SURFACES WITH CANONICAL MAPS OF DEGREE 10, 11 AND 14

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ABSTRACT

We present the first examples of complex algebraic surfaces of general type with canonical maps of degree 10, 11 and 14. They are constructed as quotients of a product of two Fermat septics by free actions of the group $\mathbb{Z}_7^2$.

Keywords: Beauville surface, abelian covers, Surface of general type, Canonical map.

BACKGROUND

It is well known that the canonical map of a curve C of genus at least two is either an embedding or of degree two. The latter happens if and only if C is hyperelliptic. For a smooth surface S of general type the situation is more difficult: suppose that the image of the canonical map ϕ_{K_S} is a surface, then *Beauville* observed:

$$d := \deg(\Phi_{K_S}) \leq 9 + \frac{27 - 9q}{p_g - 2} \leq 36.$$

In particular, $d > 27$ if and only if $q = 0$ and $p_g = 3$.

Main Question (M.Lopes-R.Pardini). For every $2 \leq d \leq 36$, does there exist any surface S with $p_g = 3$ and canonical map of degree d?

State of the art. Surfaces S with $3 \leq d \leq 9$ can be obtained as bi-double covers of del Pezzo surfaces of degree d. The only higher degrees, which have been realised, are

$$d = 12, 16, 20, 24, 27, 32 \quad \text{and} \quad 36,$$

thanks to the work of *Gleissner, Nguyen, Persson, Pignatelli, Rito and Tan.*

Our results. Fill the gaps $d = 10, 11$ and 14.

MAIN TOOLS IN THIS FIELD

Most surfaces with canonical map of high degree are constructed as branched abelian covers of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ (or modifications of them) by using Pardini's theory. This allows a precise description of the canonical system for the computation of $d = \deg(\Phi_{K_S})$. Our surfaces fit in this framework, but we can present them in an elementary way as quotients of a product of plane curves.

OUR CONSTRUCTION

We consider the *Fermat septic*

$$F = \{x_0^7 + x_1^7 + x_2^7 = 0\} \subset \mathbb{P}^2$$

together with the $\mathbb{Z}_7^2$-action

$$\varphi(a,b)(x_0 : x_1 : x_2) = (x_0 : \zeta_7^a x_1 : \zeta_7^b x_2).$$

We take a matrix $A \in \mathrm{GL}(2,7)$, such that the diagonal action $\varphi \times (\varphi \circ A)$ of $\mathbb{Z}_7^2$ on the product $F \times F$ is free. Then the quotient surface

$$S := (F \times F)/\mathbb{Z}_7^2$$

is smooth, regular, of general type and has $p_g = 3$. Its canonical system is given by three $\mathbb{Z}_7^2$-invariant holomorphic 2-forms (*bi-quartics*) on $F \times F$, defining the canonical map

$$\Phi_{K_S} : S \dashrightarrow \mathbb{P}^2.$$

We resolve the *indeterminacy* of Φ_{K_S} by a sequence of blow-ups leading to a b.p.f linear system $|M|$. The canonical map is dominant if and only if $M^2 > 0$. In this case

$$d = \deg(\Phi_{K_S}) = M^2.$$

By suitable choices of $A \in \mathrm{GL}(2,7)$, we found examples of surfaces with $d = 10, 11$ and 14.

AN EXAMPLE IN DETAIL

We look at the surface $S = (F \times F)/\mathbb{Z}_7^2$ defined by the action $\varphi \times (\varphi \circ A)$, where

$$A = \begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}.$$

The canonical map, in terms of bi-quartics, is

$$\Phi_{K_S}(x,y) = (x_1 x_2^3 y_2^4 : x_1^2 x_2^2 y_0^3 y_1 : x_0 x_1^3 y_0 y_1 y_2^2)$$

