CURVES ON SURFACES WITH TRIVIAL CANONICAL BUNDLE

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Abstract. We survey some results concerning Severi varieties and variation in moduli of curves lying on $K3$ surfaces or on abelian surfaces. A number of open problems is listed and some work in progress is mentioned.

1. Introduction

Curves on $K3$ surfaces became a hot topic in the eighties, where their relevance concerning both Brill-Noether theory and syzygies was understood. Green [G] noticed that his conjecture concerning syzygies of canonical curves would imply constancy of the Clifford index for algebraically equivalent $K3$-sections. This constancy was proved by Green and Lazarsfeld [GL], thus providing evidence for Green's Conjecture, whose validity for a general curve of any given genus was obtained only twenty years later by Voisin [V1, V2] precisely by specialization to curves on $K3$ surfaces. Coming back to the eighties, Lazarsfeld [L] used specialization to curves on $K3$ surfaces in order to exhibit the first proof of the Gieseker-Petri Theorem that avoided any type of degeneration to singular curves. In the same period, Mori and Mukai [MM] proved that a general curve of genus $p \leq 11$ and $p \neq 10$ lies on a $K3$ surface and used this in order to supply explicit parametrizations in low genus of the moduli space $\mathcal{M}_p$ classifying smooth irreducible curves of genus p [M1]. In genus 10, the locus of curves lying on $K3$ surfaces defines a divisor in $\mathcal{M}_{10}$ that was used by Farkas and Popa in order to disprove Slope's Conjecture [FP]. The investigation of Severi varieties of nodal curves on $K3$ surfaces was started by Mumford [MM, Appendix] and proves extremely useful for enumerative and modular problems. More recent applications of the study of $K3$-sections concern higher rank Brill-Noether theory [FO, LC] and rational curves on hyperkähler manifolds [CK, KLM2].

Quite surprisingly, a systematic study of curves on abelian surfaces was initiated not long ago in [KLM1], where the authors investigated their Brill-Noether theory, their variation in moduli and nonemptiness of Severi varieties. A first application towards existence of components of some Brill-Noether loci having the expected codimension in $\mathcal{M}_p$ was also provided.

This paper is aimed to survey some of the above results, without any claim of being exhaustive. The focus will be on open problems, and we will skip most of the proofs.

The first section deals with the study of Severi varieties: first the results of Mumford and Chen concerning the case of $K3$ surfaces will be recalled, and then the recent proof of nonemptiness in the abelian case will be sketched. In the $K3$ case the main proof techniques are degeneration to the union of two rational normal scrolls meeting transversally along a smooth anticanonical elliptic curve and specialization to specific elliptic $K3$ surfaces. On the other hand, in the case of abelian surfaces one may either degenerate the surface to a semiabelian surface (whose construction is reviewed in the proof of Theorem 2.5) or use isogenies from surfaces with a principal polarization. We will pose some questions and mention some work in progress.

The second section concerns variation in moduli of curves lying on $K3$ or abelian surfaces. The problem of determining the dimension of the family of $K3$ surfaces containing the same curve as (canonical) hyperplane section is strictly connected with the Gaussian map of the curve itself. This was studied by Ciliberto, Lopez and Miranda [CLM1], who proved that a general $K3$-section of any given genus $p \geq 13$ or $p = 11$ lies on a unique $K3$ surface by exploiting degeneration to cones over canonical curves. The topic is very current because of the recent characterization by Arbarello, Bruno and Sernesi [ABS2] of Brill-Noether-Petri general curves lying on $K3$ surfaces in terms of nonsurjectivity of their Gaussian map. This characterization had been conjectured by Wahl [W3] in the nineties. Another new relevant result is the accomplishment by Arbarello, Bruno and Sernesi [ABS2] of Mukai's program for reconstructing a general $K3$ surface starting from a curve lying on it. As regards curves on abelian surfaces, it was proved in [KLM1] that a general curve of genus $p \geq 2$ lying on some abelian surfaces as hyperplane section only lies on finitely many of them. The proof makes use of isogenies from principally polarized abelian surfaces along with a degeneration argument similar to the one used in [CFGK] in order to study variation in moduli of curves admitting a nodal model on a $K3$ surface. A number of open problems will be collected at the end of this section, as well.

The choice of omitting Brill-Noether theory from this survey is motivated from the abundance of literature concerning this theme. We mention Aprodu's survey [A] for the case of $K3$-sections, while we refer to [KLM1] for the Brill-Noether theory of curves on abelian surfaces.

2. Existence of nodal curves

Let S be a smooth irreducible projective surface and let $L \in \mathrm{Pic}(S)$ be a polarization on it. We consider curves in the linear system $|L|$ and define, for any fixed integer $\delta \in \mathbb{Z}^{\geq 0}$, the *Severi variety of δ-nodal curves in $|L|$* as the locally closed subscheme:

$$|L|_\delta := \{C \in |L| \,|\, C \text{ is integral and has } \delta \text{ nodes as its only singularities}\}.$$

Since having a node at a prescribed point imposes three conditions on curves in $|L|$, by moving the point on the surface one shows that

$$\mathrm{expdim}|L|_\delta = \dim |L| - \delta;$$

one says that $|L|_\delta$ is regular if it is smooth of the expected dimension. If $\omega_S \simeq \mathcal{O}_S$, then the following holds:

Proposition 2.1 (proof of Prop. 1.1 and 1.2 in [LS])**.** *If $\omega_s \simeq \mathcal{O}_S$, then $|L|_\delta$ is regular as soon as it is nonempty.*

The above result applies to both $K3$ surfaces and abelian surfaces.

