STABILITY OF RANK-3 LAZARSFELD-MUKAI BUNDLES ON $K3$ SURFACES

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ABSTRACT. Given an ample line bundle L on a $K3$ surface S, we study the slope stability with respect to L of rank-3 Lazarsfeld-Mukai bundles associated with complete, base point free nets of type g^2_d on curves C in the linear system $|L|$. When d is large enough and C is general, we obtain a dimensional statement for the variety $W^2_d(C)$. If the Brill-Noether number is negative, we prove that any g^2_d on any smooth, irreducible curve in $|L|$ is contained in a g^r_e which is induced from a line bundle on S, thus answering a conjecture of Donagi and Morrison. Applications towards transversality of Brill-Noether loci and higher rank Brill-Noether theory are then discussed.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

Many results of Brill-Noether theory regarding a general point in the moduli space M_g, which parametrizes isomorphism classes of smooth, irreducible curves of genus g, have been proved by studying curves lying on $K3$ surfaces. One of the advantages of considering an irreducible curve $C \subset S$, where S is a smooth $K3$ surface, is that some interesting properties, such as the Clifford index, do not change while moving C in its linear system (cf. [GL]). Moreover, Brill-Noether theory on C is strictly connected with the geometry of some moduli spaces of vector bundles on the $K3$ surface. Indeed, given a complete, base point free linear series A on C, one associates with the pair (C, A) a vector bundle on S, the so-called Lazarsfeld-Mukai bundle, denoted by $E_{C,A}$.

Lazarsfeld-Mukai bundles were first used by Lazarsfeld, in order to show that, given a $K3$ surface S such that $\mathrm{Pic}(S) = \mathbb{Z} \cdot L$, a general curve $C \in |L|$ satisfies the Gieseker-Petri Theorem, that is, for any line bundle $A \in \mathrm{Pic}(C)$ the Petri map

$$\mu_{0,A} : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee) \to H^0(C, \omega_C)$$

is injective (cf. [L1], [P], or [L2] for a more geometric argument).

It is natural to investigate what happens if the Picard number of S is greater than 1. In order to do this, having denoted by $|L|_s$ the locus of smooth, connected curves in the linear system $|L|$ and chosen two positive integers r, d, one studies the natural projection $\pi : \mathcal{W}^r_d(|L|) \to |L|_s$, whose fibre over C coincides with the Brill-Noether variety $W^r_d(C)$. We set $g := 1 + L^2/2$; this coincides with the genus of curves in $|L|_s$.

At first we look at the cases where $\rho(g, r, d) < 0$. Following [DM], we say that a line bundle M is *adapted* to $|L|$ whenever

(i) $h^0(S, M) \geq 2$, $h^0(S, L \otimes M^\vee) \geq 2$,
(ii) $h^0(C, M \otimes \mathcal{O}_C)$ is independent of the curve $C \in |L|_s$.

Conditions (i) and (ii) ensure that $M \otimes \mathcal{O}_C$ contributes to the Clifford index of C and $\mathrm{Cliff}(M \otimes \mathcal{O}_C)$ is the same for any $C \in |L|_s$.

Donagi and Morrison ([DM] Theorem (5.1′)) proved that, if A is a complete, base point free pencil g^1_d on a nonhyperelliptic curve $C \in |L|_s$ and $\rho(g, 1, d) < 0$, then $|A|$ is contained in the restriction to C of a line bundle $M \in \mathrm{Pic}(S)$ which is adapted to $|L|$ and such that $\mathrm{Cliff}(M \otimes \mathcal{O}_C) \leq \mathrm{Cliff}(A)$. The same is expected to hold true for any linear series of type g^r_d with $\rho(g,r,d)<0$ (compare with [DM] Conjecture (1.2)). We prove this conjecture for $r=2$ under some mild hypotheses on L.

Theorem 1.1. *Let S be a $K3$ surface and $L\in\mathrm{Pic}(S)$ be an ample line bundle such that a general curve in $|L|$ has genus g, Clifford dimension 1 and maximal gonality $k=\left\lfloor\frac{g+3}{2}\right\rfloor$. Let A be a complete, base point free g^2_d on a curve $C\in|L|_s$ such that $\rho(g,2,d)<0$.*

Then, there exists $M\in\mathrm{Pic}(S)$ adapted to $|L|$ such that the linear system $|A|$ is contained in $|M\otimes\mathcal{O}_C|$ and $\mathrm{Cliff}(M\otimes\mathcal{O}_C)\leq\mathrm{Cliff}(A)$. Moreover, one has $c_1(M)\cdot C\leq(4g-4)/3$.

We recall that the assertion that $|A|$ is contained in $|M\otimes\mathcal{O}_C|$ is equivalent to the requirement $h^0(C,A^\vee\otimes M\otimes\mathcal{O}_C)>0$. The assumption on the gonality k is used for computational reasons; however, the methods of our proof might be adapted in order to treat the cases where k is not maximal. It was proved by Ciliberto and Pareschi (cf. [CP] Proposition 3.3) that the ampleness of $L=\mathcal{O}_S(C)$ forces C to have Clifford dimension 1 with only one exception occurring for $g=10$.

The case of pencils is very particular, since it involves vector bundles of rank 2. Donagi and Morrison used the fact that any non-simple, indecomposable Lazarsfeld-Mukai bundle of rank 2 can be expressed as an extension of the image and the kernel of a nilpotent endomorphism, which both have rank 1. Their proof cannot be adapted to linear series with $r>1$, corresponding to Lazarsfeld-Mukai bundles of rank at least 3. Our techniques consist of showing that, under the hypotheses of Theorem 1.1, the rank-3 Lazarsfeld-Mukai bundle $E=E_{C,A}$ is given by an extension

$$0\to N\to E\to E/N\to 0,$$

where $N\in\mathrm{Pic}(S)$ and E/N is a μ_L-stable, torsion free sheaf of rank 2. When E is μ_L-unstable, the line bundle N coincides with its maximal destabilizing sheaf and the determinant of E/N plays the role of the line bundle M in the statement. Something similar happens if E is properly μ_L-semistable.

This suggests that the notion of stability might play a fundamental role in a general proof of the Donagi-Morrison Conjecture.

Now, we turn our attention to the cases where $\rho(g,r,d)\geq 0$. In the course of their proof of Green's Conjecture for curves on arbitrary $K3$ surfaces, Aprodu and Farkas (cf. [AF]) showed that, if L is an ample line bundle on a $K3$ surface such that a general curve $C\in|L|$ has Clifford dimension 1 and gonality k, given $d>g-k+2$, any dominating component of $\mathcal{W}^1_d(|L|)$ corresponds to simple Lazarsfeld-Mukai bundles. In particular, when the gonality is maximal this ensures that, if C is general in its linear system and the Brill-Noether number $\rho(g,1,d)$ is positive, the variety $W^1_d(C)$ is reduced and of the expected dimension. In the case $\rho(g,1,d)=0$, one finds that $W^1_d(C)$ is 0-dimensional, even though not necessarily reduced.

It is natural to wonder to what extent such a result can be expected to hold for linear series of type g^r_d with $r>1$. We prove the following theorem.

Theorem 1.2. *Let S be a $K3$ surface and $L\in\mathrm{Pic}(S)$ be an ample line bundle such that a general curve in $|L|$ has genus g, Clifford dimension 1 and maximal gonality $k=\left\lfloor\frac{g+3}{2}\right\rfloor$. Fix a positive integer d such that $\rho(g,2,d)\geq 0$ and assume $(g,d)\notin\{(2,4),(4,5),(6,6),(10,9)\}$. Then, the following hold:*

(a) *If $d>\frac{3}{4}g+2$, no dominating component of $\mathcal{W}^2_d(|L|)$ corresponds to rank-3 Lazarsfeld-Mukai bundles which are not μ_L-stable.*
(b) *If $d\leq\frac{3}{4}g+2$, let $\mathcal{W}$ be a dominating component of $\mathcal{W}^2_d(|L|)$ that corresponds to Lazarsfeld-Mukai bundles which are not μ_L-stable. Then, there exists $M\in\mathrm{Pic}(S)$*

adapted to $|L|$ such that, for a general $(C,A) \in \mathcal{W}$, the linear system $|A|$ is contained in $|M \otimes \mathcal{O}_C|$ and $\mathrm{Cliff}(M \otimes \mathcal{O}_C) \leq \mathrm{Cliff}(A)$. Moreover, $c_1(M) \cdot C \leq (4g-4)/3$.

Unlike case (a), case (b) does not exclude the existence of dominating components of $\mathcal{W}^2_d(|L|)$ which correspond to either μ_L-stable or properly μ_L-semistable Lazarsfeld-Mukai bundles. However, general points of such a component $\mathcal{W}$ give nets g^2_d, which are all contained in the restriction of the same line bundle $M \in \mathrm{Pic}(S)$ to curves in $|L|$. Furthermore, the Clifford index of $M \otimes \mathcal{O}_C$ is the same for any $C \in |L|_s$ and does not exceed $d-4$.

For a curve $C \in |L|_s$ and for a fixed value of d, we define the variety

$$\widetilde{W}^2_d(C) := \{A \in W^2_d(C) \mid A \text{ is base point free}\},$$

which is an open subscheme of $W^2_d(C)$, not necessarily dense. The following result is a direct consequence of Theorem 1.2.

Corollary 1.3. *Under the same hypotheses of Theorem 1.2, for a general $C \in |L|_s$ the following hold.*

(a) *If $d > \frac{3}{4}g + 2$, the variety $\widetilde{W}^2_d(C)$ is reduced of the expected dimension $\rho(g,2,d)$.*

(b) *If $d \leq \frac{3}{4}g + 2$, let W be an irreducible component of $\widetilde{W}^2_d(C)$ which either is non-reduced or has dimension greater than $\rho(g,2,d)$. Then, there exists an effective divisor $D \subset S$ such that $\mathcal{O}_S(D)$ is adapted to $|L|$ and, for a general $A \in W$, the linear system $|A|$ is contained in $|\mathcal{O}_C(D)|$ and $\mathrm{Cliff}(\mathcal{O}_C(D)) \leq \mathrm{Cliff}(A)$.*

Aprodu and Farkas' result follows from a parameter count for spaces of Donagi-Morrison extensions corresponding to non-simple Lazarsfeld-Mukai bundles of rank 2. The strategy used to prove Theorem 1.2 consists, instead, of counting the number of moduli of μ_L-unstable and properly μ_L-semistable Lazarsfeld-Mukai bundles of rank 3; this involves Artin stacks that parametrize the corresponding Harder-Narasimhan and Jordan-Hölder filtrations.

The plan of the paper is as follows. Sections 2 and 3 give background information on Lazarsfeld-Mukai bundles and stability of sheaves on $K3$ surfaces.

In Section 4 we present a different proof of Aprodu and Farkas' result and show that, if $\rho(g,1,d) > 0$, the Lazarsfeld-Mukai bundles corresponding to general points of any dominating component of $\mathcal{W}^1_d(|L|)$ are not only simple, but even μ_L-stable (Theorem 4.3). We introduce stacks of filtrations, studied for instance by Bridgeland in [B] and Yoshioka in [Y], and explain our parameter count in an easier case. The space of Lazarsfeld-Mukai bundles E, such that the bundles appearing in the Harder-Narasimhan filtration of E have prescribed Mukai vectors, turns out to be an Artin stack, whose dimension can be computed by using some well known facts regarding morphisms between semistable sheaves.

In Section 5 we look at the different types of possible Harder-Narashiman and Jordan-Hölder filtrations of a rank-3 Lazarsfeld-Mukai bundle E with $\det(E) = L$ and $c_2(E) = d$. If the determinants of both the subbundles E_i and the quotient sheaves E^j, given by the filtration of E, have at least 2 global sections, their restriction to a general curve $C \in |L|$ contributes to the Clifford index. This is used in order to bound from below the intersection products between the first Chern classes of the sheaves E_i and E^j.

In Sections 6, 7, 8 we estimate the number of moduli of pairs (C, A) corresponding to rank-3 Lazarsfeld-Mukai bunldes which are not μ_L-stable. The subdivision in three sections reflects the different methods necessary to treat various types of filtrations, depending on their length and on the rank of the sheaves E_i and E^j. At the end of Section 8 the proofs of both Theorem 1.1 and Theorem 1.2 are given.

In Section 9, an application towards transversality of Brill-Noether loci and Gieseker-Petri loci is presented. Recall that the Gieseker-Petri locus GP_g consists, by definition, of curves inside M_g that violate the Gieseker-Petri Theorem. For values of r, d such that $\rho(g, r, d) \geq 0$, one defines the component of GP_g of type (r, d) as

$$GP^r_{g,d} := \{[C] \in M_g \mid \exists\, (A, V) \in G^r_d(C) \text{ with } \ker \mu_{0,V} \neq 0\},$$

where $\mu_{0,V}$ is the Petri map. The subscheme

$$\widetilde{GP}^r_{g,d} := \{[C] \in M_g \mid \exists\, A \in W^r_d(C) \setminus W^{r+1}_d(C) \text{ with } \ker \mu_{0,A} \neq 0\}$$

is open in $GP^r_{g,d}$ but not necessarily dense. We prove the following:

Theorem 1.4. *Let $r \geq 3$, $g \geq 0$, $d \leq g-1$ be positive integers such that $\rho(g, r, d) < 0$ and $d - 2r + 2 \geq \lfloor (g+3)/2 \rfloor$. If $r \geq 4$, assume $d^2 > 4(r-1)(g+r-2)$. For $r = 3$, let $d^2 > 8g+1$. If -1 is not represented by the quadratic form*

$$Q(m, n) = (r-1)m^2 + mnd + (g-1)n^2, \quad m, n \in \mathbb{Z},$$

then:

(a) $M^r_{g,d} \not\subset M^1_{g,f}$ *for* $f < (g+2)/2$.
(b) $M^r_{g,d} \not\subset \widetilde{GP}^1_{g,f}$ *for* $f \geq (g+2)/2$.
(c) $M^r_{g,d} \not\subset M^2_{g,e}$ *if* $e < d - 2r + 5$ *and* $\rho(g, 2, e) < 0$.
(d) $M^r_{g,d} \not\subset \widetilde{GP}^2_{g,e}$ *if* $e < \min\left\{\frac{17}{24}g + \frac{23}{12}, d - 2r + 5\right\}$ *and* $\rho(g, 2, e) \geq 0$.

The assumption on the quadratic form Q is a mild hypothesis. For instance, it is automatically satisfied when r and g are odd and d is even.

