A CODIMENSION 2 COMPONENT OF THE GIESEKER-PETRI LOCUS

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ABSTRACT. We show that the Brill-Noether locus $M^3_{18,16}$ is an irreducible component of the Gieseker-Petri locus in genus 18 having codimension 2 in the moduli space of curves. This result disproves a conjecture predicting that the Gieseker-Petri locus is always divisorial.

1. INTRODUCTION

The Gieseker-Petri locus GP_g inside the moduli space of smooth irreducible genus g curves M_g parametrizes all those curves C that possess a line bundle A for which the Petri map $\mu_{0,A} : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee) \to H^0(C, \omega_C)$ is non-injective. By the Gieseker-Petri Theorem, GP_g is a proper subvariety of M_g and, by Clifford's Theorem and the Riemann-Roch Theorem, it breaks up as follows:

$$GP_g = \bigcup_{0<2r\leq d\leq g-1} GP^r_{g,d},$$

where $GP^r_{g,d}$ is its closed subset defined as

$$GP^r_{g,d} := \{[C] \in M_g \mid \exists\, (A, V) \in G^r_d(C) \text{ with } \ker \mu_{0,V} \neq 0\};$$

here, $\mu_{0,V}$ denotes the restriction of $\mu_{0,A}$ to $V \otimes H^0(C, \omega_C \otimes A^\vee)$. Plenty of work has been devoted to the study of the codimension of the Gieseker-Petri locus and this was partially motivated by the following controversial conjecture (cf. [CHF] for a very nice survey of the debate):

Conjecture 1.1. *The Gieseker-Petri locus GP_g has pure codimension 1 in M_g.*

The above conjecture is known to hold for low genera thanks to the work of Castorena for $g \leq 8$ (cf. [Ca]), and the author herself in the range $9 \leq g \leq 13$ (cf. [LC1]). However, in general not very much is known about the dimension of the loci $GP^r_{g,d}$ and their reciprocal position. Note that when the Brill-Noether number $\rho(g, r, d) := g - (r + 1)(g - d + r)$ is negative, the Petri map associated with a g^r_d on a genus g curve is authomatically non-injective for dimension reasons and the locus $GP^r_{g,d}$ coincides with the Brill-Noether locus

$$M^r_{g,d} := \{[C] \in M_g \mid W^r_d(C) \neq \emptyset\}.$$

This is an irreducible divisor when $\rho(g, r, d) = -1$ [EH]. However, as soon as $\rho(g, r, d) \leq -2$, the codimension of $M^r_{g,d}$ in M_g is at least 2 [St]. Hence, Conjecture 1.1 would force any Brill-Noether locus $M^r_{g,d}$ with $\rho(g, r, d) \leq -2$ to be contained in some other loci $GP^s_{g,e}$ filling up a divisorial component of GP_g. In the present paper we disprove this fact:

Theorem 1.2. *The Brill-Noether locus $M^3_{18,16}$ is an irreducible component of the Gieseker-Petri locus GP_{18} having codimension 2 in M_{18}.*

Now we summarize the results in the literature that concern the loci $GP^r_{g,d}$ with $\rho(g,r,d) \geq 0$. It was proved by Farkas [F2, F3] that they always carry a divisorial component. If moreover $\rho(g,r+1,d) < 0$, then $GP^r_{g,d}$ has pure codimension 1 outside $M^{r+1}_{g,d}$ by the work of Bruno and Sernesi [BS]. The problem remains open whether the loci $GP^r_{g,d}$ have pure codimension 1 in M_g as soon as $\rho(g,r,d) \geq 0$; this guess looks more plausible than Conjecture 1.1, even though it is known to hold in very few special cases, namely, when $\rho(g,r,d) = 0$ and for the locus $GP^1_{g,g-1}$ parametrizing curves with a vanishing theta-null.

It is worth spending some words on the reason why our counterexample occurs in genus 18 and not before. Conjecture 1.1 up to genus 13 was proved by verifying that all the loci $GP^r_{g,d}$ whose codimension is either unknown or strictly larger than 1 are contained in some divisorial components of GP_g; the proof realizes on some general inclusions holding in any genus (that we recall here in Proposition 2.1) along with a few ad hoc arguments. However, similar arguments in genus 14 fail to control the codimension 2 Brill-Noether locus $M^3_{14,13}$. This is the first case that highlights the (somehow unexpected) relevance of non-complete linear series in determining the relative position of the loci $GP^r_{g,d}$: it turns out that any genus 14 curve with a g^3_{13} also possesses a non-complete linear series g^2_{13} with non-injective Petri map. In particular, this implies that $M^3_{14,13}$ is contained in $GP^2_{14,13}$. This phenomenon involving non-complete linear series occurs any time that $d - g < \rho(g,r,d) < 0$ (cf. Proposition 2.2). All together, the results in Section §2 suggests 18 to be the lowest genus in which a Brill-Noether locus of codimension ≥ 2 may provide a counterexample to Conjecture 1.1 (cf. Remark 1). Furthermore, the same results in genus 18 reduce Theorem 1.2 to the following:

Theorem 1.3. *There exists a smooth irreducible curve $C \subset \mathbb{P}^3$ of degree 16 and genus 18 such that all the varieties $G^3_{17}(C)$, $G^2_d(C)$ for $14 \leq d \leq 17$ and $G^1_k(C)$ for $10 \leq k \leq 17$ are smooth of the expected dimension.*

A curve C as in the above statement is realized in Section 4 as a section of a smooth quartic $K3$ surface $S \subset \mathbb{P}^3$ of Picard number 2. The Brill-Noether behavior of C is analyzed by means of non-trivial techniques involving higher rank Lazarsfeld-Mukai bundles, that were partially developed in [LC2, LC3]. The definition and some basic properties of Lazarsfeld-Mukai bundles are preliminarly recalled in Section §3, where they are stated in such a way that they hold also for non-complete linear series. In fact, the most involving part in the proof of Theorem 1.3 turns out to be the control of the Petri map associated with non-complete linear series of type g^2_{16} (cf. Proposition 4.7): this because the Lazarsfeld-Mukai bundle associated with a non-complete linear series has non-vanishing h^1 and thus its automorphism group does not always govern the kernel of the Petri map (even if one chooses C to be general in its linear system).

It is very plausible that one may construct counterexamples to Conjecture 1.1 for infinitely many genera using curves on $K3$ surfaces. However, when the genus becomes higher, the number of components of the Gieseker-Petri locus GP_g increases and, given a curve $[C] \in M^r_{g,d}$, it will become more and more challenging to exclude that $[C]$ lies in some divisorial component of GP_g. In particular, if C is contained in a $K3$ surface S, Lazarsfeld-Mukai bundles of very high rank will be involved in the computation.

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2. COMPONENTS OF THE GIESEKER-PETRI LOCUS

In order to determine the irreducible components of GP_g, it is necessary to understand the reciprocal position of the loci $GP^r_{g,d}$. We summarize some inclusions holding in any genus (cf. Section 2 in [LC1]):

Proposition 2.1. *One has that:*

(i) *If* $\rho(g, r, d+1) < 0$, *then* $M^r_{g,d} \subset M^r_{g,d+1}$.
(ii) *If* $\rho(g, r-1, d-1) < 0$ *and* $r > 1$, *then* $M^r_{g,d} \subset M^{r-1}_{g,d-1}$.
(iii) *If* $\rho(g, r, d) \in \{0, 1\}$, *then* $M^r_{g,d-1} \subset GP^r_{g,d}$ *and* $M^{r+1}_{g,d+1} \subset GP^r_{g,d}$.
(iv) *If* $d < \lfloor \frac{g+3}{2} \rfloor$, *then* $M^1_{g,d}$ *is contained in the Brill-Noether divisor* $M^1_{g,(g+1)/2}$ *if* g *is odd and in the divisor* $GP^1_{g,(g+2)/2}$ *if* g *is even. Furthermore, in the latter case any curve in* $GP^1_{g,(g+2)/2}$ *has a base point free* $g^1_{(g+2)/2}$ *for which the Petri map is non-injective.*

Proof. Item (i) is straightforward: if $[C] \in M^r_{g,d}$ and $A \in \mathrm{Pic}^d(C)$ satisfies $h^0(A) \geq r+1$, then for any point $P \in C$ the line bundle $A(P) \in \mathrm{Pic}^d(C)$ satisfies the inequality $h^0(A(P)) \geq r+1$, too.