2.1. **Nodal curves on $K3$ surfaces.** Nonemptiness of Severi varieties on a general primitively polarized $K3$ surface was first proved by Mumford [MM, Appendix]; the result was then generalized by Chen to nonprimitive linear systems:

Theorem 2.2 (Chen [Ch1])**.** *Let (S, L) be a general polarized $K3$ surface and denote by p the arithmetic genus of all curves in $|L|$ (i.e., $c_1(L)^2 = 2p-2$). Then, for every $0 \leq \delta \leq \dim |L| = p$ the Severi variety $|L|_\delta$ is nonempty and regular.*

By standard deformation theory, every node of a curve in $|L|$ can be smoothed independently, and hence Theorem 2.2 amounts to proving the existence in $|L|$ of a nodal rational curve. In [Ch1] this was done by exploiting the degeneration (first introduced by Ciliberto, Lopez and Miranda in [CLM1]) of S to the union of two rational normal scrolls meeting transversally along a smooth anticanonical elliptic curve. By the same method, Chen also obtained the following stronger result in the primitive case:

Theorem 2.3 (Chen [Ch1, Ch2])**.** *Let (S, L) be a general primitively polarized $K3$ surface. Then, all rational curves in the linear system $|L|$ are nodal.*

The proof of the above theorem was simplified in [Ch2] by specialization to a $K3$ surface S' with an elliptic fibration $\pi : S' \to \mathbb{P}^1$ having a unique section σ and 24 rational nodal fibers. All curves C in the linear system $|\sigma + pF|$ (where F is the class of a fiber) have arithmetic genus p and consist of the union of σ with p fibers; if C is rational then its p fibers run among the 24 rational fibers of π, possibly counted with multiplicity. The problem then translates into showing that, even when nonreduced, such a rational C deforms to a rational nodal curve on a general genus p polarized $K3$

surface. We recall that the special $K3$ surface S' had been earlier used by Bryan and Leung [BL] for enumerative problems of curves on $K3$ surfaces.

Theorem 2.3 has been very recently applied in order to show that every singular curve in $|L|$ deforms to a nodal curve with the same geometric genus [Ch3]. The same is expected to hold for nonprimitive linear systems as well:

Conjecture 2.4 (Dedieu-Sernesi [DS]). *Let (S, L) be a (very) general polarized $K3$ surface of genus p and let $0 \leq g \leq p$. Then every singular curve in $|L|$ of geometric genus g lies in the closure of the Severi variety $|L|_{p-g}$.*

The above conjecture was proved in [DS, Thm. (B.4)] for $g > 0$ under the assumption that the normalization of a general curve of fixed geometric genus in $|L|$ is non-trigonal; however, in the nonprimitive case we are not able to verify this condition even for a general $K3$ surface.

2.2. **State of the art on abelian surfaces.** It is natural to investigate the same questions for curves on abelian surfaces.

Theorem 2.5 ([KLM1], Thms. 1.1 and 1.3). *Let (S, L) be a general polarized abelian surface of type $(1, n)$ and denote by $p := n + 1$ the arithmetic genus of all curves in $|L|$. Then:*

(i) *for any integer δ such that $0 \leq \delta \leq \dim |L| = p - 2$, the Severi variety $|L|_\delta$ is nonempty and regular;*
(ii) *for any $5 \leq g \leq p$, a genus g curve in $|L|$ can be deformed to a nodal curve in $|L|_{p-g}$.*

The inequality $g \geq 5$ in (ii) is only due to proof technique, that relies on Brill-Noether theory for singular curves in $|L|$. Indeed, by [DS, Thm. (B.6)] in order to prove point (ii) it is enough to show that the normalization of a general curve of fixed geometric genus g in $|L|$ is non-trigonal; this is done for $g \geq 5$ by bounding the dimension of families of rational curves in the generalized Kummer varieties $K^{[k]}(S)$. We spend some more words on the proof of point (i).

Sketch of proof of Theorem 2.5(i). The proof proceeds by degeneration to a $(1, n)$-polarized semiabelian surface (S_0, L_0) constructed in the following way. We start with an elliptic curve E and denote by $\oplus$ the group operation on it and by P_0 the identity element. We fix a non-torsion point $e \in E$ and define the line bundle $\mathcal{L} := \mathcal{O}_E(ne - nP_0)$. The $\mathbb{P}^1$-bundle

$$R := \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}) \xrightarrow{\pi} E$$

is a ruled surface having two sections σ_0 and σ_∞ of self-intersection 0 that are both identifiable with E. We glue the sections σ_0 and σ_∞ by means of a translation by the fixed point e; in other words we glue any point $P \in \sigma_\infty \simeq E$ to the point $P \oplus e \in \sigma_0 \simeq E$. The resulting surface S_0 is singular along $\sigma_0 \equiv \sigma_\infty$. As proved by Hulek and Weintraub in [HW], S_0 is limit of abelian surfaces and there is a line bundle L_0 on S_0 that is limit of polarizations of type $(1, n)$; if $\nu : R \to S_0$ denotes the normalization map, then $\nu^* L_0 \equiv \sigma + nF$, where F is the numerical equivalence class of a fiber of π and σ is the class of the section σ_0 (or σ_∞). The linear subsystem $|W| := \nu^*|L_0|$ then parametrizes curves $X \in |\nu^* L_0|$ such that

