GREEN'S CONJECTURE FOR CURVES ON RATIONAL SURFACES WITH AN ANTICANONICAL PENCIL

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ABSTRACT. Green's conjecture is proved for smooth curves C lying on a rational surface S with an anticanonical pencil, under some mild hypotheses on the line bundle $L = \mathcal{O}_S(C)$. Constancy of Clifford dimension, Clifford index and gonality of curves in the linear system $|L|$ is also obtained.

1. INTRODUCTION

Green's Conjecture concerning syzygies of canonical curves was first stated in [G] and proposes a generalization of Noether's Theorem and the Enriques-Babbage Theorem in terms of Koszul cohomology, predicting that for a curve C

$$K_{p,2}(C, \omega_C) = 0 \qquad \text{if and only if } p < \text{Cliff}(C). \tag{1}$$

Quite remarkably, this would imply that the Clifford index of C can be read off the syzygies of its canonical embedding. The implication $K_{p,2}(C, \omega_C) \neq 0$ for $p \geq \text{Cliff}(C)$ was immediately achieved by Green and Lazarsfeld ([G, Appendix]) and the conjectural part reduces to the vanishing $K_{c-1,2}(C, \omega_C) = 0$ for $c = \text{Cliff}(C)$, or equivalently, $K_{g-c-1,1}(C, \omega_C) = 0$.

One naturally expects the gonality k of C to be also encoded in the vanishing of some Koszul cohomology groups. In fact, Green-Lazarsfeld's Gonality Conjecture predicts that any line bundle A on C of sufficiently high degree satisfies

$$K_{p,1}(C, A) = 0 \qquad \text{if and only if } p \geq h^0(C, A) - k. \tag{2}$$

Green ([G]) and Ehbauer ([E]) have shown that the statement holds true for any curve of gonality $k \leq 3$. As in the case of Green's Conjecture, one implication is well-known (cf. [G, Appendix]); it was proved by Aprodu (cf. [A1]) that the conjecture is thus equivalent to the existence of a non-special globally generated line bundle A such that $K_{h^0(C,A)-k,1}(C, A) = 0$.

Both Green's Conjecture and Green-Lazarsfeld's Gonality Conjecture are in their full generality still open. However, by specialization to curves on $K3$ surfaces, they were proved for a general curve in each gonality stratum of M_g by Voisin and Aprodu (cf. [V1, V2, A2]). Combining this with an earlier result of Hirschowitz and Ramanan (cf. [HR]), the two conjectures follow for any curve of odd genus $g = 2k - 3$ and maximal gonality k.

In [A2], Aprodu provided a sufficient condition for a genus g curve C of gonality $k \leq (g+2)/2$ to satisfy both conjectures; this is known as the *linear growth condition* and is expressed in terms of the Brill-Noether theory of C only:

$$\dim W^1_d(C) \leq d - k \qquad \text{for } k \leq d \leq g - k + 2. \tag{3}$$

Aprodu and Farkas ([AF]) used the above characterization in order to establish Green's Conjecture for smooth curves lying on arbitrary $K3$ surfaces. It is natural to ask whether a similar strategy can solve Green's Conjecture for curves lying on anticanonical rational surfaces, since these share some common behaviour with $K3$ surfaces. The situation gets more complicated because such a surface S is in general non-minimal and its canonical bundle is non-trivial; in particular, given a line bundle $L \in \text{Pic}(S)$, smooth curves in the linear system $|L|$ do not form a family of curves with constant syzygies, as it happens instead in the case of $K3$ surfaces. Our main result is the following:

Theorem 1.1. *Let S be a smooth, projective, rational surface with an anticanonical pencil and let L be a line bundle on S such that $L \otimes \omega_S$ is nef and big. In the special case where $h^0(S, \omega_S^\vee) = \chi(S, \omega_S^\vee) = 2$, also assume that the Clifford index of a general curve in $|L|$ is not computed by the restriction of the anticanonical bundle $\omega_S^\vee$.*

Then, any smooth, irreducible curve $C \in |L|$ satisfies Green's Conjecture.

With no hypotheses on the line bundle L, we obtain Green's Conjecture and Green-Lazarsfeld's Gonality Conjecture for a general curve in $|L|_s$, where $|L|_s$ denotes the locus of smooth and irreducible curves in the linear system $|L|$ (cf. Proposition 5.2). For later use, we denote by $g(L) := 1 + (c_1(L)^2 + c_1(L) \cdot K_S)/2$ the genus of any curve in $|L|_s$.

Examples of surfaces as in Theorem 1.1 are given by all rational surfaces S whose canonical divisor satisfies $K_S^2 > 0$, or equivalently, having Picard number $\rho(S) \leq 9$, such as Del Pezzo surfaces ($-K_S$ is ample), generalized Del Pezzo surfaces ($-K_S$ is nef and big), some blow-ups of Hirzebruch surfaces. However, the class of surfaces that we are considering also includes surfaces S with $K_S^2 \leq 0$, such as rational elliptic surfaces (i.e., smooth complete complex surfaces that can be obtained by blowing up $\mathbb{P}^2$ at 9 points, which are the base locus of a pencil of cubic curves with at least one smooth member).

We also obtain the following:

Theorem 1.2. *Assume the same hypotheses as in Theorem 1.1 and let $g(L) \geq 4$. Then, all curves in $|L|_s$ have the same Clifford dimension r, the same Clifford index and the same gonality. Moreover, if the curves in $|L|_s$ are exceptional, then one of the following occurs:*

(i) *$r = 2$ and any curve in $|L|_s$ is the strict transform of a smooth, plane curve under a morphism $\phi : S \to \mathbb{P}^2$ which is the composition of finitely many blow-ups.*
(ii) *$r = 3$ and S can be realized as the blow-up of a normal cubic surface $S' \subset \mathbb{P}^3$ at a finite number of points (possibly infinitely near); any curve in $|L|_s$ is the strict transform under the blow-up map of a smooth curve in $|-3K_{S'}|$.*

This generalizes results of Pareschi (cf. [P1]) and Knutsen (cf. [K]) concerning the Brill-Noether theory of curves lying on a Del Pezzo surface S. In [K], the author proved that line bundles violating the constancy of the Clifford index only exist when $K_S^2 = 1$; they are described in terms of the coefficients of the generators of $\mathrm{Pic}(S)$ in their presentation. In fact, one can show that such line bundles are exactly those satisfying: $L \otimes \omega_S$ is nef and big and the restriction of the anticanonical bundle $\omega_S^\vee$ to a general curve in $|L|_s$ computes its Clifford index (cf. Remark 2).