In the last section we exhibit an application of our methods to higher rank Brill-Noether Theory. We give a negative answer to Question 4.2 in [LN3], which asks whether the second Clifford index $\mathrm{Cliff}_2(C)$, associated with rank-2 vector bundles on a curve C, equals $\mathrm{Cliff}(C)$ whenever C is a Petri curve. We analyze what happens in genus 11 and look at the Noether-Lefschetz divisor $\mathcal{NL}^4_{11,13}$, which consists of curves that lie on a $K3$ surface $S \subset \mathbb{P}^4$ with Picard number at least 2; this coincides with the locus of curves $[C] \in M_{11}$ such that $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$ (cf. [FO2]). We prove the following:

Theorem 1.5. *A general curve $[C] \in \mathcal{NL}^4_{11,13}$ satisfies the Gieseker-Petri Theorem.*

In other words, the Gieseker-Petri divisor GP_{11} and the Noether-Lefschetz divisor $\mathcal{NL}^4_{11,13}$ are transversal.

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2. Lazarsfeld-Mukai bundles

In this section we briefly recall the definition and the main properties of Lazarsfeld-Mukai bundles (LM bundles in the sequel) associated with complete, base point free linear series on curves lying on $K3$ surfaces. We refer to [L1], [L2], [P] for the proofs.

Let S be a $K3$ surface and $C \subset S$ a smooth connected curve of genus g. Any base point free linear series $A \in W^r_d(C)\backslash W^{r+1}_d(C)$ can be considered as a globally generated sheaf on S; therefore, the evaluation map $\mathrm{ev}_{A,S} : H^0(C,A) \otimes \mathcal{O}_S \to A$ is surjective and one defines the bundle $F_{C,A}$ to be its kernel, i.e.,

$$0 \to F_{C,A} \to H^0(C,A) \otimes \mathcal{O}_S \to A \to 0. \tag{1}$$

The LM bundle associated with the pair (C,A) is, by definition, $E_{C,A} := F^\vee_{C,A}$. By dualizing (1), one finds that $E_{C,A}$ sits in the following short exact sequence:

$$0 \to H^0(C,A)^\vee \otimes \mathcal{O}_S \to E_{C,A} \to \omega_C \otimes A^\vee \to 0; \tag{2}$$

in particular, $E_{C,A}$ is equipped with a $(r+1)$-dimensional subspace of sections. The following proposition summarizes the most important properties of $E_{C,A}$:

Proposition 2.1. *If $E_{C,A}$ is the LM bundle corresponding to a base point free linear series $A \in W^r_d(C) \setminus W^{r+1}_d(C)$, then:*

- $\mathrm{rk}\, E_{C,A} = r+1$.
- $\det E_{C,A} = L$, *where* $C \in |L|$.
- $c_2(E_{C,A}) = d$.
- *The bundle $E_{C,A}$ is globally generated off the base locus of $\omega_C \otimes A^\vee$.*
- $h^0(S,E_{C,A}) = h^0(C,A) + h^0(C,\omega_C \otimes A^\vee) = r+1+g-d+r$, $h^1(S,E_{C,A}) = h^2(S,E_{C,A}) = 0$.
- $\chi(S, E_{C,A} \otimes F_{C,A}) = 2(1-\rho(g,r,d))$.

In particular, if $\rho(g,r,d) < 0$, the LM bundle $E_{C,A}$ is non-simple.

Being a LM bundle is an open condition. Indeed, a vector bundle E of rank $r+1$ is a LM bundle whenever $h^1(S,E) = h^2(S,E) = 0$ and there exists $\Lambda \in G(r+1, H^0(S,E))$ such that the degeneracy locus of the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \to E$ is a smooth connected curve.

Analogously, given (C,A) as above, one defines a rank-r vector bundle M_A on C as the kernel of $\mathrm{ev}_{A,C} : H^0(C,A) \otimes \mathcal{O}_C \to A$. It turns out that

$$H^0(C, M_A \otimes \omega_C \otimes A^\vee) = \ker \mu_{0,A}.$$

Similarly, by tensoring (2) by $F_{C,A}$ and taking cohomology, one shows that

$$H^0(S, E_{C,A} \otimes F_{C,A}) \simeq H^0(C, F_{C,A} \otimes \omega_C \otimes A^\vee).$$

Moreover, there is the following short exact sequence:

$$0 \to \mathcal{O}_C \to F_{C,A} \otimes \omega_C \otimes A^\vee \to M_A \otimes \omega_C \otimes A^\vee \to 0. \tag{3}$$

Having denoted by $\pi : \mathcal{W}^r_d(|L|) \to |L|_s$ the natural projection and by $\mu_{1,A,S}$ the composition of the Gaussian map $\mu_{1,A} : \ker \mu_{0,A} \to H^0(C,\omega_C^2)$ with the transpose of the Kodaira-Spencer map $\delta^\vee_{C,S} : H^0(C,\omega_C^2) \to (T_C|L|)^\vee = H^1(C,\mathcal{O}_C)$, one has that $\mathrm{Im}(d\pi_{(C,A)}) \subset \mathrm{Ann}(\mathrm{Im}(\mu_{1,A,S}))$. Sard's Lemma applied to the projection π implies that, if $\mathcal{W} \subset \mathcal{W}^r_d(|L|)$ is a dominating component and C is general in its linear system, the sequence (3) is exact on the global sections for any $(C,A) \in (\pi|_{\mathcal{W}})^{-1}(C)$ such that A is base point free and $h^0(C,A) = r+1$; indeed, the coboundary map $H^0(C, M_A \otimes \omega_C \otimes A^\vee) \to H^1(C,\omega_C)$ coincides, up to a scalar factor, with $\mu_{1,A,S}$. As a consequence, the simplicity of $E_{C,A}$ is equivalent to the injectivity of $\mu_{0,A}$. In particular, if general points of $\mathcal{W}$ are complete, base point free linear series corresponding to simple LM bundles, the fiber $(\pi|_{\mathcal{W}})^{-1}(C)$ over a general $C \in |L|_s$ is reduced of the expected dimension. Standard Brill-Noether theory implies that no component of $\mathcal{W}^r_d(|L|)$ is entirely contained in $\mathcal{W}^{r+1}_d(|L|)$. Therefore, the variety $W^r_d(C)$ is reduced of the expected dimension for a general $C \in |L|_s$ if no dominating component $\mathcal{W}$ of $\mathcal{W}^r_d(|L|)$ is of one of the following types:

(a) For $(C, A) \in \mathcal{W}$ general, A is complete, base point free and $E_{C,A}$ is non-simple.
(b) For $(C, A) \in \mathcal{W}$ general, A is not base point free and $\ker \mu_{0,A} \neq 0$.

In order to exclude (b), one can proceed by induction on d because, if B denotes the base locus of A and $\ker \mu_{0,A} \neq 0$, then $\mu_{0,A(-B)} \neq 0$, too.

3. Mumford stability for sheaves on $K3$ surfaces

For later use, we recall some facts about coherent sheaves on smooth projective surfaces referring to [HL] and [Sh] for most of the proofs. Let S be a smooth, projective surface over $\mathbb{C}$ and H an ample line bundle on it. Given a torsion free sheaf E on S of rank r, the H-slope of E is defined as

$$\mu_H(E) = \frac{c_1(E) \cdot c_1(H)}{r},$$

and E is called μ_H-semistable (resp. μ_H-stable) in the sense of Mumford-Takemoto if for any subsheaf $0 \neq F \subset E$ with $\operatorname{rk} F < \operatorname{rk} E$, one has $\mu_H(F) \leq \mu_H(E)$ (resp. $\mu_H(F) < \mu_H(E)$). The Harder-Narasimhan filtration of E (HN filtration in the sequel) is the unique filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_s = E,$$

such that $E^i := E_i/E_{i-1}$ is a torsion free, μ_H-semistable sheaf for $1 \leq i \leq s$, and $\mu_H(E_{i+1}/E_i) < \mu_H(E_i/E_{i-1})$ for $1 \leq i \leq s-1$. Such a filtration always exists. It can be easily checked that, if E is a vector bundle, the sheaves E_i are locally free; moreover,

$$\mu_H(E_1) > \mu_H(E_2) > \ldots > \mu_H(E).$$

The sheaf E_1 is called the maximal destabilizing sheaf of E; the number $\mu_H(E_1)$ is the maximal slope of a proper subsheaf of E and, among the subsheaves of E of slope equal to $\mu_H(E_1)$, the sheaf E_1 has maximal rank. In particular, E_1 is μ_H-semistable.

Now, we assume E is μ_H-semistable. A Jordan-Hölder filtration of E (later on, JH filtration) is a filtration

$$0 = JH_0(E) \subset JH_1(E) \subset \ldots \subset JH_s(E) = E,$$

such that all the factors $\operatorname{gr}_i(E) := JH_i(E)/JH_{i-1}(E)$ are torsion free, μ_H-stable sheaves of slope equal to $\mu_H(E)$. This implies that $\mu_H(JH_i(E)) = \mu_H(E)$ for $1 \leq i \leq s$. The Jordan-Hölder filtration always exists but is not uniquely determined, while the graded object $\operatorname{gr}(E) := \oplus_i \operatorname{gr}_i(E)$ is.

The following result concerns morphisms between μ_H-semistable and μ_H-stable sheaves on S (cf. [Sh], [Fr]).

Proposition 3.1. *Given two torsion free sheaves E and F on S, the following holds:*

(a) *If E and F are μ_H-semistable and $\mu_H(E) > \mu_H(F)$, then $\operatorname{Hom}(E, F) = 0$.*
(b) *If E and F are μ_H-stable, $\mu_H(E) = \mu_H(F)$ and there exists $0 \neq \varphi \in \operatorname{Hom}(E, F)$, then $\operatorname{rk} E = \operatorname{rk} F$ and φ is an isomorphism in codim ≤ 1 (in particular it is injective).*

In the case where S is a $K3$ surface, by Serre duality $H^2(S, E) \simeq \operatorname{Hom}(E, \mathcal{O}_S)^\vee$; hence (a) implies that, if E is μ_H-semistable and $\mu_H(E) > 0$, then $h^2(S, E) = 0$.

From now on, we assume S to be a $K3$ surface. Throughout the paper we will often use the following fact:

Lemma 3.2. *Let $E, Q \in \mathrm{Coh}(S)$ be torsion free and $\mathrm{rk}\, E \geq 2$. If E is globally generated off a finite number of points, $h^2(S, E) = 0$ and there exists a surjective morphism $\varphi : E \to Q$, then $h^0(S, Q^{\vee\vee}) \geq 2$.*

Proof. Being a quotient of E , the sheaf Q is globally generated off a finite set. If $\mathrm{rk}\, Q \geq 2$, this trivially implies $h^0(S, Q^{\vee\vee}) \geq h^0(S, Q) \geq 2$. On the other hand, if Q has rank 1, then $Q = N \otimes I$, where $N \in \mathrm{Pic}(S)$ and I is the ideal sheaf associated with a 0-dimensional subscheme of S. Since N is a quotient of E off a finite number of points, it has no fixed components, thus it is base point free (cf. [SD]). The statement follows by remarking that $N = Q^{\vee\vee}$ cannot be trivial because $h^2(S, E) = 0$. □

Another useful result is the following one (cf. Lemma 3.1 in [GL]):

Lemma 3.3. *Let E be a vector bundle of rank r on S which is globally generated off a finite number of points. If $h^2(S, E) = 0$, then $h^0(S, \det E) \geq 2$.*

Proof. Since the natural map $\wedge^r H^0(S, E) \otimes \mathcal{O}_S \to \wedge^r E = \det E$ is surjective off a finite number of points, the line bundle $\det E$ is base point free. Therefore, it is enough to show that $\det E$ is non-trivial. This follows by remarking that, given a general $V \in G(r, H^0(S, E))$, the natural map $ev_V : V \otimes \mathcal{O}_S \to E$ is injective but is not an isomorphism since $h^2(S, E) = 0$. Therefore, $\det ev_V$ gives a section of $\det E$ vanishing on a non-zero effective divisor. □

Last but not least, we recall some notation and results from [M1]. The Mukai vector of a sheaf $E \in \mathrm{Coh}(S)$ is defined as:

$$v(E) := \mathrm{ch}(E)(1 + \omega) = \mathrm{rk}(E) + c_1(E) + (\chi(E) - \mathrm{rk}(E))\omega \in H^*(S, \mathbb{Z}) = H^{2*}(S, \mathbb{Z}),$$

where $H^4(S, \mathbb{Z})$ is identified with $\mathbb{Z}$ by means of the fundamental cocycle ω. The Mukai lattice is the pair $(H^*(S, \mathbb{Z}), \langle , \rangle)$, with $\langle , \rangle$ being the symmetric bilinear form on $H^*(S, \mathbb{Z})$ whose definition is the following:

$$\langle v, w \rangle := -\int_S v^* \wedge w,$$

where, if $v = v^0 + v^1 + v^2$ with $v^i \in H^{2i}(S, \mathbb{Z})$, we set $v^* := v^0 - v^1 + v^2$. Given $E, F \in \mathrm{Coh}(S)$, we define the Euler characteristic of the pair (E, F) as

$$\chi(E, F) := \sum_{i=0}^{2} (-1)^i \dim \mathrm{Ext}^i(E, F),$$

and it turns out that $\chi(E, F) = -\langle v(E), v(F) \rangle$.

Given a Mukai vector $v \in H^*(S, \mathbb{Z})$, let $\mathcal{M}(v)$ be the moduli stack of coherent sheaves on S of Mukai vector v. If $H \in \mathrm{Pic}(S)$ is ample, we denote by $\mathcal{M}_H(v)^{\mu ss}$ (resp. $\mathcal{M}_H(v)^{\mu s}$) the moduli stack parametrizing isomorphism classes of μ_H-semistable (resp. μ_H-stable) sheaves on S with Mukai vector v. Recall that any μ_H-stable sheaf is simple and that any irreducible component of $\mathcal{M}_H(v)^{\mu s}$ has dimension equal to $\langle v, v \rangle + 1$. Moreover, if $\gcd(v^0, v^1.H) = 1$, then μ_H-semistability and μ_H-stability coincide.

4. Stability of Lazarsfeld-Mukai bundles of rank 2

Let S be a smooth, projective $K3$ surface and consider a line bundle $L \in \mathrm{Ample}(S)$ such that a general curve $C \in |L|_s$ has genus g, Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$. In this section we prove that, if C is general in its linear system and $\rho(g,1,d)>0$, the LM bundle associated with a general complete, base point free g^1_d on C is μ_L-stable.