$$P \in X \cap \sigma_\infty \iff P \oplus e \in X \cap \sigma_0. \tag{1}$$

For a fixed $0 \leq \delta \leq n - 1$, we consider the Severi variety $|L_0|_\delta$ parametrizing curves with only nodes as singularities, exactly δ of which lie off the singular locus of S_0 and are non-disconnecting; we call such nodes the δ *marked nodes*. It is not difficult to show that a curve $X \in |W|_\delta := \nu^*|L_0|_\delta$ is the union of δ fibers $F_1, \ldots, F_\delta$ and a unique component Γ that is not a fiber. The curve Γ is a section of π and the intersection points $\Gamma \cap F_i$ for $1 \leq i \leq \delta$ are mapped to the δ marked nodes of $\nu(X)$.

A set of fibers $\{f, f \oplus e, \dots, f \oplus e^{\oplus(h-1)}\} \subset R$ is called an *h-sequence of fibers.* We associate to a curve $X = \Gamma \cup F_1 \cup \dots \cup F_\delta \in |W|_\delta$ a sequence of n integers $(\alpha_0(X), \dots, \alpha_{n-1}(X))$ by setting

$$\alpha_h(X) := \#\{h\text{-sequences of fibers in } \{F_1, \dots, F_\delta\} \text{ not contained in any } (h+1)\text{-sequence}\}, \text{ for } h \geq 1,$$

$$\alpha_0(X) := n - \sum_{i=1}^{n-1}(i+1)\alpha_i.$$

The definition of α_0 is motivated by the fact that $n = \#\{X \cap \sigma_0\}$ and a h-sequence of fibers $\{f, f \oplus e, \dots, f \oplus e^{\oplus(h-1)}\}$ contained in X determines $h+1$ intersection points of $X \cap \sigma_0$, namely, $P := f \cap \sigma_0, \dots, P \oplus e^{\oplus(h-1)} = (f \oplus e^{\oplus(h-1)}) \cap \sigma_0$ and the point $P \oplus e^{\oplus h}$, that is forced to lie in $\Gamma \oplus \sigma_0$ by (1); the integer α_0 thus coincides with the number of intersection points $Q \in \Gamma \oplus \sigma_0$ such that $Q \ominus e \in \Gamma \oplus \sigma_\infty$, where $\ominus$ denotes the inverse operation of $\oplus$.

For any $0 \leq \delta \leq n-1$ there exist n-uples of nonnegative integers $(\alpha_0, \dots, \alpha_{n-1})$ satisfying

$$\sum_{i=0}^{n-1}(i+1)\alpha_i = n, \text{ and } \sum_{i=1}^{n-1} i\alpha_i = \delta,$$

(e.g., set $\alpha_0 := n - \delta - 1$, $\alpha_\delta := 1$ and $\alpha_i = 0$ for $i \neq 0, \delta$.). One proves (cf. [KLM1, Lem. 3.6]) that for any such n-uple the variety

$$V(\alpha_0, \dots, \alpha_{n-1}) = \{\nu(X) \in |L_0|_\delta \text{ with } (\alpha_0(X), \dots, \alpha_{n-1}(X)) = (\alpha_0, \dots, \alpha_{n-1})\}$$

in nonempty and fills up one or more regular components of $|L_0|_\delta$. In order to conclude it is then enough to show that curves in $V(\alpha_0, \dots, \alpha_{n-1})$ deform with their δ marked nodes while smoothing the surface S_0. □

2.3. **Work in progress and open problems.** Theorem 2.5(i) solves the nonemptiness problem of Severi varieties on Abelian surfaces only for primitive polarizations.

Problem 2.6. *Given a general abelian surface S with polarization $L$of type (n_1, n_2) with $n_1 > 1$, when is the Severi variety $|L|_\delta$ nonempty?*

Analogously, Theorem 2.5(ii) leaves the following questions open.

Problem 2.7. *Does Theorem 2.5(ii) hold for any genus $g \geq 2$? In particular, is any genus 2 curve in the primitive linear system $|L|$ nodal?*

Problem 2.8. *Can one generalize Theorem 2.5(ii) and answer the same questions as in Problem 2.7 for nonprimitive polarizations L of type (n_1, n_2) with $n_1 > 1$?*

Concerning Problem 2.8 there is no hope to prove that for all types of polarizations all genus 2 curves on a general abelian surface are nodal, as the following example shows:

Example 2.9 ([DS], Ex. 4.17)**.** Let (A, M) be a principally polarized abelian surface, that is, $A \simeq J(C)$ for some genus 2 curve C and the unique divisor $\Theta \in |M|$ can be identified with C. Under this identification, the six Weierstrass points of C lie in the subgroup $A[2]$ of 2-torsion points of A. The kernel of the multiplication by two

$$m_2 : A \longrightarrow A \tag{2}$$

coincides with $A[2]$ and $m_2^*(M) \simeq M^{\otimes 4}$. The curve $m_2^{-1}(m_2(\Theta))$ is the union of all translates of Θ by points in $A[2]$ and thus lies in a translate of $|M^{\otimes 16}|$. Since the pullback $m_2^* : \mathrm{NS}(A) \to \mathrm{NS}(A)$ is injective and both $M^{\otimes 4}$ and $\mathcal{O}_A(m_2(\Theta))$ are symmetric, we conclude that up to translation by a 2-torsion point the curve $m_2(\Theta)$ lies in the linear system $|M^{\otimes 4}|$; furthermore, it has geometric genus 2 and has a 6-fold point at the image of the six Weierstrass points of $\Theta \simeq C$.