The proofs of Theorem 1.1 and Theorem 1.2 rely on vector bundle techniques à la Lazarsfeld (cf. [La1]); in particular, we consider rank-2 bundles $E_{C,A}$, which are the analogue of the Lazarsfeld-Mukai bundles for $K3$ surfaces. The key fact proved in Section 3 is that, if A is a complete, base point free pencil on a general curve $C \in |L|_s$, the dimension of $\ker \mu_{0,A}$ is controlled by $H^2(S, E_{C,A} \otimes E_{C,A}^\vee)$; if this is nonzero, the bundle $E_{C,A}$ cannot be slope-stable with respect to any polarization H on S.

By considering Harder-Narasimhan and Jordan-Hölder filtrations, in Section 4 we perform a parameter count for pairs (C, A) such that $E_{C,A}$ is not μ_H-stable; this gives an upper bound for the dimension of any irreducible component $\mathcal{W}$ of $\mathcal{W}^1_d(|L|)$ which dominates $|L|$ under the natural projection $\pi : \mathcal{W}^1_d(|L|) \to |L|_s$. It turns out (cf. Proposition 5.1) that, if a general curve $C \in |L|_s$ is exceptional, the same holds true for all curves in $|L|_s$ and one is either in case (i) or (ii) of Theorem 1.2; in this context we recall that Green's Conjecture for curves of Clifford dimension 2 and 3 was verified by Loose in [Lo]. If instead C has Clifford dimension 1, our parameter count ensures that it satisfies the linear growth condition (3). In order to deduce Green's

Conjecture for *every* curve in $|L|_s$, we make use of the hypotheses made on L and show that the Koszul group $K_{g-c-1,1}(C,\omega_C)$ does not depend (up to isomorphism) on the choice of $C\in|L|_s$, as soon as c equals the Clifford index of a general curve in $|L|_s$. Semicontinuity will imply the constancy of the Clifford index and the gonality.

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2. SYZYGIES AND KOSZUL COHOMOLOGY

If L is an ample line bundle on a complex projective variety X, let $S:=\mathrm{Sym}^*H^0(X,L)$ be the homogeneous coordinate ring of the projective space $\mathbb{P}(H^0(X,L)^\vee)$ and set $R(X):=\bigoplus_m H^0(X,L^m)$. Being a finitely generated S-module, $R(X)$ admits a minimal graded free resolution

$$0\to E_s\to\ldots\to E_1\to E_0\to R(X)\to 0,$$

where for $k\geq 1$ one can write $E_k=\sum_{i\geq k}S(-i-1)^{\beta_{k,i}}$. The *syzygies* of X of order k are by definition the graded components of the S-module E_k. We say that the pair (X,L) satisfies property (N_p) if $E_0=S$ and $E_k=S(-k-1)^{\beta_{k,k}}$ for all $1\leq k\leq p$. In other words, property (N_0) is satisfied whenever ϕ_L embeds X as a projectively normal variety, while property (N_1) also requires that that the ideal of X in $\mathbb{P}(H^0(X,L)^\vee)$ is generated by quadrics; for $p>1$, property (N_p) means that the syzygies of X up to order p are linear.

The most effective tool in order to compute syzygies is Koszul cohomology, whose definition is the following. Let $L\in\mathrm{Pic}(X)$ and F be a coherent sheaf on X. The Koszul cohomology group $K_{p,q}(X,F,L)$ is defined as the cohomology at the middle-term of the complex

$$\bigwedge^{p+1}H^0(L)\otimes H^0(F\otimes L^{q-1})\to\bigwedge^{p}H^0(L)\otimes H^0(F\otimes L^{q})\to\bigwedge^{p-1}H^0(L)\otimes H^0(F\otimes L^{q+1}).$$

When $F\simeq\mathcal{O}_X$, the Koszul cohomology group is conventionally denoted by $K_{p,q}(X,L)$. It turns out (cf. [G]) that property (N_p) for the pair (X,L) is equivalent to the vanishing

$$K_{i,q}(X,L)=0\qquad\text{for all } i\leq p \text{ and } q\geq 2.$$

In particular, Green's Conjecture can be rephrased by asserting that (C,ω_C) satisfies property (N_p) whenever $p<\mathrm{Cliff}(C)$.

In the sequel we will make use of the following results, which are due to Green. The first one is the Vanishing Theorem (cf. [G, Theorem (3.a.1)]), stating that

$$K_{p,q}(X,E,L)=0\qquad\text{if } p\geq h^0(X,E\otimes L^q).\tag{4}$$

The second one (cf. [G, Theorem (3.b.1)]) relates the Koszul cohomology of X to that of a smooth hypersurface $Y\subset X$ in the following way.