Fix a rank-2 LM bundle $E=E_{C,A}$ corresponding to a complete, base point free pencil $A\in W^1_d(C)$ with $C\in|L|_s$; Proposition 2.1 implies that

$$v(E)=2+c_1(L)+(g-d+1)\omega.$$

We assume E is not μ_L-stable. In the case where E is μ_L-unstable (resp. properly μ_L-semistable) we consider its HN filtration (resp. JH filtration) $0\subset M\subset E$, which gives a short exact sequence

$$0\to M\to E\to N\otimes I_\xi\to 0, \tag{4}$$

where M and N are two line bundles such that $\mu_L(M)>\mu_L(E)=g-1>\mu_L(N)$ (resp. $\mu_L(M)=\mu_L(E)=\mu_L(N)$) and I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi\subset S$ of length $l=d-c_1(N)\cdot c_1(M)$. By Lemma 3.2, we know that $h^0(S,N)\geq 2$. First of all, we prove the following:

Lemma 4.1. *In the situation above, if general curves in $|L|_s$ have Clifford dimension* 1 *and (constant) gonality k, one has $c_1(M)\cdot c_1(N)\geq k$.*

Proof. We remark that $h^2(S,M)=0$ since $\mu_L(M)>0$. Therefore, if

$$2>h^0(S,M)\geq\chi(M)=2+c_1(M)^2/2,$$

then $c_1(M)^2<0$ and the inequality $\mu_L(M)\geq g-1$ implies $c_1(M)\cdot c_1(N)\geq g+1\geq k$. From now on, we assume $h^0(S,M)\geq 2$. Since $\omega_C\otimes N^\vee|_C=M\otimes\mathcal{O}_C$, the line bundle $N|_C$ contributes to $\mathrm{Cliff}(C)$. The short exact sequence

$$0\to M^\vee\to N\to N\otimes\mathcal{O}_C\to 0$$

gives $h^0(C,N\otimes\mathcal{O}_C)\geq h^0(S,N)$. It follows that

$$\begin{aligned}\mathrm{Cliff}(N\otimes\mathcal{O}_C) &= c_1(N)\cdot(c_1(N)+c_1(M))-2h^0(C,N\otimes\mathcal{O}_C)+2\\ &\leq c_1(N)^2+c_1(N)\cdot c_1(M)-2\chi(N)-2h^1(S,N)+2\\ &= -2+c_1(N)\cdot c_1(M)-2h^1(S,N).\end{aligned}$$

Since $\mathrm{Cliff}(N\otimes\mathcal{O}_C)\geq k-2$, then $c_1(M)\cdot c_1(N)\geq k+2h^1(S,N)\geq k$. □

Our goal is to count the number of moduli of μ_L-unstable and properly μ_L-semistable LM bundles of rank 2.

Fix a nonnegative integer l and a non-trivial, globally generated line bundle N on S such that, having defined $M:=L\otimes N^\vee$, either $\mu_L(M)=\mu_L(N)=g-1$ or $\mu_L(M)>g-1>\mu_L(N)$. We consider the moduli stack $\mathcal{E}_{N,l}$ parametrizing filtrations $0\subset M\subset E$ with $[M]\in\mathcal{M}(v(M))(\mathbb{C})$ and $[E/M]\in\mathcal{M}(v(N\otimes I_\xi))(\mathbb{C})$, where $l(\xi)=l$. Note that, since both N and M are line bundles, the stack $\mathcal{M}(v(M))$ has a unique $\mathbb{C}$-point endowed with an automorphism group of dimension 1, while $\mathcal{M}(v(N\otimes I_\xi))$ is corepresented by the Hilbert scheme $S^{[l]}$ parametrizing 0-dimensional subschemes of S of length l. Two filtrations $0\subset M\subset E$ and $0\subset M'\subset E'$ are equivalent whenever there exists a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & E\\ \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2}\\ M' & \longrightarrow & E', \end{array}$$

where φ_1 and φ_2 are two isomorphisms (cf. [B] for the proof that $\mathcal{E}_{N,l}$ is algebraic). The stack $\mathcal{E}_{N,l}$ can be alternatively described as the moduli stack of extensions of type (4).

Let $p : \mathcal{E}_{N,l} \to \mathcal{M}(v(M)) \times \mathcal{M}(v(N \otimes I_\xi))$ be the natural morphism of stacks mapping the short exact sequence (4) to $(M, N \otimes I_\xi)$. The fiber of p over the $\mathbb{C}$-point $(M, N \otimes I_\xi)$ of $\mathcal{M}(v(M)) \times \mathcal{M}(v(N \otimes I_\xi))$ is the quotient stack

$$[\mathrm{Ext}^1(N \otimes I_\xi, M)/\mathrm{Hom}(N \otimes I_\xi, M)],$$

where the action of $\mathrm{Hom}(N \otimes I_\xi, M)$ over $\mathrm{Ext}^1(N \otimes I_\xi, M)$ is the trivial one (cf. [B]); it follows that in general p is not representable.

We define $\tilde{P}_{N,l}$ to be the closure of the image of $\mathcal{E}_{N,l}$ under the natural projection $q : \mathcal{E}_{N,L} \to \mathcal{M}(v(E))$, which maps the point of $\mathcal{E}_{N,L}$ given by (4) to $[E]$. The morphism q is representable (cf. proof of Lemma (4.1) in [B]) and the fiber of q over a $\mathbb{C}$-point of $\tilde{P}_{N,l}$ corresponding to E is the Quot-scheme $\mathrm{Quot}_S(E, P)$, where P is the Hilbert polynomial of $N \otimes I_\xi$. We denote by $P_{N,l}$ the open substack of $\tilde{P}_{N,l}$ whose $\mathbb{C}$-points correspond to vector bundles E satisfying $h^1(S, E) = h^2(S, E) = 0$.

Let $\mathcal{G}_{N,l} \to P_{N,l}$ be the Grassmann bundle with fiber over a point $[E] \in P_{N,l}(\mathbb{C})$ equal to $G(2, H^0(S, E))$. A $\mathbb{C}$-point of $\mathcal{G}_{N,l}$ is a pair (E, Λ) and comes endowed with an automorphism group equal to $\mathrm{Aut}(E)$. We consider the rational map

$$h_{N,l} : \mathcal{G}_{N,l} \dashrightarrow \mathcal{W}^1_d(|L|),$$

mapping a general point $(E, \Lambda) \in \mathcal{G}_{N,l}(\mathbb{C})$ to the pair (C_Λ, A_Λ), where C_Λ is the degeneracy locus of the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \to E$, which is injective for a general $\Lambda \in G(2, H^0(S, E))$, and $\omega_{C_\Lambda} \otimes A_\Lambda^\vee$ is the cokernel of ev_Λ. Notice that $d := c_1(N) \cdot c_1(M) + l$. Since while mapping to $\mathcal{W}^1_d(|L|)$ we forget the automorphisms, the fiber of $h_{N,l}$ over (C, A) is the quotient stack

$$[\mathbb{P}(\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee)^\circ)/\mathrm{Aut}(E_{C,A})],$$

where $\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee)^\circ$ denotes the open subgroup of $\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee)$ consisting of those morphisms whose kernel is isomorphic to $\mathcal{O}_S^{\oplus 2}$, and $\mathrm{Aut}(E_{C,A})$ acts on $\mathbb{P}(\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee)^\circ)$ by composition. In particular, $h_{N,l}$ is not representable. As remarked in Section 2, one has

$$\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee) \simeq H^0(S, E_{C,A} \otimes E_{C,A}^\vee);$$

it is trivial to check that

$$\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee)^\circ \simeq \mathrm{Aut}(E_{C,A}).$$

Therefore, the action of $\mathrm{Aut}(E_{C,A})$ on $\mathbb{P}(\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee)^\circ)$ is transitive and the stabilizer of any point is the subgroup generated by $\mathrm{Id}_{E_{C,A}}$; as a consequence, any fiber of $h_{N,l}$ has dimension -1 (cf. [Go] for the definition of the dimension of a locally Noetherian algebraic stack). We denote by $\mathcal{W}_{N,l}$ the closure of the image of $h_{N,l}$. The following holds:

Proposition 4.2. *Assume that $P_{N,l}$ be non-empty and let $\mathcal{W}$ be an irreducible component of $\mathcal{W}_{N,l}$. Then*

$$\dim \mathcal{W} \leq g + d - k,$$

where k is the gonality of any curve in $|L|_s$.

Proof. Proposition 3.1, together with the fact that $h^0(S, I_\xi) = 0$ if $l > 0$, implies that

$$\dim \mathrm{Hom}(M, N \otimes I_\xi) = \begin{cases} 1 & \text{if } M \simeq N,\ \xi = \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

It follows that the dimension of the fibers of p is constant and equals $-\chi(M, N \otimes I_\xi)$, unless $M \simeq N$ and $l = 0$, in which case it is $-\chi(M, N \otimes I_\xi) + 1$.

Regarding the fibers of q, it is well known (cf. [HL] Proposition 2.2.8) that, given $[\varphi : E \to N \otimes I_\xi] \in \mathrm{Quot}_S(E,P)$, the following holds:

$$\begin{aligned} \dim \mathrm{Hom}(K, N\otimes I_\xi) - \dim \mathrm{Ext}^1(K, N\otimes I_\xi) &\leq \dim_{[\xi]} \mathrm{Quot}_S(E,P) \\ &\leq \dim \mathrm{Hom}(K, N\otimes I_\xi), \end{aligned} \tag{5}$$

where $K = \ker \varphi$; moreover, if $\mathrm{Ext}^1(K, N\otimes I_\xi) = 0$, then $\mathrm{Quot}_S(E,P)$ is smooth in $[\varphi]$ of dimension equal to $\dim \mathrm{Hom}(K, N\otimes I_\xi)$. Since $K \simeq M$, if $M \simeq N$ and $l = 0$, the fibers of q are smooth of dimension 1; indeed, $\mathrm{Ext}^1(N,N) \simeq H^1(S, \mathcal{O}_S) = 0$. Otherwise, the fibers of q are 0-dimensional. It follows that, if $P_{N,l}$ is non-empty, then:

$$\begin{aligned} \dim \mathcal{G}_{N,l} &= \dim P_{N,l} + 2(g-d+1) \\ &= \dim \mathcal{M}(v(M)) + \dim \mathcal{M}(v(N\otimes I_\xi)) + \langle v(M), v(N\otimes I_\xi)\rangle + 2(g-d+1) \\ &= 2l - 2 + c_1(M)\cdot c_1(N) - \frac{c_1(M)^2}{2} - \frac{c_1(N)^2}{2} - 2 + l + 2(g-d+1) \\ &= 3l + 2g - 2d - 2 - (g-1) + 2c_1(M)\cdot c_1(N) \\ &= g + d - 1 - c_1(N)\cdot c_1(M) \\ &\leq g + d - 1 - k, \end{aligned}$$

where we have used that $c_1(M) + c_1(N) = c_1(L)$ and $d = c_1(M)\cdot c_1(N) + l$, and the last inequality follows from Lemma 4.1. The statement is a consequence of the fact that the fibers of $h_{N,l}$ are quotient stacks of dimension equal to -1. □

We can finally prove the following result:

Theorem 4.3. *Assume that general curves in $|L|_s$ have Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$.*

- *If $\rho(g,1,d) > 0$, any dominating component of $\mathcal{W}^1_d(|L|)$ corresponds to μ_L-stable LM bundles. In particular, if $C \in |L|_s$ is general, the variety $W^1_d(C)$ is reduced and has the expected dimension $\rho(g,1,d)$.*
- *If $\rho(g,1,k) = 0$ and $C \in |L|_s$ is general, then $W^1_k(C)$ has dimension 0.*

Proof. When $\rho(g,1,d) > 0$, we show that no component $\mathcal{W}$ of $\mathcal{W}^1_d(|L|_s)$ corresponding to either μ_L-unstable or properly μ_L-semistable LM bundles dominates $|L|$. Proposition 4.2 gives:

$$\dim \mathcal{W} \leq g + d - k \leq g + d - \frac{g+2}{2}.$$

Our claim follows by remarking that any dominating component of $\mathcal{W}^1_d(|L|)$ has dimension at least $g + \rho(g,1,d)$ and that $\rho(g,1,d) > d - \frac{g+2}{2}$ whenever $d > \frac{g+2}{2}$.

If $k = \frac{g+2}{2}$, that is, $\rho(g,1,k) = 0$, our parameter count shows that any dominating component of $\mathcal{W}^1_k(|L|)$ has dimension g; hence, if $C \in |L|$ is general, $W^1_k(C)$ is 0-dimensional, even though not necessarily reduced. By induction on d, one excludes the existence of components of $\mathcal{W}^1_d(|L|)$ whose general points correspond to linear series which are not base point free. □

5. Lazarsfeld-Mukai bundles of rank 3 which are not μ_L-stable

We fix a LM bundle $E = E_{C,A}$ associated with a complete, base point free g^2_d on a smooth connected curve $C \in |L|_s$ with $L \in \mathrm{Ample}(S)$. By Proposition 2.1, we have

$$v(E) = 3 + c_1(L) + (2 + g - d)\omega,$$

where $g = g(C)$. We assume that E is not μ_L-stable and, in the case where it is μ_L-unstable, we look at its HN filtration:

$$0 = E_0 \subset E_1 \subset \ldots \subset E_s = E.$$

On the other hand, if E is properly μ_L-semistable, we consider its JH filtration:

$$0 = JH_0(E) \subset JH_1(E) \subset \ldots \subset JH_s(E) = E.$$

We first consider the cases where either E is properly μ_L-semistable and $JH_1(E)$ has rank 2, or E is μ_L-unstable, $\operatorname{rk} E_1 = 2$ and E_1 is μ_L-stable. Under these hypotheses, E sits in the following short exact sequence:

$$0 \to M \to E \to N \otimes I_\xi \to 0, \tag{6}$$

where $M = JH_1(E)$ (resp. $M = E_1$) is a μ_L-stable vector bundle of rank 2, N is a line bundle and I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$. Moreover,

$$\mu_L(M) \geq \mu_L(E) = \frac{2g-2}{3} \geq \mu_L(N \otimes I_\xi) = \mu_L(N), \tag{7}$$

with the former inequality being strict whenever the latter one is. We have that $c_1(L) = c_1(E) = c_1(M) + c_1(N)$ and $d = c_2(E) = c_1(N) \cdot c_1(M) + l(\xi) + c_2(M)$, where $l(\xi)$ denotes the length of ξ. We prove the following:

Lemma 5.1. *Assume a general curve $C \in |L|_s$ has Clifford dimension 1 and gonality k. In the above situation, one has $c_1(N) \cdot c_1(M) \geq k$ and*

$$d \geq \frac{3}{4}k + \frac{7}{6} + \frac{g}{3}. \tag{8}$$

Proof. As E is globally generated off a finite number of points, N is base point free and non-trivial, thus $h^0(S, N) \geq 2$ and $\mu_L(N) > 0$. The inequality $\mu_L(M) > 0$ implies that $h^2(S, M) = 0$ and, since $\mu_L(\det M) = 2\mu_L(M)$, we have that $h^2(S, \det M) = 0$, too. Therefore, $h^0(S, \det M) \geq \chi(\det M) = 2 + c_1(M)^2/2$ and, if $h^0(S, \det M) < 2$, then $c_1(M)^2 \leq -2$ and $c_1(N) \cdot c_1(M) \geq (4g+2)/3 > k$ by the first inequality in (7), which gives

$$c_1(M)^2 + c_1(N) \cdot c_1(M) \geq \frac{4g-4}{3}. \tag{9}$$

On the other hand, if $h^0(S, \det M) \geq 2$, then $N|_C$ contributes to $\operatorname{Cliff}(C)$ and one shows, as in the proof of Lemma 4.1, that $c_1(N) \cdot c_1(M) \geq k + 2h^1(S, N) \geq k$.

