# Metric stability of the planetary N-body problem

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**Abstract.** The "solution" of the N-body problem (NBP) has challenged astronomers and mathematicians for centuries. In particular, the "metric stability" (i.e., stability in a suitable measure theoretical sense) of the planetary NBP is a formidable achievement in this subject completing an intricate path paved by mathematical milestones (by Newton, Weierstrass, Lindstedt, Poincarè, Birkhoff, Siegel, Kolmogorov, Moser, Arnold, Herman,...). In 1963 V.I. Arnold gave the following formulation of the metric stabiliy of the planetary problem:

*If the masses of n planets are sufficiently small in comparison with the mass of the central body, the motion is conditionally periodic for the majority of initial conditions for which the eccentricities and inclinations of the Kepler ellipses are small.*

Arnold gave a proof of this statement in a particular case (2 planets in a plane) and outlined a strategy (turned out to be controversial) for the general case. Only in 2004 J. Féjoz, completing work by M.R. Herman, published the first proof of Arnold's statement following a different approach using a "first order KAM theory" (developed by Rüssmann, Herman et al., and based on weaker non-degeneracy conditions) and removing certain secular degeneracies by the aid of an auxiliary fictitious system. Arnold's more direct and powerful strategy – including proof of torsion, Birkhoff normal forms, explicit measure estimates – has been completed in 2011 by the authors introducing new symplectic coordinates, which allow, after a proper symplectic reduction of the phase space, a direct check of classical non–degeneracy conditions.

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## 1. Introduction

On July 5th, 1687 Sir Isaac Newton published his *Philosophiae Naturalis Principia Mathematica*, one of the most influential book in the history of modern science. The main impulse for its publication came from Edmond Halley, who urged Newton to write the mathematical solution of the two–body (Kepler) problem.

In general, the N–body problem (NBP) consists in determining the motion of $N \geq 2$ point–masses (i.e., ideal bodies with no physical dimensions identified with points in the Euclidean three–dimensional space) interacting only through Newton's law of gravitational attraction.

After his complete mathematical description of the general solution for the two body case, Newton immediately turned to the three–body problem (Sun, Earth and Moon) but got

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discouraged, describing it as a “head–aching problem”. The immense difficulty in trying to obtain explicitly the general solution of the NBP (something that, later, was proved to be impossible) drove, then, mathematicians to focus on the issue of convergence of formal power series for solutions of the planetary problem, the smallness expansion parameter being the mass ratio between planets and Sun. Many eminent personalities in the mid 1800’s, such as Weierstrass and Dirichlet (who claimed to have a proof, which was never found), were convinced that the series were convergent. The question become a major mathematical issue and King Oscar II of Sweden and Norway, enlightened ruler, issued, in 1885, a prize for solving the problem or, in absence of a complete solution, for the best contribution. The prize was finally awarded on the occasion of the king’s 60th birthday (21 January, 1889) to Henri Poincaré[1] , who came to the belief (albeit not to the proof) that the series were divergent. The convergence problem was exported into a more general (and less degenerate) setting, namely, perturbation theory for non–degenerate nearly–integrable Hamiltonian systems. The breakthrough came in 1954 at the Amsterdam ICM, where N.N. Kolmogorov announced and gave a sketchy proof of his theorem on the preservation of (maximal) quasi–periodic motions[2] in nearly–integrable systems. In his amazing 6–page long article [22] Kolmogorov set the foundation of KAM (Kolmogorov–Arnold–Moser) theory, outlining a (super–exponentially) convergent perturbation theory for real–analytic systems, able to deal with the small divisor problems arising in the formal solutions of quasi–periodic motions: one of the crucial (and ingenious) idea was to fix the frequencies of the final motions rather than initial data[3] . With additions by Moser and Arnold, Kolmogorov’s strategy could be used to show, indirectly[4] , convergence of the formal (Lindstedt) series for “general” solutions, where “general” means that the phase space region corresponding to (linearly) stable quasi–periodic motions tends to fill a Cantor set of asymptotic measure density equal to one (as the smallness parameter goes to zero). Thus, a way of rephrasing the main outcome of KAM theory is that analytic nearly–integrable (non–degenerate) Hamiltonian systems are asymptotically metrically stable.

However, in view of the strong degeneracies of the Kepler problem (i.e., of the integrable limit of the planetary NBP), the main hypothesis of Kolmogorov’s theorem did not apply to the planetary problem. Besides the real–analyticity assumption, the main hypothesis of Kolmogorov’s theorem is that the limit integrable Hamiltonian depends only on $d$ action variables, $d$ being the number of degrees of freedom (:= half of phase–space dimension) and that its gradient map is a local diffeomorphism. In the planetary problem the integrable limit depends only on $n$ actions while the number of degrees of freedom (after reducing the total linear momentum; see below) is $3n$.

In 1963 Arnold, 26, took up the question of extending Kolmogorov’s theorem to systems modeling the main features of the planetary problem, namely, Hamiltonian systems with $n+m$ degrees of freedom, whose integrable limit depends only on $n$ action variables[5]

[1] At first Poincarè submitted a contribution containing a serious mistake, which he amended in a feverish effort: the outcome was the famous 270 page memoir [25], by now, regarded as the birth of modern theory of dynamical systems and chaos; compare [3].

[2] In general, a “quasi–periodic” (or “conditionally periodic”) orbit with (rationally independent) frequencies $(\omega_1, ..., \omega_d) = \omega \in \mathbb{R}^d$ is a trajectory conjugated to a linear flow, $\theta \to \theta + \omega t$ on a $d$ dimensional torus; if $d$ equals the number of degrees of freedom (i.e., half dimension of the pahse space), the quasi–periodic orbit is called maximal.

[3] For generalities on KAM theory, see, e.g., [2] or [6].

[4] Direct proofs of convergence of Lindstedt series came much later; see [8, 16, 19].

(which, in the planetary problem, are the square roots of the semimajor axes of the decoupled 2BP planet–Sun). This implies that the $n$ conjugated angles (the mean anomalies of the 2BP's, in the planetary problem) are *fast angles*, bringing naturally in play averaging theory, according to which the leading dynamics is governed by the average of the Hamiltonian over the fast angles; the resulting Hamiltonian is thus the sum of the integrable limit and the average over the fast angles of the perturbation function (the "secular Hamiltonian"). Now, what happens in the planetary problem is that the secular Hamiltonian has an *elliptic equilibrium* in the origin of the remaining $2m$ symplectic variables, corresponding physically to circular orbits revolving in the same plane. Arnold formulated and gave a detailed proof of a generalization of Kolmogorov's theorem working for properly–degenerate systems with secular Hamiltonian possessing an elliptic equilibrium; he called such theorem the "Fundamental Theorem". The non–degeneracy hypotheses involve, now, not only the integrable limit (which, as in Kolmogorov's theorem, is assumed to define through the gradient map an $n$–diffeomorphism), but also the *Birkhoff normal form*[6] ("BNF" for short) of the secular $2m$ variables, and in particular the first order Birkhoff invariants (the eigenvalues associated to the elliptic equilibrium) and the second order invariants, which may be viewed as an $(m \times m)$ matrix. The "full" *torsion* (or "twist") hypothesis is guaranteed if such matrix is non–singular. After giving the (long and beautiful) proof of his Fundamental Theorem, Arnold checks the torsion hypothesis in the simpler non–trivial case, namely, 2 planets constrained on a plane. He then discusses how to generalize first to the planar case with $n$ planets, and, from there, to the spacial general case[7] .

However, various serious problems prevented, for long time, to carry over Arnold's strategy. In first place, the standard hypotheses for constructing the BNF is that the first order Birkhoff invariants are non–resonant (i.e., do not have vanishing non–trivial integer coefficient linear combinations) up to a certain order. But indeed, besides a well know resonance related to rotation invariance, which Arnold was aware of, a second rather mysterious resonance was discovered by Herman in the 1990's, namely, that the sum of the first order Birkhoff invariants, in the general spatial case, vanishes identically; such resonance is now known as "Herman resonance". A second and more important problem is related to the torsion hypothesis. Indeed, in the full $6n$ dimensional phase space, the planetary Hamiltonian has an identically vanishing torsion (a fact, proved only recently in [12], ignored by Arnold and only suspected by Herman, compare [20]). Finally, there is a rather vague suggestion by Arnold to check non–degeneracies "bifurcating" from the planar problem, i.e., viewing the planar problem as a limit of the spacial one, which is a fact hard to justify analytically.

Herman's approach is rather different. After convincing himself that in the spatial case there might be a serious torsion problem, he turned to a different KAM technique, based on a different and somewhat weaker non–degeneracy condition, a condition which involves only the first order Birkhoff invariants and the gradient map of the limiting integrable Hamiltonian. Such condition is that the first order Birkhoff invariants – which are parameterized by the semimajor axes – do not lie identically in a fixed plane ("non–planarity" condition). However, as mentioned above, this is not true in the planetary problem since the invariants lie in the intersection of two planes corresponding to the rotational and the Herman's resonances. To overcome this problem, following a trick introduced by Poincaré, Herman

[5] Such systems are sometimes called "properly–degenerate".

[6] For generalities on Birkhoff normal form theory, see [21]; for a Birkhoff normal form theory adapted to the NBP, see Proposition B.1 below.

[7] In Appendix C we report *verbatim*, some of Arnold's claims and suggestions as given in [1].

modifies the planetary Hamiltonian by adding a term proportional to a function Poisson–commuting with the planetary Hamiltonian; he manages to do that so that the modified Hamiltonian is non–degenerate (i.e., the modified Birkhoff invariants are non–planar). Now, by an abstract argument, two Poisson–commuting Hamiltonians have the same Lagrangian transitive invariant tori, therefore the invariant tori gotten by applying the weaker KAM theory to the modified Hamiltonian are invariant also for the planetary problem[8] . This scheme was worked out, clarified and published by Jacques Féjoz in [17]; see also [18].

Finally, in 2011, the original strategy of Arnold has been reconsidered, from a different point of view, in the paper[9] [11], where, thanks to new symplectic coordinates (called RPS for RegularizedPlanetarySymplectic), it is proven that *in a "partially reduced setting" the planetary problem has indeed non–vanishing torsion.* Recall that the "natural" phase space (after linear momentum reduction) of the planetary $(1+n)$–body problem is $6n$–dimensional and that standard symplectic coordinates are given by Poincaré variables; this setting has been used by Arnold (with minor modifications) and by Herman and Féjoz. In this setting the planetary Hamiltonian is still rotation invariant and admits, therefore, besides energy, other three global analytic integrals, which are the three components of the total angular momentum. Now, while in three dimensions it is customary to use the celebrated Jacobi's classical reduction of the nodes[10] in higher dimensions the reduction of the nodes is not so popular, even though it was knonw since the early 1980's thanks to the work of Deprit [15]. In [11], (an action–angle version of) Deprit variables replace Delaunay variables and, after a Poincaré regularization, one is lead to the new RPS variables. A main feature of these variables is that one symplectic couple of the secular cartesian variables (related to the inclination of the total angular momentum), say $(p_n, q_n)$ are *both* cyclic coordinates (i.e., invariants), which means that the planetary Hamiltonian in such coordinates *does not depend on this couple of variables.* The significance of this fact is that *the phase space is foliated by* $(6n-2)$*–dimensional symplectic submanifold* $\{(p_n, q_n) = \mathrm{const}\}$ *on which the planetary Hamiltonian has the same form.* In this partially reduced[11] setting the original Arnold's strategy can be carried out, torsion explicitely checked and all its dynamical consequences drawn: All this will be described below.