Theorem 2.1. *Let X be a smooth irreducible projective variety and assume $L,N\in\mathrm{Pic}(X)$ satisfy*

$$H^0(X,N\otimes L^\vee)\;=\;0\tag{5}$$

$$H^1(X,N^q\otimes L^\vee)\;=\;0,\qquad\forall\, q\geq 0.\tag{6}$$

Then, for every smooth integral divisor $Y\in|L|$, there exists a long exact sequence

$$\cdots\to K_{p,q}(X,L^\vee,N)\to K_{p,q}(X,N)\to K_{p,q}(Y,N\otimes\mathcal{O}_Y)\to K_{p-1,q+1}(X,L^\vee,N)\to\cdots.$$

3. Petri map via vector bundles

Let S be a smooth rational surface with an anticanonical pencil and $C \subset S$ be a smooth, irreducible curve of genus g. We set $L := \mathcal{O}_S(C)$. If A is a complete, base point free g^r_d on C, as in the case of $K3$ surfaces, let $F_{C,A}$ be the vector bundle on S defined by the sequence

$$0 \to F_{C,A} \to H^0(C,A) \otimes \mathcal{O}_S \stackrel{ev_{A,S}}{\longrightarrow} A \to 0,$$

and set $E_{C,A} := F^\vee_{C,A}$. Since $N_{C|S} = \mathcal{O}_C(C)$, by dualizing the above sequence we get

$$0 \to H^0(C,A)^\vee \otimes \mathcal{O}_S \to E_{C,A} \to \mathcal{O}_C(C) \otimes A^\vee \to 0. \tag{7}$$

This trivially implies that:

- $\chi(S, E_{C,A} \otimes \omega_S) = h^0(S, E_{C,A} \otimes \omega_S) = g - d + r$,
- $\mathrm{rk}\, E_{C,A} = r+1$, $c_1(E_{C,A}) = L$, $c_2(E) = d$,
- $h^2(S, E_{C,A}) = 0$, $\chi(S, E_{C,A}) = g - d + r - c_1(L) \cdot K_S$.

Being a bundle of type $E_{C,A}$ is an open condition. Indeed, a vector bundle E of rank $r+1$ is of type $E_{C,A}$ whenever $h^1(S, E \otimes \omega_S) = h^2(S, E \otimes \omega_S) = 0$ and there exists $\Lambda \in G(r+1, H^0(S,E))$ such that the degeneracy locus of the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \to E$ is a smooth connected curve.

Notice that the dimension of the space of global sections of $E_{C,A}$ depends not only on the type of the linear series A but also on $A \otimes \omega_S$. In particular, one has

$$\begin{aligned} h^0(S, E_{C,A}) &= r + 1 + h^0(C, \mathcal{O}_C(C) \otimes A^\vee), \\ h^1(S, E_{C,A}) &= h^0(C, A \otimes \omega_S). \end{aligned}$$

Moreover, if the line bundle $\mathcal{O}_C(C) \otimes A^\vee$ has sections, then $E_{C,A}$ is generated off its base points. In the case $r = 1$, we prove the following.

Lemma 3.1. *Let A be a complete, base point free g^1_d on $C \subset S$. If either*

- $h^0(S, \omega_S^\vee) > 2$, *or*
- $h^0(S, \omega_S^\vee) = 2$ *and* $A \not\simeq \omega_S^\vee \otimes \mathcal{O}_C$

holds, then $h^0(C, A \otimes \omega_S) = 0$.

Proof. Since $L \otimes \omega_S$ is effective, the short exact sequence

$$0 \to L^\vee \otimes \omega_S^\vee \to \omega_S^\vee \to \omega_S^\vee \otimes \mathcal{O}_C \to 0$$

implies $h^0(C, \omega_S^\vee \otimes \mathcal{O}_C) \geq h^0(S, \omega_S^\vee)$ and the statement follows trivially if $h^0(S, \omega_S^\vee) > 2$. Let $h^0(S, \omega_S^\vee) = 2$ and $h^0(C, A \otimes \omega_S) > 0$. Then necessarily $h^0(C, \omega_S^\vee \otimes \mathcal{O}_C) = 2$ and $A \otimes \omega_S$ is the fixed part of the linear system of sections of A. Since A is base point free by hypothesis, then $A \simeq \omega_S^\vee \otimes \mathcal{O}_C$. □

Under the hypotheses of the above Lemma, the bundle $E_{C,A}$ is globally generated off a finite set and $\chi(S, E_{C,A}) = h^0(S, E_{C,A}) = g - d + 1 - c_1(L) \cdot K_S$. We remark that the vanishing of $h^1(S, E_{C,A})$ turns out to be crucial in most of the following arguments and this is why the assumptions on the anticanonical linear system of S are needed.

The following proposition gives a necessary and sufficient condition for the injectivity of the Petri map $\mu_{0,A} : H^0(C,A) \otimes H^0(C, \omega_C \otimes A^\vee) \to H^0(C, \omega_C)$.

Proposition 3.2. *If $C \in |L|_s$ is general and either $h^0(S, \omega_S^\vee) > 2$ or $h^0(S, \omega_S^\vee) = 2$ and $A \not\simeq \omega_S^\vee \otimes \mathcal{O}_C$, then for any complete, base point free pencil A on C one has:*

$$\ker \mu_{0,A} \simeq H^2(S, E_{C,A} \otimes E^\vee_{C,A}).$$

In particular, the vanishing of the one side implies the vanishing of the other.

Proof. The proof proceeds as in [P2], hence we will not enter into details. As A is a pencil, the kernel of the evaluation map $ev_{A,C} : H^0(C,A) \otimes \mathcal{O}_C \twoheadrightarrow A$ is ismomorphic to $A^\vee$ and $\ker \mu_{0,A} \simeq H^0(C, \omega_C \otimes A^{-2})$. Since $\det F_{C,A} = L^\vee$, by adjunction one finds the following short exact sequence:

$$0 \to \omega_S \otimes \mathcal{O}_C \to F_{C,A} \otimes \omega_C \otimes A^\vee \to \omega_C \otimes A^{-2} \to 0. \tag{8}$$

