Non-simple polarised abelian surfaces

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based on a joint work with Robert Auffarth

Background

A general complex abelian surface does not contain any elliptic curves. In the end of 19th century G.Humbert investigated, among others, principally polarised abelian surfaces containing elliptic curves. He has shown that, in the moduli $\mathcal{A}_2$, there are countably many irreducible components $\mathcal{E}(c)$ of loci of surfaces that contain elliptic curves. They are parametrised by the discriminant $\Delta = c$ of the *singular equation*. Note that $\Delta = 1$ is the locus of products of elliptic curves.

In coordinates of the Siegel space

$$\mathfrak{h}_2 = \{Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} : Im(Z) > 0\}$$

the loci $\mathcal{E}(c)$ satisfy singular relations $a_1z_1 + a_2z_2 + a_3z_3 + a_4(z_2^2 - z_1z_3) + a_5 = 0$ for some integers $(a_1, a_2, a_3, a_4, a_5)$ without a common divisor satisfying $\Delta = a_2^2 - 4a_1a_3 - 4a_4a_5 = c^2$ for some integer c.

In such a case, the surface $X_Z = \mathbb{C}^2/(\mathbb{Z}^2 + Z\mathbb{Z}^2)$ contains two complementary elliptic curves both of exponent c.

Examples in non principally-polarised case

A possible naive approach is to say that being non-simple is invariant under isogeny, hence a $(1, d)$ polarised case is similar to the principal case. One only need to tweak a singular equation accordingly. The following two examples show that the tweaking behaves quite badly and the non-principal setting is actually quite interesting. Firstly we define a moduli space

$$\mathcal{E}_d(m, n) := \left\{(A, \mathcal{L}) \in \mathcal{A}_2(d) : \begin{array}{c} A \text{ contains a pair of complementary} \\ \text{elliptic curves of exponents } m, n \end{array}\right\}.$$

1. For $d = 6$, we have two components of products of elliptic curves, i.e. the equations $z_2 = 0$ and $6z_1 - 5z_2 + z_3 = 0$ (both having $\Delta = 1$) yield abelian surfaces containing complementary elliptic curves of exponents $1, 6$ and $2, 3$ respectively.
2. As a corollary from the Main theorem, one can show that for $d = 90$, the equations $1620z_1 - 81z_2 + z_3 = 0$ (with $\Delta = 9^2$) and $180z_1 - 27z_2 + z_3 = 0$ (with $\Delta = 3^2$) induce the same locus $\mathcal{E}_{90}(18, 45)$ in $\mathcal{A}_2(90)$.

Questions we would like to answer

The exponent of an elliptic (sub)curve $E \subset X$ can be defined as the degree of the restricted polarising line bundle from X to E. For every elliptic curve on a surface, one can find so called *complementary elliptic curve*. In the p.p. case, the complementary exponents coincide (see [3]). This is not the case in general. hence the first question is:

- What are possible complementary exponents on a $(1, d)$ polarised surface? (When $\mathcal{E}_d(m, n)$ is non-empty?)

In the p. p. case Humbert described irreducible components of the moduli of non-simple surfaces. Hence the question:

- Is the moduli $\mathcal{E}_d(m, n)$ irreducible (if non-empty)?

Main theorem

Theorem [1] For $d, m, n \in \mathbb{Z}_+$, the moduli space $\mathcal{E}_d(m, n)$ is a (non-empty) irreducible subvariety of dimension 2 of $\mathcal{A}_2(d)$ if and only if

$$mn \cdot gcd(m, n, d) = gcd(m, n)^2 d.$$

In such a case, we can write $m = \frac{cd}{a}, n = \frac{cd}{b}$ for some pairwise coprime $a, b, c \in \mathbb{Z}_+$. Then, there exist $u, v \in \mathbb{Z}$ satisfying $au - bv = c$ and $\mathcal{E}_d(m, n)$ is the image of the set of period matrices

$$\left\{Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} : Im(Z) > 0, z_3 = \left(\frac{du}{b} + \frac{dv}{a}\right) z_2 - \frac{d^2uv}{ab} z_1\right\}.$$

Idea of the proof

The idea is to use what we know in the principally polarised case and to control the kernel of an isogeny and especially its intersection with elliptic curves.

Let $X \in \mathcal{E}(c)$, i. e. there exist complementary elliptic curves E, F of exponent c in X, and let $P \in X$ be of order d. Let $A = X/P$ and $f : X \to A$ be the quotient isogeny to a $(1, d)$ polarised surface. Note that every non-simple surface arises in this way.

Firstly, we reduce possible cases by showing that we can assume that $ker(f) \cap E \cap F = \{0\}$.

Then, we denote $a = |ker(f) \cap E|$, $b = |ker(f) \cap F|$ and we show that a, b, c need to be pairwise coprime.

Now, we assume $c = 1$ and show that any a, b (coprime divisors of d) can appear. This follows from the fact that the sum of independent elements of order $\frac{d}{a}$ and $\frac{d}{b}$ is of order d. With this in mind, it is not hard to show that all such points are equivalent under the action of product of symplectic groups of E and F.

Taking a general c is quite technical. Using the symplectic action on the set of $\frac{cd}{GCD(c,d)}$ torsion points we show that we get exactly one orbit of the action that has properties that we look for.

The equations follows from direct computation and knowledge of Humbert's singular equations.

Remarks concerning higher dimensions

We can answer the questions for non-simple p.p. abelian varieties of any dimension, see [4]. It is also known (see [5]) that the irreducible components of the moduli of non-simple abelian varieties corresponds to orbits of some actions of groups on spaces of torsion points (if non-empty). In [2] we have generalisation of Humbert's singular equations. None of these results can answer the questions we stated.

Our construction does not generalise well, either. Firstly, not every polarisation can be obtained via a cyclic isogeny. Secondly, the subvarieties are of higher dimensions and not p.p. so their torsion points are no longer 'homogenous' under the action of a symplectic group.

Moreover, in general we do not expect any sort of uniqueness of orbits hence irreducible components will depend on more refine invariants (as defined in [5]).

References

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