## 2. The classical Hamiltonian of the planetary NBP

In this section (and in Appendix A) we review the classical Hamiltonian description of the planetary NBP due, essentially, to Delaunay and Poincaré.

Newton's equations for $1+n$ bodies (point masses), which interact only through gravi-

[8] However, besides not having information about the normal form around the tori of the original Hamiltonian (which is intrinsic in this first order KAM theory), this abstract argument does not allow to read back the KAM structure in the unmodified setting.

[9] This paper is based on the PhD thesis [23].

[10] For a symplectic description of Jacobi's reduction of the nodes, see [4].

[11] Indeed, in these $(6n-2)$–symplectic submanifold, the planetary Hamiltonian still admits an energy–commuting integral, namely the Euclidean length of the total angular momentum. It is possible (and done in [11]) to further reduce to a fully rotationally reduced $(6n-4)$–dimensional phase space, however in such totally reduced setting many symmetries and nice feature shared by Poincaré and RPS variables (such as D'Alembert rules, parities in the secular variables, etc.) are lost and the symplectic description becomes somewhat more clumsy.

tational attraction, are given by:

$$\ddot{u}^{(i)} = \sum_{\substack{0\le j\le n\\ j\neq i}} \mathrm{m}_j \frac{u^{(j)}-u^{(i)}}{|u^{(i)}-u^{(j)}|^3}\ , \qquad i=0,1,...,n\ , \tag{2.1}$$

where $u^{(i)} = \big(u_1^{(i)}, u_2^{(i)}, u_3^{(i)}\big) \in \mathbb{R}^3$ are the cartesian coordinates of the $i^{\rm th}$ body of mass $\mathrm{m}_i > 0$, $|u| = \sqrt{u\cdot u} = \sqrt{\sum_i u_i^2}$ is the standard Euclidean norm, “dots" over functions denote time derivatives, and the gravitational constant has been set to one (which is possible by rescaling time $t$). These equations are equivalent to the (standard) Hamilton equations associated to the Hamiltonian function[12]

$$\widehat{\mathcal{H}}_{\rm N} := \sum_{i=0}^{n} \frac{|U^{(i)}|^2}{2\mathrm{m}_i} - \sum_{0\le i<j\le n} \frac{\mathrm{m}_i \mathrm{m}_j}{|u^{(i)}-u^{(j)}|}\ ,$$

where $(U^{(i)}, u^{(i)})$ are standard symplectic variables ($U^{(i)} = \mathrm{m}_i \dot{u}^{(i)}$ is the momentum conjugated to $u^{(i)}$) and the phase space is the “collisionless" open domain in $\mathbb{R}^{6(n+1)}$ given by

$$\widehat{\mathcal{M}} := \{U^{(i)}, u^{(i)} \in \mathbb{R}^3 :\ u^{(i)} \neq u^{(j)}\ ,\ 0 \le i \neq j \le n\} \tag{2.2}$$

endowed with the standard symplectic form

$$\sum_{i=0}^{n} dU^{(i)} \wedge du^{(i)} := \sum_{\substack{0\le i\le n\\ 1\le k\le 3}} dU_k^{(i)} \wedge du_k^{(i)}\ . \tag{2.3}$$

Exploiting the invariance of Newton’s equation by change of inertial frames, or, equivalently, the existence of the vector–valued integral[13] given by the total linear momentum $\sum_{i=0}^{n} U^{(i)}$, Poincaré showed how to make a “symplectic reduction” lowering by three units the number of degrees of freedom. Indeed, the dynamics generated by $\widehat{\mathcal{H}}_{\rm N}$ on $\widehat{\mathcal{M}}$ is equivalent to the dynamics on

$$\mathcal{M} := \Big\{(X,x) = (X^{(1)},...,X^{(n)},x^{(1)},...,x^{(n)}) \in \mathbb{R}^{6n} : 0 \neq x^{(i)} \neq x^{(j)}\ , \forall\, i\neq j\Big\},$$

(endowed with the standard symplectic form $\sum_{i=1}^{n} dX^{(i)} \wedge dx^{(i)}$) by the Hamiltonian

$$\begin{aligned} \mathcal{H}_{\rm plt}(X,x) \ &:= \ \sum_{i=1}^{n} \frac{|X^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|} + \mu \sum_{1\le i<j\le n} \frac{X^{(i)}\cdot X^{(j)}}{m_0} - \frac{m_i m_j}{|x^{(i)}-x^{(j)}|} \\ &=: \ \mathcal{H}_{\rm plt}^{(0)}(X,x) + \mu \mathcal{H}_{\rm plt}^{(1)}(X,x)\ , \end{aligned} \tag{2.4}$$

where the mass of the Sun is[14] $\mathrm{m}_0 = m_0$ and the mass of the planets are $\mathrm{m}_i = \mu m_i$ ($1 \le i \le n$), $\mu$ being a small parameter, while $M_i := \frac{m_0 m_i}{m_0+\mu m_i}$ and $\bar{m}_i := m_0 + \mu m_i$. In

[12] I.e., the equations $\dot{U}_j^{(i)} = -\partial_{u_j^{(i)}} \widehat{\mathcal{H}}_{\rm N}$, $\dot{u}_j^{(i)} = \partial_{U_j^{(i)}} \widehat{\mathcal{H}}_{\rm N}$, $0\le i\le n$, $1\le j\le 3$; for general information on Hamiltonian systems, see, e.g., [2].

[13] Recall that $F(X,x)$ is an integral for $\mathcal{H}(X,x)$ if $\{F,\mathcal{H}\}=0$ where $\{F,G\} = F_X\cdot G_x - F_x \cdot G_X$ denotes the (standard) Poisson bracket; in particular an integral $F$ for $\mathcal{H}$ is constant for the $\mathcal{H}$ flow, i.e., $F\circ \phi_{\mathcal{H}}^t \equiv$ const., where $\phi_{\mathcal{H}}^t$ denotes the Hamiltonian flow generated by $\mathcal{H}$.

[14] Note the different character: upright for unscaled and italic for rescaled masses.

such description $\mathcal{M}$ corresponds to the (symplectic) submanifold of $\widehat{\mathcal{M}}$ of zero total linear momentum and zero total center of mass and $x^{(i)} = u^{(i)} - u^{(0)}$, for $i \geq 1$, are heliocentric coordinates; full details are given in Appendix A.

Obviously, in such variables, there is no more a conserved total linear momentum[15] , however, the system is *still invariant under rotations* and the total angular momentum

$$\mathrm{C} = (\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3) := \sum_{i=1}^{n} \mathrm{C}^{(i)} , \qquad \mathrm{C}^{(i)} := x^{(i)} \times X^{(i)} , \tag{2.5}$$

is still a (vector–valued) integral for $\mathcal{H}_{\rm plt}$. The integrals $\mathrm{C}_i$, however, do not commute (i.e., their Poisson brackets do not vanish[16] ) but, for example, $|\mathrm{C}|$ and $\mathrm{C}_3$ are two commuting, independent integrals, a remark that will be crucial in what follows.

Next, by regularizing the Delaunay action–angle coordinates for the $n$ decoupled two–body problems with Hamiltonian $\mathcal{H}^{(0)}_{\rm plt}$ in a neighborhood of co–circular and co–planar motions, Poincaré brings out in a neat way the nearly–integrable structure of planetary NBP. The real–analytic symplectic variables doing the job are usually known as *Poincaré variables*: in such variables the Hamiltonian $\mathcal{H}_{\rm plt}(X, x)$ takes the form

$$\mathcal{H}_{\rm P}(\Lambda, \lambda, \mathrm{z}) = h_{\rm K}(\Lambda) + \mu f_{\rm P}(\Lambda, \lambda, \mathrm{z}) \ , \ \ (\Lambda, \lambda) \in \mathbb{R}^n_+ \times \mathbb{T}^n \ , \ \mathrm{z} := (\eta, \mathrm{p}, \xi, \mathrm{q}) \in \mathbb{R}^{4n} \tag{2.6}$$

where the "Kepler" unperturbed term $h_{\rm K}$ is given by

$$h_{\rm K}(\Lambda) := -\sum_{i=1}^{n} \frac{M_i^3 \bar m_i^2}{2\Lambda_i^2} , \qquad \Lambda_i := M_i \sqrt{\bar m_i a_i}, \tag{2.7}$$

$a_i$ being the semimajor axis of the instantaneous two–body system formed by the $i^{\rm th}$ planet and the Sun; as phase space, we consider a collisionless domain around the "secular origin" $\mathrm{z} = 0$ (which corresponds to co–planar, co–circular motions) of the form

$$(\Lambda, \lambda, \mathrm{z}) = (\Lambda, \lambda, \eta, \mathrm{p}, \xi, \mathrm{q}) \in \mathcal{M}^{6n}_{\rm P} := \mathcal{A} \times \mathbb{T}^n \times B^{4n} \tag{2.8}$$

endowed with the symplectic form $\displaystyle\sum_{i=1}^{n} d\Lambda_i \wedge \lambda_i + \sum_{i=1}^{n} \eta_i \wedge d\xi_i + \sum_{i=1}^{n} d\mathrm{p}_i \wedge d\mathrm{q}_i$; $\mathcal{A}$ is a set of "well separated" semimajor axes

$$\mathcal{A} := \big\{\Lambda : \ \underline{a}_j < a_j < \overline{a}_j \quad \text{for} \quad 1 \leq j \leq n\big\} \tag{2.9}$$

where $\underline{a}_1, \cdots, \underline{a}_n, \overline{a}_1, \cdots, \overline{a}_n$, are positive numbers verifying $\underline{a}_j < \overline{a}_j < \underline{a}_{j+1}$ for any $1 \leq j \leq n$, $\overline{a}_{n+1} := \infty$, and $B^{4n}$ is a $4n$–dimensional ball around the secular origin $\mathrm{z} = 0$. A complete description of Delaunay and Poincaré variables is given in Appendix A.
Here, let us point out that the Hamiltonian (2.4) retains rotation and reflection invariance and, in particular, invariance by rotation with respect the $k^{(3)}$–axis and invariance by reflection with respect to the coordinate planes. This implies that the perturbation $f_{\rm P}$ in (2.6) satisfies (classical) symmetry relations known as *d'Alembert rules*, which are given by the following