The coboundary map $\delta : H^0(C, \omega_C \otimes A^{-2}) \to H^1(C, \omega_S \otimes \mathcal{O}_C)$ coincides, up to multiplication by a nonzero scalar factor, with the composition of the Gaussian map

$$\mu_{1,A} : \ker \mu_{0,A} \to H^0(C, \omega_C^2)$$

and the dual of the Kodaira spencer map

$$\rho^\vee : H^0(C, \omega_C^2) \to (T_C|L|)^\vee = H^0(C, N_{C|S})^\vee = H^1(C, \omega_S \otimes \mathcal{O}_C).$$

Indeed, as in [P2, Lemma 1], one finds a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_S \otimes \mathcal{O}_C & \longrightarrow & F_{C,A} \otimes \omega_C \otimes A^\vee & \longrightarrow & \omega_C \otimes A^{-2} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow s & & \\ 0 & \longrightarrow & \omega_S \otimes \mathcal{O}_C & \longrightarrow & \Omega_S^1 \otimes \omega_C & \longrightarrow & \omega_C^2 & \longrightarrow & 0, \end{array}$$

where the homomorphism induced by s on global sections is $\mu_{1,A}$ and the coboundary map $H^0(C, \omega_C^2) \to H^1(C, \omega_S \otimes \mathcal{O}_C)$ equals (up to a scalar factor) $\rho^\vee$.

If A has degree d, look at the natural projection $\pi : \mathcal{W}_d^1(|L|) \to |L|_s$. First order deformation arguments (see, for instance, [ACG, p. 722]) imply that

$$\mathrm{Im}(d\pi_{(C,A)}) \subset \mathrm{Ann}(\mathrm{Im}(\rho^\vee \circ \mu_{1,A})).$$

Therefore, by Sard's Lemma, if $C \in |L|_s$ is general, the short exact sequence (8) is exact on the global sections for any base point free $A \in W_d^1(C) \setminus W_d^2(C)$, and $\ker \mu_{0,A} \simeq H^0(C, F_{C,A} \otimes \omega_C \otimes A^\vee)$. By tensoring short exact sequence (7) with $F_{C,A} \otimes \omega_S$, one finds that

$$H^0(C, F_{C,A} \otimes \omega_C \otimes A^\vee) \simeq H^0(S, E_{C,A}^\vee \otimes E_{C,A} \otimes \omega_S)$$

because $H^i(S, F_{C,A} \otimes \omega_S) \simeq H^{2-i}(S, E_{C,A})^\vee = 0$ for $i = 0, 1$. The statement follows now by Serre duality. □

Corollary 3.3. *Let H be any polarization on S and $\mathcal{W}$ be an irreducible component of $\mathcal{W}_d^1(|L|)$ which dominates $|L|$ and whose general points correspond to μ_H-stable bundles; in the special case where $h^0(S, \omega_S^\vee) = 2$, assume that general points of $\mathcal{W}$ are not of the form $(C, \omega_S^\vee \otimes \mathcal{O}_C)$.*

Then, $\rho(g,1,d) \geq 0$ and $\mathcal{W}$ is reduced of dimension equal to $\dim |L| + \rho(g,1,d)$.

Proof. Let (C,A) be a general point of $\mathcal{W}$. If $E_{C,A}$ is stable, $E_{C,A} \otimes \omega_S$ also is. The inequality $\mu_H(E_{C,A}) > \mu_H(E_{C,A} \otimes \omega_S)$ implies that $H^2(S, E_{C,A}^\vee \otimes E_{C,A}) \simeq \mathrm{Hom}(E_{C,A}, E_{C,A} \otimes \omega_S)^\vee = 0$. As a consequence, $\mathcal{W}$ is smooth in (C,A) of the expected dimension. □

Remark 1. Corollary 3.3 can be alternatively proved by arguing in the following way.
Let $M := M_H^{\mu s}(c)$ be the moduli space of μ_H-stable vector bundles on S of total Chern class $c = 2 + c_1(L) + d\omega \in H^{2*}(S, \mathbb{Z})$, where ω is the fundamental cocycle. Since every $[E] \in M$ satisfies $\mathrm{Ext}^2(E,E)_0 = 0$, it turns out that M is a smooth, irreducible projective variety of dimension $4d - c_1(L)^2 - 3$ (cf. [CoMR, Remark 2.3]), as soon as it is non-empty. Let M^0 be the open subset of M parametrizing vector bundles $[E]$ of type $E_{C,A}$; if this is nonempty, define $\mathcal{G}$ as the Grassmann bundle on M^0 with fiber over $[E]$ equal to $G(2, H^0(S,E))$. Look at the rational map $h : \mathcal{G} \dashrightarrow \mathcal{W}_d^1(|L|)$ sending a general $(E, \Lambda) \in \mathcal{G}$ to the pair (C_Λ, A_Λ), where C_Λ is the degeneracy locus of the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \to E$ and $\mathcal{O}_{C_\Lambda}(C_\Lambda) \otimes A^\vee$ is its cokernel. Since any

$[E] \in M^0$ is simple, one easily checks that h is birational onto its image, that is denoted by $\mathcal{W}$. As a consequence, the dimension of $\mathcal{W}$ equals:

$$4d - c_1(L)^2 - 3 + 2(g - d - 1 - c_1(L) \cdot K_S) = 2d - 3 - c_(L) \cdot K_S \leq \dim |L| + \rho(g, 1, d).$$

4. Parameter count

By the analysis performed in the previous section, given a polarization H on S, the linear growth condition for a general curve in $|L|_s$ can be verified by controlling the dimension of every dominating component $\mathcal{W} \subset \mathcal{W}^1_d(|L|)$, whose general points are pairs (C, A) such that $A \not\simeq \omega_S^\vee \otimes \mathcal{O}_C$ and the bundle $E_{C,A}$ is not μ_H-stable. Indeed, if $A \simeq \omega_S^\vee \otimes \mathcal{O}_C$ for a general point of $\mathcal{W}$, then $\omega_S^\vee \otimes \mathcal{O}_{C'}$ is an isolated point of $W^1_d(C')$ for every $C' \in |L|_s$.