The μ_L-stability of M implies that

$$-2 \leq \langle v(M), v(M) \rangle = c_1(M)^2 - 4\chi(M) + 8 = 4c_2(M) - c_1(M)^2 - 8.$$

Therefore, we have

$$d = c_1(N) \cdot c_1(M) + c_2(M) + l(\xi) \geq c_1(N) \cdot c_1(M) + \frac{c_1(M)^2}{4} + \frac{6}{4} \geq \frac{3}{4}k + \frac{7}{6} + \frac{g}{3};$$

this concludes the proof. □

Now, we assume that either E is μ_L-unstable, $\operatorname{rk} E_1 = 1$ and E/E_1 is μ_L-stable, or E is properly μ_L-semistable and its JH filtration is of type $0 \subset JH_1(E) \subset E$ with $\operatorname{rk} JH_1(E) = 1$. Denoting by N the line bundle E_1 (resp. $JH_1(E)$), one has a short exact sequence:

$$0 \to N \to E \to E/N \to 0, \tag{10}$$

where E/N is a rank-2, μ_L-stable, torsion free sheaf on S such that

$$\mu_L(N) \geq \mu_L(E) \geq \mu_L(E/N),$$

and either both inequalities are strict, or none is. We prove the following:

Lemma 5.2. *In the above situation, if a general curve $C \in |L|_s$ has Clifford dimension 1 and gonality k, then $c_1(N) \cdot c_1(E/N) \geq k$.*

Proof. As in the proof of Lemma 3.2 one shows that $h^0(S, E/N) \geq 2$. Since E/N is stable, then $\mu_L(E/N) > 0$ and $h^2(S, E/N) = 0$. Moreover, the vector bundle $(E/N)^{\vee\vee}$ is globally generated off a finite number of points and $h^0(S, \det(E/N)) \geq 2$ by Lemma 3.3 because $\det(E/N) := \det(E/N)^{\vee\vee}$.

Since $\mu_L(N) = c_1(N) \cdot (c_1(N) + c_1(E/N)) \geq (2g-2)/3 > 0$, we have $h^2(S, N) = 0$. Hence, if $h^0(S, N) < 2$, then $c_1(N)^2 < 0$ and $c_1(N) \cdot c_1(E/N) \geq (2g+4)/3 > k$. Otherwise, $N \otimes \mathcal{O}_C$ contributes to the Clifford index and this implies $c_1(N) \cdot c_1(E/N) \geq k$, too. □

The cases still to be considered are the following ones:

(i) E is μ_L-unstable with HN filtration $0 \subset E_1 \subset E_2 \subset E$.
(ii) E is properly μ_L-semistable with JH filtration $0 \subset JH_1(E) \subset JH_2(E) \subset E$.
(iii) E is μ_L-unstable with HN filtration $0 \subset E_1 \subset E$ and E_1 is a properly μ_L-semistable vector bundle of rank 2.
(iv) E is μ_L-unstable with HN filtration $0 \subset E_1 \subset E$ and E_1 is a line bundle such that E/E_1 is a properly μ_L- semistable torsion free sheaf of rank 2.

In all these cases one has four short exact sequences:

$$0 \to N \to E \to E/N \to 0 \tag{11}$$

$$0 \to M \to E \to N_1 \otimes I_{\xi_1} \to 0, \tag{12}$$

$$0 \to N \to M \to N_2 \otimes I_{\xi_2} \to 0, \tag{13}$$

$$0 \to N_2 \otimes I_{\xi_2} \to E/N \to N_1 \otimes I_{\xi_1} \to 0, \tag{14}$$

where N, N_1, N_2 are line bundles, I_{ξ_1} and I_{ξ_2} denote the ideal sheaves of two 0-dimensional subschemes $\xi_1, \xi_2 \subset S$, the sheaf E/N has rank-2 and no torsion, while M is a vector bundle of rank 2. Moreover, the following inequalities hold:

$$\mu_L(N) \geq \mu_L(N_2) \geq \mu_L(N_1), \tag{15}$$

$$\mu_L(N) \geq \frac{2g-2}{3} \geq \mu_L(N_1); \tag{16}$$

in particular, $\mu_L(N) = \mu_L(N_2)$ (resp. $\mu_L(N_1) = \mu_L(N_2)$) whenever M (resp. E/N) is properly μ_L-semistable, that is, in cases (ii) and (iii) (resp. in cases (ii) and (iv)). Analogously, equalities in (16) force E to be properly μ_L-semistable with JH-filtration $0 \subset N \subset M \subset E$, that is, one is in case (ii).

Lemma 5.3. *In the above situation, $N_1 \otimes \mathcal{O}_C$ always contributes to the Clifford index of $C \in |L|_s$. Moreover, one of the following occurs:*

(a) *Both $N \otimes \mathcal{O}_C$ and $N_2 \otimes \mathcal{O}_C$ contribute to the Clifford index of $C \in |L|_s$.*
(b) *The inequality $c_1(N) \cdot (c_1(N_1) + c_1(N_2)) \geq \frac{2g+4}{3}$ holds and either $N_2 \otimes \mathcal{O}_C$ contributes to the Clifford index of C or $c_1(N_2) \cdot (c_1(N) + c_1(N_1)) \geq g$.*

(c) *The linear series $N \otimes \mathcal{O}_C$ contributes to the Clifford index of $C \in |L|_s$ and one has the inequality $c_1(N_2) \cdot c_1(N) > \frac{1}{2} c_1(N) \cdot (c_1(N_1) + c_1(N_2))$.*

(d) *The inequality $c_1(N) \cdot c_1(N_2) \geq \frac{g+5}{3}$ holds.*

In particular, if a general $C \in |L|_s$ has Clifford dimension 1 and gonality k, then

$$d \geq c_1(N) \cdot c_1(N_1) + c_1(N) \cdot c_1(N_2) + c_1(N_1) \cdot c_1(N_2) \geq \frac{3}{2} k. \tag{17}$$

Proof. Being a quotient of E off a finite set, N_1 is base point free and non-trivial, thus $h^0(S, N_1) \geq 2$ and $\mu_L(N_1) > 0$. By the "Strong Bertini' s Theorem" (cf. [SD]), N_1 is nef. Proposition 3.1 implies $h^2(S, N) = h^2(S, N_2) = 0$ because of (15). Analogously, $\mu_L(N_2 \otimes N) = \mu_L(N_2) + \mu_L(N) > 0$ and $h^2(S, N_2 \otimes N) = 0$. Moreover, the following holds:

$$\begin{aligned} c_1(N_2 \otimes N)^2 &= c_1(N_2)^2 + c_1(N)^2 + 2c_1(N_2) \cdot c_1(N) \\ &\geq c_1(N)^2 + c_1(N_2) \cdot c_1(N) + c_1(N_1) \cdot c_1(N) + c_1(N_1)^2 \\ &= \mu_L(N) + c_1(N_1)^2 > 0, \end{aligned}$$

where we have used that, since $\mu_L(N_2) \geq \mu_L(N_1)$, then

$$c_1(N_2)^2 + c_1(N_2) \cdot c_1(N) \geq c_1(N_1)^2 + c_1(N_1) \cdot c_1(N), \tag{18}$$

and that $c_1(N_1)^2 \geq 0$ because N_1 is nef. We obtain that

$$h^0(S, N_2 \otimes N) \geq \chi(N_2 \otimes N) = 2 + \frac{1}{2} c_1(N_2 \otimes N)^2 > 2,$$

thus $N_1 \otimes \mathcal{O}_C$ always contributes to the Clifford index of $C \in |L|_s$.

If both $h^0(S, N_2) \geq 2$ and $h^0(S, N) \geq 2$, we are in case (a).

If $h^0(S, N_2) \geq 2$ and $h^0(S, N) < 2$, we show that (b) occurs. Since $\chi(N) < 2$, one has $c_1(N)^2 < 0$ and $c_1(N) \cdot (c_1(N_1) + c_1(N_2)) \geq \mu_L(E) + 2 = (2g+4)/3$ by the first inequality in (16). Since $\mu_L(N \otimes N_1) > 0$, then $h^2(S, N \otimes N_1) = 0$. Moreover, one can show that

$$c_1(N \otimes N_1)^2 \geq \mu_L(N_1) + c_1(N_2)^2 > c_1(N_2)^2.$$

It follows that, if $c_1(N \otimes N_1)^2 < 0$, then $c_1(N_2)^2 < 0$ and

$$2g - 2 < 2c_1(N) \cdot c_1(N_2) + 2c_1(N_1) \cdot c_1(N_2),$$

that is, $c_1(N_2) \cdot (c_1(N) + c_1(N_1)) \geq g$. On the other hand, if $c_1(N \otimes N_1)^2 \geq 0$, then $h^0(S, N \otimes N_1) \geq 2$ and $N_2 \otimes \mathcal{O}_C$ contributes to the Clifford index.

From now on, assume $h^0(S, N_2) < 2$, hence $c_1(N_2)^2 < 0$. Since $\det E/N \simeq N_1 \otimes N_2$, Lemma 3.3 implies $h^0(S, N_1 \otimes N_2) \geq 2$. Thus, if $h^0(S, N) \geq 2$, the linear series $N \otimes \mathcal{O}_C$ contributes to the Clifford index of $C \in |L|_s$. Furthermore, inequality (18), together with the fact that $c_1(N_2)^2 < 0 \leq c_1(N_1)^2$, implies that $c_1(N_2) \cdot c_1(N) > c_1(N_1) \cdot c_1(N)$. We obtain

$$c_1(N_2) \cdot c_1(N) > \frac{1}{2} c_1(N) \cdot (c_1(N_1) + c_1(N_2)),$$

and we are in case (c)

It remains to treat the case where both $h^0(S, N_2) < 2$ and $h^0(S, N) < 2$. Under these hypotheses, $c_1(N_2)^2 < 0$ and $c_1(N)^2 < 0$ and we obtain

$$\begin{aligned} 2g - 2 &\leq c_1(N_1)^2 + 2c_1(N_1) \cdot c_1(N) + 2c_1(N_1) \cdot c_1(N_2) + 2c_1(N) \cdot c_1(N_2) - 4 \\ &= 2c_1(N) \cdot c_1(N_2) + 2\mu_L(N_1) - c_1(N_1)^2 - 4 \\ &\leq 2c_1(N) \cdot c_1(N_2) + \frac{4g-4}{3} - 4. \end{aligned}$$

As a consequence, $c_1(N) \cdot c_1(N_2) \geq \frac{g+5}{3}$ and we are in case (d).

Now, we assume that C has Clifford dimension 1 and gonality k and prove inequality (17). One shows, as in Lemma 4.1, that

$$c_1(N_1) \cdot (c_1(N) + c_1(N_2)) \geq k, \tag{19}$$

because $N_1 \otimes \mathcal{O}_C$ always contributes to the Clifford index of $C \in |L|_s$. Analogously, if $N \otimes \mathcal{O}_C$ (resp. $N_2 \otimes \mathcal{O}_C$) contributes to $\mathrm{Cliff}(C)$, then $c_1(N) \cdot (c_1(N_1) + c_1(N_2)) \geq k$ (resp. $c_1(N_2) \cdot (c_1(N) + c_1(N_1)) \geq k$); therefore, the last part of the statement is proved if either (a) or (b) occurs (use that $(2g + 4)/3 \geq k$).

In case (c), one arrives at the same conclusion by adding inequality (19) and

$$c_1(N) \cdot c_1(N_2) > \frac{1}{2} c_1(N) \cdot (c_1(N_1) + c_1(N_2)) \geq \frac{k}{2}. \tag{20}$$

Similarly, in case (d), one uses that $c_1(N) \cdot c_1(N_2) \geq (g + 5)/3 \geq k/2$.

□

Corollary 5.4. *Assume $C \in |L|_s$ has Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$ and let E be the Lazarsfeld-Mukai bundle associated with a complete, base point free net $A \in W^2_d(C)$. If E is not μ_L-stable, $d < \frac{3}{4}k + \frac{7}{6} + \frac{g}{3}$ and $(g, d) \neq (6, 6)$, then E is given by an extension of type (10), with $N \in \mathrm{Pic}(S)$ and E/N a μ_L-stable, torsion free sheaf of rank 2 such that $\mu_L(N) \geq (2g - 2)/3 \geq \mu_L(E/N)$.*

Proof. Apply Lemma 5.1 and Lemma 5.3 and remark that $\lceil \frac{3}{4}k + \frac{7}{6} + \frac{g}{3} \rceil \leq \lceil \frac{3}{2}k \rceil$ unless $g = 6$. □

6. Cases with a μ_L-stable subbundle of rank 2 and L-slope $\geq \mu_L(E)$

We assume that a general curve in $|L|$ has Clifford dimension 1 and maximal gonality. In this section we show that, if $C \in |L|_s$ is general, the LM bundle E corresponding to a general, complete, base point free g^2_d on C is neither properly μ_L-semistable with JH filtration $0 \subset JH_1(E) \subset E$ and $\mathrm{rk}\, JH_1(E) = 2$, nor μ_L-unstable with a μ_L-stable, rank-2 vector bundle E_1 as maximal destabilizing sheaf .

Fix a positive integer d. Choose $l \in \mathbb{N}$ and a non-trivial, globally generated line bundle N such that

$$\mu_L(N) \leq \frac{2g - 2}{3} \leq \frac{(c_1(L) - c_1(N)) \cdot c_1(L)}{2}, \tag{21}$$

and impose that these are either two equalities or two strict inequalities. Set

$$\begin{aligned} c_1 &:= c_1(L) - c_1(N), \\ c_2 &:= d - c_1.c_1(N) - l, \\ \chi &:= g - d + 5 - \chi(N) + l, \end{aligned}$$

and define the vector $v := 2 + c_1 + (\chi - 2)\omega \in H^*(S, \mathbb{Z})$. The following construction is analogous to that of Section 4.