Item (ii) is [LC1, Lem. 2.2], while item (iv) is proved in [LC1, Lem. 2.3 and Cor. 2.4].

To obtain item (iii) one can proceed as in the proof of [LC1, Lem. 2.5] having the foresight to use a general point of C in order to construct a g^r_d from the g^r_{d-1} or the g^{r+1}_{d+1}, in case the latter is not primitive. □

The above inclusions have been used in [LC1] in order to prove that the Gieseker-Petri locus has pure codimension 1 in M_g for $g \leq 13$. However, already in genus 14 they do not imply the inclusion of the Brill-Noether locus $M^3_{14,13}$ (which has codimension 2 in M_{14}) neither in a component of type $GP^r_{14,d}$ with $\rho(14, r, d) \geq 0$ nor in a Brill-Noether divisor. The following result takes care of this component and thus motivates why our counterexample occurs in genus 18 and not before.

Proposition 2.2. *Let* g, r, d *be integers such that* $\rho(g, r, d) < d - g < 0$. *Then any genus* g *curve with a complete* g^r_d *also possesses a non-complete* g^{r-1}_d *for which the Petri map is non-injective. In particular, this implies the inclusion* $M^r_{g,d} \setminus M^{r+1}_{g,d} \subset GP^{r-1}_{g,d}$.

Proof. The statement is trivial if $\rho(g, r-1, d) < 0$, so we may assume $\rho(g, r-1, d) \geq 0$. We consider a curve $[C] \in M^r_{g,d}$ possessing a complete linear series A of type g^r_d. The kernel of the Petri map $\mu_{0,A}$ has dimension $\geq -\rho(g, r, d) > g - d$. On the other hand, the space Z_r of tensors in $H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee)$ that do not have maximal rank is a Zariski closed subset of codimension equal to $h^0(\omega_C \otimes A^\vee) - r = g - d$; hence, for any linear subspace X of $H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee)$ one has

$$\mathrm{codim}_X(X \cap Z_r) \leq g - d,$$

cf. [Ei]. One obtains the statement setting $X = \ker \mu_{0,A}$. □

Remark 1. When $\rho(g, r, d) = -2$, the inequalities $\rho(g, r, d) < d - g < 0$ imply $d = g - 1$. Furthermore, for $\rho(g, r, d) = -2$ the locus $M^r_{g,d}$ has codimension 2 in M_g, while the codimension of $M^{r+1}_{g,d}$ is strictly larger [St]; hence, no irreducible components of

$M^r_{g,d}$ is contained in $M^{r+1}_{g,d}$ in this case and Proposition 2.2 yields the inclusion $M^r_{g,d} \subset GP^{r-1}_{g,d}$. For instance, we obtain that $M^3_{14,13} \subset GP^2_{14,13}$. In genus 14 also all the other Brill-Noether loci $M^r_{14,d}$ with $\rho(14,r,d) < -1$ are included in some loci $GP^s_{14,e}$ with $\rho(14,s,e) \geq 0$ or in some Brill-Noether divisors thanks to Proposition 2.1.

Analogously, Propositions 2.1 and 2.2 enable us to control all the Brill-Noether loci of codimension ≥ 2 in genus $g \in \{15, 16, 17\}$. In particular, in these genera the Gieseker-Petri locus decomposes as

$$GP_g = \bigcup_{\substack{0<2r\leq d\leq g-1 \\ \rho(g,r,d)\geq -1}} GP^r_{g,d}.$$

However, this does not prove Conjecture 1.1 up to genus 17 since it may still fail for some loci $GP^r_{g,d}$ with $\rho(g,r,d) \geq 0$.

Now we concentrate on the case $g = 18$, where Proposition 2.1 provides the following decomposition of the Gieseker-Petri locus:

$$GP_{18} = GP^3_{18,17} \cup M^3_{18,16} \cup \bigcup_{d=14}^{17} GP^2_{18,d} \cup \bigcup_{k=10}^{17} GP^1_{18,k}; \tag{1}$$

here we have used that $\rho(18,2,14) = 0$ in order to conclude that $M^2_{18,13} \subset GP^2_{18,14}$. Since $\rho(18,3,16) = -2$, then $\mathrm{codim}_{M_{18}} M^3_{18,16} = 2$ (cf. [St]). In order to prove that $M^3_{18,16}$ is an irreducible component of GP_{18}, it is enough to verify that

$$M^3_{18,16} \not\subset GP^3_{18,17} \cup \bigcup_{d=14}^{17} GP^2_{18,d} \cup \bigcup_{k=10}^{17} GP^1_{18,k}. \tag{2}$$

Equivalently, one has to provide a curve C as in Theorem 1.3.

3. Lazarsfeld-Mukai bundles

In this section we recall the definition and basic properties of Lazarsfeld-Mukai bundles extending them to non-complete linear series. Let C be a smooth genus g curve lying on a $K3$ surface S and let (A, V) be a base point free g^r_d on C. The Lazarsfeld-Mukai bundle $E_{C,(A,V)}$ is defined as the dual of the kernel of the evaluation map $V \otimes \mathcal{O}_S \to A$, and thus sits in the following short exact sequence:

$$0 \to V^\vee \otimes \mathcal{O}_S \to E_{C,(A,V)} \to \omega_C \otimes A^\vee \to 0. \tag{3}$$

In particular, $E_{C,(A,V)}$ is globally generated off the base locus of $\omega_C \otimes A^\vee$ and both its Chern classes and cohomology are easily computed from (3):

- $\mathrm{rk} E_{C,(A,V)} = r+1$, $c_1(E_{C,(A,V)}) = C$, $c_2(E_{C,(A,V)}) = d$;
- $h^0(E_{C,(A,V)}) = r+1+h^1(A)$, $h^1(E_{C,(A,V)}) = h^0(A) - r - 1$, $h^2(E_{C,(A,V)}) = 0$.

In particular, $h^1(E_{C,(A,V)}) = 0$ as soon as the linear series (A, V) is complete, that is, $V = H^0(A)$; in this case one denotes $E_{C,(A,V)}$ simply by $E_{C,A}$. If instead (A, V) is non-complete, the vector bundle constructed as universal extension of $E_{C,(A,V)}$ is naturally isomorphic to $E_{C,A}$, as one can easily check by the very definition of Lazarsfeld-Mukai bundles; in other words, the universal extension looks as follows:

$$0 \to H^1(E_{C,(A,V)}) \otimes \mathcal{O}_S \to E_{C,A} \to E_{C,(A,V)} \to 0 \tag{4}$$

with cocyle $\mathrm{id} \in \mathrm{Hom}(H^1(E_{C,(A,V)}), H^1(E_{C,(A,V)}))$.

The following remark will be useful in the sequel:

Remark 2. Let $E_{C,(A,V)}$ be a Lazarsfeld-Mukai bundle on a $K3$ surface S and assume there is a surjective morphism $E_{C,(A,V)} \to N \otimes I_\xi$ for some line bundle $N \in \mathrm{Pic}(S)$ and some 0-dimensional subscheme $\xi \subset S$. Since $E_{C,(A,V)}$ is globally generated off a finite set and $h^2(E_{C,(A,V)}) = 0$, the line bundle N shares the same properties and in particular N is nontrivial. By [SD, Cor. 3.2], line bundles on $K3$ surfaces have no base points outside their fixed components and thus N is globally generated.

This observation can be generalized to higher rank torsion free sheaves using [LC2, Lem. 3.3] in the following way. Let Q be a torsion free sheaf on S endowed with a surjection $E_{C,(A,V)} \to Q$. The sheaf Q is then globally generated off a finite set and satisfies $h^2(Q) = 0$. We can thus apply [LC2, Lem. 3.3] stating that such a Q satisfies $h^0(\det Q) \geq 2$. If the $K3$ surface S contains no (-2)-curves then $\det Q$ has no fixed components and hence is globally generated again by [SD, Cor. 3.2].