Let A be a complete, base point free g^1_d on a curve $C \in |L|_s$ such that the bundle $E := E_{C,A}$ is not μ_H-stable and $A \not\simeq \omega_S^\vee \otimes \mathcal{O}_C$ if $h^0(S, \omega_S^\vee) = 2$. The maximal destabilizing sequence of E has the form

$$0 \to M \to E \to N \otimes I_\xi \to 0, \tag{9}$$

where $M, N \in \mathrm{Pic}(S)$ satisfy

$$\mu_H(M) \geq \mu_H(E) \geq \mu_H(N), \tag{10}$$

with both or none of the inequalities being strict, and I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$ of length $l = d - c_1(N) \cdot c_1(M)$.

Lemma 4.1. *In the above situation, assume that general curves in $|L|_s$ have Clifford index c. If $\mu_{0,A}$ is non-injective and C is general in $|L|$, then the following inequality holds:*

$$c_1(M) \cdot c_1(N) + c_1(N) \cdot K_S \geq c. \tag{11}$$

Proof. Being a quotient of $E := E_{C,A}$ off a finite set, N is base component free and is nontrivial since $H^2(S, N \otimes \omega_S) = 0$. As a consequence, $h^0(S, N) \geq 2$. Proposition 3.2 implies that $\mathrm{Hom}(E, E \otimes \omega_S) \neq 0$. Applying $\mathrm{Hom}(E, -)$ to the short exact sequence (9) twisted with ω_S, we obtain

$$0 \to \mathrm{Hom}(E, M \otimes \omega_S) \to \mathrm{Hom}(E, E \otimes \omega_S) \to \mathrm{Hom}(E, N \otimes \omega_S \otimes I_\xi) \to \cdots .$$

Apply now $\mathrm{Hom}(-, N \otimes \omega_S \otimes I_\xi)$ (respectively $\mathrm{Hom}(-, M \otimes \omega_S)$) to exact sequence (9), and find that $\mathrm{Hom}(E, N \otimes \omega_S \otimes I_\xi) = 0$ (resp. $\mathrm{Hom}(E, M \otimes \omega_S) \simeq \mathrm{Hom}(N \otimes I_\xi, M \otimes \omega_S)$), hence $N^\vee \otimes M \otimes \omega_S$ is effective and $h^0(S, M \otimes \omega_S) \geq 2$. This ensures that $N \otimes \mathcal{O}_C$ contributes to the Clifford index of C and

$$\begin{aligned} c \leq \mathrm{Cliff}(N \otimes \mathcal{O}_C) &= c_1(N) \cdot (c_1(N) + c_1(M)) - 2h^0(C, N \otimes \mathcal{O}_C) + 2 \\ &\leq c_1(N)^2 + c_1(N) \cdot c_1(M) - 2h^0(S, N) + 2 \\ &= c_1(N) \cdot c_1(M) + c_1(N) \cdot K_S. \end{aligned}$$

□

Now, upon fixing a nonnegative integer l and a line bundle N such that (10) is satisfied for $M := L \otimes N^\vee$, we want to estimate the number of moduli of pairs (C, A) such that the bundle $E_{C,A}$ sits in a short exact sequence like (9). The following construction is analogous to the one performed in [LC, Section 4] in the case of $K3$ surfaces. Let $\mathcal{E}_{N,l}$ be the moduli stack of extensions of type (9), where $l(\xi) = l$. Having fixed $c \in H^{2*}(S, \mathbb{Z})$, we denote by $\mathcal{M}(c)$ the moduli stack of coherent sheaves of total Chern class c. We consider the natural maps $p : \mathcal{E}_{N,l} \to \mathcal{M}(c(M)) \times \mathcal{M}(c(N \otimes I_\xi))$ and $q : \mathcal{E}_{N,L} \to \mathcal{M}(c(E))$, which send the $\mathbb{C}$-point of $\mathcal{E}_{N,l}$ corresponding to extension (9) to the classes of $(M, N \otimes I_\xi)$ and E respectively. Notice that, since M, N lie in $\mathrm{Pic}(S)$, the stack $\mathcal{M}(c(M))$ has a unique $\mathbb{C}$-point endowed with a

1-dimensional space of automorphisms, while $\mathcal{M}(c(N\otimes I_\xi))$ is corepresented by the Hilbert scheme $S^{[l]}$ parametrizing 0-dimensional subschemes of S of length l.

We denote by $\tilde{Q}_{N,l}$ the closure of the image of q and by $Q_{N,l}$ its open substack consisting of vector bundles of type $E_{C,A}$ for some $C\in|L|_s$ and $A\in W^1_d(C)$, with $d:=l+c_1(M)\cdot c_1(N)$ and $A\not\simeq\omega_S^\vee\otimes\mathcal{O}_C$ if $h^0(S,\omega_S^\vee)=2$. Let $\mathcal{G}_{N,l}\to Q_{N,l}$ be the Grassmann bundle whose fiber over $[E]\in Q_{N,l}(\mathbb{C})$ is $G(2,H^0(S,E))$. By construction, a $\mathbb{C}$-point of $\mathcal{G}_{N,l}$ corresponding to a pair $([E],\Lambda)$, with $\Lambda\in G(2,H^0(S,E))$, comes endowed with an automorphism group equal to $\mathrm{Aut}(E)$. We define $\mathcal{W}_{N,l}$ to be the closure of the image of the rational map $\mathcal{G}_{N,l}\dashrightarrow\mathcal{W}^1_d(|L|)$, which sends a general point $([E],\Lambda)\in\mathcal{G}_{N,l}(\mathbb{C})$ to the pair (C_Λ,A_Λ) where the evaluation map $ev_\Lambda:\Lambda\otimes\mathcal{O}_S\hookrightarrow E$ degenerates on C_Λ and has $\mathcal{O}_{C_\Lambda}(C_\Lambda)\otimes A_\Lambda^\vee$ as cokernel. The following proposition gives an upper bound for the dimension of $\mathcal{W}_{N,l}$.