The above questions are being stressed in a work in progress with Nicolò Sibilla. The strategy is to consider isogenies from principally polarized abelian surfaces. Indeed, given any polarized abelian surface (S, L) of type (n_1, n_2), there exists a principally polarized abelian surface (A, M) along with an isogeny:

$$0 \longrightarrow \mathbb{Z}/n_1 \oplus \mathbb{Z}/n_2 \longrightarrow A \xrightarrow{\alpha} S \longrightarrow 0, \tag{3}$$

such that $\alpha^* L \simeq M^{\otimes n_1 n_2}$. Hence, up to translation by a 2-torsion point, the image $\alpha(\Theta)$ of the theta divisor $\Theta \in |M|$ is a genus 2 curve lying in the linear system $|L|$. Problems 2.6 and 2.7 can then be addressed by analyzing whether, under suitable assumption on the generality of (S, L) and on the integers n_1 and n_2, all singularities of $\alpha(\Theta)$ are nodes. Note that, if S itself is principally polarized and L is four times a principal polarization, the isogeny α in (3) is different from (2) and it is still plausible that $\alpha(\Theta)$ is nodal.

In this context, we recall that any polarization L of type (n_1, n_2) on an abelian surface S induces an isogeny

$$\phi_L : S \longrightarrow \widehat{S} := \mathrm{Pic}^0(S)$$

assigning to a point $x \in S$ the line bundle $L \otimes t_x^*(L)$, where t_x denotes the translation by x on S. The map ϕ_L has kernel $K(L) \simeq \mathbb{Z}_{n_1}^{\oplus 2} \oplus \mathbb{Z}_{n_2}^{\oplus 2}$. Furthermore, given any curve $C \in |L|$ of geometric genus 2, the universal property of the Jacobian $J(\widetilde{C})$ of the normalization $\widetilde{C}$ of C provides an isogeny

$$\lambda : J(\widetilde{C}) \longrightarrow S. \tag{4}$$

The kernel of the dual isogeny $\widehat{\lambda} : \widehat{S} \longrightarrow J(\widetilde{C})$ is a maximal totally isotropic subgroup of the kernel $K(\widehat{L})$ of the isogeny $\phi_{\widehat{L}}$ induced by the dual polarization $\widehat{L}$ on $\widehat{S}$. Concerning Problems 2.7 and 2.8, we expect that the following holds:

Conjecture 2.10. *Let (S, L) be a general polarized abelian surface of type (n_1, n_2) and assume in the linear system $|L|$ there is a genus 2 curve C that is not nodal. Then, the isogeny λ in (4) factors through the multiplication by 2 on $J(\widetilde{C})$. In particular, as soon as 4 does not divide n_1, all genus two curves in $|L|$ are nodal.*

3. Moduli maps

The moduli space $\mathcal{F}_p$ of polarized $K3$ surfaces of genus p is irreducible of dimension 19. We denote by $\mathcal{P}_p$ the parameter space for pairs $((S, L), C)$ where $(S, L) \in \mathcal{F}_p$ and $C \in |L|$ is smooth and irreducible, along with the natural forgetful morphisms:

$$\begin{array}{ccc} & \mathcal{P}_p & \\ {}^{q_p}\swarrow & & \searrow^{f_p} \\ \mathcal{F}_p & & \mathcal{M}_p \, . \end{array} \tag{5}$$

The map q_p realizes $\mathcal{P}_p$ as an open subset of a $\mathbb{P}^p$-bundle over $\mathcal{F}_p$, while f_p sends $((S, L), C)$ to the class of C in the moduli space $\mathcal{M}_p$ of genus p curves.

Analogously, let $\mathcal{A}(1, n)$ be the moduli space of $(1, n)$-polarized abelian surfaces with a suitable level structure that makes it a fine moduli space, and let $\mathcal{P}(1, n)$ be the open subset of a tautological $\mathbb{P}^{n-1}$-bundle over $\mathcal{A}(1, n)$ parametrizing pairs $((S, L), C)$ where (S, L) lies in $\mathcal{A}(1, n)$ and $C \in |L|$ is a smooth and irreducible curve of genus $p := n + 1$. As above, we consider the natural morphisms:

$$\begin{array}{ccc} & \mathcal{P}(1, n) & \\ {}^{q_n}\swarrow & & \searrow^{a_n} \\ \mathcal{A}(1, n) & & \mathcal{M}_p \, . \end{array} \tag{6}$$

3.1. **The $K3$ case and the Gaussian map.** First of all, we recall that $\dim \mathcal{P}_p = 19 + p \geq \dim \mathcal{M}_p = 3p - 3$ if and only if $p \leq 11$; thus, the last inequality is a necessary condition for a general curve of genus p to lie on a $K3$ surface. In fact, Mori and Mukai proved that for $p \neq 10$ this condition is also sufficient:

Theorem 3.1 (Mori-Mukai [MM]). *The map f_p is dominant for $p \leq 11$ and $p \neq 10$.*

In the case $p = 11$, even more is true: f_{11} is birational and, if C is a general curve of genus 11, the unique $K3$ surface on which it lies can be recovered as a Brill-Noether locus in the moduli space of rank-two vector bundles on C with canonical determinant [M2]. We will come back to this construction in the next subsection.