For any r, d we denote by $\mathcal{G}^r_d(|C|)$ the variety parametrizing pairs $(C', (A', V'))$ such that $C' \subset S$ is a smooth curve linearly equivalent to C, and $(A', V') \in G^r_d(C')$; there is a natural forgetful map $\pi : \mathcal{G}^r_d(|C|) \to |C|$.

Lazarsfeld-Mukai bundles are used in order to control the injectivity of the Petri map.

Proposition 3.1. *If C is general in its linear system and $(A, V) \in G^r_d(C)$ is base point free, then:*

$$\dim \ker \mu_{0,V} = h^0(E^\vee_{C,(A,V)} \otimes \omega_C \otimes A^\vee) - 1.$$

If moreover (A, V) is complete, then

$$\dim \ker \mu_{0,A} = h^0(E^\vee_{C,A} \otimes E_{C,A}) - 1$$

and $\mu_{0,A}$ is injective if and only if $E_{C,A}$ is simple.

Proof. The statement is proved for complete linear series in [P]. We briefly sketch the proof in order to convince the reader that it works for non-complete linear series, as well. The kernel of $\mu_{0,V}$ is isomorphic to $H^0(C, M_{A,V} \otimes \omega_C \otimes A^\vee)$, where $M_{A,V}$ is the kernel of the evaluation map on the curve $V \otimes \mathcal{O}_C \to A$. On the other hand, one has the following exact sequence:

$$0 \to \mathcal{O}_C \to E^\vee_{C,(A,V)} \otimes \omega_C \otimes A^\vee \to M_{A,V} \otimes \omega_C \otimes A^\vee \to 0. \tag{5}$$

If C is general in its linear system, the latter remains exact when we pass to global section. Indeed, the vanishing of the coboundary map $\delta : H^0(M_{A,V} \otimes \omega_C \otimes A^\vee) \to H^1(\mathcal{O}_C)$ turns out to be equivalent to the surjectivity of the differential of the projection map $\pi : \mathcal{G}^r_d(|L|) \to |L|$ at the point $(C, (A, V))$; the result thus follows from Sard's Lemma. The last part of the statement follows tensoring (3) with $E^\vee_{C,(A,V)}$ and only holds for complete linear series as it requires $h^1(E_{C,(A,V)}) = 0$. □

We now recall the structure of the Lazarsfeld-Mukai bundle associated with a linear series that is obtained restricting a line bundle $M \in \mathrm{Pic}(S)$ to a curve $C \subset S$.

Lemma 3.2 ([LC3] Lemma 4.1). *Let $N \in \mathrm{Pic}(S)$ satisfy $h^0(N) \geq 2$ and $h^1(N) = 0$; also assume that $M := \mathcal{O}_S(C) \otimes N^\vee$ is globally generated and satisfies $h^1(M) = 0$. Then the Lazarsfeld-Mukai bundle $E_{C,M \otimes \mathcal{O}_C}$ sits in the following short exact sequence*

$$0 \to N \to E_{C,M\otimes\mathcal{O}_C} \to E_{D,\omega_D} \to 0, \tag{6}$$

where D is any smooth curve in the linear system $|M|$.

We will also need the following:

Remark 3 ([LC3] Remark 6)**.** Assume there exists a line bundle $N \in \mathrm{Pic}(S)$ such that $h^0(N) \geq 2$ together with an injective morphism $N \hookrightarrow E_{C,A}$ to some Lazarsfeld-Mukai bundle $E_{C,A}$. Then, the linear series $|A|$ is contained in $|(L \otimes N^\vee) \otimes \mathcal{O}_C|$; this coincides with the restriction of $|L \otimes N^\vee|$ to C if $h^1(N) = 0$.

Concerning the Lazarsfeld-Mukai bundle associated with the canonical line bundle, we state the following:

Lemma 3.3. *Let E_{D,ω_D} be the Lazarsfeld-Mukai bundle associated with the canonical line bundle on a smooth irreducible curve $D \subset S$. Then the following hold:*

(i) *E_{D,ω_D} is simple;*
(ii) *E_{D,ω_D} does not depend on the choice of D in its linear system.*

Proof. Sequence (5) along with the obvious vanishing $0 = \ker \mu_{0,\omega_D} \simeq H^0(M_{\omega_D})$ implies that $\mathrm{Hom}(E_{D,\omega_D}, \mathcal{O}_D) = 0$; hence, (i) follows from (3). Having fixed D, we consider the Grassmannian

$$G\left(g(D), H^0(E_{D,\omega_D})\right) \simeq \mathbb{P}(H^0(E_{D,\omega_D})^\vee) \simeq \mathbb{P}^{g(D)}.$$

For a general $\Lambda \in G\left(g(D), H^0(E_{D,\omega_D})\right)$ the cokernel of the evaluation $\Lambda \otimes \mathcal{O}_S \to E_{D,\omega_D}$ is isomorphic to $\mathcal{O}_{D_1}$ for some smooth curve $D_1 \in |D|$; hence, $E_{D,\omega_D} \simeq E_{D_1,\omega_{D_1}}$. The rational map $h : G\left(g(D), H^0(E_{D,\omega_D})\right) \dashrightarrow |D| \simeq \mathbb{P}^{g(D)}$ constructed in this way is injective since E_{D,ω_D} is simple. Hence, it is birational and its image coincides with the open subset of $|D|$ parametrizing smooth and irreducible curves; this proves (ii). □

Remark 4. By [Mu], the moduli space $Sp(c(E_{D,\omega_D}))$ of sheaves on S with the same Chern classes as E_{D,ω_D} is smooth of dimension 0; our remark is equivalent to the statement that $Sp(c(E_{D,\omega_D}))$ only contains one Lazarsfeld-Mukai bundle, namely, E_{D,ω_D} itself.

4. THE GIESEKER-PETRI LOCUS IN GENUS 18

In this section we prove the following theorem that clearly implies Theorem 1.3.

Theorem 4.1. *There exists a smooth $K3$ surface $S \subset \mathbb{P}^3$ such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where H is a hyperplane section of S and C is a smooth curve of genus 18 and degree 16.*

If C is general in its linear system, then all the Brill-Noether varieties $G^3_{17}(C)$, $G^2_d(C)$ for $14 \leq d \leq 17$ and $G^1_k(C)$ for $10 \leq k \leq 17$ are smooth of the expected dimension.

We recall Mori's Theorem (cf. [Mo]) stating that, if $e > 0$ and $g \geq 0$ there is a smooth genus g curve C lying on a smooth quartic surface $S \subset \mathbb{P}^3$ such that $\deg(C) = e$ if and only if either $g = e^2/8 + 1$, or $g < e^2/8$ and $(e, g) \neq (5, 3)$. This ensures the existence of a $K3$ surface S as in the statement since $g = 18 < (\deg(C))^2/8 = (H \cdot C)^2/8 = 32$. Using the intersection numbers $H^2 = 4$, $H \cdot C = 16$ and $C^2 = 2g - 2 = 34$, one can easily verify that S contains neither curves of genus 1 nor (-2)-curves, or equivalently (cf. [F1]), that 0 and -1 are not represented by the quadratic form

$$Q(a, b) := 2a^2 + 16ab + 17b^2. \tag{7}$$

Remark 5. In particular, any effective line bundle L on our $K3$ surface S satisfies $c_1(L)^2 > 0$ and is globally generated by [SD]. Something even stronger holds, namely, L is automatically ample (cf., e.g., [Hu, Corollary 8.1.6]).

From now on, we assume C to be general in its linear system , so that we can apply Proposition 3.1 in order to translate the injectivity of Petri maps on C in terms of Lazarsfeld-Mukai bundles on S. We will study the simplicity of such bundles by analyzing their slope-stability with respect to the line bundle $\mathcal{O}_S(C - H)$.