[15] In particular, $\sum_{i=1}^{n} X^{(i)}$ is *not* an integral for $\mathcal{H}_{\rm plt}$
[16] Indeed, $\{\mathrm{C}_1, \mathrm{C}_2\} = \mathrm{C}_3$, $\{\mathrm{C}_2, \mathrm{C}_3\} = \mathrm{C}_1$ and$\{\mathrm{C}_3, \mathrm{C}_1\} = \mathrm{C}_2$.

transformations:

$$\begin{cases} (\eta,\xi,\mathrm{p},\mathrm{q}) \to (-\xi,-\eta,\mathrm{q},\mathrm{p})\,, & (\Lambda,\lambda) \to \Big(\Lambda, \frac{\pi}{2}-\lambda\Big) \\ (\eta,\xi,\mathrm{p},\mathrm{q}) \to (\eta,\xi,-\mathrm{p},-\mathrm{q})\,, & (\Lambda,\lambda) \to (\Lambda,\lambda) \\ (\eta,\xi,\mathrm{p},\mathrm{q}) \to (-\eta,\xi,\mathrm{p},-\mathrm{q})\,, & (\Lambda,\lambda) \to (\Lambda,\pi-\lambda) \\ (\eta,\xi,\mathrm{p},\mathrm{q}) \to (\eta,-\xi,-\mathrm{p},\mathrm{q})\,, & (\Lambda,\lambda) \to (\Lambda,-\lambda) \\ (\Lambda,\lambda,\mathrm{z}) \to (\Lambda,\lambda_1+g,\ldots,\lambda_n+g,\mathcal{S}^g\mathrm{z}) \end{cases} \tag{2.10}$$

where, for any $g \in \mathbb{T}$, $\mathcal{S}^g$ acts as synchronous clock–wise rotation by the angle $g$ in the symplectic $\mathrm{z}_i$–planes:

$$\mathcal{S}^g : \mathrm{z} \to \mathcal{S}^g\mathrm{z} = \Big(\mathcal{S}_g\mathrm{z}_1, ..., \mathcal{S}_g\mathrm{z}_{2n}\Big)\,, \qquad \mathcal{S}_g := \begin{pmatrix} \cos g & \sin g \\ -\sin g & \cos g \end{pmatrix}\;; \tag{2.11}$$

compare (3.26)–(3.31) in [12]. By such symmetries, in particular, the averaged perturbation

$$f_{\mathrm{P}}^{\mathrm{av}}(\Lambda,\mathrm{z}) := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f_{\mathrm{P}}(\Lambda,\lambda,\mathrm{z})d\lambda\,, \tag{2.12}$$

which is called the *secular Hamiltonian*, is even in $z$ around the origin $\mathrm{z}=0$ and its expansion in powers of z has the form

$$f_{\mathrm{P}}^{\mathrm{av}} = C_0(\Lambda) + \mathcal{Q}_h(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \mathcal{Q}_v(\Lambda)\cdot\frac{\mathrm{p}^2+\mathrm{q}^2}{2} + \mathrm{O}(|\mathrm{z}|^4)\,, \tag{2.13}$$

where $\mathcal{Q}_h$, $\mathcal{Q}_v$ are suitable quadratic forms and $\mathcal{Q}\cdot u^2$ denotes the 2–index contraction $\sum_{i,j}\mathcal{Q}_{ij}u_iu_j$ ($\mathcal{Q}_{ij}$, $u_i$ denoting, respectively, the entries of $\mathcal{Q}$, $u$). This shows that $\mathrm{z}=0$ *is an elliptic equilibrium for the secular dynamics* (i.e, the dynamics generated by $f_{\mathrm{P}}^{\mathrm{av}}$). The explicit expression of such quadratic forms can be found, *e.g.* , in (36), (37) of [17] (revised version).

The *truncated averaged Hamiltonian*

$$\overline{\mathcal{H}}_{\mathrm{P}}^{\mathrm{av}}(\Lambda,\lambda,\mathrm{z}) := h_{\mathrm{K}} + \mu\Big(C_0(\Lambda) + \mathcal{Q}_h(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \mathcal{Q}_v(\Lambda)\cdot\frac{\mathrm{p}^2+\mathrm{q}^2}{2}\Big)$$

is integrable, with $3n$ commuting integrals given by

$$\Lambda_i\,, \qquad \rho_i = \frac{{\eta_i}^2+{\xi_i}^2}{2}\,, \qquad \mathrm{r}_i = \frac{{\mathrm{p}_i}^2+{\mathrm{q}_i}^2}{2}\,, \qquad (1\le i\le n)\,.$$

The general trajectory of this system fills a $3n$–dimensional torus with $n$ fast frequencies $\partial_{\Lambda_i}h_{\mathrm{K}}(\Lambda_i)$ and $2n$ slow frequencies given by

$$\mu\Omega = \mu(\sigma,\varsigma) = \mu(\sigma_1,\cdots,\sigma_n,\varsigma_1,\cdots,\varsigma_n)\,, \tag{2.14}$$

$\sigma_i$ and $\varsigma_i$ being the real eigenvalues of $\mathcal{Q}_h(\Lambda)$ and $\mathcal{Q}_v(\Lambda)$, respectively. Such tori correspond to $n$ nearly co–planar and co–circular planets rotating around the Sun with Keplerian frequencies $\partial_{\Lambda_i}h_{\mathrm{K}}(\Lambda_i)$ and with small eccentricities and inclinations slightly and slowly oscillating with frequencies $\mu\sigma$ and $\mu\varsigma$ .

A fundamental problem in the planetary NBP concerns the perturbative analysis of the integrable dynamics governed by $\overline{\mathcal{H}}_{\mathrm{P}}^{\mathrm{av}}$, when the full planetary Hamiltonian $\mathcal{H}_{\mathrm{P}}$ is considered. The main technical tool is Kolmogorov's 1954 Theorem [22] (which, incidentally, was clearly motivated by Celestial Mechanics) on the persistence under perturbation of quasi–periodic motions for nearly–integrable system with real–analytic Hamiltonian in *action–angle variables* given by

$$H_\mu(I,\varphi) := h(I) + \mu f(I,\varphi)\,, \qquad (I,\varphi) \in \mathbb{R}^d \times \mathbb{T}^d. \tag{2.15}$$

Kolmogorv's Theorem, however, holds in a neighborhoods of points $I_0$ where the integrable Hamiltonian is *non–degenerate* in the sense that $\det h''(I_0) \neq 0$, where $h''$ denotes the Hessian matrix of $h$ (equivalently, the frequency map $I \to h'(I)$ is a local diffeomorphism). This conditions is strongly violated by the planetary Hamiltonian since for $\mu = 0$ the integrable (Keplerian) limit depends only on $n$ action variables (the $\Lambda$'s), while the number of degrees of freedom is $d = 3n$. A nearly–integrable system with Hamiltonian as in (2.15) for which $h$ does not depend upon all the actions $I_1$,...,$I_d$ is called *properly–degenerate*[17] .

In the next section we recall Arnold's statement on the planetary NBP and outline his strategy of proof based on a generalization of Kolmogorov's theory to properly–degenerate system.

## 3. Arnold's theorem on the planetary NBP (1963)

In the 1963 paper [1] Arnold – probably in his deeper contribution to KAM theory and Celestial Mechanics – formulated his main result as follows ([1, p. 127]):

**Theorem 3.1.** *If the masses, eccentricities and inclinations of the planets are sufficiently small, then for the majority of initial conditions the true motion is conditionally periodic and differs little from Lagrangian motion*[18] *with suitable initial conditions throughout an infinite interval of time* $-\infty < t < +\infty$.

**Proper degeneracies and Arnold's "Fundamental Theorem".** As mentioned above, Kolmogorov opened the route to a rigorous proof of (maximal) quasi–periodic trajectories in Hamiltonian systems, but the planetary system violates drastically the main hypotheses of his theorem. This was a main challenge for his young and brilliant student Vladimir Igorevich Arnold, who at 26 gave a major impulse and draw the path which, eventually, would lead to a complete solution of the metric stability problem for the NBP.
One of the main steps – a result that in [1] Arnold called "The Fundamental Theorem" – is to extend Kolmogorov's Theorem to properly–degenerate systems, and, more specifically, to properly–degenerate systems with "secular" elliptic equilibria (or, more precisely, elliptic lower dimensional tori).
Let us proceed to formulate Arnold's Fundamental Theorem.

Let $\mathcal{M}$ denote the phase space $\mathcal{M} := \big\{(I,\varphi,p,q) :\ (I,\varphi) \in V \times \mathbb{T}^n \text{ and } (p,q) \in B\big\}$

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[17] In general, maximal quasi–periodic solutions (i.e., quasi–periodic solutions with $d$ rationally–independent frequencies) for properly–degenerate systems do not exist: trivially, any unperturbed properly–degenerate system on a $2d$ dimensional phase space with $d \geq 2$ will have motions with frequencies not rationally independent over $\mathbb{Z}^d$. But they may exist under further conditions on the perturbation $f$, as we shall see.

[18] Arnold defines the "Lagrangian motions", at p. 127 as follows: *the Lagrangian motion is conditionally periodic and to the $n$ "rapid" frequencies of the Kepler motion are added $n$ (in the planar problem) or $2n-1$ (in the space problem) "slow" frequencies of the secular motions.* This dynamics corresponds, essentially, to the above "truncated integrable planetary dynamics". The missing frequency in the space problem is because one of the spatial secular frequency, say, $\varsigma_n$ vanishes identically; compare Eq. (3.3) below.

where $V$ is an open bounded region in $\mathbb{R}^n$ and $B$ is a ball around the origin in $\mathbb{R}^{2m}$; $\mathcal{M}$ is equipped with the standard symplectic form

$$dI \wedge d\varphi + dp \wedge dq = \sum_{i=1}^{n} dI_i \wedge d\varphi_i + \sum_{i=1}^{m} dp_i \wedge dq_i \ .$$

Let, also, $H_\mu$ be a real analytic Hamiltonian on $\mathcal{M}$ of the form $H_\mu(I,\varphi,p,q) := h(I) + \mu f(I,\varphi,p,q)$, and denote by $f^{\rm av}$ the average of $f$ over the "fast angles" $\varphi$: $f^{\rm av}(I,p,q) := \int_{\mathbb{T}^n} f(I,\varphi,p,q)\frac{d\varphi}{(2\pi)^n}$.