Inspired by the genus 11 case, it makes sense to investigate whether for $p > 11$ the map f_p is birational onto its image. For $p \geq 13$ the answer is affirmative:

Theorem 3.2 (Ciliberto-Lopez-Miranda [CLM1]). *The map f_p is birational onto its image for $p \geq 13$ and $p = 11$.*

The above result relies on properties of the Gaussian map

$$\nu_C : \quad \bigwedge^2 H^0(C, \omega_C) \longrightarrow H^0(C, \omega_C^{\otimes 3}) \tag{7}$$

$$f \wedge g \longrightarrow f \cdot dg - g \cdot df\,.$$

The name Gaussian map comes from the relation between ν_C and the Gauss map

$$\varphi : C \to G(1, p-1),$$

where $G(1, p-1)$ denotes the Grassmannian of lines in $\mathbb{P}^{p-1}$, mapping each point P to the tangent line at P to $C \subset \mathbb{P}^{p-1}$. Let $\phi : C \to \mathbb{P}^N$ denote the composition of φ with the Plücker embedding of the Grassmannian $G(1, p-1)$ into $\mathbb{P}^N$. Then, after applying the isomorphisms

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \simeq \bigwedge^2 H^0(\mathbb{P}^{p-1}, \mathcal{O}_{\mathbb{P}^{p-1}}(1)) \simeq \bigwedge^2 H^0(C, \omega_C),$$

the morphism ν_C coincides with the restriction $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(C, \mathcal{O}_C(1))$ induced by ϕ.

The morphism ν_C is also named after Wahl, who first realized its relevance in the study of curves on $K3$ surfaces by noticing that ν_C cannot be surjective if C lies in the image of f_p [W1]. On the other hand, a general curve of genus p has a surjective Gaussian map if $p = 10$ or $p \geq 12$ [CHM]. Generic injectivity of f_p is translated in terms of Gaussian maps thanks to the following:

Proposition 3.3 (Ciliberto-Lopez-Miranda [CLM1], §5.3). *If a general curve in the image of f_p has a corank one Gaussian map, then f_p is birational onto its image.*

Sketch of proof. The forgetful maps in (5) lift to morphisms at the Hilbert scheme level:

$$\begin{array}{ccc} & \mathfrak{P}_p & \\ {\scriptstyle \mathfrak{q}_p}\swarrow & & \searrow{\scriptstyle \mathfrak{f}_p} \\ \mathfrak{H}_p & & \mathfrak{C}_p\,. \end{array}$$

where $\mathfrak{H}_p$ is the Hilbert scheme of $K3$ surfaces of genus p in $\mathbb{P}^p$, the space $\mathfrak{P}_p$ denotes the flag Hilbert scheme of pairs $C \subset S \subset \mathbb{P}^p$ with $[S \subset \mathbb{P}^p] \in \mathfrak{H}_p$ and C a smooth hyperplane section of it, while $\mathfrak{C}_p$ is the Hilbert scheme of canonical curves of genus p in $\mathbb{P}^p$ (all living in some hyperplane). The fibers of $\mathfrak{f}_p$ have dimension at least $p + 1$, which is the dimension of the space of projectivities fixing an hyperplane. Since a fiber of f_p is the quotient of a fiber of $\mathfrak{f}_p$ by the projective group, it is enough to show that a general fiber of $\mathfrak{f}_p$ is irreducible of dimension $p + 1$.

We recall that for a general $[C \subset S \subset \mathbb{P}^p] \in \mathfrak{P}_p$, any cone X_C over C defines a point of $\mathfrak{H}_p$ (cf. [P]). More precisely, there exists a flat family $\psi : \chi \to \mathbb{P}^1$ such that a general fiber of ψ is isomorphic to an embedding of S in $\mathbb{P}^p$ and a special fiber of ψ is isomorphic to $X_C \subset \mathbb{P}^p$. In particular, the point $x := [C \subset X_C \subset \mathbb{P}^p]$ lies in any component of the fiber of $\mathfrak{f}_p$ over $c := [C \subset \mathbb{P}^p]$. The tangent space of the fiber of $\mathfrak{f}_p$ at x is isomorphic to the space of global sections

$$H^0(X_C, N_{X_C/\mathbb{P}^p}(-C)) \simeq \oplus_{k\geq 1} H^0(C, N_{C/\mathbb{P}^{p-1}}(-k)),$$

where the last isomorphism follows by restricting to the cone minus its vertex and then projecting to C. If the ideal of C is generated by quadrics then $H^0(C, N_{C/\mathbb{P}^{p-1}}(-k)) = 0$ for $k \geq 3$ and, if C is a general curve in the image of f_p, the vanishing $H^0(C, N_{C/\mathbb{P}^{p-1}}(-2)) = 0$ holds as well [CLM1, Lem. 4]. On the other hand, one has

$$H^0(C, N_{C/\mathbb{P}^{p-1}}(-1)) = p + \operatorname{cork} \nu_C. \tag{8}$$

In conclusion, if $\mathrm{cork}\nu_C = 1$, then

$$p + 1 = \dim\, T_x \mathfrak{f}_p^{-1}(c) \geq \dim\, \mathfrak{f}_p^{-1}(c) \geq p + 1,$$

and hence equalities hold. Furthermore, the fiber $\mathfrak{f}_p^{-1}(c)$ is irreducible because otherwise it would be singular at x (which lies in the intersection of all its components). . □

In order to provide curves on $K3$ surfaces as in Proposition 3.3, Ciliberto, Lopez and Miranda [CLM1] performed a double degeneration, first to the union of two rational normal scrolls meeting along an elliptic normal curve and then to a suitable union of planes whose general hyperplane section is a union of lines with corank one Gaussian map.