Lemma 4.2. *Let S be a $K3$ surface as in Theorem 4.1. Then, the line bundle $\mathcal{O}_S(C-H)$ is ample and the following hold:*

(i) *the slope of any line bundle on S with respect to $C-H$ is divisible by 6;*
(ii) *if an effective line bundle $L \in \mathrm{Pic}(S)$ satisfies $\mu_{C-H}(L) = 6$, then $c_1(L) = C-H$;*
(iii) *if an effective line bundle$L \in \mathrm{Pic}(S)$ satisfies $\mu_{C-H}(L) = 12$, then $c_1(L) \in \{H, 2(C-H), 4C-5H\}$.*

Proof. Since $(C-H)^2 > 0$ and $H \cdot (C-H) > 0$, then $\mathcal{O}_S(C-H)$ is effective and thus automatically ample by Remark. Item (i) follows trivially from the intersection numbers $(C-H)\cdot C = 18$ and $(C-H)\cdot H = 12$. Now let L be an effective line bundle (hence, $c_1(L)^2 > 0$) and write $c_1(L) = aH + bC$ for some integers a and b. First assume that $c_1(L)\cdot(C-H) = 6$ and $c_1(L) \neq C-H$. Since $(C-H)^2 = 6$, the Hodge Index Theorem yields either $c_1(L)^2 = 2$ or $c_1(L)^2 = 4$. The former case does no occur since 1 is not represented by the quadratic form (7); the latter case can also be excluded since the system of diophantine equations $2a^2 + 17b^2 + 16ab - 2 = 12a + 18b - 6 = 0$ has no integral solutions. This proves (ii).

We now assume $c_1(L)\cdot(C-H) = 12$ as in (iii), or equivalently, $a = 1+3k$, $b = -2k$ with $k \in \mathbb{Z}$. This contradicts the inequality $c_1(L)^2 > 0$ unless $k \in \{-2,-1,0\}$, thus proving (iii). □

We recall that any slope-stable (with respect to any polarization) coherent sheaf E on S moves in a smooth moduli space of dimension

$$(1-\mathrm{rk}E)c_1(E)^2 + 2\mathrm{rk}E c_2(E) - 2(\mathrm{rk}E)^2 + 2, \tag{8}$$

cf. [Mu]; the Chern classes of E thus satisfy the inequality

$$c_2(E) \geq -\frac{1}{\mathrm{rk}E} + \mathrm{rk}E + \frac{\mathrm{rk}E - 1}{2\mathrm{rk}E}c_1(E)^2, \tag{9}$$

that is slightly stronger than Bogomolov's inequality.

First of all, we study complete pencils on a curve $C \subset S$ as in Theorem 4.1.

Proposition 4.3. *Let $S \subset \mathbb{P}^3$ be a $K3$ surface as in Theorem 4.1. If C is general in its linear system, then C has maximal gonality 10 and, for $10 \leq k \leq 17$, the Brill-Noether variety $G^1_k(C)$ is smooth at all points corresponding to complete pencils.*

Proof. By Theorem 3 in [F1], C has maximal gonality 10. Let A be a complete g^1_k on C with $10 \leq k \leq 17$. By induction on k, we may assume A is base point free. By contradiction, we suppose that the rank 2 Lazarsfeld-Mukai bundle $E = E_{C,A}$ is non-simple. Hence, it cannot be μ_{C-H}-stable and there is a destabilizing short exact sequence:

$$0 \to M \to E \to N \otimes I_\xi \to 0, \tag{10}$$

where $N, M \in \mathrm{Pic}(S)$ satisfy

$$\mu_{C-H}(M) \geq \mu_{C-H}(E) = 9 \geq \mu_{C-H}(N) > 0, \tag{11}$$

with the last inequality following from the fact that N is globally generated and non-trivial by Remark 2. By Lemma 4.2 (i)-(ii), the only possibility is $c_1(N) = C-H$ and $c_1(M) = H$. Since $(C-2H)^2 < 0$, then both $\mathrm{Hom}(M, N\otimes I_\xi) = 0$ and $\mathrm{Hom}(N\otimes I_\xi, M) = 0$. The non-simplicity of E thus yields $E \simeq \mathcal{O}_S(H) \oplus \mathcal{O}_S(C-H)$.

We consider the rational map $h_E : G(2, H^0(E)) \dashrightarrow \mathcal{G}^1_k(|C|)$ mapping a general 2 dimensional subspace $\Lambda \subset H^0(E)$ to the pair (C_Λ, A_Λ), where C_Λ is the degeneracy locus of the (injective) evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \to E$ and $\omega_{C_\Lambda} \otimes A_\Lambda^\vee$ is the cokernel of ev_Λ. The fiber of h_E over $(C, A) \in \operatorname{Im} h_E \subset \mathcal{G}_k^1(|C|)$ is isomorphic to

$$\mathbb{P}\mathrm{Hom}(E, \omega_C \otimes A^\vee) \simeq \mathbb{P}H^0(S, E \otimes E^\vee),$$

which is 1-dimensional. It follows that

$$\dim \operatorname{Im} h_E = 2(h^0(E) - 2) - 1 = 2(g - k + 1) - 1 < g,$$

as $k \geq 10 > (g+1)/2$; in particular, the image of h_E does not dominate the linear system $|C|$. This implies that, if C is general in its linear system and $10 \leq k \leq 17$, the Lazarsfeld-Mukai bundle associated with *any* complete, base point free g_k^1 on C is simple, and thus the statement follows from Proposition 3.1. □

We now treat complete linear series of type g_d^2.

Proposition 4.4. *Let $S \subset \mathbb{P}^3$ be a $K3$ surface as in Theorem 4.1. If C is general in its linear system, then C has no linear series of type g_{13}^2. Furthermore, for $14 \leq d \leq 17$ the Brill-Noether variety $G_d^2(C)$ is smooth at all points parametrizing complete nets .*

Proof. Let C be general in its linear system and $A \in \operatorname{Pic}^d(C)$ be a complete g_d^2 on C with $d \leq 17$; by induction on d, we may assume it to be base point free. By contradiction, suppose that the rank 3 Lazarsfeld-Mukai bundle $E = E_{C,A}$ is non-simple, and hence not μ_{C-H}-stable. We separately analyze two cases.

CASE A: The maximal destabilizing subsheaf of E is a μ_{C-H}-stable rank 2 vector bundle E_1.

We consider the short exact sequence

$$0 \to E_1 \to E \to N \otimes I_\xi \to 0, \tag{12}$$

where $\xi \subset S$ is a 0-dimensional subscheme, $N \in \operatorname{Pic}(S)$ is globally generated and non-trivial by Remark 2, and the following inequalities are satisfied:

$$\mu_{C-H}(E_1) \geq \mu_{C-H}(E) = 6 \geq \mu_{C-H}(N) > 0.$$

Lemma 4.2 (i)-(ii) yields $c_1(N) = C - H$ and $c_1(E_1) = H$. Since $\mu_{C-H}(E_1) = \mu_{C-H}(N)$ and E_1 is stable, then $\operatorname{Hom}(E_1, N \otimes I_\xi) = \operatorname{Hom}(N \otimes I_\xi, E_1) = 0$ (cf. [Fr]). As E is non-simple, then $\xi = \emptyset$ and (12) splits, that is, $E \simeq E_1 \oplus \mathcal{O}_S(C - H)$. By Remark 3, the linear series $|A|$ is then contained in the restriction of $|H|$ to C and thus $d \leq H \cdot C = 16$.