**Theorem 3.2** ("The Fundamental Theorem"; [1]). *Assume that $f^{\rm av}$ is of the form*

$$f^{\rm av} = f_0(I) + \sum_{j=1}^{m} \Omega_j(I) r_j + \frac{1}{2}\tau(I) r \cdot r + o_4 \ , \qquad r_j := \frac{p_j^2 + q_j^2}{2} \ , \tag{3.1}$$

*where $\tau$ is a symmetric $(m \times m)$–matrix and $\lim_{(p,q)\to 0} |o_4|/|(p,q)|^4 = 0$. Assume, also, that $I_0 \in V$ is such that*

$$\det h''(I_0) \neq 0 \qquad (*)\ ; \qquad\qquad \det \tau(I_0) \neq 0 \qquad (**)\ . \tag{3.2}$$

*Then, in any neighborhood of $\{I_0\} \times \mathbb{T}^d \times \{(0,0)\} \subseteq \mathcal{M}$ there exists a positive measure set of phase points belonging to analytic "KAM tori" spanned by maximal quasi–periodic solutions with $n+m$ rationally–independent (Diophantine[19] ) frequencies, provided $\mu$ is small enough.*

Let us make some remarks.

(i) The function $f^{\rm av}$ in (3.1) is said to be in *Birkhoff normal form* (with respect to the variables $p, q$) up to order 4 (compare [21] and Appendix B below). Actually, Arnold requires that $f^{\rm av}$ is in Birkhoff normal form up to order 6 (instead of 4); but such condition can be relaxed and (3.1) is sufficient: compare [9], where Arnold's Fundamental Theorem is revisited and various improvements obtained.

(ii) Condition (3.2)–$(*)$ is immediately seen to be satisfied in the general planetary problem; the correspondence with the planetary Hamiltonian in Poincaré variables (2.6) being the following: $m = 2n$, $I = \Lambda$, $\varphi = \lambda$, $\mathrm{z} = (p,q)$, $h = h_{\rm K}$, $f = f_{\rm P}$.

(iii) Condition (3.2)–$(**)$ is a "twist" or "torsion" condition on the secular Hamiltonian. It is actually possible to develop a weaker KAM theory where no torsion is required. This theory is due to Rüssmann [27], Herman and Féjoz [17], where $f^{\rm av}$ is assumed to be in Birkhoff normal form up to order 2, $f^{\rm av} = f_0(I) + \sum_{j=1}^{m} \Omega_j(I) r_j + o_2$, and the secular frequency map $I \to \Omega(I)$ is assumed to be *non–planar*, meaning that no neighborhood of $I_0$ is mapped into an hyperplane.

(iv) The ingenious idea of Arnold in order to remove the proper degeneracy of the system goes roughly as follows. Instead of $h(I)$, consider $\hat{h}(I,r) := h(I) + \mu f_2^{\rm av}(I,r)$ as a new unperturbed part viewed as a function of the actions $(I,r)$, $f_2^{\rm av}(I,r)$ being the

[19] A vector $\omega \in \mathbb{R}^d$ is Diophantine if there exist positive constants $\gamma$ and $c$ such that $|\omega \cdot k| \geq \gamma/|k|^c$, $\forall\, k \in \mathbb{Z}^d \backslash \{0\}$.

truncation of $f^{\rm av}$ in (3.1) up to degree two in the variables $r$. By averaging theory, the original Hamiltonian can be symplectically conjugated to a new "effective" nearly–integrable system $\tilde h(I, r) + \mu^a \hat f(I, r, \varphi.\psi)$ ($(\varphi, \psi) \in \mathbb{T}^n \times \mathbb{T}^m$) with $a \in \mathbb{N}$ large enough and $\tilde h$ close to $\hat h$: this is the starting point for constructing Kolmogorov ($n+m$–dimensional) tori (note that the full torsion condition mentioned in the introduction corresponds to the Kolmogorov non–degeneracy of $\hat h$).

(v) The elliptic secular equilibrium $(p, q) = 0$ plays a fundamental rôle in this construction. The density of the tori is closer and closer to one as soon as the variables $(p, q)$ (eccentricities and inclinations, in the planetary problem) approach the origin; see also Theorem 5.3 below. Arnold however noticed that, at least in the case of the planar three–body problem, a stronger result holds: $f^{\rm av}$ is *integrable* and one can replace $f_2^{\rm av}$ with $f^{\rm av}$ in the definition of $\hat h$ (see the previous item); this yields a more global and astronomically relevant result. Indeed, the density of the tori depends *only* on $\mu$ and *not* on eccentricities and inclinations. The independence of the Kolmogorov tori from eccentricities (in such cases inclinations are not independent quantities[20] ) has been proved also for the spatial three–body case and the planar general case [24] (notwithstanding the fact that $f^{\rm av}$ is no longer integrable).

(vi) Actually, the torsion assumption (3.2)–$(**)$ implies stronger results:

– It is possible to give explicit and accurate bounds on the measure of the "Kolmogorov set", i.e., the set covered by the closure of quasi–periodic motions ([9]);

– The quasi–periodic motions found belong to a smooth family of *non–degenerate Kolmogorov tori*, which means, essentially, that the dynamics can be linearized in a neighborhood of each torus.

– The above Kolmogorov tori are cumulation sets for periodic orbits with longer and longer periods. Thus the measure of the closure of periodic orbits tends to fill a set of full measure as the distance from the secular origin $\mathrm{z} = 0$ tends to zero, showing that a "metric asymptotic" version of Poincaré's conjecture about the density of periodic orbits in phase space holds in the general planetary NBP around co–planar and co–circular motions; see [7].

On the basis of Theorem 3.2, Arnold's strategy is to compute the Birkhoff normal form (3.1) of the secular Hamiltonian $f_{\rm P}^{\rm av}$ in (2.12) and to check the non–vanishing of the torsion (3.2)–$(**)$, a program which he carried out completely only in the planar three–body case ($n = 2$).

**The planar three–body case (Arnold, 1963).** In the planar case the Poincaré variables become simply $(\Lambda, \lambda, \mathrm{z}) := (\Lambda, \lambda, \eta, \xi) \in \mathbb{R}_+^n \times \mathbb{T}^n \times \mathbb{R}^{2n}$, with the $\Lambda$'s as in (2.7) and

$$\lambda_i = \ell_i + \mathrm{g}_i\ , \qquad \left\{ \begin{array}{l} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)}\ \cos \mathrm{g}_i \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)}\ \sin \mathrm{g}_i \end{array} \right. ,$$

where, referring to the instantaneous $i^{\rm th}$ two–body system planet–Sun, $\ell_i$ is the mean anomaly, $g_i$ the argument of the perihelion and $\Gamma_i$ the absolute value of the $i^{\rm th}$ angular

[20] In the spatial three–body problem completely reduced by rotations, the mutual inclination is a function of eccentricities.

momentum (compare Appendix A for more details). The planetary, planar Hamiltonian, is given by

$$\mathcal{H}_{\text{P,pln}}(\Lambda, \lambda, \mathrm{z}) = h_{\text{K}}(\Lambda) + \mu f_{\text{P,pln}}(\Lambda, \lambda, \mathrm{z}) \ , \ \ \mathrm{z} := (\eta, \xi) \in \mathbb{R}^{2n}$$

with $\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f_{\text{P,pln}} =: f^{\text{av}}_{\text{P,pln}} = C_0(\Lambda) + \mathcal{Q}_h(\Lambda) \cdot \frac{\eta^2+\xi^2}{2} + \mathrm{O}(|\mathrm{z}|^4)$. In Eq. (3.4.31), p.138 of [1], Arnold computed the first and second order Birkhoff invariants for $n = 2$ finding, in the asymptotics $a_1 \ll a_2$:

$$\begin{cases} \Omega_1 = -\dfrac{3}{4} m_1 m_2 \Big(\dfrac{a_1}{a_2}\Big)^2 \dfrac{1}{a_2 \Lambda_1}\Big(1 + \mathrm{O}\Big(\dfrac{a_1}{a_2}\Big)\Big) \\[2ex] \Omega_2 = -\dfrac{3}{4}\, m_2^2 \,\dfrac{1}{a_2 \Lambda_2}\Big(1 + \mathrm{O}\Big(\dfrac{a_1}{a_2}\Big)^2\Big) \end{cases}$$

$$\tau = m_1 m_2 \frac{a_1^2}{a_2^3} \begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2} \\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2} \end{pmatrix} (1 + \mathrm{O}(a_2^{-5/4})) \ ,$$

which shows that the $\Omega_j$'s are non resonant up to any finite order (in a suitable $\Lambda$–domain), so that the planetary, planar Hamiltonian can be put in Birkhoff normal form up to order 4 and that the second order Birkhoff invariants are non–degenerate in the sense that[21]

$$\det \tau = -(m_1 m_2)^2 \ \frac{117}{16} \ \frac{a_1^4}{a_2^6 (\Lambda_1 \Lambda_2)^2} \ (1 + o(1)) = -\frac{117}{16} \frac{1}{m_0^2} \frac{a_1^3}{a_2^7} \ (1 + o(1)) \neq 0.$$

This allow to apply Theorem 3.2 and to prove Arnold's planetary theorem in the planar three–body ($n = 2$) case.

An extension of this method to the *spatial three–body problem*, exploiting Jacobi's reduction of the nodes and its symplectic realization, is due to P. Robutel [26].

**Obstacles to the generalization of Arnold's project: Secular degeneracies.** In the general spatial case it is customary to call $\sigma_i$ the eigenvalues of $\mathcal{Q}_h(\Lambda)$ and $\varsigma_i$ the eigenvalues of and $\mathcal{Q}_v(\Lambda)$, so that $\Omega = (\sigma, \varsigma)$; compare (2.14).

It turns out that such invariants satisfy identically the following two *secular resonances*

$$\varsigma_n = 0 \ , \qquad\qquad \sum_{i=1}^n (\sigma_i + \varsigma_i) = 0 \qquad\qquad (3.3)$$

and, actually, it can be shown that *these are the only exact resonances identically satisfied by the first order Birkhoff invariants*; compare [17, Prop. 78 at p. 1575].

The first resonance was well known to Arnold, while the second one was apparently discovered by M. Herman in the 1990's and is now known as *Herman resonance.*

Both resonances violate Birkhoff's non–resonance condition (compare Eq. (B.1) below) but *do not violate* a more special Birkhoff condition sufficient for rotational invariant systems, as explained in Appendix B (compare, in particular Eq. (B.3)).

There is, however, a much more serious problem for Arnold's approach, namely, a strong degeneracy of the *second order Birkhoff invariance*, still a reflection of rotational invariance. Indeed, the torsion matrix $\tau$ is degenerate, as clarified in [12], where it is proven that $\tau$ is

[21] In [1] the $\tau_{ij}$ are defined as $1/2$ of the ones defined here.

equivalent to a matrix of the form

$$\begin{pmatrix} \bar\tau & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4}$$

$\bar\tau$ being a matrix of order $(2n-1)$.