It is worth spending a few words concerning the exceptional case of genus $p = 10$, where the map f_p has three-dimensional fibers and thus fails to be surjective. It was proved by Cukierman and Ulmer [CU] that the closure of the image of f_{10} coincides with the closure of the divisor of curves in $\mathcal{M}_{10}$ having a nonsurjective Gaussian map. It is natural to investigate whether the nonsurjectivity of the Gaussian map also characterizes curves on $K3$ surfaces in any genus $p \geq 12$. This question was raised in [W3] by Wahl, who related the cokernel of ν_C to deformations of the affine cone over C. A canonical curve $C \subset \mathbb{P}^{p-1}$ is called *extendable* if it is a hyperplane section of a surface $S \subset \mathbb{P}^p$ that is not a cone. Wahl [W3] conjectured that the extendability of a canonical curve of Clifford index at least 3 is equivalent to the nonsurjectivity of the Gaussian map. This conjecture has been proved only very recently by Arbarello, Bruno and Sernesi:

Theorem 3.4 (Arbarello-Bruno-Sernesi [ABS2]). *Let $C \subset \mathbb{P}^{p-1}$ be a canonical curve of genus $p \geq 11$ and* $\mathrm{Cliff}(C) \geq 3$. *Then C is extendable if and only if ν_C is not surjective.*

Note that the extendability of a curve $C \subset \mathbb{P}^{p-1}$ does not ensure that $[C]$ lies in the closure of the image of f_p, as it might be a hyperplane section of a surface S that has some isolated singularities and is not the limit of smooth $K3$ surfaces.

Example 3.5 (Wahl [W3]). Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 7$. It is not difficult to show that ν_C has corank $10 = \dim\, H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2})$ ([W2, Rmk. 4.9]) and hence C is extendable by Theorem 3.4. However, curves of Clifford dimension two cannot lie on any $K3$ surface [CP, Kn]. In this particular case, the surface $S \subset \mathbb{P}^p$ on which C lies can be explicitly constructed as follows. Let ξ denote the set of $3d$ intersection points of C with a general smooth cubic curve $\Gamma \subset \mathbb{P}^2$. The linear system $|I_\xi(d)|$ contains both the smooth curve C and all reducible curves of the form $\Gamma \cup D$ where D is a plane curve of degree $d - 3$. In fact, one has:

$$\dim\, |I_\xi(d)| = \dim\, |\mathcal{O}_{\mathbb{P}^2}(d-3)| + 1 = \frac{(d-1)(d-2)}{2} = g(C) =: p,$$

and $|I_\xi(d)|$ defines a morphism $\varphi : Bl_\xi\mathbb{P}^2 \to S \subset \mathbb{P}^p$, whose image S is a surface having C as (canonically embedded) hyperplane section. It turns out that S has a unique (nonsmoothable) elliptic singularity at the point that is the image under φ of the proper transform $\widetilde{\Gamma}$ of Γ in $Bl_\xi\mathbb{P}^2$.

Smooth plane curves are very special from a Brill-Noether viewpoint. This led Wahl to conjecture that an extendable genus p curve C satisfying the Gieseker-Petri Theorem (i.e., such that for all integers $r, d \geq 1$ the Brill-Noether variety

$$W^r_d(C) := \{A \in \mathrm{Pic}^d(C) \mid \dim\, H^0(C, A) \geq r + 1\}$$

is smooth of dimension equal to the Brill-Noether number $\rho(p, r, d) := p - (r + 1)(p - d + r)$) always lies in the image of f_p. Very recently, Arbarello, Bruno and Sernesi [ABS2] proved a slightly weaker statement: under Wahl's assumption,$[C]$ lies in the closure of the image of f_p. Along with Theorem 3.4, this yields:

Theorem 3.6 (Arbarello-Bruno-Sernesi [ABS2])**.** *Let $C \subset \mathbb{P}^{p-1}$ be a canonical curve of genus $p \geq 12$ satisfying the Gieseker-Petri Theorem. Then C lies on a polarized $K3$ surface of genus p, or on a limit thereof, if and only if ν_C is not surjective.*

The above statement is optimal; indeed, there exist some plane curves (the Du Val curves) that are extendable ([ABS2]), satisfy the Gieseker-Petri Theorem ([ABFS]) but do not lie on any smooth $K3$ surface ([AB]).

3.2. **Mukai's program for curves on $K3$ surfaces.** As already mentioned, in the case of genus 11, Mukai [M2] proved birationality of the map f_{11} by exhibiting a rational inverse of it. Let C be a general curve of genus 11 and consider the moduli space $M_C(2, K_C)$ of semistable rank-two vector bundles on C with canonical determinant. The Brill-Noether locus $M_C(2, K_C, 5)$ of vector bundles in $M_C(2, K_C)$ with a space of global sections of dimension at least 5 turns out to be a Fourier-Mukai transform of the unique $K3$ surface on which C lies. In [M3] Mukai also suggested that a similar procedure should make it possible to reconstruct a general polarized $K3$ surfaces of any genus $p \equiv 3 \bmod 4$ starting from a general hyperplane section of it. This program was recently carried out by Arbarello, Bruno and Sernesi [ABS1].