Let $\mathcal{E}_{N,l}$ be the moduli stack of filtrations $0 \subset M \subset E$, where $[M] \in \mathcal{M}_L(v)^{\mu s}(\mathbb{C})$ and $[E/M] \in \mathcal{M}(v(N \otimes I_\xi))(\mathbb{C})$ with $l(\xi) = l$. This is alternatively described as the moduli stack of extensions

$$0 \to M \to E \to N \otimes I_\xi \to 0, \tag{22}$$

with M and ξ as above.

If $p : \mathcal{E}_{N,l} \to \mathcal{M}_L(v)^{\mu s} \times \mathcal{M}(v(N \otimes I_\xi))$ denotes the morphism of Artin stacks mapping the short exact sequence (22) to $(M, N \otimes I_\xi)$, the fiber of p over the point of $M_L(v)^{\mu s} \times \mathcal{M}(v(N \otimes I_\xi))$ corresponding to the pair $(M, N \otimes I_\xi)$ is the quotient stack

$$[\mathrm{Ext}^1(N \otimes I_\xi, M)/\mathrm{Hom}(N \otimes I_\xi, M)].$$

Define $\tilde{P}_{N,l}$ to be the closure of the image of $\mathcal{E}_{N,L}$ under the natural projection

$$q : \mathcal{E}_{N,L} \to \mathcal{M}(v(E)),$$

which sends the isomorphism class of extension (22) to $[E] \in \mathcal{M}(v(E))(\mathbb{C})$. The morphism q is representable and the fiber of q over the point of $\tilde{P}_{N,l}$ corresponding to $[E]$ is the Quot-scheme $\mathrm{Quot}_S(E, P)$, where by P we denote the Hilbert polynomial of $N \otimes I_\xi$. We consider the open substack $P_{N,l} \subset \tilde{P}_{N,l}$, whose $\mathbb{C}$-points are isomorphism classes of vector bundles E such that $h^1(S, E) = h^2(S, E) = 0$.

Lemma 6.1. *The stack $P_{N,l}$, if nonempty, has dimension*

$$\dim P_{N,l} = 2l + \langle v, v\rangle + \langle v(N \otimes I_\xi), v\rangle.$$

Proof. We claim that the dimension of the fibers of p is constant. Indeed, Serre duality and Proposition 3.1 imply that $\dim \mathrm{Ext}^2(N \otimes I_\xi, M) = \dim \mathrm{Hom}(M, N \otimes I_\xi) = 0$ for any $[M] \in \mathcal{M}_L(v)^{\mu s}(\mathbb{C})$ and $\xi \in S^{[l]}$. This shows that $\mathcal{E}_{N,l}$, if nonempty, has dimension equal to

$$\dim(M_L(v)^{\mu s} \times \mathcal{M}(v(N \otimes I_\xi))) - \chi(N \otimes I_\xi, M) = 2l - 1 + 1 + \langle v, v\rangle + \langle v(N \otimes I_\xi), v\rangle;$$

note that this coincides with the dimension computed by Yoshioka (cf. Lemma 5.2 in [Y]). The statement follows by remarking that, if $P_{N,l}$ is nonempty, then $\dim P_{N,l} = \dim \tilde{P}_{N,l} = \dim \mathcal{E}_{N,l}$ because the Quot-schemes corresponding to the fibers of q are 0-dimensional (use inequalities analogous to (5)). □

We consider the Grassmann bundle $\mathcal{G}_{N,l} \to P_{N,l}$, whose fiber over $[E] \in P_{N,l}(\mathbb{C})$ is $G(3, H^0(S, E))$, and the rational map $h_{N,l} : \mathcal{G}_{N,l} \dashrightarrow \mathcal{W}^2_d(|L|)$. The fiber of $h_{N,l}$ over a pair (C, A) is the quotient stack

$$[\mathbb{P}(\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee)^\circ)/\mathrm{Aut}(E_{C,A})],$$

where $\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee)^\circ \subset \mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee)$ consists, by definition, of morphisms with kernel isomorphic to $\mathcal{O}_S^{\oplus 3}$. This quotient stack has dimension equal to -1, as in Section 4. Our goal is to estimate the dimension of the closure of the image of $h_{N,l}$, which is denoted by $\mathcal{W}_{N,l}$. We first prove the following:

Lemma 6.2. *If $\mathcal{G}_{N,l}$ is nonempty, then*

$$\dim \mathcal{G}_{N,l} = g + \rho(g, 2, d) + \chi(M, N \otimes I_\xi).$$

Moreover, $\chi(M, N \otimes I_\xi) \leq \frac{4}{3}g + \frac{8}{3} - d - \frac{3}{2}c_1(N) \cdot c_1$.

Proof. We use that

$$\begin{aligned} 2(\rho(g,2,d) - 1) = \langle v(E), v(E)\rangle &= \langle v(N \otimes I_\xi), v(N \otimes I_\xi)\rangle + \langle v, v\rangle + 2\langle v(N \otimes I_\xi), v\rangle \\ &= 2l - 2 + \langle v, v\rangle + 2\langle v(N \otimes I_\xi), v\rangle; \end{aligned}$$

this implies that

$$\begin{aligned} \dim \mathcal{G}_{N,l} = \dim P_{N,l} + 3(h^0(S,E) - 3) &= 2\rho(g,2,d) - \langle v(N \otimes I_\xi), v\rangle + 3(g - d + 2) \\ &= g + \rho(g,2,d) + \chi(M, N \otimes I_\xi), \end{aligned}$$

as soon as $\mathcal{G}_{N,l}$ is nonempty.

Since $\chi(M, N\otimes I_\xi) = -\langle v(N\otimes I_\xi), v\rangle = 2\chi(N\otimes I_\xi)+\chi-4-c_1(N)\cdot c_1$, the last part of the statement follows by remembering that $\chi(E)=\chi+\chi(N\otimes I_\xi) = g-d+5$ and that

$$\frac{c_1(N)^2}{2} \leq \frac{g-1}{3} - \frac{c_1(N)\cdot c_1}{2}$$

because $\mu_L(E)\geq \mu_L(N\otimes I_\xi)$. □

In conclusion, we prove the following:

Proposition 6.3. *Assume that a general curve in $|L|_s$ has Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2}\right\rfloor$. Let $\mathcal{W}\subset\mathcal{W}_{N,l}$ be an irreducible component of $\mathcal{W}^2_d(|L|)$; then, $\rho(g,2,d)>0$ and $\mathcal{W}$ does not dominate the linear system $|L|$.*

Proof. Lemma 5.1 gives $c_1(N)\cdot c_1\geq k\geq (g+2)/2$ and $d\geq \frac{3}{4}k+\frac{7}{6}+\frac{g}{3}\geq \frac{17}{24}g+\frac{23}{12}$; in particular, $\rho(g,2,d)>0$. By Lemma 6.2, we have

$$\begin{aligned}\dim\mathcal{G}_{N,l} &\leq g+\rho(g,2,d)+\frac{4}{3}g+\frac{8}{3}-d-\frac{3}{2}k\\ &\leq g+\rho(g,2,d)+\frac{4}{3}g+\frac{8}{3}-d-\frac{3}{4}g-\frac{3}{2}\\ &= g+\rho(g,2,d)+\frac{7}{12}g+\frac{7}{6}-d.\end{aligned}$$

Since any fiber of $h_{N,l}$ is an algebraic stack of dimension -1, then

$$\dim\mathcal{W}\leq g+\rho(g,2,d)+\frac{7}{12}g+\frac{13}{6}-d.$$

The right hand side is strictly smaller than $g+\rho(g,2,d)$ because $d>\frac{7g+26}{12}$. It follows that $\mathcal{W}$ cannot dominate $|L|$. □

7. Cases with a μ_L-stable quotient sheaf of rank 2 and L-slope $\leq\mu_L(E)$

In this section we count the number of moduli of rank-3 LM bundles E, which are either properly μ_L-semistable with JH filtration $0\subset JH_1(E)\subset E$ where $JH_1(E)$ is a line bundle, or μ_L-unstable with maximal destabilizing sheaf E_1 such that E/E_1 is a μ_L-stable, torsion free sheaf of rank 2.

Fix an integer $d\geq 4$. Choose $N\in\mathrm{Pic}(S)$ such that

$$\mu_L(N)\geq\frac{2g-2}{3}\geq\frac{(c_1(L)-c_1(N))\cdot c_1(L)}{2}, \tag{23}$$

with equality holding either everywhere or nowhere.

As before, we set $c_1' := c_1(L)-c_1(N)$, $c_2' := d-c_1'\cdot c_1(N)$, $\chi' := g-d+5-\chi(N)$, $v' := 2+c_1'+(\chi'-2)\omega\in H^*(S,\mathbb{Z})$.

We denote by $\mathcal{F}_N$ the algebraic stack of extensions

$$0\to N\to E\to E/N\to 0, \tag{24}$$

where E/N defines a point of $\mathcal{M}^{\mu s}_L(v')$. Equivalently, $\mathcal{F}_N$ is the moduli stack of filtrations $0\subset N\subset E$ such that $[E/N]\in\mathcal{M}^{\mu s}_L(v')(\mathbb{C})$. Consider the two projections $p:\mathcal{F}_N\to\mathcal{M}^{\mu s}_L(v')\times\mathcal{M}(v(N))$ and $q:\mathcal{F}_N\to\mathcal{M}(v(E))$ and define $\tilde{R}_N$ to be the closure of the image of q. The open substack $R_N\subset\tilde{R}_N$ consists, by definition, of points corresponding to bundles E such that $h^1(S,E)=h^2(S,E)=0$. We look at the Grassmann bundle $\mathcal{G}_N\to R_N$ with fiber over $[E]\in R_N(\mathbb{C})$ equal to $G(3,H^0(S,E))$. The closure of the image of $\mathcal{G}_N$ under the rational map $h_N : \mathcal{G}_N \dashrightarrow \mathcal{W}^2_d(|L|)$ is denoted by $\mathcal{W}_N$. As before, the fibers of h_N are quotient stacks of dimension -1.

Lemma 7.1. *The stack $\mathcal{G}_N$, if nonempty, has dimension*

$$\dim \mathcal{G}_N = g + \rho(g, 2, d) + \chi(E/N, N).$$

Proof. The fiber of p over a point of $\mathcal{M}^{\mu s}_L(v') \times \mathcal{M}(v(N))$ corresponding to $(E/N, N)$ is the quotient stack $[\mathrm{Ext}^1(E/N, N)/\mathrm{Hom}(E/N, N)]$. Since $\mu_L(N) \geq \mu_L(E/N)$ and E/N is μ_L-stable, Serre duality and Proposition 3.1 imply that $\mathrm{Ext}^2(E/N, N) = 0$; hence, the dimension of the fibers of p is constantly equal to $-\chi(E/N, N) = \langle v(N), v'\rangle$. The morphism q is representable and, as in the previous sections, one shows that its fibers are Quot-schemes of dimension 0. Therefore, if R_N is nonempty, one has:

$$\dim R_N = \dim \tilde{R}_N = \dim \mathcal{F}_N = \langle v', v'\rangle + \langle v(N), v'\rangle.$$

The statement follows by proceeding as in the proof of Lemma 6.2. □

The next Lemma gives an upper bound for $\chi(E/N, N)$.

Lemma 7.2. *Assume that a general curve $C \in |L|_s$ has Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$. If R_N is nonempty, then $\chi(E/N, N) \leq \frac{3}{2}g - 2d + 3$ for any E/N corresponding to a point of $\mathcal{M}^{\mu s}_L(v')$.*

Proof. Consider the extension (24), where $[E] \in R_N(\mathbb{C})$. Since $\mu_L(N) > 0$, one has $h^1(S, E/N) = h^2(S, N) = 0$. As in Lemma 3.2 one obtains $\chi(E/N) = h^0(S, E/N) \geq 2$, hence $\chi(N) = \chi(E) - \chi(E/N) \leq g - d + 3$. As a consequence:

$$\begin{aligned}
\chi(E/N, N) &= 2\chi(N) + \chi' - 4 - c_1(N) \cdot c_1' \\
&= g - d + 1 + \chi(N) - c_1(N) \cdot c_1' \\
&\leq 2g - 2d + 4 - c_1(N) \cdot c_1' \\
&\leq \frac{3}{2}g - 2d + 3,
\end{aligned}$$

where the last inequality follows from Lemma 5.2. □

Finally, we prove the following:

Proposition 7.3. *We assume that a general curve in $|L|$ has Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$. If $d > \frac{3}{4}g + 2$, no irreducible component $\mathcal{W}$ of $\mathcal{W}^2_d(|L|)$ which is contained in $\mathcal{W}_N$ dominates the linear system $|L|$.*

Proof. Let $\mathcal{W} \subset \mathcal{W}_N$ be an irreducible component of $\mathcal{W}^2_d(|L|)$. Since any fiber of h_N is an Artin stack of dimension equal to -1, Lemma 7.1 and Lemma 7.2 imply that

$$\dim \mathcal{W} \leq g + \rho(g, 2, d) + \frac{3}{2}g - 2d + 4.$$

If $\rho(g, 2, d) \geq 0$, the condition $d > \frac{3}{4}g + 2$ prevents the map $\mathcal{W} \to |L|$ from being dominant. □

Now we show that, if d is small enough and $C \in |L|_s$, any complete base point free g^2_d on C, whose LM bundle is given by an extension of type (24), is contained in a linear series which is induced from a line bundle on S.

Proposition 7.4. *Let S and L be as in the hypotheses of Proposition 7.3 and A be a complete, base point free g^2_d on a curve $C \in |L|_s$. If $d < (5g+13)/6$ and the LM bundle $[E_{C,A}] \in R_N(\mathbb{C})$ for some $N \in \mathrm{Pic}(S)$, the linear system $|A|$ is contained in the restriction to C of the linear system $|L \otimes N^\vee|$ on S. Moreover, $L \otimes N^\vee$ is adapted to $|L|$ and $\mathrm{Cliff}(L \otimes N^\vee \otimes \mathcal{O}_C) \leq \mathrm{Cliff}(A) = d-4$.*

Proof. By hypothesis, $E = E_{C,A}$ sits in a short exact sequence like (24), where E/N is μ_L-stable and $\mu_L(N) \geq (2g-2)/3 \geq \mu_L(E/N)$. Since $\mu_L(N) > 0$, then $h^2(S,N) = 0$.

The μ_L-stability of E/N implies

$$-2 \leq \langle v', v' \rangle = 4c_2' - (c_1')^2 - 8,$$

thus $c_2' \geq 3/2 + (c_1')^2/4$.

If $h^0(S,N) < 2$, then $c_1(N)^2 \leq -2$, which implies $(c_1')^2 + 2c_1(N) \cdot c_1' \geq 2g$ and $c_1' \cdot c_1(N) \geq (2g+4)/3$. In particular,

$$d = c_1' \cdot c_1(N) + c_2' \geq c_1' \cdot c_1(N) + \frac{3}{2} + \frac{(c_1')^2}{4} \geq \frac{g}{2} + \frac{3}{2} + \frac{g+2}{3} = \frac{5g+13}{6},$$

thus a contradiction. Therefore, one has both $h^0(S,N) \geq 2$ and $h^0(S, \det E/N) \geq 2$.