## 4. Proofs of Arnold's theorem

**Herman-Fejóz proof (2004).** In 2004 J. Fejóz [17] published the first complete proof of a general version of Arnold's planetary theorem: this proof completed a long project carried out by M. Herman. In order to avoid fourth order computations, Herman (also because seemed to suspect the degeneracy of the matrix of the second order Birkhoff invariant; compare the Remark towards the end of p. 24 of [20]), turned to a weaker KAM theory, which makes use of a "first order KAM condition" based on the non–planarity of the frequency map. But, the resonances (3.3) show that the frequency map lies in the intersection of two planes, violating the non–planarity condition. To overcome this problem Herman and Féjoz use a trick by Poincarè, consisting in modifying the Hamiltonian by adding a commuting Hamiltonian, so as to remove the degeneracy. By a Lagrangian intersection theory argument, if two Hamiltonian commute and $\mathcal{T}$ is a Lagrangian invariant transitive torus for one of them, then $\mathcal{T}$ is invariant (but not necessarily transitive) also for the other Hamiltonian; compare [17, Lemma 82, p. 1578]. Thus, the KAM tori constructed for the modified Hamiltonian are indeed invariant tori also for the original system. Now, the expression of the vertical component of the total angular momentum $\mathrm{C}_3$ has a particular simple expression in Poincaré variables: indeed, $\mathrm{C}_3 = \sum_{j=1}^n \Big(\Lambda_j - \frac{1}{2}(\eta_j^2 + \xi_j^2 + \mathrm{p}_j^2 + \mathrm{q}_j^2)\Big)$, so that the modified Hamiltonian $\mathcal{H}_\delta := \mathcal{H}_{\mathrm{P}}(\Lambda, \lambda, \mathrm{z}) + \delta \mathrm{C}_3$ is easily seen to have a non–planar frequency map (first order Birlhoff invariants), and the above abstract remark applies.

Herman's KAM theory (as given in [17]) works in the $C^\infty$ category, so that the tori obtained in [17] are proven to be $C^\infty$, on the other hand, since the planetary Hamiltonian flow is real–analytic, it is natural to expect that also their maximal quasi–periodic solutions (and the tori they span) are real–analytic. This is proven in [13], where Rüßmann first–order KAM theory [27] is extended to properly–degenerate systems.

**Completion of Arnold's project (2011).** In [11] Arnold's original strategy is reconsidered and full torsion of the planetary problem is proved by introducing new symplectic variables (called RPS–variables standing for Regularized Planetary Symplectic variables), which allow for a symplectic partial reduction of rotations eliminating one degree of freedom (i.e., lowering by two units the dimension of the phase space). In such reduced setting the first resonance in (3.3) disappears (but not the second one) and the question about the torsion is reduced to study the determinant of $\bar\tau$ in (3.4), which, in fact, is shown to be non–singular; compare [11, §8] and [12] (where a precise connection is made between the Poincaré and the RPS–variables compare also Theorem 5.1 below).

In the next section we shall review the main ideas and techniques discussed in [11].

## 5. A new symplectic view of the planetary phase space and completion of Arnold's project

We start by describing the new set of symplectic variables, which allow to have a new insight on the symplectic structure of the phase space of the planetary model, or, more in general, of any rotational invariant model.

The idea is to start with action–angle variables having, among the actions, two independent commuting integrals related to rotations, for example, the Euclidean length of the total angular momentum C and its vertical component $\mathrm{C}_3$, and then (imitating Poincaré) to regularize around co–circular and co–planar configurations.

The variables that do the job are a "planetary" action–angle version of certain variables introduced by A. Deprit in[22] 1983 [15].

**The Regularized planetary symplectic (RPS) variables.** Let $n \geq 2$, $1 \leq i \leq n$, and consider the "partial angular momenta" $S^{(i)} := \sum_{j=1}^{i} \mathrm{C}^{(j)}$, (note that $S^{(n)} = \sum_{j=1}^{n} \mathrm{C}^{(j)} =: \mathrm{C}$) and define the "Deprit nodes"

$$\begin{cases} \nu_i := S^{(i)} \times \mathrm{C}^{(i)} \, , & 2 \leq i \leq n \\ \nu_1 := \nu_2 & \\ \nu_{n+1} := k^{(3)} \times \mathrm{C} =: \bar{\nu} \, ; & \end{cases}$$

(recall the definition of the "individual" and total angular momenta in (2.5)).

The *Deprit action–angle variables* $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ are defined as follows. Let $P_i$ denote the coordinates of the $i^{th}$ instantaneous perihelion (relatively to the instantaneous planet–Sun 2–body system), let $(k^{(1)}, k^{(2)}, k^{(3)})$ be the standard orthonormal basis in $\mathbb{R}^3$, and, for $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a non–vanishing vector $w$, denote by $\alpha_w(u, v)$ the positively oriented angle (mod $2\pi$) between $u$ and $v$ (orientation follows the "right hand rule", the thumb being $w$).

The Deprit variables $\Lambda$, $\Gamma$ and $\ell$ are in common with the Delaunay variables (compare (A.4) in Appendix A), while

$$\gamma_i := \alpha_{\mathrm{C}^{(i)}}(\nu_i, P_i) \qquad \Psi_i := \begin{cases} |S^{(i+1)}| \, , & 1 \leq i \leq n-1 \\ \mathrm{C}_3 := \mathrm{C} \cdot k^{(3)} & i = n \end{cases}$$

$$\psi_i := \begin{cases} \alpha_{S^{(i+1)}}(\nu_{i+2}, \nu_{i+1}) & 1 \leq i \leq n-1 \\ \zeta := \alpha_{k^{(3)}}(k^{(1)}, \bar{\nu}) & i = n. \end{cases}$$

Define also $G := |\mathrm{C}| = |S^{(n)}|$.

The "Deprit inclinations" $\iota_i$ are defined through the relations

$$\cos \iota_i := \begin{cases} \dfrac{\mathrm{C}^{(i+1)} \cdot S^{(i+1)}}{|\mathrm{C}^{(i+1)}||S^{(i+1)}|} \, , & 1 \leq i \leq n-1 \, , \\[2ex] \dfrac{\mathrm{C} \cdot k^{(3)}}{|\mathrm{C}|} \, , & i = n \, . \end{cases}$$

Similarly to the case of the Delaunay variables, the Deprit action–angle variables are not defined when the Deprit nodes $\nu_i$ vanish or the eccentricitiy $e_i \notin (0, 1)$, but on the do-

[22] See also [10] and [11].

main where they are well defined they yield a real–analytic set of symplectic variables, i.e., $\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)} = \sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\gamma_i + d\Psi_i \wedge d\psi_i$; for a proof, see [10] or §3 of [11].

The RPS variables are given by[23] $(\Lambda, \lambda, z) := (\Lambda, \lambda, \eta, \xi, p, q)$ with (again) the $\Lambda$'s as in (2.7) and, for $1 \le i \le n$,

$$\lambda_i = \ell_i + \gamma_i + \psi_{i-1}^n \ , \qquad \begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)}\ \cos\left(\gamma_i + \psi_{i-1}^n\right) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)}\ \sin\left(\gamma_i + \psi_{i-1}^n\right) \end{cases}$$

$$\begin{cases} p_i = \sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\ \cos\psi_i^n \\ q_i = -\sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\ \sin\psi_i^n \end{cases}$$

where $\Psi_0 := \Gamma_1$, $\Gamma_{n+1} := 0$, $\psi_0 := 0$, $\psi_i^n := \sum_{i \le j \le n} \psi_j$. On the domain of definition, the RPS variables are symplectic:

$$\sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\gamma_i + d\Psi_i \wedge d\psi_i = \sum_{i=1}^n d\Lambda_i \wedge d\lambda_i + d\eta_i \wedge d\xi_i + dp_i \wedge dq_i \ ;$$

for a proof, see [23] or [11, §4].

As phase space, consider a set of the same form as in (2.8), (2.9), namely

$$(\Lambda, \lambda, z) \in \mathcal{M}^{6n}_{\text{RPS}} := \mathcal{A} \times \mathbb{T}^n \times B^{4n} \tag{5.1}$$

with $B$ a $4n$–dimensional ball around the origin (origin, which corresponds, as in Poincaré variables, to planar co–circular motions).

Poincaré and RPS variables are intimately connected: If we denote by

$$\phi_{\text{P}}^{\text{RPS}} : \quad (\Lambda, \lambda, z) \to (\Lambda, \uplambda, \mathrm{z}) \tag{5.2}$$

the symplectic trasformation between RPS and Poincaré variables, then the following result holds.

**Theorem 5.1** ([12]). *The symplectic map $\phi_{\text{P}}^{\text{RPS}}$ in* (5.2) *has the form*

$$\uplambda = \lambda + \varphi(\Lambda, z) \qquad \mathrm{z} = \mathcal{Z}(\Lambda, z)$$

*where $\varphi(\Lambda, 0) = 0$ and, for any fixed $\Lambda$, the map $\mathcal{Z}(\Lambda, \cdot)$ is 1:1, symplectic (i.e., it preserves the two form $d\eta \wedge d\xi + dp \wedge dq$) and its projections verify, for a suitable $\mathcal{V} = \mathcal{V}(\Lambda) \in \mathrm{SO}(n)$, with $O_3 = O(|z|^3)$,*

$$\Pi_{\upeta} \mathcal{Z} = \eta + O_3 \ , \ \Pi_{\upxi} \mathcal{Z} = \xi + O_3 \ , \ \Pi_{\mathrm{p}} \mathcal{Z} = \mathcal{V} p + O_3 \ , \ \Pi_{\mathrm{q}} \mathcal{Z} = \mathcal{V} q + O_3 \ .$$

*where $O_3 = O(|z|^3)$.*

**Partial reduction of rotations.** Recalling that $\Gamma_{n+1} = 0$, $\Psi_{n-1} = |S^{(n)}| = |\mathrm{C}|$, $\Psi_n = \mathrm{C}_3$, $\psi_n = \alpha_{k^{(3)}}(k^{(1)}, k_3 \times \mathrm{C})$ one sees that

$$\begin{cases} p_n = \sqrt{2(|\mathrm{C}| - \mathrm{C}_3)}\ \cos\psi_n \\ q_n = -\sqrt{2(|\mathrm{C}| - \mathrm{C}_3)} \sin\psi_n \ , \end{cases}$$

[23] Beware of notations: we use upright characters for Poincaré variables $(\Lambda, \uplambda, \mathrm{z}) := (\Lambda, \uplambda, \upeta, \mathrm{p}, \upxi, \mathrm{q})$ and standard italic for RPS variables $(\Lambda, \lambda, z) := (\Lambda, \lambda, \eta, \xi, p, q)$.