We perform a parameter count like in [LC2] contradicting the generality of C. The stable sheaf E_1 moves in a moduli space $\mathcal{M}_1$ of dimension $4d - 58$, cf. (8). Let $\mathcal{M}_1^\circ$ denote the oben subset of $\mathcal{M}_1$ parametrizing generically generated vector bundles with vanishing H^1 and H^2, and let $p : \mathcal{G}_1 \to \mathcal{M}_1^\circ$ be the Grassmann bundle whose fiber over a point $[E_1] \in \mathcal{M}_1^\circ$ is $G(3, H^0(E_1 \oplus \mathcal{O}_S(C - H)))$. We define a rational map $h_1 : \mathcal{G}_1 \dashrightarrow \mathcal{G}_d^2(|C|)$ mapping a general point $(E_1 \oplus \mathcal{O}_S(C - H), \Lambda) \in \mathcal{G}_1$ to the pair (C_Λ, A_Λ), where C_Λ is the degeneracy locus of the evaluation map

$$ev_\Lambda : \Lambda \otimes \mathcal{O}_S \to E_1 \oplus \mathcal{O}_S(C - H),$$

which is injective for a general $\Lambda \in G(3, H^0(S, E_1 \oplus \mathcal{O}_S(C - H)))$, and $\omega_{C_\Lambda} \otimes A_\Lambda^\vee$ is the cokernel of ev_Λ. A general fiber of p has dimension $60 - 3d$ and the fiber of h_1 over a general point $(C_\Lambda, A_\Lambda) \in \operatorname{Im} h_1$ is isomorphic to the projective line

$$\mathbb{P}\mathrm{Hom}(E_1 \oplus \mathcal{O}_S(C - H), \omega_C \otimes A^\vee) \simeq \mathbb{P}\mathrm{Hom}(E_1 \oplus \mathcal{O}_S(C - H), E_1 \oplus \mathcal{O}_S(C - H));$$

thus, the image of h_1 has dimension $d + 1 \leq 17 < g$ and does not dominate the linear system $|C|$.

CASE B: There is a line bundle $M \in \mathrm{Pic}(S)$ destabilizing E and having maximal slope.

Having maximal slope, the line subbundle $M \subset E$ is saturated and we have a short exact sequence

$$0 \to M \to E \to E/M \to 0, \tag{13}$$

where E/M is a rank 2 torsion free sheaf such that

$$\mu_{C-H}(M) \geq \mu_{C-H}(E) = 6 \geq \mu_{C-H}(E/M).$$

The line bundle $\det E/M$ is globally generated and non-trivial by Remark 2 and thus satisfies $0 < \mu_{C-H}(\det E/M) = 2\mu_{C-H}(E/M) \leq 12$. In particular, by Lemma 4.2(i) either $\mu_{C-H}(\det E/M) = 6$, or $\mu_{C-H}(\det E/M) = 12$.

SUBCASE B1: The bundle E/M in (13) *satisfies $\mu_{C-H}(\det E/M) = 6$.*

Lemma 4.2(ii) yields $c_1(E/M) = C - H$ and $c_1(M) = H$. Since E/M is generically generated, $H^2(E/M) = 0$ and $\mu_{C-H}(E/M) = 3$, then E/M is μ_{C-H}-stable by Lemma 4.2(i). The inequality $\mu_{C-H}(E/M) < \mu_{C-H}(M)$ implies $\mathrm{Hom}(M, E/M) = 0$. We now show that $\mathrm{Hom}(E/M, M) = 0$, too. By contradiction, assume the existence of a non-zero morphism $\alpha : E/M \to M$. The image of α equals $\mathcal{O}_S(H - D) \otimes I_\xi$ for some effective divisor D such that $\mathcal{O}_S(H-D)$ is globally generated and some 0-dimensional subscheme $\xi \subset S$. The stability of E/M yields

$$3 = \mu_{C-H}(E/M) < \mu_{C-H}(H-D)) \leq \mu_{C-H}(H) = 12.$$

Since $\mathcal{O}_S(2H - C)$ is non-effective, Lemma 4.2(ii) implies that $D = 0$. Equivalently, $\mathrm{Im}\alpha \simeq \mathcal{O}_S(H) \otimes I_\xi$ and $\ker\alpha \simeq \mathcal{O}_S(C - 2H) \otimes I_\eta$ for some 0-dimensional subscheme $\eta \subset S$. One gets the contradiction

$$d = c_2(E) = H \cdot (C - H) + c_2(E/M) \geq H \cdot (C - H) + H \cdot (C - 2H) = 20.$$

Therefore, $\mathrm{Hom}(E/M, M) = 0$ and the fact that E is non-simple forces (13) to split, that is, $E \simeq M \oplus E/M$. However, a parameter count as the one performed in Case A (using the fact that E/M moves in a moduli space of dimension $4d - 60$) shows that such a splitting Lazarsfeld-Mukai bundle cannot be associated with a general curve in the linear system $|C|$ as soon as $d \leq 17$.

SUBCASE B2: The bundle E/M in (13) *satisfies $\mu_{C-H}(\det E/M) = 12$.*

Equivalently, we have $\mu_{C-H}(M) = \mu_{C-H}(E) = \mu_{C-H}(E/M) = 6$. Since E/M is generically generated and $H^2(E/M) = 0$ by Remark 2, it is μ_{C-H}-semistable by Lemma 4.2(i). More strongly, Lemma 4.2(ii) ensures that E/M is μ_{C-H}-stable as soon as the vanishing $\mathrm{Hom}(E/M, \mathcal{O}_S(C - H)) = 0$ holds.

By contradiction, let $\alpha \in \mathrm{Hom}(E/M, \mathcal{O}_S(C - H))$ be non-zero. The semistabily of E/M yields $c_1(\mathrm{Im}\alpha) = C - H$. One gets a short exact sequence

$$0 \to \det E/M \otimes (H - C) \otimes I_\eta \to E/M \xrightarrow{\alpha} (C - H) \otimes I_\xi \to 0$$

for some 0-dimensional subschemes ξ and η, and thus

$$c_2(E/M) \geq c_1(E/M) \cdot (C - H) - 6 = 6;$$

hence, by (13), we obtain

$$d = c_1(M) \cdot c_1(E/M) + c_2(E/M) \geq c_1(E/M) \cdot (C - c_1(E/M)) + 6. \tag{14}$$

On the other hand, Lemma 4.2(iii) yields $c_1(E/M) \in \{H, 2(C - H), 4C - 5H\}$. In all the three cases inequalities $d \leq 17$ and (14) are in contradiction. This proves that

E/M is μ_{C-H}-stable and hence $\mathrm{Hom}(M, E/M) = \mathrm{Hom}(E/M, M) = 0$. Since E is non-simple, then $E \simeq M \oplus E/M$ and one falls under Case A. $\square$

The next step consists in studying linear series of type g^3_d.

Proposition 4.5. *Let $S \subset \mathbb{P}^3$ be a $K3$ surface as in Theorem 4.1. If C is general in its linear system, then the following hold:*

- (i) *the Brill-Noether variety $G^3_{17}(C)$ is smooth of the expected dimension;*
- (ii) *the Brill-Noether variety $G^3_{16}(C)$ consists of a unique isolated point corresponding to the line bundle $\mathcal{O}_C(H)$.*

We will first prove the following weaker result:

Lemma 4.6. *Let $S \subset \mathbb{P}^3$ be a $K3$ surface as in Theorem 4.1. If C is general in its linear system, then the Petri map associated with any base point free g^3_{17} on C is injective and the only g^3_{16} is $\mathcal{O}_C(H)$.*

Proof. It is enough to consider complete linear series of type g^3_d for $d = 16, 17$. Indeed, if C admits a g^r_d with $r \geq 4$ and $d = 16, 17$, then it is easy to show that it admits a positive dimensional family of complete (but not necessarily base point free) g^3_{16}. Furthermore, instead of considering complete g^3_{16} and g^3_{17} with base points, we will study complete, base point free linear series of type g^3_d for all values of $d \leq 17$.

Let $E := E_{C,A}$ be a non-simple rank 4 Lazarsfeld-Mukai bundle associated with a complete base point free A on C of type g^3_d for $d \leq 17$ such that the Petri map $\mu_{0,A}$ is non-injective (the last request is automatically satisfied if $d \leq 16$). Since $\mu_{C-H}(E) = 9/2$, Lemma 4.2(i) excludes that E is destabilized by a vector subbundle of rank 3. Hence, only two cases need to be taken in consideration.