Remark that $(E/N)^{\vee\vee}$ is globally generated off a finite set and

$$h^i(S,(E/N)^{\vee\vee}) = h^i(S,E/N) = 0 \text{ for } i = 1,2.$$

Since $\det E/N = \det(E/N)^{\vee\vee}$ is base point free and non trivial, if $h^1(S, \det E/N) \neq 0$, then $(c_1')^2 = 0$ and Proposition (1.1) in [GL] implies the existence of a smooth elliptic curve $\Sigma \subset S$ such that

$$(E/N)^{\vee\vee} = \mathcal{O}_S(\Sigma) \oplus \mathcal{O}_S(\Sigma).$$

Such equality would contradict the stability of E/N, thus we conclude that $(c_1')^2 \geq 2$ (and $c_2' \geq 2$) and

$$h^1(S, \det E/N) = 0. \tag{25}$$

This ensures that $h^0(C, \det E/N \otimes \mathcal{O}_C)$ does not depend on the curve $C \in |L|_s$ (cf. [DM] Lemma (5.2)). Hence, the line bundle $\det E/N = L \otimes N^\vee$ is adapted to $|L|$.

We obtain:

$$\begin{aligned}
\mathrm{Cliff}(\det E/N \otimes \mathcal{O}_C) &= c_1(E/N)^2 + c_1(N) \cdot c_1(E/N) - 2h^0(C, \det E/N \otimes \mathcal{O}_C) + 2 \\
&\leq c_1(E/N)^2 + c_1(N) \cdot c_1(E/N) - 2h^0(S, \det E/N) + 2 \\
&= c_1(N) \cdot c_1(E/N) - 2 - 2h^1(S, \det E/N) \\
&= d - c_2(E/N) - 2 \\
&\leq d - 4.
\end{aligned}$$

It remains only to prove that $h^0(C, \det E/N \otimes \mathcal{O}_C \otimes A^\vee) > 0$. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(C,A)^\vee \otimes \mathcal{O}_S & \longrightarrow & E & \xrightarrow{\alpha} & \omega_C \otimes A^\vee & \longrightarrow & 0. \\
 & & & & \uparrow{\scriptstyle \gamma} & & & & \\
 & & & & N & & & &
\end{array}$$

Since $h^2(S,N) = 0$, the composition $\alpha \circ \gamma \neq 0$. This implies $\mathrm{Hom}(N, \omega_C \otimes A^\vee) \neq 0$ and we have finished because $N^\vee \otimes \omega_C \otimes A^\vee \simeq \det E/N \otimes \mathcal{O}_C \otimes A^\vee$.

□

8. Remaining cases

In this section we consider rank-3 LM bundles E of type (i), (ii), (iii), (iv) on page 12, such that $\det E = L$ and $c_2(E) = d$ is fixed.

Choose $l_2 \in \mathbb{N}$ and two line bundles $N, N_2 \in \mathrm{Pic}(S)$ such that $N_1 := L \otimes (N \otimes N_2)^\vee$ is globally generated and non-trivial, and the following holds:

$$\mu_L(N) \geq \mu_L(N_2) \geq \mu_L(N_1), \tag{26}$$

$$\mu_L(N) \geq \frac{2g-2}{3} \geq \mu_L(N_1), \tag{27}$$

where in (27) either both the inequalities are strict, or none is.

Set $v := v(N)$, $v_1 := v(N_1 \otimes I_{\xi_1})$ and $v_2 := v(N_2 \otimes I_{\xi_2})$, with $l(\xi_2) = l_2$ and

$$l(\xi_1) = l_1 := d - l_2 - c_1(N) \cdot c_1(N_1) - c_1(N) \cdot c_1(N_2) - c_1(N_1) \cdot c_1(N_2).$$

Define $\mathcal{F}_{N,N_2,l_2}$ to be the moduli stack of extensions

$$0 \to N_2 \otimes I_{\xi_2} \to E/N \to N_1 \otimes I_{\xi_1} \to 0,$$

where $\xi_i \subset S$ is a 0-dimensional subscheme of length l_i for $i = 1, 2$. We consider the projections $p_2 : \mathcal{F}_{N,N_2,l_2} \to \mathcal{M}(v_2) \times \mathcal{M}(v_1)$ and $q_2 : \mathcal{F}_{N,N_2,l_2} \to \mathcal{M}(v(E/N))$, and we denote by Q_{N,N_2,l_2} the closure of the image of q_2.

If $\mathcal{E}_{N,N_2,l_2}$ is the moduli stack of extensions

$$0 \to N \to E \to E/N \to 0,$$

where $[E/N] \in Q_{N,N_2,l_2}(\mathbb{C})$, consider the morphisms $p_1 : \mathcal{E}_{N,N_2,l_2} \to \mathcal{M}(v) \times Q_{N,N_2,l_2}$ and $q_1 : \mathcal{E}_{N,N_2,l_2} \to \mathcal{M}(v(E))$. The closure of the image of q_1 is denoted by $\tilde{P}_{N,N_2,l_2}$ and its open substack, consisting of points which correspond to vector bundles E such that $h^1(S,E) = h^2(S,E) = 0$, by P_{N,N_2,l_2}.

Remark that, if E is a LM bundle of type (i), (ii), (iii) or (iv), there exist N, N_2 and l_2 such that $[E]$ defines a point of P_{N,N_2,l_2}. In order to count the number of moduli of such bundles, we start by proving the following:

Lemma 8.1. *The stack Q_{N,N_2,l_2}, if nonempty, has dimension*

$$\dim Q_{N,N_2,l_2} = 2l_1 + 2l_2 - 2 + \langle v_1, v_2 \rangle,$$

unless $N_1 \simeq N_2$, $l_2 \neq 0$ and $l_1 = 0$. In this case, for any component $Q \subset Q_{N,N_2,l_2}$, the following inequality holds:

$$\dim Q \leq 2l_1 + 2l_2 - 1 + \langle v_1, v_2 \rangle.$$

Proof. The fiber of p_2 over the point of $\mathcal{M}_L(v_2) \times \mathcal{M}_L(v_1)$ given by $(N_2 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1})$ is the quotient stack

$$[\mathrm{Ext}^1(N_1 \otimes I_{\xi_1}, N_2 \otimes I_{\xi_2})/\mathrm{Hom}(N_1 \otimes I_{\xi_1}, N_2 \otimes I_{\xi_2})].$$

Since $\mu_L(N_2) \geq \mu_L(N_1)$, if either $N_1 \not\simeq N_2$ or $N_1 \simeq N_2$, $l_1 \neq 0$ and $l_2 = 0$, one finds that

$$\mathrm{Hom}(N_2 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1}) = 0.$$

In these cases, the fibers of p_2 have constant dimension equal to $-\chi(N_1 \otimes I_{\xi_1}, N_2 \otimes I_{\xi_2})$ while the fibers of q_2 are 0-dimensional Quot-schemes, hence the statement follows.

If $N_1 \simeq N_2$ and $l_1 = l_2 = 0$, the conclusion is the same because the fibers of p_2 have constant dimension equal to $-\chi(N_1 \otimes I_{\xi_1}, N_2 \otimes I_{\xi_2}) + 1$ and the fibers of q_2 are smooth Quot-schemes of dimension 1. Indeed, $\mathrm{Hom}(N, N) = 1$ and $\mathrm{Ext}^1(N, N) = 0$.

On the other hand, if $N_1 \simeq N_2$ and $l_2 \neq 0$, the fibers of p_2 do not necessarily have constant dimension; indeed, $\dim \mathrm{Hom}(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1})$ depends on the reciprocal position of ξ_1 and ξ_2. Since $\mathcal{H}om(I_{\xi_2}, \mathcal{O}_S) \simeq \mathcal{H}om(I_{\xi_2}, I_{\xi_2}) \simeq \mathcal{O}_S$ (cf. [OSS]), one shows that

$$\mathcal{H}om(I_{\xi_2}, I_{\xi_1}) \simeq \{f \in \mathcal{O}_S \mid f \cdot I_{\xi_2} \subseteq I_{\xi_1}\} =: (I_{\xi_1} : I_{\xi_2}) = I_{\xi_1 \setminus (\xi_1 \cap \xi_2)};$$

hence, one finds that

$$\dim \mathrm{Hom}(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1}) = H^0(S, \mathcal{H}om(I_{\xi_2}, I_{\xi_1})) = \begin{cases} 1 & \text{if } \xi_1 \subseteq \xi_2 \\ 0 & \text{otherwise} \end{cases}. \tag{28}$$

As in [Y], let $\mathcal{N}^0_{N,N_2,l_2}$ (resp. $\mathcal{N}^1_{N,N_2,l_2}$) be the substack of $\mathcal{M}(v_2) \times \mathcal{M}(v_1)$ whose points correspond to pairs $(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1})$ such that $\xi_1 \not\subseteq \xi_2$ (resp. $\xi_1 \subseteq \xi_2$), that is, $\dim \mathrm{Hom}(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1}) = 0$ (resp. $\dim \mathrm{Hom}(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1}) = 1$). Remark that $\mathcal{N}^0_{N,N_2,l_2}$ and $\mathcal{N}^1_{N,N_2,l_2}$ are complementary and that, being open, $\mathcal{N}^0_{N,N_2,l_2}$ is dense in $\mathcal{M}(v_2) \times \mathcal{M}(v_1)$ provided $l_1 \neq 0$.

We define $\mathcal{F}^0_{N,N_2,l_2} := (p_2)^{-1}(\mathcal{N}^0_{N,N_2,l_2})$ and $\mathcal{F}^1_{N,N_2,l_2} := (p_2)^{-1}(\mathcal{N}^1_{N,N_2,l_2})$ and we denote by Q^0_{N,N,l_2} and Q^1_{N,N,l_2} the closures of the images under q_2 of $\mathcal{F}^0_{N,N_2,l_2}$ and $\mathcal{F}^1_{N,N_2,l_2}$ respectively. Since the fibers of q_2 are Quot-schemes, we obtain that:

$$\begin{aligned} \dim Q^0_{N,N_2,l_2} &= \dim \mathcal{F}^0_{N,N_2,l_2} = \dim \mathcal{N}^0_{N,N_2,l_2} + \langle v_1, v_2 \rangle \leq 2l_1 + 2l_2 - 2 + \langle v_1, v_2 \rangle, \\ \dim Q^1_{N,N_2,l_2} &\leq \dim \mathcal{F}^1_{N,N_2,l_2} = \dim \mathcal{N}^1_{N,N_2,l_2} + \langle v_1, v_2 \rangle + 1 \leq 2l_1 + 2l_2 - 1 + \langle v_1, v_2 \rangle, \end{aligned}$$

where the last inequality in the second row is strict, unless the stack $\mathcal{N}^1_{N,N_2,l_2}$ is dense in $\mathcal{M}(v_2) \times \mathcal{M}(v_1)$, that is, $l_1 = 0$.

The statement follows because every component of Q_{N,N,l_2} is contained either in Q^0_{N,N,l_2} or in Q^1_{N,N,l_2}. □

By proceeding as in Lemma 8.1, one proves the following:

Proposition 8.2. *Let Z be a nonempty irreducible component of P_{N,N_2,l_2}. We have that*

$$\dim Z = 2l_1 + 2l_2 + \langle v_2, v \rangle + \langle v_1, v \rangle + \langle v_1, v_2 \rangle - \alpha, \tag{29}$$

where α satisfies:

(a) *If N, N_1, N_2 are all non-isomorphic, then $\alpha = 3$.*
(b) *Assume $N \simeq N_1 \simeq N_2$. If $l_2 \neq 0$ and $l_1 = 0$, then $\alpha \in \{1, 2, 3\}$. If $l_1 \neq 0$ and $l_2 = 0$, one has $\alpha \in \{2, 3\}$. In all the other cases, $\alpha = 3$. If $N \simeq N_1 \not\simeq N_2$, one has $\alpha = 3$ unless $l_1 = 0$, in which case $\alpha \in \{2, 3\}$.*
(c) *If $N \simeq N_2 \not\simeq N_1$, then $\alpha = 3$ unless $l_2 = 0$, in which case $\alpha \in \{2, 3\}$.*
(d) *Assume $N_1 \simeq N_2 \not\simeq N$. Then $\alpha = 3$ except when $l_2 \neq 0$ and $l_1 = 0$; in this case $\alpha \in \{2, 3\}$.*

Note that LM bundles of type (i) lie in some P_{N,N_2,l_2} with N, N_1, N_2 as in case (a). Analogously, if E is a LM bundle of type (iii) (resp. of type (iv)), there exist $N, N_2, N_1 = L \otimes (N \otimes N_2)^\vee$ as in (a) or (c) (resp. as in (a) or (d)) and $l_2 \in \mathbb{N}$ such that $[E] \in P_{N,N_2,l_2}(\mathbb{C})$. On the other hand, if a bundle of type (ii) defines a point of $P_{N,N_2,l}$, then $\mu_L(N) = \mu_L(N_2) = \mu_L(N_1)$ and any case of the previous proposition may occur.

Now, we consider the Grassmann bundle $\psi : \mathcal{G}_{N,N_2,l_2} \to P_{N,N_2,l_2}$ with fiber over a point of P_{N,N_2,l_2} corresponding to a bundle E equal to $G(3, H^0(S, E))$ and denote by $\mathcal{W}_{N,N_2,l_2}$ the closure of the image of the rational map $h_{N,N_2,l_2} : \mathcal{G}_{N,N_2,l_2} \dashrightarrow \mathcal{W}^2_d(|L|)$.

Lemma 8.3. *Assume that general curves in $|L|$ have Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$. Then, for any irreducible component $\mathcal{W}$ of $\mathcal{W}_{N,N_2,l_2}$, one has*

$$\dim \mathcal{W} \leq \frac{1}{4}g + d + \frac{3}{2} - \alpha,$$

where α is as in Proposition 8.2.