showing that *the conjugated variables* $p_n$ *and* $q_n$ *are both integrals and hence both cyclic* for the planetary Hamiltonian, which, therefore, in such variables, will have the form

$$\mathcal{H}_{\text{RPS}}(\Lambda, \lambda, \bar{z}) = h_{\text{K}}(\Lambda) + \mu f_{\text{RPS}}(\Lambda, \lambda, \bar{z}) \ , \tag{5.3}$$

where $\bar{z}$ denotes the set of variables

$$\bar{z} := (\eta, \xi, \bar{p}, \bar{q}) := \big((\eta_1, \ldots, \eta_n), (\xi_1, \ldots, \xi_n), (p_1, \ldots, p_{n-1}), (q_1, \ldots, q_{n-1})\big) \ .$$

In other words, *the phase space* $\mathcal{M}^{6n}_{\text{rps}}$ *in* (5.1) *is foliated by* $(6n-2)$*–dimensional invariant manifolds*

$$\mathcal{M}^{6n-2}_{p_n,q_n} := \mathcal{M}^{6n}_{\text{RPS}}|_{p_n,q_n=\text{const}} \ , \tag{5.4}$$

and since the restriction of the standard symplectic form on such manifolds is simply $d\Lambda \wedge d\lambda + d\eta \wedge d\xi + d\bar{p} \wedge d\bar{q}$, such submanifolds are symplectic and the planetary flow is the standard Hamiltonian flow generated by $\mathcal{H}_{\text{RPS}}$ in (5.3). The submanifolds depend upon a particular orientation of the total angular momentum: in particular, $\mathcal{M}^{6n-2}_0$ correspond to the total angular momentum parallel to the vertical $k_3$–axis. Notice, also, that *the analytic expression of the planetary Hamiltonian* $\mathcal{H}_{\text{rps}}$ *is the same on each submanifold.*

In view of these observations, it is enough to study the planetary flow of $\mathcal{H}_{\text{RPS}}$ on, say, the vertical submanifold $\mathcal{M}^{6n-2}_0$.

**Planetary Birkhoff normal forms and torsion.** The RPS variables share with Poincaré variables classical *D'Alembert symmetries*, i.e., $\mathcal{H}_{\text{RPS}}$ is invariant under the transformations (2.10), $\mathcal{S}$ being as in (2.11); compare also Remark 3.3 of [12].

This implies that the averaged perturbation $f^{\text{av}}_{\text{RPS}} := \dfrac{1}{(2\pi)^n}\displaystyle\int_{\mathbb{T}^n} f_{\text{RPS}}\ d\lambda$ also enjoys D'Alembert rules and thus has an expansion analogue to (2.13), but independent of $(p_n, q_n)$:

$$f^{\text{av}}_{\text{RPS}}(\Lambda, \bar{z}) = C_0(\Lambda) + \mathcal{Q}_h(\Lambda) \cdot \frac{\eta^2 + \xi^2}{2} + \bar{\text{Q}}_{\text{v}}(\Lambda) \cdot \frac{\bar{p}^2 + \bar{q}^2}{2} + \text{O}(|\bar{z}|^4) \tag{5.5}$$

with $\mathcal{Q}_h$ of order $n$ and $\bar{\text{Q}}_{\text{v}}$ of order $(n-1)$. Notice that the matrix $\mathcal{Q}_h$ in (5.5) is the same as in (2.13), since, when $p = (\bar{p}, p_n) = 0$ and $q = (\bar{q}, q_n) = 0$, Poincaré and RPS variables coincide.

Using Theorem 5.1, one can also show that $\text{Q}_{\text{v}} := \begin{pmatrix} \bar{\text{Q}}_{\text{v}} & 0 \\ 0 & 0 \end{pmatrix}$ is conjugated (by a unitary matrix) to $\mathcal{Q}_v$ in (2.13), so that the eigenvalues $\bar{\varsigma}_i$ of $\bar{\text{Q}}_{\text{v}}$ coincide with $(\varsigma_1, \ldots, \varsigma_{n-1})$, as one naively would expect.

In view of the remark after (3.3), and of rotation–invariant Birkhoff theory[24] , one sees that one can construct, in an open neighborhood of co–planar and co–circular motions, the Birkhoff normal form of $f^{\text{av}}_{\text{RPS}}$ at any finite order.

More precisely, for $\epsilon > 0$ small enough, denoting

$$\mathcal{P}_\epsilon := \mathcal{A} \times \mathbb{T}^n \times B^{4n-2}_\epsilon \ , \qquad B^{4n-2}_\epsilon := \{\bar{z} \in \mathbb{R}^{4n-2} : |\bar{z}| < \epsilon\} \ ,$$

an $\epsilon$–neighborhood of the co–circular, co–planar region, one can find a real–analytic symplectic transformation $\phi_\mu : (\Lambda, \breve{\lambda}, \breve{z}) \in \mathcal{P}_\epsilon \to (\Lambda, \lambda, \bar{z}) \in \mathcal{P}_\epsilon$ such that $\breve{\mathcal{H}} := \mathcal{H}_{\text{RPS}} \circ \phi_\mu =$

[24] According to which the only forbidden frequencies for constructing the Birkhoff normal form are generated by those integer vectors $k$ such that $\sum k_i = 0$; compare Proposition B.2, Appendix B below.

$h_{\rm K}(\Lambda) + \mu f(\Lambda, \breve\lambda, \breve z)$ with

$$\breve f_{\rm av}(\Lambda, \breve z) := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f\, d\breve\lambda = C_0(\Lambda) + \Omega \cdot \breve{\rm R} + \frac12\, \bar\tau\, \breve{\rm R}\cdot\breve{\rm R} + \breve{\mathcal P}(\Lambda, \breve z)$$

where

$$\begin{cases} \Omega = (\sigma, \bar\varsigma) \\ \breve z := (\breve\eta, \breve\xi, \breve p, \breve q)\ , \quad \breve{\rm R} = (\breve\rho, \breve r)\ , \quad \breve{\mathcal P}(\Lambda, \breve z) = O(|\breve z|^6)\ , \\ \breve\rho = (\breve\rho_1, \cdots, \breve\rho_n)\ , \quad \breve r = (\breve r_1, \cdots, \breve r_{n-1})\ , \\ \breve\rho_i := \frac{\breve\eta_i^2 + \breve\xi_i^2}{2}\ , \quad \breve r_i = \frac{\breve p_i^2 + \breve q_i^2}{2} \end{cases}$$

With straightforward (but not trivial!) computations, one can then show full torsion for the planetary problem.
More precisely, one finds (compare Proposition 8.1 of [11]):

**Theorem 5.2.** *For $n \geq 2$ and $0 < \delta_\star < 1$ there exist $\bar\mu > 0$, $0 < \underline{a}_1 < \overline{a}_1 < \cdots < \underline{a}_n < \overline{a}_n$ such that, on the set $\mathcal A$ defined in (2.9) and for $0 < \mu < \bar\mu$, the matrix $\bar\tau = (\tau_{ij})$ is non–singular* $\det\bar\tau = d_n(1+\delta_n)$*, where $|\delta_n| < \delta_\star$ and*

$$d_n := (-1)^{n-1}\frac35\Big(\frac{45}{16}\,\frac{1}{m_0^2}\Big)^{n-1}\frac{m_2}{m_1 m_0}a_1\Big(\frac{a_1}{a_n}\Big)^3\prod_{2\leq k\leq n}\Big(\frac{1}{a_k}\Big)^4 .$$

**Kolmogorov tori for the planetary problem.** At this point one can apply to the planetary Hamiltonian in normalized variables $\breve{\mathcal H}(\Lambda, \breve\lambda, \breve z)$ Arnold's Theorem 3.2 above completing Arnold's project on the planetary $N$–body problem.

Indeed, by using the refinements of Theorem 3.2 as given in [9], from Theorem 5.2 there follows

**Theorem 5.3.** *There exists positive constants $\epsilon_*$, $c_*$ and $C_*$ such that the following holds. If $0 < \epsilon < \epsilon_*$ and $0 < \mu < \epsilon^6/(\log\epsilon^{-1})^{c_*}$ then each symplectic submanifold $\mathcal M^{6n-2}_{p_n,q_n}$* (5.4) *contains a positive measure $\mathcal H_{\rm RPS}$–invariant Kolmogorov set $\mathcal K_{p_n,q_n}$, which is actually the suspension of the same Kolmogorov set $\mathcal K \subseteq \mathcal P_\epsilon$, which is $\breve{\mathcal H}$–invariant. Furthermore, $\mathcal K$ is formed by the union of $(3n-1)$–dimensional Lagrangian, real–analytic tori on which the $\breve{\mathcal H}$–motion is analytically conjugated to linear Diophantine quasi–periodic motions with frequencies $(\omega_1, \omega_2) \in \mathbb R^n \times \mathbb R^{2n-1}$ with $\omega_1 = O(1)$ and $\omega_2 = O(\mu)$. Finally, $\mathcal K$ satisfies the bound*[25] $\ {\rm meas}\,\mathcal P_\epsilon \geq {\rm meas}\,\mathcal K \geq \Big(1 - C_*\sqrt\epsilon\Big)\,{\rm meas}\,\mathcal P_\epsilon$.

**Conley-Zehnder stable periodic orbits.** The tori $\mathcal T \in \mathcal K$ form a (Whitney) smooth family of *non–degenerate Kolmogorov tori*, which means the following. The tori in $\mathcal K$ can be parameterized by their frequency $\omega \in \mathbb R^{3n-1}$ (i.e., $\mathcal T = \mathcal T_\omega$) and there exist a real–analytic symplectic diffeomorphism $\nu : (y, x) \in B^m \times \mathbb T^m \to \nu(y, x; \omega) \in \mathcal P_\epsilon$, $m := 3n-1$, uniformly Lipschitz in $\omega$ (actually $C^\infty$ in the sense of Whitney) such that, for each $\omega$

- $\breve{\mathcal H}\circ\nu = E + \omega\cdot y + Q$; (Kolmogorov's normal form)
- $E \in \mathbb R$ (the energy of the torus); $\omega \in \mathbb R^m$ is a Diophantine vector;
- $Q = O(|y|^2)$ and $\det\displaystyle\int_{\mathbb T^m}\partial_{yy}Q(0,x)\,dx \neq 0$ , (non–degeneracy)

[25] In particular, ${\rm meas}\,\mathcal K \simeq \epsilon^{4n-2} \simeq {\rm meas}\,\mathcal P_\epsilon$.

- $\mathcal{T}_\omega = \nu(0, \mathbb{T}^m)$.