CASE A: The maximal destabilizing subsheaf of E is a μ_{C-H}-stable rank 2 vector bundle E_1.

We have a short exact sequence:

$$0 \to E_1 \to E \to E_2 \to 0, \tag{15}$$

where E_2 is a torsion free sheaf of rank 2 satisfying $\mu_{C-H}(E_2) \leq \mu_{C-H}(E) = 9/2$. Again Lemma 4.2(i) forces E_2 to be stable. Therefore, $c_2(E_i) \geq \frac{3}{2} + \frac{1}{4}c_1(E_i)^2$ for $i = 1, 2$ by (9). Furthermore, $\det E_2$ is globally generated and non-trivial by Remark 2 and its slope is bounded above by $2\mu_{C-H}(E) = 9$. Lemma 4.2(i)-(ii) thus implies $c_1(E_2) = C - H$ and $c_1(E_1) = H$. One gets the contradiction

$$17 \geq d = c_2(E) = H \cdot (C - H) + c_2(E_1) + c_2(E_2) \geq 15 + \frac{1}{4}(H^2 + (C - H)^2) = \frac{35}{2}.$$

CASE B: There is a line bundle $N \in \mathrm{Pic}(S)$ destabilizing E and having maximal slope.

The line bundle N is a saturated subsheaf of E and thus sits in a short exact sequence

$$0 \to N \to E \to E/N \to 0, \tag{16}$$

where E/N is a torsion free sheaf of rank 3 such that $\mu_{C-H}(E/N) \leq \mu_{C-H}(E) = 9/2$. By Remark 2, $\det E/N$ is a non-trivial globally generated line bundle whose slope is bounded above by $3\mu_{C-H}(E) = 27/2$. Lemma 4.2 yields that either $c_1(E/N) = C - H$ or $\mu_{C-H}(\det E/N) = 12$.

SUBCASE B1 : The sheaves in (16) *satisfy $c_1(N) = H$ and $c_1(E/N) = C - H$.*

Since $\mu_{C-H}(E/N) = 2$, then E/N is stable by Lemma 4.2 (i) (as it cannot admit a destabilizing quotient sheaf of smaller rank for slope reasons). In particular, from (9) we get $c_2(E/N) \geq \frac{8}{3} + \frac{1}{3}c_1(E/N)^2 = \frac{14}{3}$, and hence:

$$17 \geq d = c_2(E/N) + c_1(N) \cdot c_1(E/N) = c_2(E/N) + H \cdot (C - H) \geq 12 + \frac{14}{3}.$$

The only possibility is thus $c_2(E/N) = 5$ and $d = 17$. We first show that, if C is general in its linear system, then (16) cannot split. The sheaf E/N moves in a moduli space $\mathcal{M}$ of dimension 2, cf. (8); let $\mathcal{M}^\circ$ be its open subset parametrizing generically generated torsion free sheaves with vanishing H^1 and H^2. Over $\mathcal{M}^\circ$ we consider the Grassmann bundle $\mathcal{G}$ whose fiber over a general $[F] \in \mathcal{M}^\circ$ is the 16-dimensional Grassmannian $G(4, H^0(\mathcal{O}_S(H) \oplus F))$. It is enough to remark that the image of the rational map $h : \mathcal{G} \dashrightarrow \mathcal{G}^4_{17}(|C|)$ defined as in the proof of Proposition 4.4 does not dominate $|C|$; this follows because $\dim \mathcal{G} = 18$ and the fibers of h have positive dimension.

Hence, (16) does not split. Note that $\mathrm{Hom}(N, E/N) = 0$ as E/N is μ_{C-H}-stable. The non-simplicity of E implies the existence of a morphism $0 \neq \alpha : E/N \to N \simeq \mathcal{O}_S(H)$. Write $\mathrm{Im}\,\alpha = \mathcal{O}_S(H - D) \otimes I_\xi$ for some effective divisor D and 0-dimensional subscheme ξ. As E/N is μ_{C-H}-stable and globally generated, then $\mathcal{O}_S(H - D)$ is a globally generated line bundle satisfying

$$2 = \mu_{C-H}(E/N) < \mu_{C-H}(H - D)) \leq \mu_{C-H}(H) = 12.$$

By Lemma 4.2, either $D = 0$ or $H - D \equiv C - H$; the latter case can be excluded since it implies $D \sim 2H - C$, which is not effective. We conclude that $D = 0$ and get the following short exact sequence

$$0 \to \mathcal{O}_S(C - 2H) \otimes I_\eta \to E/N \xrightarrow{\alpha} \mathcal{O}_S(H) \otimes I_\xi \to 0$$

for some 0-dimensional subschemes $\xi, \eta \subset S$. This leads to the contradiction $5 = c_2(E/N) \geq H \cdot (C - 2H) = 8$.

SUBCASE B2 : The sheaf E/N in (16) *satisfies* $\mu_{C-H}(\det E/N) = 12$.

We apply Lemma 4.2(iii). The case $c_1(E/N) = 4C - 5H$ does not occur because it would imply $c_1(N) = 5H - 3C$ and thus the contradiction

$$d = c_2(E) \geq (4C - 5H) \cdot (5H - 3C) = 52.$$

Now, assume $c_1(E/N) = 2(C - H)$ and $c_1(N) = 2H - C$. It follows that

$$c_2(E/N) = d - 2(C - H)(2H - C) = d - 12 \leq 5. \tag{17}$$

In particular, E/N cannot be μ_{C-H}-stable because otherwise (9) would imply $c_2(E/N) \geq \frac{8}{3} + \frac{1}{3}c_1(E/N)^2 = \frac{32}{3}$. However, since $\mu_{C-H}(E/N) = 4$ and $h^2(E/N) = 0$, Lemma 4.2(i) excludes that E/N is destabilized by any subsheaf of rank 2. The sheaf E/N is thus destabilized by a subsheaf M of maximal slope and rank 1 and the quotient $Q := (E/N)/M$ is a generically generated torsion free sheaf of rank 2 satisfying $H^2(Q) = 0$. By Remark 2, $\det Q$ is a non-trivial globally generated line bundle such that $\mu_{C-H}(\det Q) = 2\mu_{C-H}(Q) \leq 2\mu_{C-H}(E/N) = 8$; again Lemma 4.2(i)-(ii) yields $c_1(Q) = C - H = c_1(M)$ and $c_2(E/N) \geq (C - H)^2 = 6$, contradicting (17). This excludes the case $c_1(E/N) = 2(C - H)$.

It remains to consider the case $c_1(E/N) = H$ and $c_1(N) = C - H$. Remark 3 then implies that the linear series $|A|$ is contained in $|\mathcal{O}_C(H)|$, which is a complete base point free g^3_{16}. The only possibility is thus $A \simeq \mathcal{O}_C(H)$. □

Proof of Proposition 4.5. Lemma 4.6 implies that any linear series of type g^3_{16} or g^3_{17} is complete. Furthermore, $G^3_{17}(C)$ is smooth at all points corresponding to base point free linear series. In order to conclude, it remains to show that $\dim\ker\mu_{0,\mathcal{O}_C(H)}=2$, or, equivalently by Proposition 3.1, that the Lazarsfeld-Mukai bundle $E=E_{C,\mathcal{O}_C(H)}$ satisfies $h^0(E\otimes E^\vee)=3$. By Lemma 3.2, E sits in the short exact sequence

$$0\to\mathcal{O}_S(C-H)\to E\to E_{H,\omega_H}\to 0, \tag{18}$$

where E_{H,ω_H} is the rank 3 Lazarsfeld-Mukai bundle associated with the canonical sheaf ω_H on any smooth hyperplane section H of S (cf. Lemma 3.3).