Proof. Let $\mathcal{G}$ be an irreducible component of $\mathcal{G}_{N,N_2,l_2}$ such that $\mathcal{W} = \overline{h_{N,N_2,l_2}(\mathcal{G})}$. Since $\mathcal{G} = \psi^{-1}(Z)$ for some irreducible component Z of P_{N,N_2,l_2}, Proposition 8.2 implies that:

$$\begin{aligned}
\dim \mathcal{G} &= 3(g-d+2) + \dim Z \\
&= 3(g-d) + 12 - \alpha - 2\chi(E) + 2l_1 + 2l_2 + \\
&\quad c_1(N)\cdot c_1(N_1) + c_1(N)\cdot c_1(N_2) + c_1(N_1)\cdot c_1(N_2) \\
&= g-d+2-\alpha+2(l_1+l_2) + c_1(N)\cdot(c_1(N_1)+c_1(N_2)) + c_1(N_1)\cdot c_1(N_2) \\
&= g+d+2-\alpha - c_1(N)\cdot c_1(N_1) - c_1(N)\cdot c_1(N_2) - c_1(N_1)\cdot c_1(N_2) \\
&\leq g+d+2-\alpha-\frac{3}{2}k \\
&\leq \frac{1}{4}g + d + \frac{1}{2} - \alpha,
\end{aligned}$$

where we have used Lemma 5.3 and the fact that $k \geq (g+2)/2$. The statement follows since the fibers of h_{N,N_2,l_2} are quotient stacks of dimension -1. □

Finally, we prove the following:

Proposition 8.4. *Assume that general curves in $|L|$ have Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$. Fix a positive integer d such that $(g,d) \notin \{(2,4),(4,5),(6,6),(10,9)\}$. Let $\mathcal{W} \subset \mathcal{W}_{N,N_2,l_2}$ be an irreducible component of $\mathcal{W}^2_d(|L|)$. Then $\rho(g,2,d) \geq 0$ and $\mathcal{W}$ does not dominate $|L|$.*

Proof. Lemma 5.3 implies $d \geq \frac{3}{2}k$, hence $\rho(g,2,d) \geq 0$. Lemma 8.3 gives:

$$\dim \mathcal{W} \leq \frac{1}{4}g + d + \frac{3}{2} - \alpha.$$

Therefore, $\mathcal{W}$ cannot dominate $|L|$ if

$$\frac{1}{4}g + d + \frac{3}{2} - \alpha < g + \rho(g,2,d) = -g + 3d - 6,$$

that is, $d > \frac{5}{8}g + \frac{15}{4} - \frac{\alpha}{2}$. In particular, as $\alpha \geq 1$, it is enough to require $d > \frac{5}{8}g + \frac{13}{4} =: h$. Such inequality is satisfied always except for

$$(g,d) \in \{(2,4),(3,5),(4,5),(5,6),(6,6),(6,7),(8,8),(10,9),(14,12)\}.$$

If $(g,d) = (6,6)$, the linear system $|L|$ can be dominated by $\mathcal{W}$. In all the other cases $d = \lfloor h \rfloor$ and we check whether $\alpha > 2h - 2\lfloor h \rfloor + 1$, which would prevent $\mathcal{W}$ from being dominant. This holds true if $(g,d) \notin \{(2,4),(4,5),(10,9)\}$ (use that the case $\alpha = 1$ may occur only when parametrizing LM bundles of type (ii) and that, if $\gcd(2g-2,3) = 1$, there do not exist properly μ_L-semistable bundles of Mukai vector $v(E)$). □

Remark 1. The four cases which are not covered by Proposition 8.4 might be treated by "ad hoc" arguments but this is not the purpose of the paper.

Proofs of Theorem 1.1 and Theorem 1.2 are now straightforward.

Proof of Theorem 1.1. Being non-simple, the LM bundle $E_{C,A}$ is not μ_L-stable. Since $d < \frac{2}{3}g + 2$, Corollary 5.4 implies the existence of a line bundle $N \in \mathrm{Pic}(S)$ such that $E_{C,A} \in R_N(\mathbb{C})$. The statement thus follows directly from Proposition 7.4. □

Proof of Theorem 1.2. Case (a) trivially follows from Proposition 6.3, Proposition 7.3 and Proposition 8.4.

Now, let $\frac{2}{3}g + 2 \leq d \leq \frac{3}{4}g + 2$. Given $\mathcal{W}$ an irreducible component of $\mathcal{W}^2_d(|L|)$ which dominates $|L|$ and whose general point corresponds to a LM bundle that is not μ_L-stable, Proposition 6.3 and Proposition 8.4 imply the existence of a line bundle $N \in \mathrm{Pic}(S)$ such that $\mathcal{W} \subset \mathcal{W}_N$. The statement follows from Proposition 7.4. □

9. TRANSVERSALITY OF SOME BRILL-NOETHER LOCI

We apply our results in order to prove Theorem 1.4 in the introduction.

Theorem 9.1. *Let $r \geq 3$, $g \geq 0$, $d \leq g - 1$ be positive integers such that $\rho(g, r, d) < 0$ and $d - 2r + 2 \geq \lfloor (g + 3)/2 \rfloor$. If $r \geq 4$, assume $d^2 > 4(r - 1)(g + r - 2)$. For $r = 3$, let $d^2 > 8g + 1$. If -1 is not represented by the quadratic form*

$$Q(m, n) = (r - 1)m^2 + mnd + (g - 1)n^2 \quad m, n \in \mathbb{Z},$$

there exists a smooth curve $C \subset \mathbb{P}^r$ of genus g, degree d and maximal gonality $\left\lfloor \frac{g+3}{2} \right\rfloor$. Moreover, one can choose C such that for any complete, base point free g^1_e on C with $\rho(g, 1, e) \geq 0$ the Petri map is injective.

Proof. Notice that the inequalities $d \leq g - 1$ and $d^2 > 4(r - 1)(g - 1)$ trivially imply $d > 4(r - 1)$.

In order to prove the first part of the statement, we proceed as in [Fa] Theorem 3 paying special attention to our slightly different hypotheses. Rathmann's Theorem implies the existence of a $2r - 2$-degree $K3$ surface $S \subset \mathbb{P}^r$ and a smooth curve $C \subset S$ of degree d and genus g such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where H is the hyperplane section of S. Our assumption on Q implies that S does not contain (-2)-curves. As in [Fa], one shows that the line bundle $L := \mathcal{O}_S(C)$ is ample by Nakai-Moishezon criterion (if $D \subset S$ is an effective divisor, use that $D^2 \geq 0$ and $D \cdot H > 2$, in order to show that $C \cdot D > 0$). Hence, C has Clifford dimension 1 (cf. [CP] Proposition 3.3).

Assume that C has gonality $k < \left\lfloor \frac{g+3}{2} \right\rfloor$. We reach a contradiction by showing that $k \geq d - 2r + 2$. If A is a complete, base point free pencil g^1_k on C, by [DM] Theorem (4.2) there exists an effective divisor $D \equiv mH + nC$ on S, such that $|A|$ is contained in the linear system $|\mathcal{O}_C(D)|$ and the following conditions are satisfied:

$$h^0(S, \mathcal{O}_S(D)) \geq 2, \; h^0(S, \mathcal{O}_S(C - D)) \geq 2, \; C \cdot D \leq g - 1, \; \mathrm{Cliff}(D|_C) = \mathrm{Cliff}(A).$$

In particular, as remarked in [DM] page 60, the last equality implies that

$$h^1(S, \mathcal{O}_S(D)) = h^1(S, \mathcal{O}_S(C - D)) = 0,$$

thus $c_1(D)^2 > 0$ and $c_1(C - D)^2 > 0$. Moreover, one has

$$k = 2 + \mathrm{Cliff}(D|_C) = D \cdot (C - D).$$

We show that

$$f(m, n) = D \cdot C - D^2 = -(2r - 2)m^2 + d(1 - 2n)m + (n - n^2)(2g - 2) \geq d - 2r + 2,$$

for values of m and n satisfying the following inequalities:

(i) $(r - 1)m^2 + mnd + n^2(g - 1) > 0$,
(ii) $(r - 1)m^2 + (mn - m)d + (1 - n)^2(g - 1) > 0$,
(iii) $2 < (2r - 2)m + nd < d - 2$,
(iv) $md + (2n - 1)(g - 1) \leq 0$.

Assume first that $n = 1$, and set $a = -m$. Then (iii) implies $0 < a < (d-2)/(2r-2)$. Inequality (i) is equivalent to $(r-1)a^2 - ad + g - 2 \geq 0$, whence

$$a \leq \frac{d - \sqrt{d^2 - 4(r-1)(g-2)}}{2r-2}.$$

We have $f(-a, 1) \geq d-2r+2$ whenever $1 \leq a \leq d/(2r-2)-1$. For either $r \geq 4$ or $r = 3$ and $d^2 - 8g \geq 8$, this holds true because $d^2 - 4(r-1)(g-2) \geq 4r(r-1) > 4(r-1)^2$. If $r = 3$ and $d^2 - 8g < 8$, then $d^2 - 8g = 4$ and $d \equiv 2 \mod 4$. Hence, (iii) implies that $1 \leq a < (d-4)/4$. Remark that $f(-a, 1) = d - 2r + 2$ whenever $a = 1$, that is, $C \equiv C - H$. The case $n = 0$ can be treated similarly by using (ii) instead of (i), and one obtains that $f(m, 0) \geq d - 2r + 2$ with equality holding only for $m = 1$, that is, $D \equiv H$.

If $n < 0$, inequalities (i), (iii) and (iv) imply that $-\alpha n < m \leq (g-1)(1-2n)/d$, where

$$\alpha = \frac{d + \sqrt{d^2 - 4(r-1)(g-1)}}{2r-2}.$$

Therefore, one has

$$f(m, n) \geq \min \left\{ f(-\alpha n, n), f\left(\frac{(g-1)(1-2n)}{d}, n \right) \right\}.$$

Analogously, if $n \geq 2$, then $\max\{-\beta n, (2-nd)/(2r-2)\} < m \leq (g-1)(1-2n)/d$, where

$$\beta = \frac{d - \sqrt{d^2 - 4(r-1)(g-1)}}{2r-2};$$

this gives

$$f(m, n) \geq \min \left\{ f\left(\frac{(g-1)(1-2n)}{d}, n \right), \max \left\{ f(-\beta n, n), f\left(\frac{2-nd}{2r-2}, n \right) \right\} \right\}.$$

Computations in [Fa] give $\max\left\{f(-\beta n, n), f\left((2-nd)/(2r-2), n\right)\right\} > d - 2r + 2$ if $n \geq 2$, and $f(-\alpha n, n) > d-2r+2$ when $n < 0$, unless $r = 3$, $n = -1$ and $d^2 - 8g = 4$. In this case, $d \equiv 2 \mod 4$ and $m \geq (d+4)/4$ by (iii); one uses that $f((d+4)/4, -1) > d-4$. In order to conclude the proof that C has maximal gonality, it is enough to remark that the function

$$h(n) := f\left(\frac{(g-1)(1-2n)}{d}, n \right) = \frac{g-1}{2} \left[\frac{(2n-1)^2(d^2 - 4(r-1)(g-1))}{d^2} + 1 \right]$$

reaches its minimum for $n = 1/2$ and $h(0) \geq d - 2r + 2$ by direct computation.

Concerning the last part of the statement, assume C is general in its linear system and let A be a complete, base point free pencil g^1_e on C such that $\rho(g, 1, e) \geq 0$ and $\ker \mu_{0,A} \neq 0$. The bundle $E = E_{C,A}$ is non-simple, hence it cannot be μ_L-stable. As a consequence, there exists a short exact sequence

$$0 \to M \to E \to N \otimes I_\xi \to 0, \tag{30}$$

where M, N are line bundles, I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$ and $c_1(M) \cdot C \geq \mu_L(E) = g - 1 \geq c_1(N) \cdot C$. If sequence (30) does not split, then

$$h^0(S, E \otimes E^\vee) \leq 1 + \dim \operatorname{Hom}(M, N \otimes I_\xi) + \dim \operatorname{Hom}(N \otimes I_\xi, M).$$

Since $\mu_L(M) \geq \mu_L(N)$, if $\operatorname{Hom}(M, N \otimes I_\xi) \neq 0$ then $M \simeq N$ and $C = 2c_1(M)$, which is absurd. It follows that $N^\vee \otimes M$ is non-trivial and effective. Since S does not contain (-2)-curves, one has

$$c_1(N^\vee \otimes M)^2 = C^2 - 4c_1(N) \cdot c_1(M) = 2g - 2 - 4c_1(N) \cdot c_1(M) \geq 0;$$

this contradicts Lemma 4.1, which states that $c_1(N)\cdot c_1(M)\geq k\geq (g+2)/2$. Thus, $\xi=\emptyset$ and sequence (30) splits. We have to show that, if $E=N\oplus M$ is a splitting LM bundle, the rational map $\chi: G(2,H^0(S,E))\dashrightarrow |L|$ cannot be dominant. Remark that χ factors through the rational map $h_E: G(2,H^0(S,E))\dashrightarrow \mathcal{W}^1_e(|L|)$, whose fiber over a point $(C,A)\in \operatorname{Im} h_E$ is at least 1-dimensional since it is isomorphic to $\mathbb{P}(\operatorname{Hom}(E_{C,A},\omega_C\otimes A^\vee)^\circ)$, where $\operatorname{Hom}(E_{C,A},\omega_C\otimes A^\vee)^\circ$ is an open subgroup of $\operatorname{Hom}(E_{C,A},\omega_C\otimes A^\vee)\simeq H^0(S,E_{C,A}\otimes E^\vee_{C,A})$. This is enough to conclude because $\rho(g,1,e)\geq 0$, hence $\dim G(2,H^0(S,E))=2(g-e+1)\leq g$. □

Theorem 9.2. *Let g,r,d satisfy the hypotheses of Theorem 9.1. The curve C can be chosen such that, if*

$$e<\min\left\{d-2r+5,\frac{17}{24}g+\frac{23}{12}\right\},$$

then C does not have any complete, base point free net g^2_e for which the Petri map is non-injective.

Proof. Let $S\subset\mathbb{P}^r$ be as in the proof of Theorem 9.1 and C be general in its linear system. Let A be a complete, base point free net on C of degree $d_A<\frac{17}{24}g+\frac{23}{12}$; if $\rho(g,2,d_A)\geq 0$, assume moreover that $\ker\mu_{0,A}\neq 0$. Corollary 5.4 and Proposition 7.4 imply that $|A|$ is contained in the linear system $|\mathcal{O}_C(D)|$ for some effective divisor $D\equiv mH+nC$ on S such that:

$$h^0(S,\mathcal{O}_S(D))\geq 2,\ h^0(S,\mathcal{O}_S(C-D))\geq 2,\ C\cdot D\leq\frac{4g-4}{3},\ \operatorname{Cliff}(D|_C)\leq\operatorname{Cliff}(A).$$

In fact, the Lazarsfeld-Mukai bundle $E:=E_{C,A}$ is given by an extension:

$$0\to N\to E\to E/N\to 0,$$

where $N:=\mathcal{O}_S(C-D)$ and E/N is a μ_L-stable torsion free sheaf of rank 2 on S. As in the proof of Proposition 7.4, one shows that $D^2>0$, hence $h^1(S,\mathcal{O}_S(D))=0$. Moreover, one obtains that $h^1(S,N)=0$ because the equality $(C-D)^2=0$ would imply $d\geq(5g+4)/6$, which is absurd. As a consequence, one has

$$d_A-4=\operatorname{Cliff}(A)\geq\operatorname{Cliff}(D|_C)=D\cdot C-2h^0(S,\mathcal{O}_C(D|_C))+2=D\cdot(C-D)-2, \tag{31}$$

and equality holds only if $D^2=2$ and $c_2(E/N)=2$ (cf. proof of Proposition 7.4); in particular, for $D\equiv H$, the inequality is strict. We show that

$$f(m,n):=D\cdot(C-D)\geq d-2r+2, \tag{32}$$

and, if equality holds, then either $D\equiv H$ or $D\equiv C-H$. Computations are similar to those in Theorem 9.1, but now, instead of having $D\cdot C\leq g-1$, we only know that $D\cdot C\leq(4g-4)/3$. Therefore, inequality (iv) must be replaced with

(iv′) $md+(2n-\frac{4}{3})(g-1)\leq 0$.