Let C be a general curve in the image of f_p for $p = 2s + 1$ with $s \geq 6$. The first key observation is that the Brill-Noether locus $M_C(2, K_C, s)$ of rank-two vector bundles having canonical determinant and a space of global sections of dimension at least s is always nonempty and of positive dimension. This fact was first realised by Voisin and is unforeseen for $p > 11$, since the expected dimension of $M_C(2, K_C, s)$ is ≤ 0 as soon as $p \geq 13$. The curve C has a one-dimensional family of line bundles A of degree $s + 2$ such that $\dim H^0(C, A) = s + 2$. A general such A is base point free and thus the evaluation map

$$\mathrm{ev}_{S,A} : H^0(C, A) \otimes \mathcal{O}_S \longrightarrow A$$

is surjective. One defines the Lazarsfeld-Mukai bundle $E_{S,A}$ to be the rank-two vector bundle on S dual to the kernel of $\mathrm{ev}_{S,A}$, and the Voisin bundle E_A as the restriction of $E_{S,A}$ to C. If S and C are general, then both $E_{S,A}$ and E_A are stable ([ABS1, Lem. 2.5, Prop. 3.1]). More precisely, $E_{S,A}$ defines a point in the moduli space $M_v(S)$ of $[C]$-stable vector bundles E on S with Mukai vector $v = (2, [C], s)$ (i.e., $\mathrm{rk}\, E = 2$, $c_1(E) = [C]$, $\chi(E) = s + 2$), while $[E_A] \in M_C(2, K_C, s)$. All Voisin's bundles E_A lie in the same irreducible component of $M_C(2, K_C, s)$, that is denoted by $V_C(2, K_C, s)$ and has dimension ≥ 1. We recall that $\dim M_v(S) = 2$ and $M_v(S)$ is a smooth $K3$ surface.

Theorem 3.7 (Arbarello-Bruno-Sernesi [ABS1])**.** *Let $p = 2s + 1$ with s an odd integer ≥ 5, and let $(S, C) \in \mathcal{P}_p$ be general. Then the following hold:*

(a) *restriction to C defines an isomorphism*

$$\sigma : M_v(S) \longrightarrow V := V_C(2, K_C, s)_{\mathrm{red}} \subseteq M_C(2, K_C, s) \tag{9}$$

and hence V is a smooth $K3$ surface;

(b) *there is a polarization h on V such that any polarized abelian surface on which C lies is isomorphic to the Fourier-Mukai transform $M_{\widehat{v}}(V)$ of V where $\widehat{v} = (2, h, s)$.*

The proof strategy is specialization to a $K3$ surface S with Picard number two for which $S \simeq M_v(S)$ (i.e., S is Fourier-Mukai self-dual) by an explicit isomorphism, that clarifies both the isomorphism (9) and the embedding $C \hookrightarrow M_v(S)$. The statement is then proved for a general polarized $K3$ surface by a deformation theoretical argument.

3.3. **Generic finiteness in the abelian case.** Quite surprisingly, the moduli maps in the abelian case (that is, the maps appearing in the diagram (6)) have been studied only a short while ago in [KLM1]. The main result is the following:

Theorem 3.8 ([KLM1])**.** *For every $n \geq 1$, the moduli map a_n in (6) is generically finite. In other words, a general curve of genus $p := n+1 \geq 2$ lying on some $(1,n)$-polarized abelian surfaces only lies on finitely many of them.*

Proof. Let $\overline{\mathcal{P}(1,n)}$ be the partial compactification of $\mathcal{P}(1,n)$ parametrizing pairs $((S,L),C)$ where $(S,L) \in \mathcal{A}(1,n)$ and C is a nodal curve in the linear system $|L|$. The map a_n in (6) extends to a morphism

$$\overline{a_n} : \overline{\mathcal{P}(1,n)} \to \overline{\mathcal{M}_p},$$

that applies $((S,L),C)$ to the class of the stable curve $[C] \in \overline{\mathcal{M}_p}$. By Theorem 2.5 there are points $((S,L),C) \in \overline{\mathcal{P}(1,n)}$ with C of geometric genus 2. We recall (4) and the discussion after that. If $\widetilde{C}$ is the normalization of C, we obtain an isogeny $\lambda : J(\widetilde{C}) \longrightarrow S$. The kernel of the dual isogeny $\widehat{\lambda}$ is a maximal totally isotropic subgroup of the kernel $K(\widehat{L}) \simeq \mathbb{Z}_n \oplus \mathbb{Z}_n$ of the morphism $\phi_{\widehat{L}} : \widehat{S} \longrightarrow S$ induced by $\widehat{L}$. Finiteness (of the number of subgroups) of $K(\widehat{L})$ then yields finiteness of the fiber $\overline{a_n}^{-1}([C])$ and the statement follows by upper semicontinuity. □

Remark 3.9. In [KLM1, Thm. 1.2] a stronger version of Theorem 3.8 is actually achieved. For $2 \leq g \leq p = n+1$, denote by $\mathcal{U}_{g,n} \subset \mathcal{M}_g$ the locus of curves admitting a $(p-g)$-nodal model as hyperplane section of some $(1,n)$-polarized abelian surface. There exists an irreducible component $\mathcal{U}$ of $\mathcal{U}(g,n)$ such that, if $[\widetilde{C}] \in \mathcal{U}$ is general, then both the number of $(p-g)$-nodal models of $\widetilde{C}$ occuring as hyperplane section of some $(1,n)$-polarized abelian surfaces and the number of abelian surfaces on which any such model lies are finite. An earlier, but still quite recent, analogous result for curves on $K3$ surfaces is due to Kemeny [Ke] and, independently, Ciliberto, Flamini, Galati, Knutsen [CFGK].