We first claim that E_{H,ω_H} is μ_{C-H}-stable. As E_{H,ω_H} is globally generated and satisfies $H^2(E_{H,\omega_H})=0$ and $\mu_{C-H}(E_{H,\omega_H})=4$, then Lemma 4.2(i) implies that it cannot be destabilized by any vector bundle of rank 2. If it is not stable, there exists a destabilizing sequence

$$0\to N\to E_{H,\omega_H}\to E_{H,\omega_H}/N\to 0,$$

with $N\in\mathrm{Pic}(S)$ and $Q:=E_{H,\omega_H}/N$ a globally generated torsion free sheaf of rank 2 satisfying $h^2(Q)=0$. In particular, the line bundle $\det Q$ is globally generated and non-trivial by Remark 2 and satisfies

$$0<\mu_{C-H}(\det Q)=2\mu_{C-H}(Q)\le 2\mu_{C-H}(E_{H,\omega_H})=8;$$

hence, $c_1(Q)=C-H$ by Lemma 4.2(i)-(ii) and $c_1(N)=2H-C$. One gets

$$4=c_2(E_{H,\omega_H})=(C-H)\cdot(2H-C)+c_2(Q)=6+c_2(Q),$$

and this is a contradiction since the second Chern class of a globally generated rank 2 torsion free sheaf on S is always positive. Therefore, E_{H,ω_H} is μ_{C-H}-stable as claimed.

By applying first $\mathrm{Hom}(E,-)$ and then $\mathrm{Hom}(-,\mathcal{O}_S(C-H))$ and $\mathrm{Hom}(-,E_{H,\omega_H})$ to (18) and by remarking that $\mathrm{Hom}(\mathcal{O}_S(C-H),E_{H,\omega_H})=0$ for slope reasons, one shows that

$$2\le\dim\ker\mu_{0,\mathcal{O}_C(H)}=h^0(S,E\otimes E^\vee)-1\le 1+\dim\mathrm{Hom}(E_{H,\omega_H},\mathcal{O}_S(C-H)),$$

and the inequality is strict unless the sequence (18) splits. Therefore, if we prove that $\dim\mathrm{Hom}(E_{H,\omega_H},\mathcal{O}_S(C-H))\le 1$, then $E\simeq\mathcal{O}_S(C-H)\oplus E_{H,\omega_H}$ and $\dim\ker\mu_{0,\mathcal{O}_C(H)}=2$, as desired. Given $0\ne\alpha:E_{H,\omega_H}\to\mathcal{O}_S(C-H)$, there exist an effective divisor D and a 0-dimensional subscheme $\xi\subset S$ such that $\mathrm{Im}\,\alpha=\mathcal{O}_S(C-H-D)\otimes I_\xi$. The line bundle $\mathcal{O}_S(C-H-D)$ is globally generated and its slope is bounded below by $\mu_{C-H}(E_{H,\omega_H})=4$ and above by $\mu_{C-H}(C-H)=6$. Lemma 4.2 thus yields $D=0$ and E_{H,ω_H} sits in the following short exact sequence:

$$0\to K\to E_{H,\omega_H}\to\mathcal{O}_S(C-H)\otimes I_\xi\to 0, \tag{19}$$

where K is a vector bundle of rank 2 such that $c_1(K)=2H-C$, $\chi(K)=l(\xi)-1$ and $c_2(K)=-2-l(\xi)$. Moreover, K is μ_{C-H}-stable because otherwise it would be destabilized by a line bundle N such that

$$3=\mu_{C-H}(K)\le\mu_{C-H}(N)<\mu_{C-H}(E_{H,\omega_H})=4,$$

thus contradicting Lemma 4.2(i). The stability of K implies $c_2(K)\ge\frac{3}{2}+\frac{1}{4}c_1(K)^2=-2$ by (9), hence $l(\xi)=0$. By applying $\mathrm{Hom}(-,\mathcal{O}_S(C-H))$ to the sequence (19), one finds that

$$\dim\mathrm{Hom}(E_{H,\omega_H},\mathcal{O}_S(C-H))\le 1+\dim\mathrm{Hom}(K,\mathcal{O}_S(C-H)).$$

We will now show that $\mathrm{Hom}(K, \mathcal{O}_S(C-H)) = 0$, which concludes the proof. If there exists $0 \neq \beta : K \to \mathcal{O}_S(C-H)$, then $\mathrm{Im}\,\beta = \mathcal{O}_S(C-H-D_1) \otimes I_{\xi_1}$ for some divisor $D_1 \geq 0$ and 0-dimensional subscheme $\xi_1 \subset S$. Since K is stable, then

$$3 = \mu_{C-H}(K) \leq \mu_{C-H}(C-H-D_1) \leq \mu_{C-H}(C-H) = 6,$$

thus $D_1 = 0$ by Lemma 4.2(i) and one gets the following short exact sequence:

$$0 \to \mathcal{O}_S(3H-2C) \to K \xrightarrow{\beta} \mathcal{O}_S(C-H) \otimes I_{\xi_1} \to 0.$$

One gets a contradiction since $-2 = c_2(K) = (3H-2C)\cdot(C-H) + l(\xi_1) \geq 0$. This concludes the proof. □

In order to conclude the proof of Theorem 4.1, it only remains to show that the Brill-Noether varieties of C are smooth of the expected dimension at the points parametrizing non-complete linear series.

Proposition 4.7. *Let $S \subset \mathbb{P}^3$ be a $K3$ surface as in Theorem 4.1. If C is general in its linear system, then any non-complete linear series on C of degree $\leq g-1 = 17$ has injective Petri map.*

Proof. The only complete linear series on C with non-injective Petri-map is $\mathcal{O}_C(H)$. Therefore, it is enough to prove the statement for non-complete linear series of the form $(\mathcal{O}_C(H), V)$ with $\dim V = 3$. Let $E_V := E_{C,(\mathcal{O}_C(H),V)}$ be the associated Lazarsfeld-Mukai bundle. By §3, E_V satisfies $h^1(E_V) = 1$ and sits in the following universal extension, cf. (4):

$$0 \to \mathcal{O}_S \to E_{C,\mathcal{O}_C(H)} \to E_V \to 0. \tag{20}$$

By Proposition 3.1, we need to show that $\mathrm{Hom}(E_V, \mathcal{O}_C(C-H)) = 0$. As an intermediate step, we will first prove that E_V is simple. Short exact sequences (18) and (20) fit in the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S(C-H) & \longrightarrow & E_V & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S(C-H) & \longrightarrow & E_{C,\mathcal{O}_C(H)} & \longrightarrow & E_{H,\omega_H} & \longrightarrow & 0. \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_S & = & \mathcal{O}_S & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the right hand side of the diagram, one deduces that Q coincides with the Lazarsfeld-Mukai bundle $E_W := E_{H,(\omega_H,W)}$ associated with some non complete linear series (ω_H, W) on some hyperplane section H of S. In particular, the bundle E_W is globally generated and $\mu_{C-H}(E_W) = 6$. If E_W were not μ_{C-H}-stable, by Lemma 4.2 it would lie in a short exact sequence

$$0 \to \mathcal{O}_S(2H-C) \to E_W \to \mathcal{O}_S(C-H) \otimes I_\xi \to 0,$$

for some 0-dimensional subscheme $\xi \subset S$, and thus the contradiction

$$4 = c_2(E_W) = (2H - C) \cdot (C - H) + l(\xi) \geq 6.$$

Hence, E_W is μ_{C-H}-stable and $\mathrm{Hom}(\mathcal{O}_S(C-H), E_W) = \mathrm{Hom}(E_W, \mathcal{O}_S(C-H)) = 0$. It follows that E_V is simple unless $E_V \simeq E_W \oplus \mathcal{O}_S(C-H)$. We will now show that, if C is general in its linear system, the Lazarsfeld-Mukai bundle E_V associated with any non-complete linear series $(\mathcal{O}_C(H), V)$ does not split in this way.

Remember that E_{H,ω_H} is rigid by Remark 4. We consider the Quot-scheme $\mathcal{Q} := \mathrm{Quot}_S(E_{H,\omega_H}, P)$, where P is the Hilbert polynomial of E_W. It is well known (cf. [HL] Proposition 2.2.8) that, for any $[E_W] \in \mathcal{Q}$, the following holds:

$$\dim_{[E_W]} \mathcal{Q} \leq \dim \mathrm{Hom}(\mathcal{O}_S, E_W) = h^0(E_W) = 3; \tag{21}$$

and hence the dimension of any component of the Quot-scheme is ≤ 3.

