\|_{s,r,\mathtt{a}/2,\mathcal{O}_\rho} \overset{\text{Lemma7.4}}{\lessdot} \left(\frac{r}{R}\right)^{-1}\|G_3\|_{R/2,\mathtt{a}/2}\\ &\overset{(7.13)}{\lessdot} \left(\frac{r}{R}\right)^{-1} R^4 = \frac{R^5}{r}\,.\end{aligned} \tag{7.31}$$

In conclusion, by (7.28), (7.29), (7.30), (7.31) we get $\|\mathcal{P}_*\|^T_{s,r,\mathtt{a}/2,N,2,2,\mathcal{O}_\rho} \lessdot r^2 + R^5 r^{-1}$. In order to prove the estimates (7.25) we have to prove Lipschitz estimates (see (2.20), (3.26)). We first note that the vector fields $\mathcal{P}^y\partial_y$ and $\mathcal{P}_*$ are *analytic* in the parameters $\xi \in \mathcal{O}_\rho$. Then we apply Cauchy estimates in the subdomain $\mathcal{O} = \mathcal{O}(\rho) \subset \mathcal{O}_\rho$ (see (7.26)), noting that $\rho \lessdot \mathrm{dist}(\mathcal{O}, \partial\mathcal{O}_\rho)$. Then

$$\|\mathcal{P}_*\|^{\mathrm{lip}}_{s,r,\mathtt{a}/2,\mathcal{O}} \lessdot \rho^{-1}\|\mathcal{P}_*\|_{s,r,\mathtt{a}/2,\mathcal{O}_\rho} \quad \text{and} \quad \|\mathcal{P}^y\partial_y\|^{\mathrm{lip}}_{s,r,\mathtt{a}/2,\mathcal{O}} \lessdot \rho^{-1}\|\mathcal{P}^y\partial_y\|_{s,r,\mathtt{a}/2,\mathcal{O}_\rho}$$

and (7.25) is proved. □

We now verify that the assumptions of Theorems 4.1–4.2 are fulfilled by $\mathcal{N}+\mathcal{P}$ in (7.20) with parameters $\xi \in \mathcal{O}(\rho)$ defined in (7.26). Note that the sets $\mathcal{O} = [\rho/2, \rho]^n$ defined in Theorem 4.2 and $\mathcal{O}(\rho)$ defined in (7.26) are diffeomorphic through $\xi_i \mapsto (7\rho + 2\xi_i)/12$. The frequency $\vec{\omega}$ (recall (4.7)) defined in (7.21) has the form $\vec{\omega} = \bar{\omega} + A\xi$ in (4.14) with

$$A := \sum\nolimits_{1\leqq l\leqq 3}\kappa_l A^{(l)}, \quad A^{(l)} := \left(\mathrm{diag}_{i\in\mathcal{I}^+}\lambda_i^{-1}\right)\left(-\frac{1}{4}\mathrm{Id}_{n/2} + \mathbf{1}_{n/2}\right)\left(\mathrm{diag}_{i\in\mathcal{I}^+}\vec{a}_i^{(l)}\right), \tag{7.32}$$

denoting by $\mathbf{1}_{n/2}$ the $(n/2)\times(n/2)$ matrix with all entries equal to 1. Then $\vec{\omega}$ and Ω_j, defined in (7.21) satisfy (4.14) and hypotheses (A1)–(A2) follow. Moreover (A3)-(A4) and the quantitative bound (4.8) follow by (7.25), choosing

$$\begin{aligned} &s = 1,\ r = R^{1+\frac{3}{4}},\ \rho = C_* R^2 \text{ as in (7.17)}, \quad N \text{ as in Lemma 7.5},\ \theta = 2, \\ &\quad \mu = 2,\ \gamma = R^{3+\frac{1}{5}} \end{aligned} \tag{7.33}$$

and taking R *small enough*. Hence Theorem 4.1 applies.

Let us verify that also the assumptions of Theorem 4.2 are fulfilled. Since $\mathbf{1}_{n/2}^2 = (n/2)\mathbf{1}_{n/2}$ by (7.32) we get that the matrix A is invertible with

$$A^{-1} := \left(\mathrm{diag}_{i\in\mathcal{I}^+} 1/\vec{a}_i\right)\left(-4\mathrm{Id}_{n/2} + \frac{16}{2n-1}\mathbf{1}_{n/2}\right)\left(\mathrm{diag}_{i\in\mathcal{I}^+}\lambda_i\right), \tag{7.34}$$

for all $\kappa_1, \kappa_2, \kappa_3$ such that $\vec{a}_i \overset{(7.22)}{:=} \sum_{1\leqq l\leqq 3}\kappa_l \vec{a}_i^{(l)} = -(\kappa_1 + \kappa_2 i^2 + \kappa_3\lambda_i^2)\lambda_i^{-2} \neq 0, \forall i \in \mathcal{I}^+$, see (1.12). Moreover, by (7.22) and (7.34) we have $(A^T)^{-1}\vec{a} = 4\bar{\omega}/(2n-1)$ and then condition (4.15) is equivalent to (1.13) (note that $(A^T)^{-1}\vec{a}$ does *not* depend on $\kappa_1, \kappa_2, \kappa_3$).

Finally we deduce that the Cantor set of parameters $\mathcal{O}_\infty \subset \mathcal{O}$ in (4.11) has asymptotically full density because

$$\frac{|\mathcal{O}\setminus\mathcal{O}_\infty|}{|\mathcal{O}|} \overset{(4.16)}{\lessdot} \rho^{-1}\gamma^{2/3} \overset{(7.33)}{\lessdot} R^{-2}R^{\frac{2}{3}(3+\frac{1}{5})} = R^{\frac{2}{15}} \to 0\,.$$

The proof of Theorem 1.1 is now completed.

Remark 7.1. If $g = g^{(=3)} + g^{(\geqq 4)}$ (unlike (1.10)) then $\|G_3\|^T_{R/2,\mathtt{a}/2,N_0',7/4,3} \lessdot R^3$ which does not fit the smallness condition of Theorem 4.1. The term of order four should be removed by a further step of Birkhoff normal form. If the term $g^{(=3)}$ depends on the space variable $\mathtt{x}$ nothing changes except to check the twist condition, see (7.21), (7.34). For simplicity, we did not pursue these points.

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