3.4. **Open problems and work in progress.** Even though Theorem 3.2 establishes generic injectivity of f_p for any genus $p \geq 13$ and $p = 11$, a rational inverse of it has been constructed only for $p \equiv 3 \bmod 4$.

Problem 3.10. *Find a generalization of Mukai's program to all genera $p \geq 13$ and $p = 11$.*

The first step in Mukai's program (i.e., Theorem 3.7(a)) actually works for any odd genus $p = 2s+1$, and the requirement concerning the parity of s is only used in the construction of the polarization h on the surface $V \simeq M_v(S)$. Hence, a solution to Problem 3.10 in odd genus should be attainable and is essentially a technical matter. On the other hand, in order to approach the same problem for even genus, one needs some new ideas since in this case there is no natural candidate for a rational inverse of f_p, at least up to the author's knowledge.

It is sometimes interesting to study the restriction of f_p to the preimage under q_p of a proper closed subset of $\mathcal{F}_p$. For instance, in [FV] the 11-dimensional moduli space $\mathcal{F}_p^{\mathcal{N}}$ of genus p Nikulin surfaces (that is, $K3$ surfaces endowed with a non-trivial double cover branched along eight disjoint rational curves) is considered, along with the restriction $f_p^{\mathcal{N}} : \mathcal{P}_p^{\mathcal{N}} \longrightarrow \mathcal{M}_p$ of f_p to $\mathcal{P}_p^{\mathcal{N}} := q_p^{-1}(\mathcal{F}_p^{\mathcal{N}})$. This map factorizes through a morphism $n_p : \mathcal{P}_p^{\mathcal{N}} \longrightarrow \mathcal{R}_p$ to the Prym moduli space $\mathcal{R}_p$ parametrizing étale double covers of smooth genus p curves; in [FV], the authors showed that the map n_p is dominant for $p \leq 7$ and $p \neq 6$. In a joint work in progress with Knutsen and Verra [KLV], we prove finiteness of n_p in the remaining cases.

Similarly, one may restrict the map f_p to the preimage under q_p of the locus $\mathcal{F}_p^K$ of Kummer surfaces (that is, $K3$ surfaces that arise as desingularization of the quotient of an abelian surface by the involution -1) and ask the following question:

Problem 3.11. *When is the restriction $f_p^K : \mathcal{P}_p^K \longrightarrow \mathcal{M}_p$ of f_p to $\mathcal{P}_p^K := q_p^{-1}(\mathcal{F}_p^K)$ generically finite/generically injective?*

The interest in the above question comes from its likely connection to the problem of generic injectivity of the map a_n in the abelian case, which is still open:

Problem 3.12. *Let $n \geq 2$ and let a_n be the moduli map appearing in (6). Is a_n birational onto its image? If yes, can one explicitily construct a rational inverse of it?*

Let (S, L) be a $(1, n)$-polarized abelian surface and let $K(S)$ be its Kummer surface, that is, the blow up of $S/\langle -1\rangle$ at its 16 singular points. A general curve $C \in |L|$ is neither invariant nor anti-invariant under the involution -1 and hence the restriction of the quotient map $\pi : S \dashrightarrow K(S)$ to C is birational; the image $\overline{C} := \pi(C)$ has $C^2 = 2n$ (ordinary) double points. Assume one may prove that $K(S)$ is the only Kummer surface containing $\overline{C}$ and that there are no other $(2n)$-nodal models of C lying on some Kummer surfaces; then the first part of Problem 2.8 would reduce to investigating whether $\overline{C}$ may lie on some Fourier-Mukai partner of S non-isomorphic to S, that is, another abelian surface with the same Kummer surface as S. Concerning the second part of Problem 2.8, we expects a construction similar to Mukai's program to work for some genera in the abelian case, as well.

Theorems 3.1 and 3.2 have been generalized to nonprimitive linear systems on $K3$ surfaces in [CLM2] again by exploiting the Wahl map and deformation to cones. The careful reader will have noticed that in the abelian case the requirement for the polarization to be primitive is used in the proof of Theorem 3.8 only when applying the non-emptiness result for Severi varieties of genus 2 curves. Therefore, a generalization of Theorem 2.5(i) would solve not only Problem 2.6 but also the following:

Problem 3.13. *Let*

$$\begin{array}{ccc} & \mathcal{P}(n_1,n_2) & \\ {}^{q_{(n_1,n_2)}}\swarrow & & \searrow^{a_{(n_1,n_2)}} \\ \mathcal{A}(n_1,n_2) & & \mathcal{M}_p, \qquad p = n_1 n_2 + 1, \end{array} \tag{10}$$

generalize diagram (6) for nonprimitive linear system of type (n_1, n_2) with $n_1 > 1$. When is the moduli map $a_{(n_1,n_2)}$ dominant/generically finite?

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