Let $\mathcal{G}_{\mathcal{Q}} \to \mathcal{Q}$ be the Grassmann bundle whose fiber over a general $[E_W] \in \mathcal{Q}$ is the 15-dimensional Grassmannian $G(3, H^0(E_W \oplus \mathcal{O}_S(C-H)))$. We define $h_{\mathcal{Q}} : \mathcal{G}_{\mathcal{Q}} \dashrightarrow |C|$ mapping a general point $(E_W, \Lambda) \in \mathcal{G}_{\mathcal{Q}}$ to the degeneracy locus of the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \to E_W \oplus \mathcal{O}_S(C-H)$, which is a smooth curve $C_\Lambda \in |C|$. The fibers of $h_{\mathcal{Q}}$ are at least 1-dimensional because the composition of ev_Λ with any autormophism of $E_W \oplus \mathcal{O}_S(C-H)$ has the same degeneracy locus C_Λ. Therefore, the image of $h_{\mathcal{Q}}$ has dimension $\leq \dim \mathcal{G}_{\mathcal{Q}} - 1 = \dim \mathcal{Q} + 15 - 1 \leq 17$ and $h_{\mathcal{Q}}$ is not dominant. This shows that, if C is general, the Lazarsfeld-Mukai bundle E_V associated with any non-complete linear series $(\mathcal{O}_C(H), V)$ of type g^2_{16} on C is simple.

In order to conclude the proof, we now show that the simplicity of E_V implies that $\dim \mathrm{Hom}(E_V, \mathcal{O}_C(C-H)) = 1$ and thus the injectivity of the Petri map $\mu_{0,(\mathcal{O}_C(H),V)}$. Consider the short exact sequence defining E_V:

$$0 \to V^\vee \otimes \mathcal{O}_S \to E_V \xrightarrow{f} \mathcal{O}_C(C-H) \to 0. \tag{22}$$

Let $f' : E_V \to \mathcal{O}_C(C-H)$ be a morphism different from f; we may assume that f' is surjective since this is true for f and thus for a general morphism from E_V to $\mathcal{O}_C(C-H)$. We want to show that f' is obtained by composing f with an automorphism of E_V, and is thus a scalar multiple of f since E_V is simple. Equivalently, if we consider the long exact sequence

$$0 \to H^0(E_V \otimes E_V^\vee) \to \mathrm{Hom}(E_V, \mathcal{O}_C(C-H)) \xrightarrow{\delta} \mathrm{Ext}^1(E_V, V^\vee \otimes \mathcal{O}_S)$$

obtained applying $\mathrm{Hom}(E_V, -)$ to (22), we need to to prove that $\delta(f') = 0$. By contradiction, assume $\delta(f') \in \mathrm{Ext}^1(E_V, V^\vee \otimes \mathcal{O}_S)$ is the class of a nontrivial extension

$$0 \to V^\vee \otimes \mathcal{O}_S \to E_1 \to E_V \to 0 \tag{23}$$

that fits (by construction) in the following commutative diagram:

(24)
$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & V^\vee \otimes \mathcal{O}_S & \longrightarrow & E_V & \xrightarrow{f} & \mathcal{O}_C(C-H) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle f'} & & \\
0 & \longrightarrow & V^\vee \otimes \mathcal{O}_S & \longrightarrow & E_1 & \xrightarrow{h} & E_V & \longrightarrow & 0. \\
 & & & & {\scriptstyle g}\uparrow & & \uparrow & & \\
 & & & & K & = & K & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $\mathrm{Ext}^1(E_V, V^\vee \otimes \mathcal{O}_S) \simeq H^1(E_V)^\vee \otimes V^\vee$, the element $\delta(f')$ also corresponds to a non-zero morphism from $H^1(E_V)$ to $V^\vee$. This implies that the extension (23) fits in the following commutative diagram:

(25)
$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_S & = & \mathcal{O}_S & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & V^\vee \otimes \mathcal{O}_S & \longrightarrow & E_1 & \xrightarrow{h} & E_V & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & H^1(E_V) \otimes \mathcal{O}_S & \longrightarrow & E_{C,\mathcal{O}_C(H)} & \longrightarrow & E_V & \longrightarrow & 0, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where the lowest row is (20).

Since $H^1(E_{C,\mathcal{O}_C(H)}) = 0$, the second column in the diagram splits, that is, $E_1 \simeq \mathcal{O}_S \oplus E_{C,\mathcal{O}_C(H)}$. We will obtain a contradiction looking at the maps in diagram (24). Since $h \circ g$ is injective and the $\mathcal{O}_S$-factor of E_1 is contained in the kernel of h, the image of $g : K \to E_1 \simeq \mathcal{O}_S \oplus E_{C,\mathcal{O}_C(H)}$ is contained in $E_{C,\mathcal{O}_C(H)}$ and its cokernel E_V thus splits as $\mathcal{O}_S \oplus \mathrm{Coker}(q \circ g)$, where $q : E_1 \to E_{C,\mathcal{O}_C(H)}$ is the obvious projection. This is a contradiction as $H^2(E_V) = 0$. Therefore, $\delta(f') = 0$ as required.

□

References

[BS] A. Bruno, E. Sernesi, *A note on the Petri loci*, Manuscripta Math. **136** (2011), 439-443.

[Ca] A. Castorena, *Remarks on the Gieseker-Petri divisor in genus eight*, Rend. Circ. Mat. Palermo **59** (2010), 143-150.

[CHF] D. Chen, G. Farkas, I. Morrison, Effective divisors on moduli spaces of curves and abelian varieties, Clay Math. Proc. **18** (2013), 131–169.

[Ei] D. Eisenbud, *Linear sections of determinantal varieties,* Amer. J. Math. **110** (1988), 541–575.

[EH] D. Eisenbud, J. Harris, *Irreducibility of some families of linear series with Brill-Noether Number* -1, Ann. Scient. Éc. Norm. Sup. 4e série **22** (1989), 33-53.

[F1] G. Farkas, *Brill-Noether Loci and the Gonality Stratification of M_g*, J. Reine Angew. Math. **539** (2001), 185-200.

[F2] G. Farkas, *Gaussian maps, Gieseker-Petri loci and large theta-characteristics*, J. Reine Angew. Math. **581** (2005), 151-173.

[F3] G. Farkas, *Rational maps between moduli space of curves and Gieseker-Petri divisor*, J. Algebraic Geom. **19** (2010), 243-284.

[Fr] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer-Verlag Nw York (1998).

[Hu] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge University Press (2016).

[HL] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves (second edition)*, Cambridge University Press (2010).

[LC1] M. Lelli-Chiesa, *The Gieseker-Petri divisor in M_g for $g \leq 13$*, Geom. Ded. **158** (2012), 149-165.

[LC2] M. Lelli-Chiesa, *Stability of rank-3 Lazarsfeld-Mukai bundles on $K3$ surfaces*, Proc. Lond. Math. Soc. **107** (2013), 451–479.

[LC3] M. Lelli-Chiesa, *Generalized Lazarsfeld-Mukai bundles and a conjecture of Donagi and Morrison*, with an Appendix joint with A. L. Knutsen, Adv. Math. **268** (2015), 529–563.

[Mo] S. Mori, *On degrees and genera of curves on smooth quartic surfaces in $\mathbb{P}^3$*, Nagoya Math. J. **96** (1984), 127-132.

[Mu] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface*, Invent. Math. **77** (1984), 101-116.

[P] G. Pareschi, *A proof of Lazarsfeld's Theorem on curves on $K3$ surfaces*, J. Algebraic Geom. **4** (1995), 195-200.

[SD] B. Saint-Donat, *Projective models of $K3$ surfaces*, Amer. J. Math. **96 (1974)**, 602-639.

[St] F. Steffen, *A generalized principal ideal theorem with an application to Brill-Noether theory*, Invent. Math. **132** (1998), 73-89.

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