Proposition 4.2. *Assume that general curves in $|L|_s$ have Clifford index c. Then, every irreducible component $\mathcal{W}$ of $\mathcal{W}^1_d(|L|_s)$ which dominates $|L|$ and is contained in $\mathcal{W}_{N,l}$ satisfies*

$$\dim\mathcal{W}\le\dim|L|+d-c-2.$$

Proof. The fiber of p over a $\mathbb{C}$-point of $\mathcal{M}(c(M))\times\mathcal{M}(c(N\otimes I_\xi))$ corresponding to $(M,N\otimes I_\xi)$ is the quotient stack

$$[\mathrm{Ext}^1(N\otimes I_\xi,M)/\mathrm{Hom}(N\otimes I_\xi,M)],$$

where the action of the Hom group on the Ext group is trivial. The fiber of q over $[E]\in\tilde{Q}_{N,l}(\mathbb{C})$ is the Quot-scheme $\mathrm{Quot}_S(E,P)$, where P is the Hilbert polynomial of $N\otimes I_\xi$. The condition $\mu_H(M)\ge\mu_H(N)$ implies that $\mathrm{Ext}^2(N\otimes I_\xi,M)\simeq\mathrm{Hom}(M,N\otimes\omega_S\otimes I_\xi)^\vee=0$, hence the dimension of the fibers of p is constant and equals

$$-\chi(S,N\otimes M^\vee\otimes\omega_S\otimes I_\xi)=-g+2c_1(N)\cdot c_1(M)+c_1(M)\cdot K_S+l.$$

By looking at the tangent and obstruction spaces at any point, one shows that the Quot schemes constituting the fibers of q are either all 0-dimensional or all smooth of dimension 1; indeed, $\mathrm{Hom}(M,N\otimes I_\xi)=0$ unless $M\simeq N$ and $l=0$, in which case $\mathrm{Ext}^1(M,N\otimes I_\xi)=H^1(S,\mathcal{O}_S)=0$. As a consequence, if nonempty, $Q_{N,l}$ has dimension at most $3l-2-g+2c_1(N)\cdot c_1(M)+c_1(M)\cdot K_S$.

Since the map $h_{N,l}$ forgets the automorphisms, its fiber over a pair $(C,A)\in\mathcal{W}_{N,l}$ is the quotient stack

$$[U/\mathrm{Aut}(E_{C,A})],$$

where U is the open subscheme of $\mathbb{P}(\mathrm{Hom}(E_{C,A},\mathcal{O}_C(C)\otimes A^\vee))$ whose points correspond to morphisms with kernel equal to $\mathcal{O}_S^{\oplus 2}$, and $\mathrm{Aut}(E_{C,A})$ acts on U by composition. Using the vanishing $h^i(S,E_{C,A}\otimes\omega_S)=0$ for $i=1,2$, one checks that

$$\mathrm{Hom}(E_{C,A},\mathcal{O}_C(C)\otimes A^\vee)\simeq H^0(S,E_{C,A}\otimes E_{C,A}^\vee),$$

and U is isomorphic to $\mathbb{P}\mathrm{Aut}(E_{C,A})$. Hence, the fibers of $h_{N,l}$ are stacks of dimension -1 and

$$\begin{aligned}\dim\mathcal{W}_{N,l}&\le 3l-1-g+2c_1(N)\cdot c_1(M)+c_1(M)\cdot K_S+2(g-d-1-c_1(L)\cdot K_S)\\&= d+g-3-c_1(N)\cdot c_1(M)-c_1(N)\cdot K_S-c_1(L)\cdot K_S.\end{aligned}$$

The conclusion follows now from the fact that $\dim|L|\ge g-1-c_1(L)\cdot K_S$, along with Lemma 4.1. □

5. Proof of the main results

We recall some facts about exceptional curves, that is, curves of Clifford dimension greater than 1. Coppens and Martens ([CM]) proved that, if C is an exceptional curve of gonality k and Clifford dimension r, then $\mathrm{Cliff}(C)=k-3$ and C possesses a 1-dimensional family of g^1_k. Furthermore, if $r\le 9$, there exists a unique line bundle computing $\mathrm{Cliff}(C)$ (cf. [ELMS]);

this is conjecturally true for any r. Curves of Clifford dimension 2 are precisely the smooth plane curves of degree ≥ 5, while the only curves of Clifford dimension 3 are the complete intersections of two cubic surfaces in $\mathbb{P}^3$ (cf. [Ma]). We will use these results in the proof of the following:

Proposition 5.1. *Let L be a line bundle on a smooth, rational surface S with an anticanonical pencil. If $g(L) \geq 4$ and a general curve $C \in |L|_s$ is exceptional, then any other curve inside $|L|_s$ has the same Clifford dimension r of C and either case (i) or (ii) of Theorem 1.2 occurs.*

Proof. Since any curve of odd genus and maximal gonality has Clifford dimension 1 (cf. [A3, Corollary 3.11]), we can assume that general curves in $|L|_s$ have gonality $k \leq (g+2)/2$ and are exceptional. There exists a component $\mathcal{W}$ of $\mathcal{W}^1_k(|L|)$ of dimension at least $\dim|L|+1$ and, by Corollary 3.3, this is contained in $\mathcal{W}_{N,l}$ for some N and l. Notice that the line bundle N is nef since it is globally generated off a finite set. Furthermore, it follows from the proof of Proposition 4.2 that N and $M := L \otimes N^\vee$ satisfy equality in (11), that is,

$$k-3 = c_1(M)\cdot c_1(N) + c_1(N)\cdot K_S = k - l + c_1(N)\cdot K_S;$$

in particular, $N \otimes \mathcal{O}_C$ computes the Clifford index of a general $C \in |L|_s$ and $h^1(S, M^\vee) = 0$. Having at least a 2-dimensional space of sections, the line bundle $\omega_S^\vee \otimes \mathcal{O}_C$ has degree $\geq k$, thus $-c_1(M)\cdot K_S \geq k-3+l$.