The cases $n\in\{0,1\}$ can be treated exactly as before. For $n<0$, we have

$$f(m,n)\geq\min\left\{f(-\alpha n,n),f\left(\frac{(g-1)(\frac{4}{3}-2n)}{d},n\right)\right\}.$$

If $n\geq 2$, then

$$f(m,n)\geq\min\left\{f\left(\frac{(g-1)(\frac{4}{3}-2n)}{d},n\right),\max\left\{f(-\beta n,n),f\left(\frac{2-nd}{2r-2},n\right)\right\}\right\}.$$

Therefore, it is enough to show that

$$g(n) := f\left(\frac{(g-1)(\frac{4}{3}-2n)}{d}, n\right) - d + 2r - 2 > 0 \text{ for } n < 0 \text{ or } n \geq 2.$$

One can write $g(n) = an^2 + bn + c$, with

$$\begin{aligned} a &= -4(2r-2)\left(\frac{g-1}{d}\right)^2 + 2d\left(\frac{g-1}{d}\right), \\ b &= \frac{16}{3}(2r-2)\left(\frac{g-1}{d}\right)^2 - \frac{8}{3}d\left(\frac{g-1}{d}\right), \\ c &= -\frac{16}{9}(2r-2)\left(\frac{g-1}{d}\right)^2 + \frac{4}{3}d\left(\frac{g-1}{d}\right) - d + 2r - 2. \end{aligned}$$

Since $a > 0$ and $0 < -b/(2a) < 1$, our claim follows if $g(0) = c > 0$, or equivalently, if

$$\frac{3}{4} < \frac{g-1}{d} < \frac{3}{8}\left(\frac{d-2(r-1)}{r-1}\right).$$

The left inequality is trivial since $d \leq g-1$. The right inequality is equivalent to the condition $8(g-1)(r-1) < 3d^2 - 6d(r-1)$, which is satisfied as well (if $r \geq 4$, use that $8(g-1)(r-1) < 2d^2 - 8(r-1)^2$ and $d > 4(r-1)$; if $r = 3$, use that $d^2 > 8g+1$ and either $(g,d) = (12,11)$ or $d \geq 12$ by manipulation of the hypotheses).

We conclude that $d_A \geq d - 2r + 4$ and the inequality is strict unless equalities hold both in (31) and (32), thus $D \equiv C - H$ and $(C-H)^2 = 2$. This case can be excluded since it would imply $d = g + r - 3 \geq g$. □

Remark that the condition $e < \frac{17}{24}g + \frac{23}{12}$ is automatically satisfied if $\rho(g, 2, e) < 0$.

The proof of Theorem 1.4 is now trivial: apply Theorem 9.1 and Theorem 9.2 and proceed by induction on f and e in order to deal with pencils g^1_f and nets g^2_e which have a nonempty base locus.

10. Noether-Lefschetz divisor and Gieseker-Petri divisor in genus 11

The Clifford index $\mathrm{Cliff}(C)$ is one of the most important invariants of an algebraic curve C. In [LN1] Lange and Newstead defined the analogue of the Clifford index for higher rank vector bundles in the following way. If $\mathcal{U}_C(n,d)$ denotes the moduli space of semistable rank-n vector bundles of degree d on a genus-g curve C, given $E \in \mathcal{U}_C(n,d)$, the Clifford index of E is

$$\gamma(E) := \mu(E) - \frac{2}{n}h^0(C,E) + 2 \geq 0,$$

where $\mu(E)$ denotes the slope of E. For any positive integer n, one defines the higher Clifford index of C

$$\mathrm{Cliff}_n(C) := \min\{\gamma(E) \,|\, E \in \mathcal{U}_C(n,d),\ h^0(C,E) \geq 2n,\ \mu(E) \leq g-1\}.$$

A natural question is whether higher Clifford indices are new invariants, different from the ones arising in classical Brill-Noether theory. In [LN1] Lange and Newstead reformulated a conjecture of Mercat (cf. [Me]) in a slightly weaker form predicting:

$$\mathrm{Cliff}_n(C) = \mathrm{Cliff}(C); \tag{33}$$

remark that trivially $\mathrm{Cliff}_n(C) \leq \mathrm{Cliff}(C)$, while the opposite inequality is largely non-trivial. When $n = 2$, the conjecture has been proved for a general curve in M_g if $g \leq 16$ by Farkas and Ortega (cf. [FO1]) and the same is expected to hold true in any genus. However, if $g \geq 11$, there are examples of curves with maximal Clifford index $\mathrm{Cliff}(C) = \left\lfloor \frac{g-1}{2} \right\rfloor$ that violate (33) for $n = 2$. These have been constructed in [FO1], [FO2], [LN1], [LN2], [LN3] as sections of $K3$ surfaces with Picard number at least 2. We recall that the $K3$ locus

$$\mathcal{K}_g := \{[C] \in M_g \,|\, C \subset S,\ S \text{ is a } K3 \text{ surface}\}$$

is irreducible of dimension $19 + g$ if either $g = 11$ or $g \geq 13$ (cf. [CLM]). In particular, $\mathcal{K}_{11} = M_{11}$ and a general curve $[C] \in M_{11}$ lies on a unique $K3$ surface with Picard number 1 (cf [M2]). Given two positive integers r, d such that $d^2 > 4(r-1)g$ and d does not divide $2r - 2$, one defines the Noether-Lefschetz divisor inside $\mathcal{K}_g$ as

$$\mathcal{NL}^r_{g,d} := \left\{ [C] \in \mathcal{K}_g \;\middle|\; \begin{array}{l} C \subset S,\ S \text{ is a } K3 \text{ surface},\ \mathrm{Pic}(S) \supset \mathbb{Z}C \oplus \mathbb{Z}H, \\ H \text{ nef },\ H^2 = 2r-2,\ C^2 = 2g-2,\ C \cdot H = d \end{array} \right\}.$$

In [FO2] it is proved that a curve C of genus 11 violates Mercat's conjecture for $n = 2$ whenever $[C] \in \mathcal{NL}^4_{11,13}$.

Since some of the curves exhibited in [LN1], [LN2], [LN3] do not satisfy the Gieseker-Petri Theorem, Lange and Newstead asked whether $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$ whenever C is a Petri curve (Question 4.2 in [LN3]). We prove Theorem 1.5, which gives a negative answer to this question.

Let $S \subset \mathbb{P}^4$ be a $K3$ surface such that $\mathrm{Pic}(S) = \mathbb{Z}C \oplus \mathbb{Z}H$, where H is the hyperplane section, $H^2 = 6$, $C^2 = 20$ and $C \cdot H = 13$. Denote by L the line bundle $\mathcal{O}_S(C)$. We show that, if $C \in |L|$ is general, then $[C]$ does not lie in the Gieseker-Petri locus GP_{11}. Recall that GP_{11} has pure codimension 1 in M_{11} (cf. [LC]) and decomposes in the following way:

$$GP_{11} = M^2_{11,9} \cup GP^2_{11,10} \cup \bigcup_{d=7}^{10} GP^1_{11,d},$$

where $M^2_{11,9}$ is a Brill-Noether divisor. Therefore, proving the transversality of $\mathcal{NL}^4_{11,13}$ and GP_{11} is equivalent to showing that in the above situation, if $C \in |L|$ is general, then C has no g^2_9 and the varieties $G^2_{10}(C)$ and $G^1_d(C)$ for $7 \leq d \leq 10$ are smooth of the expected dimension.

We proceed as in the previous section; since the hypotheses of Theorem 9.1 are not satisfied, explicit computations must be performed. Direct calculations imply that S does not contain any (-2)-curve. Moreover, C is an ample line bundle on S by Proposition 2.1 in [LN3]. As a consequence, C has Clifford dimension 1 (cf. Proposition 3.3 in [CP]) and $\mathrm{Cliff}(C) = 5$ (cf. Proposition 3.3 in [FO2]). In particular, C has maximal gonality $k = 7$ and has no g^2_d for $d \leq 8$. Hence, in order to prove that $G^2_9(C) = \emptyset$, it is enough to exclude the existence of complete, base point free g^2_9 on C. Similarly, the condition $[C] \notin GP^2_{11,10}$ is equivalent to the requirement for $G^2_{10}(C)$ to be smooth of the expected dimension $\rho(11, 2, 10)$ in the points corresponding to complete, base point free linear series. Analogously, by induction on d, if the Petri map associated with any complete, base point free pencil of degree $7 \leq d \leq 10$ is injective, then $[C] \notin \cup_{d=7}^{10} GP^1_{11,d}$.

For any $A \in G^2_9(C)$, the Petri map $\mu_{0,A}$ is non-injective for dimension reasons and the bundle $E = E_{C,A}$ is non-simple, hence it cannot be μ_L-stable. Since

$$\gcd(rk\,E, c_1(E)^2) = \gcd(3, 20) = 1,$$

there are no properly semistable sheaves of Mukai vector $v(E) = (3, C, 4)$; hence, E is μ_L-unstable. By Corollary 5.4, E sits in the short exact sequence

$$0 \to N \to E \to E/N \to 0, \tag{34}$$

where $N \in \mathrm{Pic}(S)$ is its maximal destabilizing sheaf and the quotient E/N is a μ_L-stable torsion free sheaf of rank 2. Having denoted by D the first Chern class of E/N, Proposition 7.4 implies that the line bundle $\mathcal{O}_C(D)$ contributes to $\mathrm{Cliff}(C)$. Moreover, as in the proof of the aforementioned proposition, one shows that

$$D^2 \geq 2, \tag{35}$$

$$c_2(E/N) \geq \frac{3}{2} + \frac{1}{4}D^2. \tag{36}$$

Furthermore, Lemma 5.2 gives

$$c_1(N) \cdot c_1(E/N) = (C - D) \cdot D \geq k = 7. \tag{37}$$

Since

$$9 = c_2(E) = c_2(E/N) + (C - D) \cdot D \geq \frac{3}{2} + \frac{1}{4}D^2 + (C - D) \cdot D,$$

the divisor $D \equiv mH + nC$ must satisfy

$$\begin{cases} C \cdot D = 13m + 20n = 9 \\ D^2 = 6m^2 + 20n^2 + 26mn = 2(m+n)(3m+10n) = 2 \end{cases}.$$

One shows that this system admits no integral solution. As a consequence, a general curve in $|L|$ has no linear series of type g^2_9.

Analogously, given a complete, base point free $A \in G^2_{10}(C)$ with $\ker \mu_{0,A} \neq 0$, the LM bundle $E = E_{C,A}$ is μ_L-unstable and its maximal destabilizing sheaf is a line bundle N such that E/N is μ_L-stable by Corollary 5.4. With the same notation as above, inequalities (35), (36), (37) still hold true and the following cases must be considered:

$$(a) \begin{cases} C \cdot D = 10 \\ D^2 = 2 \\ (c_2(E/N) = 2) \end{cases} \qquad (b) \begin{cases} C \cdot D = 9 \\ D^2 = 2 \\ (c_2(E/N) = 3) \end{cases}$$
$$(c) \begin{cases} C \cdot D = 11 \\ D^2 = 4 \\ (c_2(E/N) = 3) \end{cases} \qquad (d) \begin{cases} C \cdot D = 13 \\ D^2 = 6 \\ (c_2(E/N) = 3) \end{cases}.$$

These systems have no integral solutions except for (d), which is satisfied by

$$(m, n) = (1, 0).$$

Therefore, $N = \mathcal{O}_S(C - H)$ and $v(E/N) = (2, H, 2)$. Since $\langle v(E/N), v(E/N)\rangle = -2$, the sheaf E/N is uniquely determined.

By applying first $\mathrm{Hom}(E, -)$ and then $\mathrm{Hom}(-, N)$ and $\mathrm{Hom}(-, E/N)$ to the short exact sequence (34), one shows that

$$h^0(S, E \otimes E^\vee) \leq 2 + \dim \mathrm{Hom}(N, E/N) + \dim \mathrm{Hom}(E/N, N)$$

and the inequality is strict if the sequence does not split. Since $\mu_L(N) > \mu_L(E/N)$, Proposition 3.1 implies that $\mathrm{Hom}(N, E/N) = 0$. Let $0 \neq \alpha \in \mathrm{Hom}(E/N, N)$. Since both $\mathrm{Im}\,\alpha$ and $\ker \alpha$ are torsion free sheaves of rank 1, there exists an effective divisor D_1 on S and two 0-dimensional subschemes $\xi_1, \xi_2 \subset S$ such that E/N is given by an extension

$$0 \to \mathcal{O}_S(2H - C + D_1) \otimes I_{\xi_1} \to E/N \to \mathcal{O}_S(C - H - D_1) \otimes I_{\xi_2} \to 0.$$

The μ_L-stability of E/N implies that

$$13/2 = \mu_L(E/N) < (C - H - D_1) \cdot C = -D_1 \cdot C + 7;$$

since C has positive intersection with any non-trivial effective divisor, $D_1 = 0$. It follows that

$$3 = c_2(E/N) = (2H - C) \cdot (C - H) + l(\xi_1) + l(\xi_2) \geq 7,$$

which is absurd. Hence, $\mathrm{Hom}(E/N, N) = 0$ and (34) splits. As a consequence, the bundle $E = N \oplus E/N$ is uniquely determined.

We look at the rational map $\chi : G(3, H^0(S, E)) \dashrightarrow |L|$; this cannot be dominant since $\dim G(3, H^0(S, E)) = 9$. Therefore, a general curve $C \in |L|$ does not lie in $GP^2_{11,10}$.

It remains to show that, if $C \in |L|$ is general, then $[C] \notin \cup_{d=7}^{10} GP^1_{11,d}$. It is enough to prove that for any complete, base point free g^1_d on C the Petri map is injective. One can proceed exactly as in the last part of the proof of Theorem 9.1 since S does not contain (-2)-curves.

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