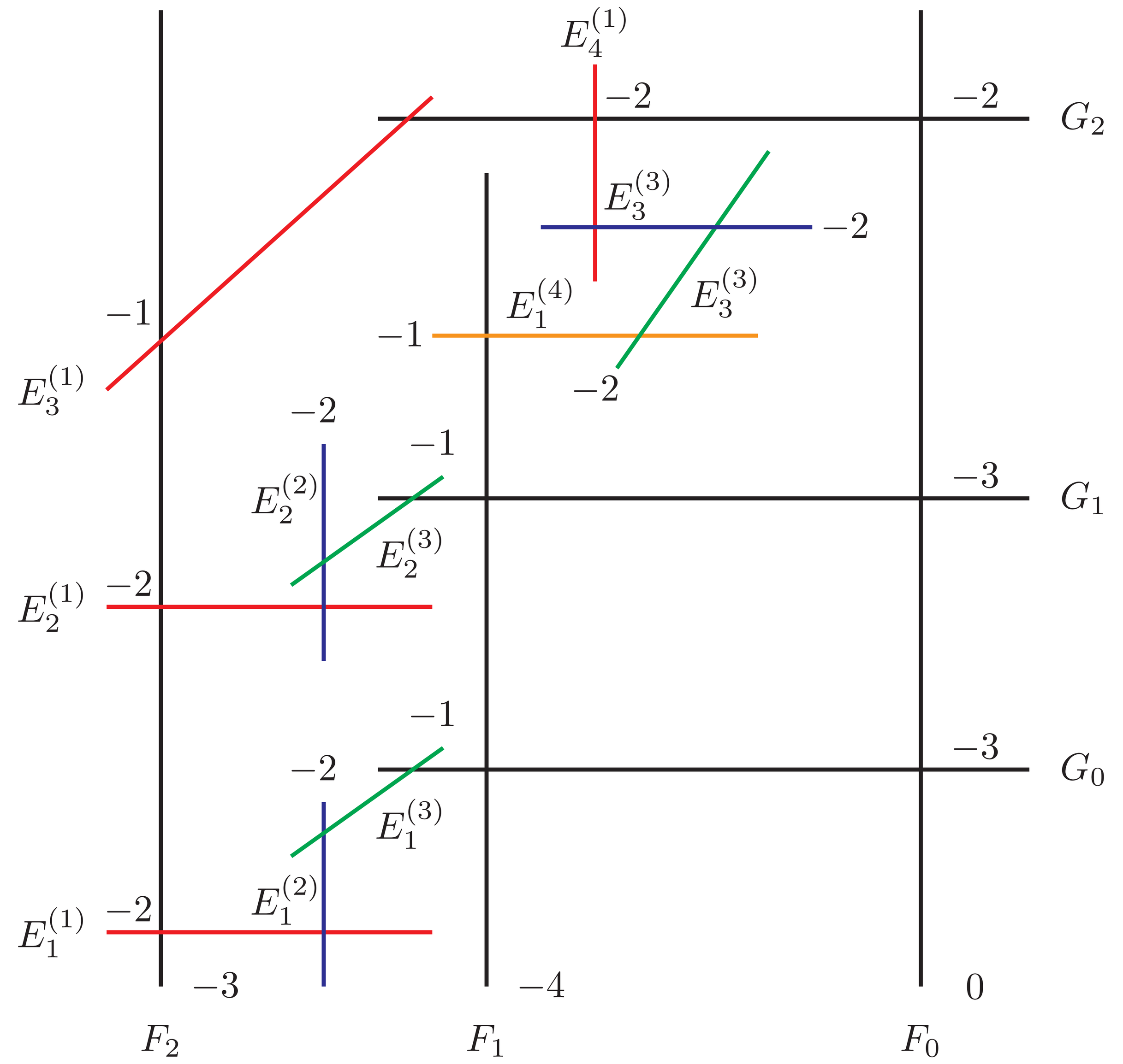
In order to analyse the canonical system, it is convenient to define the curves

$$F_i := \mathrm{div}(x_i), \quad G_j := \mathrm{div}(y_j) \quad \text{on } S.$$

They intersect transversally at only one point. The fixed part of $|K_S|$ is F_1 and the mobile one has 4 base points. In addition to blowing up these points, we need 7 further blow-ups to get a b.p.f. linear system $|M|$. An explicit computation yields

$$d = \deg(\Phi_{K_S}) = M^2 = 10.$$

The configuration of the 11 exceptional curves $E_i^{(j)}$ and the strict transforms of F_i and G_j are illustrated in the picture below.



CONCLUSION AND FINAL REMARKS

- Our surfaces are *Beauville surfaces*. Indeed, $\mathbb{Z}_7^2$ acts on the Fermat septic as the Galois group of the cover

  $$\pi : F \to \mathbb{P}^1, \qquad (x_0 : x_1 : x_2) \mapsto (x_1^7 : x_2^7),$$

  which is branched over $0, -1$ and ∞.
- *Beauville surfaces* are rigid, i.e. they admit no non-trivial deformations of their complex structure.
- There are precisely seven isomorphism classes of *Beauville surfaces* with $p_g = 3$ and abelian Galois group. We could classify them by using a modified version of the MAGMA algorithm from the paper [2].
- All of the seven surfaces can be realised exactly in the same way as we sketched, but for different choices of matrices $A \in \mathrm{GL}(2,7)$.
- One of these surfaces has canonical map composed by a pencil, more precisely the image is the conic section

  $$\{z^2 = xy\} \subset \mathbb{P}^2.$$

  Two of them have canonical map of degree 14. For the remaining four surfaces the degree is

  $$d = 5, 7, 10 \quad \text{and} \quad 11.$$

REFERENCES

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FUTURE RESEARCH & ONGOING PHD OF F. FALLUCCA

- Find further examples and realize new degrees, especially prime degrees.
- Understand the canonical map of product-quotients with non-abelian Galois groups.
- Study the image of the canonical map for surfaces with $p_g \geq 4$.

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