Assume $h^0(S, N\otimes\omega_S) \geq 2$; the restriction of M to a general curve $C \in |L|_s$ contributes to its Clifford index and

$$k-3 \leq \mathrm{Cliff}(M\otimes\mathcal{O}_C) = c_1(M)\cdot c_1(N) + c_1(M)\cdot K_S \leq 3-2l.$$

As $k \geq 2r$ (cf. [ELMS, Proposition 3.2]), we have $r \leq 3$; if $r=3$, then $l=0$, while $r=2$ implies $l \leq 1$. Let $r=2$; since $\chi(S,N) = h^0(S,N) = h^0(C, N\otimes\mathcal{O}_C) = 3$ and $h^i(S, N\otimes\omega_S) = 0$ for $i=1,2$ (as one can check twisting (9) with ω_S and taking cohomology), then $c_1(N)^2 = l+1$ and $h^0(S, N\otimes\omega_S) = l \leq 1$, contradicting our assumption. Hence, the inequality $h^0(S, N\otimes\omega_S) \geq 2$ implies $r=3$ and $l=0$.

Assume instead that $h^0(S, N\otimes\omega_S) \leq 1$; we get $c_1(N)^2 \leq 3-l$ and $h^0(C, N\otimes\mathcal{O}_C) = h^0(S,N) = \chi(S,N) \leq 4-l$. Since $N\otimes\mathcal{O}_C$ computes the Clifford index of C, then $r \leq 3$ holds in this case as well. Moreover, $l=0$ when $r=3$, and $l\leq 1$ if $r=2$.

Let $r=2$ and $l=1$; then, $c_1(N)^2 = -c_1(N)\cdot K_S = 2$. By [Ha, Lemma 2.6, Theorem 2.11], N is base point free and not composed with a pencil, hence it defines a generically $2:1$ morphism $\phi := \phi_N : S \to \mathbb{P}^2$ splitting into the composition of a birational morphism $\psi : S \to S'$, which contracts the finitely many curves $E_1, \cdots, E_h$ having zero intersection with $c_1(N)$, and a ramified double cover $\pi : S' \to \mathbb{P}^2$. Set $N' := \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$; since $N = \psi^*(N')$ and ψ^* preserves both the intersection products and the dimensions of cohomology groups, we have $c_1(N')^2 = -c_1(N')\cdot K_{S'} = 2$ and

$$1 = h^0(S, N\otimes\omega_S) \geq h^0(S, N\otimes\omega_S(-E_1-\cdots-E_h)) = h^0(S', N'\otimes\omega_{S'}).$$

We apply Theorem 3.3. in [Ha] and get $N' = \omega_{S'}^\vee$ and $K_{S'}^2 = 2$ (cases (b), (c), (d) of the aforementioned theorem cannot occur since they would contradict $c_1(N')^2 = 2$). The line bundle $N\otimes\mathcal{O}_C$ is very ample because it computes $\mathrm{Cliff}(C)$ (cf. [ELMS, Lemma 1.1]); hence, C is isomorphic to $C' = \psi(C)$ and $\omega_{S'}^\vee \otimes \mathcal{O}_{C'}$ is also very ample. Proceeding as in the proof of [P1, Lemma 2.6] (where the ampleness of $\omega_{S'}^\vee$ is not really used), one shows that $\phi(C') \in |-2K_{S'}|$. This gives a contradiction because it implies $g(C') = g(C) = 3$.

Up to now, we have shown that $r \leq 3$ and $l = 0$, hence $-c_1(N)\cdot K_S = 3$ and $c_1(N)^2 > 0$. By [Ha, Proposition 3.2], the line bundle N defines a morphism $\phi_N : S \to \mathbb{P}^r$ which is birational to its image and only contracts the finitely many curves which have zero intersection with $c_1(N)$. If $r=2$, then ϕ_N is the blow-up of $\mathbb{P}^2$ at finitely many points (maybe infinitely near) and any curve in $|L|_s$ is the strict transform of a smooth plane curve. For $r = 3$, the image of ϕ_N is a normal cubic surface $S' \subset \mathbb{P}^3$ and any curve in $|L|_s$ is the strict transform of a smooth curve in $|-3K_{S'}|$, hence has Clifford dimension 3. □

The following result is now straightforward.

Proposition 5.2. *Let C be a smooth, irreducible curve lying on a rational surface S with an anticanonical pencil. If C is general in its linear system, then C satisfies Green's Conjecture; if moreover C is not isomorphic to the complete intersection of two cubics in $\mathbb{P}^3$, then it satisfies Green-Lazarsfeld's Gonality Conjecture as well.*

Proof. We assume that C has genus $g \geq 4$, Clifford dimension 1, Clifford index c and gonality $k \leq (g+2)/2$. Having fixed $k \leq d \leq g-k+2$, Corollary 3.3 and Proposition 4.2 imply that every dominating component $\mathcal{W}$ of $\mathcal{W}^1_d(|L|)$ has dimension $\leq \dim |L| + d - k$, hence C satisfies the linear growth condition (3). Green's Conjecture for smooth plane curve and complete intersection of two cubics in $\mathbb{P}^3$ was established by Loose in [Lo], while Aprodu proved Green-Lazarsfeld's Gonality Conjecture for curves of Clifford dimension 2 in [A1]. □

We proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. We can assume $g(L) \geq 4$. By Proposition 5.2, if $C \in |L|_s$ is general then $K_{g-c-1,1}(C, \omega_C) = 0$, where $c = \text{Cliff}(C)$. If we show that the group $K_{g-c-1,1}(C, \omega_C)$ does not depend (up to isomorphism) on the choice of C in its linear system, by semicontinuity of the Clifford index, Green's Conjecture follows for any curve in $|L|_s$.