Now, in the first paragraph of [14] Conley and Zehnder, putting together KAM theory (and in particular exploiting Kolmogorv's normal form for KAM tori) together with Birkhoff–Lewis fixed–point theorem show that long–period periodic orbits cumulate densely on Kolmogorov tori so that, in particular, *the Lebesgue measure of the closure of the periodic orbits can be bounded below by the measure of the Kolmogorov set.* Notwithstanding the proper degeneracy, this remark applies also in the present situation and as a consequence of Theorem 5.3 and of the fact that the tori in $\mathcal{K}$ are non–degenerate Kolmogorov tori it follows ([7]) that *in the planetary model the measure of the closure of the periodic orbits in $\mathcal{P}_\epsilon$ can be bounded below by a constant times $\epsilon^{4n-2}$.*

## A. Details on the classical Hamiltonian structure

**Inertial manifold.** Equations (2.1) are invariant by change of "inertial frames", i.e., by change of variables of the form $u^{(i)} \to u^{(i)} - (a + ct)$ with fixed $a, c \in \mathbb{R}^3$. This allows to restrict the attention to the manifold of "initial data" given by

$$\sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(0) = 0 \,, \qquad \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}(0) = 0 \,; \tag{A.1}$$

indeed, just replace the coordinates $u^{(i)}$ by $u^{(i)} - (a + ct)$ with

$$a := \mathrm{m}_{\rm tot}^{-1} \sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(0) \qquad \text{and} \qquad c := \mathrm{m}_{\rm tot}^{-1} \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}(0) \,, \qquad \mathrm{m}_{\rm tot} := \sum_{i=0}^{n} \mathrm{m}_i \,.$$

The total linear momentum $\mathrm{M}_{\rm tot} := \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}$ does not change along the flow of (2.1), i.e., $\dot{\mathrm{M}}_{\rm tot} = 0$ along trajectories; therefore, by (A.1), $\mathrm{M}_{\rm tot}(t)$ vanishes for all times. But, then, also the position of the total center of mass $B(t) := \sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(t)$ is constant ($\dot{B} = 0$) and, again by (A.1), $B(t) \equiv 0$. In other words, the manifold of initial data (A.1) is invariant under the flow generated by (2.1).

**The Linear momentum reduction.** In view of the invariance properties discussed above, in the variables $(U^{(i)}, u^{(i)}) \in \widehat{\mathcal{M}}$, (recall (2.2) and that $U^{(i)} := \mathrm{m}_i \dot{u}^{(i)}$), it is enough to consider the submanifold $\widehat{\mathcal{M}}_0 := \{(U, u) \in \widehat{\mathcal{M}} : \sum_{i=0}^{n} \mathrm{m}_i u^{(i)} = 0 = \sum_{i=0}^{n} U^{(i)}\}$, which corresponds to the manifold described in (A.1).

The submanifold $\widehat{\mathcal{M}}_0$ is symplectic, i.e., the restriction of the form (2.3) to $\widehat{\mathcal{M}}_0$ is again a symplectic form; indeed: $\Big(\sum_{i=0}^{n} dU^{(i)} \wedge du^{(i)}\Big)\Big|_{\widehat{\mathcal{M}}_0} = \sum_{i=1}^{n} \frac{\mathrm{m}_0 + \mathrm{m}_i}{\mathrm{m}_0} dU^{(i)} \wedge du^{(i)}$.

Poincaré's *symplectic reduction* ("reduction of the linear momentum") goes as follows. Let $\phi_{\rm he} : (R, r) \to (U, u)$ be the linear transformation given by

$$\phi_{\rm he} : \left\{ \begin{array}{lll} u^{(0)} = r^{(0)} \,, & u^{(i)} = r^{(0)} + r^{(i)} \,, & (i = 1, ..., n) \\ U^{(0)} = R^{(0)} - \sum_{i=1}^{n} R^{(i)} \,, & U^{(i)} = R^{(i)} \,, & (i = 1, ..., n) \,; \end{array} \right. \tag{A.2}$$

such transformation is symplectic, i.e., $\sum_{i=0}^n dU^{(i)} \wedge du^{(i)} = \sum_{i=0}^n dR^{(i)} \wedge dr^{(i)}$. recall that this means, in particular, that in the new variables the Hamiltonian flow is again standard: more precisely, one has that $\phi^t_{\widehat{\mathcal{H}}_{\rm N}} \circ \phi_{\rm he} = \phi_{\rm he} \circ \phi^t_{\widehat{\mathcal{H}}_{\rm N} \circ \phi}$.

Letting $\mathrm{m}_{\rm tot} := \sum_{i=0}^n \mathrm{m}_i$ one sees that, in the new variables, $\widehat{\mathcal{M}}_0$ reads

$$\big\{(R,r) \in \mathbb{R}^{6(n+1)} : \ R^{(0)} = 0, r^{(0)} = -\mathrm{m}_{\rm tot}^{-1} \sum_{i=1}^n \mathrm{m}_i r^{(i)} \ , \ 0 \neq r^{(i)} \neq r^{(j)} \ \forall\, 1 \leq i \neq j \leq n\big\}.$$

The restriction of the 2–form (2.3) to $\widehat{\mathcal{M}}_0$ is simply $\displaystyle\sum_{i=1}^n dR^{(i)} \wedge dr^{(i)}$ and

$$\mathcal{H}_{\rm N} := \widehat{\mathcal{H}}_{\rm N} \circ \phi_{\rm he}|_{\mathcal{M}_0} = \sum_{i=1}^n \frac{|R^{(i)}|^2}{2\frac{\mathrm{m}_0 \mathrm{m}_i}{\mathrm{m}_0+\mathrm{m}_i}} - \frac{\mathrm{m}_0 \mathrm{m}_i}{|r^{(i)}|} + \sum_{1 \leq i<j \leq n} \frac{R^{(i)} \cdot R^{(j)}}{\mathrm{m}_0} - \frac{\mathrm{m}_i \mathrm{m}_j}{|r^{(i)} - r^{(j)}|} \ .$$

The dynamics generated by $\widehat{\mathcal{H}}_{\rm N}$ on $\widehat{\mathcal{M}}_0$ is equivalent to the dynamics generated by the Hamiltonian $(R,r) \in \mathbb{R}^{6n} \to \mathcal{H}_{\rm N}(R,r)$ on

$$\mathcal{M}_0 := \Big\{(R,r) = (R^{(1)}, ..., R^{(n)}, r^{(1)}, ..., r^{(n)}) \in \mathbb{R}^{6n} : 0 \neq r^{(i)} \neq r^{(j)} \ , \forall i \neq j\Big\}$$

with respect to the standard symplectic form $\sum_{i=1}^n dR^{(i)} \wedge dr^{(i)}$; to recover the full dynamics on $\widehat{\mathcal{M}}_0$ from the dynamics on $\mathcal{M}_0$ one will simply set $R^{(0)}(t) \equiv 0$ and $r^{(0)}(t) := -\mathrm{m}_{\rm tot}^{-1} \displaystyle\sum_{i=1}^n \mathrm{m}_i r^{(i)}(t)$.

Since we are interested in the planetary case, we perform the trivial rescaling by a small positive parameter $\mu$:

$$\begin{aligned} &m_0 := \mathrm{m}_0 \ , \ \mathrm{m}_i = \mu m_i \ (i \geq 1) \ , \quad X^{(i)} := \frac{R^{(i)}}{\mu} \ , \ x^{(i)} := r^{(i)} \ , \\ &\mathcal{H}_{\rm plt}(X,x) := \frac{1}{\mu} \mathcal{H}_{\rm N}(\mu X, x) \ , \end{aligned}$$

a transformation which leaves unchanged Hamilton's equations.

**Delaunay and Poincaré variables..** The Hamiltonian $\mathcal{H}^{(0)}_{\rm plt}$ in (2.4) governes the motion of $n$ decoupled two–body problems with Hamiltonian

$$h^{(i)}_{\rm 2B} = \frac{|X^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|} \ , \qquad (X^{(i)}, x^{(i)}) \in \mathbb{R}^3 \times \mathbb{R}^3_* := \mathbb{R}^3 \times (\mathbb{R}^3 \backslash \{0\}) \ .$$

Such two–body sytems are, as well known, integrable. The explicit "symplectic integration" is done by means of the *Delaunay variables*, whose construction we, now, briefly, recall (for full details and proofs, see, e.g., [5]).

Assume that $h^{(i)}_{\rm 2B}(X^{(i)}, x^{(i)}) < 0$ so that the Hamiltonian flow $\phi^t_{h^{(i)}_{\rm 2B}}(X^{(i)}, x^{(i)})$ evolves on a Keplerian ellipse $\mathfrak{E}_i$ and assume that the eccentricity $e_i \in (0,1)$.

Let $a_i$, $P_i$ denote, respectively, the *semimajor axis* and the *perihelion* of $\mathfrak{E}_i$.

Let $\mathrm{C}^{(i)}$ denote the $i^{\rm th}$ angular momentum $\mathrm{C}^{(i)} := x^{(i)} \times y^{(i)}$.

Let us, also, introduce the "Delaunay nodes"

$$\bar\nu_i := k^{(3)} \times \mathrm{C}^{(i)} \quad 1 \le i \le n\,, \tag{A.3}$$

where $(k^{(1)}, k^{(2)}, k^{(3)})$ is the standard orthonormal basis in $\mathbb{R}^3$. Finally, for $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a non–vanishing vector $w$, let $\alpha_w(u, v)$ denote the positively oriented angle (mod $2\pi$) between $u$ and $v$ (orientation follows the "right hand rule").

The *Delaunay action–angle variables* $(\Lambda_i, \Gamma_i, \Theta_i, \ell_i, \mathrm{g}_i, \theta_i)$ are, then, defined as

$$\left\{\begin{array}{l} \Lambda_i := M_i\sqrt{\bar m_i a_i}\\ \ell_i := \text{mean anomaly of } x^{(i)} \text{ on } \mathfrak{E}_i \end{array}\right. , \quad \left\{\begin{array}{l} \Gamma_i := |\mathrm{C}^{(i)}| = \Lambda_i\sqrt{1-e_i^2}\\ \mathrm{g}_i := \alpha_{\mathrm{C}^{(i)}}(\bar\nu_i, P_i) \end{array}\right.$$
$$\left\{\begin{array}{l} \Theta_i := \mathrm{C}^{(i)}\cdot k^{(3)}\\ \theta_i := \alpha_{k^{(3)}}(k^{(1)}, \bar\nu_i) \end{array}\right. \tag{A.4}$$

Notice that the Delaunay variables are defined on an open set of full measure of the Cartesian phase space $\mathbb{R}^{3n} \times \mathbb{R}^{3n}_*$, namely, on the set where $e_i \in (0, 1)$ and the nodes $\bar\nu_i$ in (A.3) are well defined; on such set the "Delaunay inclinations" $\mathrm{i}_i$ defined through the relations

$$\cos \mathrm{i}_i := \frac{\mathrm{C}^{(i)}\cdot k^{(3)}}{|\mathrm{C}^{(i)}|} = \frac{\Theta_i}{\Gamma_i}\,, \tag{A.5}$$

are well defined and we choose the branch of $\cos^{-1}$ so that $\mathrm{i}_i \in (0, \pi)$.