Set $N := L \otimes \omega_S$; since N is nef and big, the hypotheses of Theorem 2.1 are satisfied. Indeed, (5) and (6) for $q = 1$ follow directly from the fact that S is regular and has geometric genus 0. We remark that this also implies that

$$H^0(C, \omega_C) \simeq H^0(S, L \otimes \omega_S), \quad \forall\, C \in |L|_s.$$

Equality (6) for $q = 0$ is trivial since $|L|$ contains a smooth, irreducible curve. For $q \geq 2$, the line bundle N^{q-1} is nef and big and the Kawamata-Viehweg Vanishing Theorem (cf. [La2, Theorem 4.3.1]) implies that

$$0 = H^1(S, N^{-(q-1)})^\vee \simeq H^1(S, (L \otimes \omega_S)^{q-1} \otimes \omega_S) = H^1(S, N^q \otimes L^\vee).$$

By adjunction, for any curve $C \in |L|_s$, we obtain the following long exact sequence

$$\begin{aligned} \cdots \;\to\; & K_{g-c-1,1}(S, L^\vee, L \otimes \omega_S) \to K_{g-c-1,1}(S, L \otimes \omega_S) \to K_{g-c-1,1}(C, \omega_C) \\ \to\; & K_{g-c-2,2}(S, L^\vee, L \otimes \omega_S) \to \cdots. \end{aligned}$$

The group $K_{g-c-1,1}(S, L^\vee, L \otimes \omega_S)$ trivially vanishes since $H^0(S, \omega_S) = 0$. By the Vanishing Theorem (4) applied to $K_{g-c-2,2}(S, L^\vee, L \otimes \omega_S)$, we can conclude that

$$K_{g-c-1,1}(S, L \otimes \omega_S) \simeq K_{g-c-1,1}(C, \omega_C), \tag{12}$$

provided that $g - c - 2 \geq h^0(S, L \otimes \omega_S^2)$. We can assume $h^0(S, L \otimes \omega_S^2) \geq 2$ and we are under the hypothesis that the anticanonical system contains a pencil. Hence, $\omega_S^\vee \otimes \mathcal{O}_C$ contributes to the Clifford index and, if $C \in |L|_s$ is general, then:

$$\begin{aligned} c = \text{Cliff}(C) \leq \text{Cliff}(\omega_S^\vee \otimes \mathcal{O}_C) &= -c_1(L) \cdot K_S - 2h^0(C, \omega_S^\vee \otimes \mathcal{O}_C) + 2 \\ &= -c_1(L) \cdot K_S - 2h^0(S, \omega_S^\vee) + 2. \end{aligned} \tag{13}$$

Since $H^1(S, L \otimes \omega_S^2) \simeq H^1(S, L^\vee \otimes \omega_S^\vee)^\vee = 0$, we have

$$\begin{aligned} h^0(S, L \otimes \omega_S^2) = \chi(S, L \otimes \omega_S^2) &= g + c_1(L) \cdot K_S + K_S^2 \\ &\leq g - c + 1 - h^0(S, \omega_S^\vee) - h^1(S, \omega_S^\vee). \end{aligned}$$

The conclusion is straightforward unless $\chi(S,\omega_S^\vee) = h^0(S,\omega_S^\vee) = 2$; in this case the hypothesis that $\mathrm{Cliff}(\omega_S^\vee \otimes \mathcal{O}_C) > c$ for a general $C \in |L|_s$ is necessary in order to get to the conclusion. ☐

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. By Proposition 5.1, we can assume that general curves in $|L|_s$ have Clifford dimension 1; we denote by c their Clifford index.

The isomorphism (12), valid for any curve $C \in |L|_s$, together with Green and Lazarsfeld's result stating that $K_{p,1}(C,\omega_C) \neq 0$ for $p \leq g - \mathrm{Cliff}(C) - 2$, implies the constancy of the Clifford index. By semicontinuity of the gonality, all curves in $|L|_s$ have Clifford dimension 1 and the same gonality. ☐

Remark 2. Knutsen [K] proved that, if a line bundle L on a Del Pezzo surface S satisfies $g(L) \geq 4$, then the Clifford index curves in $|L|_s$ is constant unless S has degree 1, the line bundle L is ample, $c_1(L)\cdot E \geq 2$ for every (-1)-curve E if $c_1(L)^2 \geq 8$, and there is an integer $k \geq 3$ such that $-c_1(L)\cdot K_S = k$, $c_1(L)^2 \geq 5k-8 \geq 7$ and $c_1(L)\cdot\Gamma \geq k$ for every smooth rational curve such that $\Gamma^2 = 0$. In this case, the curves through the base point of $\omega_s^\vee$ form a family of codimension 1 in $|L|_s$, have gonality $k-1$ and Clifford index $k-3$, while a general curve $C \in |L|_s$ has gonality k and Clifford index $k-2$; in particular, $\omega_S^\vee \otimes \mathcal{O}_C$ computes $\mathrm{Cliff}(C)$. The easiest example where the gonality is not constant is provided by $L = \omega_S^{-n}$ for $n \geq 3$.

Vice versa, if S has degree 1 and $\mathrm{Cliff}(\omega_S^\vee \otimes \mathcal{O}_C) = \mathrm{Cliff}(C)$ for a general $C \in |L|_s$, one recovers Knutsen's conditions because, if Γ is a smooth rational curve with $\Gamma^2 = 0$, then $\mathcal{O}_S(\Gamma)$ cuts out a base point free pencil on C and, if $c_1(L)^2 \geq 8$ and E is a (-1)-curve, then $\mathcal{O}_C(-K_S + E)$ defines a net on C which contributes to its Clifford index. This shows that the extra hypothesis we make when $\chi(S,\omega_S^\vee) = h^0(S,\omega_S^\vee) = 2$ is unavoidable.

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