The Delaunay variables become singular when $\mathrm{C}^{(i)}$ is vertical (the Delaunay node is no more defined) and in the circular limit (the perihelion is not unique). In these cases different variables have to been used (see below).

On the set where the Delaunay variables are well posed, they define a *symplectic set of action–angle variables*, i.e., $\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)} = \sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathrm{g}_i + d\Theta_i \wedge d\theta_i$, for a proof, see §3.2 of [5].

In Delaunay action–angle variables $((\Lambda, \Gamma, \Theta), (\ell, \mathrm{g}, \theta))$ the Hamiltonian $\mathcal{H}^{(0)}_{\rm plt}$ takes the form (2.7). We shall restrict our attention to the collisionless phase space

$$\mathcal{M}_{\rm plt} := \Big\{\Lambda_i > \Gamma_i > \Theta_i > 0\,, \quad \frac{\Lambda_i}{M_i\sqrt{\bar m_i}} \ne \frac{\Lambda_j}{M_j\sqrt{\bar m_j}}\,,\ \forall\, i \ne j\Big\} \times \mathbb{T}^{3n}\,,$$

endowed with the standard symplectic form $\sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathrm{g}_i + d\Theta_i \wedge d\theta_i$.

Notice that the $6n$–dimensional phase space $\mathcal{M}_{\rm plt}$ is foliated by $3n$–dimensional $\mathcal{H}^{(0)}_{\rm plt}$–invariant tori $\{\Lambda, \Gamma, \Theta\} \times \mathbb{T}^3$, which, in turn, are foliated by $n$–dimensional tori $\{\Lambda\} \times \mathbb{T}^n$, expressing geometrically the degeneracy of the integrable Keplerian limit of the $(1 + n)$–body problem.

A regularization of the Delaunay variables in their singular limit was introduced by Poincaré, in such a way that the set of action–angle variables $((\Gamma, \Theta), (\mathrm{g}, \theta))$ is mapped onto cartesian variables regular near the origin, which corresponds to co–circular and co–planar motions, while the angles conjugated to $\Lambda_i$, which remains invariant, are suitably shifted.

More precisely, the *Poincaré variables* are given by $(\Lambda, \lambda, \mathrm{z}) := (\Lambda, \lambda, \eta, \xi, \mathrm{p}, \mathrm{q}) \in \mathbb{R}^n_+ \times \mathbb{T}^n \times \mathbb{R}^{4n}$, with the $\Lambda$'s as in (A.4) and

$$\lambda_i = \ell_i + \mathrm{g}_i + \theta_i, \quad \left\{\begin{array}{l} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)}\, \cos(\theta_i + \mathrm{g}_i)\\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)}\, \sin(\theta_i + \mathrm{g}_i) \end{array}\right. , \quad \left\{\begin{array}{l} \mathrm{p}_i = \sqrt{2(\Gamma_i - \Theta_i)}\, \cos\theta_i\\ \mathrm{q}_i = -\sqrt{2(\Gamma_i - \Theta_i)}\, \sin\theta_i \end{array}\right.$$

Notice that $e_i = 0$ corresponds to $\eta_i = 0 = \xi_i$, while $\mathrm{i}_i = 0$ corresponds to $\mathrm{p}_i = 0 = \mathrm{q}_i$; compare (A.4) and (A.5).

On the domain of definition, the Poincaré variables are symplectic

$$\sum_{i=1}^{n} d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathrm{g}_i + d\Theta_i \wedge d\theta_i = \sum_{i=1}^{n} d\Lambda_i \wedge d\lambda_i + d\eta_i \wedge d\xi_i + d\mathrm{p}_i \wedge d\mathrm{q}_i \ ;$$

for a proof, see Appendix C of [4].

## B. Birkhoff normal forms

In this appendix we recall a few known and less known facts about the general theory of Birkhoff normal forms.

Consider as phase space a $2m$ ball $B^{2m}_\delta$ around the origin in $\mathbb{R}^{2m}$ and a real–analytic Hamiltonian of the form $H(w) = c_0 + \Omega \cdot \mathrm{r} + o(|w|^2)$ where

$$\begin{cases} w = (u_1, \ldots, u_m, v_1, \ldots, v_m) \in \mathbb{R}^{2m} \,, \\ \mathrm{r} = (\mathrm{r}_1, \ldots, \mathrm{r}_m) \,, \quad \mathrm{r}_j = \frac{u_j^2 + v_j^2}{2} \,. \end{cases}$$

the symplectic form being $\sum du_i \wedge dv_i$. The components $\Omega_j$ of $\Omega$ are called the first order Birkhoff invariants. The following is a classical result due to G.D. Birkhoff.

**Proposition B.1.** *Assume that the first order Birkhoff invariants $\Omega_j$ verify, for some $a > 0$ and integer $s$,*

$$|\Omega \cdot k| \geq a > 0, \quad \forall\, k \in \mathbb{Z}^m : \ 0 < |k|_1 := \sum_{j=1}^{m} |k_j| \leq 2s \,. \tag{B.1}$$

*Then, there exists $0 < \delta' \leq \delta$ and a symplectic transformation $\breve\phi : \breve w \in B^{2m}_{\delta'} \to w \in B^{2m}_\delta$ which puts $H$ into Birkhoff normal form up to the order $2s$, i.e.,*

$$H \circ \breve\phi = c_0 + \Omega \cdot \breve{\mathrm{r}} + \sum_{2 \leq h \leq s} P_h(\breve{\mathrm{r}}) + o(|\breve w|^{2s})] \,, \tag{B.2}$$

*where $P_h$ are homogeneous polynomials in $\breve{\mathrm{r}}_j = |\breve w_j|^2/2 := (\breve u_j^2 + \breve v_j^2)/2$ of degree $h$.*

Less known is that the hypotheses of this proposition may be loosened in the case of *rotation invariant Hamiltonians*: this fact, for example, has been used neither in [1] nor in [17].

First, let us generalize the class of Hamiltonian functions so as to include the secular Hamiltonian (2.13): let us consider an open, bounded, connected set $U \subseteq \mathbb{R}^n$ and consider the phase space $\mathcal{D} := U \times \mathbb{T}^n \times B^{2m}_\delta$, endowed with the standard symplectic form $dI \wedge d\varphi + du \wedge dv$.

We say that *a Hamiltonian $H(I, \varphi, w)$ on $\mathcal{D}$ is rotation invariant* if $H \circ \mathcal{R}^g = H$ for any $g \in \mathbb{T}$, where $\mathcal{R}^g$ is a *symplectic rotation by an angle $g \in \mathbb{T}$* on $\mathcal{D}$, i.e., a symplectic map of the form $\mathcal{R}^g : (I, \varphi, w) \to (I', \varphi', w') \quad \text{with} \quad I'_i = I_i, \ \varphi'_i = \varphi_i + g, \ w' = \mathcal{S}^g w$, with $\mathcal{S}^g$ definined in (2.11).

Now, consider a $\varphi$–independent real–analytic Hamiltonian $H\ :\ (I,\varphi,w)\in\mathcal{D}\to H(I,w)\in\mathbb{R}$ of the form $H(I,w)=c_0(I)+\Omega(I)\cdot\mathrm{r}+o(|w|^2;I)$, by $f=o(|w|^2;I)$ we mean that $f=f(I,w)$ and $|f|/|w|^2\to 0$ as $w\to 0$.

Then, it can be proven the following

**Proposition B.2.** *Assume that $H$ is rotation–invariant and that the first order Birkhoff invariants $\Omega_j$ verify, for all $I\in U$, for some $a>0$ and integer $s$*

$$|\Omega\cdot k|\geq a>0,\quad \forall\, 0\neq k\in\mathbb{Z}^m:\sum_{i=1}^n k_i=0\ \text{ and }\ |k|_1\leq 2s\,. \tag{B.3}$$

*Then, there exists $0<\delta'\leq\delta$ and a symplectic transformation $\breve\phi\ :\ (I,\breve\varphi,\breve w)\in\breve{\mathcal{D}}:=U\times\mathbb{T}^n\times B^{2m}_{\delta'}\to(I,\varphi,w)\in\mathcal{D}$ which puts $H$ into Birkhoff normal form up to the order $2s$ as in* (B.2) *with the coefficients of $P_h$ and the reminder depending also on $I$. Furthermore, $\breve\phi$ leaves the $I$–variables fixed, acts as a $\breve\varphi$–independent shift on $\breve\varphi$, is $\breve\varphi$–independent on the remaining variables and is such that*

$$\breve\phi\circ\mathcal{R}^g=\mathcal{R}^g\circ\breve\phi\,. \tag{B.4}$$

The proof of Proposition B.2 may be found in §7.2 in [11].

## C. Arnold's statements (from [1])

- *Conditionally periodic motions in the many–body problem have been found. If the masses of $n$ "planets" are sufficiently small in comparison with the mass of the central body, the motion is conditionally periodic for the majority of initial conditions for which the eccentricities and inclinations of the Kepler ellipses are small. Further, the major semiaxis perpetually remain close to their original values and the eccentricities and inclinations remain small.* [1, p. 87]
- *With the help of the fundamental theorem*[26] *of Chapter IV, we investigate in this chapter the class of "planetary" motions in the three–body and many–body problems. We show that, for the majority of initial conditions under which the instantaneous orbits of the planets are close to circles lying in a single plane, perturbation of the planets on one another produces, in the course of an infinite interval of time, little change on these orbits provided the masses of the planets are sufficiently small.*
  In particular, it follows from our results that *in the $n$-body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities of the bodies belong to this set, the distances of the bodies from each other will remain perpetually bounded.* [1, p.125]
- At p. 127 one finds Theorem 3.1 reported at the beginning of § 3 above.
- As mentioned in the introduction, Arnold provides a full detailed proof, checking the non–degeneracy conditions of his fundamental theorem, only for the two–planet model ($n=2$) in the planar regime. As for generalizations, he states:

[26] I.e., Theorem 3.2 above.

- **The plane problem of $n > 2$ planets.** *The arguments of §2 and 3 easily carry over to the case of more than two planets. [$\cdots$] We shall not dwell on the details of the calculations which lead to the results of §1, 4.* [1, p. 139]
- Finally, for the spatial general case:

  *The rather lengthy calculations involved in the solution of (3.5.9), the construction of variables satisfying conditions 1)–4), and the verification of non–degeneracy conditions analogous to the arguments of § 4 will not be discussed here.* [1, p. 142]

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