IRREDUCIBILITY OF SEVERI VARIETIES ON $K3$ SURFACES

ANDREA BRUNO AND MARGHERITA LELLI-CHIESA

Abstract. Let (Y, L) be a general primitively polarized $K3$ surface of genus g. For every $0 \leq \delta \leq g$ we consider the Severi variety parametrizing integral curves in $|L|$ with exactly δ nodes as singularities. We prove that its closure in $|L|$ is connected as soon as $\delta \leq g-1$. If $\delta \leq g-4$, we obtain the stronger result that the Severi variety is irreducible, as predicted by a well-known conjecture. The results are obtained by degeneration to Halphen surfaces.

1. Introduction.

Let L be a polarization on a smooth irreducible projective surface S defined over the field of complex numbers, and denote by g the arithmetic genus of all curves in $|L|$. For any fixed integer $0 \leq \delta \leq g$ the *Severi variety* of δ-nodal curves in $|L|$ is the locally closed subscheme of $|L|$ defined as

$$V_\delta(S, L) := \{C \in |L| \text{ s.t. } C \text{ is integral with exactly } \delta \text{ nodes as singularities}\};$$

the same definition applies to singular surfaces S with the further requirement that the curves C lie in the smooth locus of S. These varieties are named after Severi, who introduced them in the case $S = \mathbb{P}^2$ [Se], where he proved that they are nonempty and smooth of the expected dimension, namely, $\dim |L| - \delta$. Severi also claimed that they are irreducible, but a rigorous proof of this fact was accomplished only some sixty years later by Harris [Ha]. Since then, Severi varieties were thoroughly investigated for many types of surfaces, in particular as regards their nonemptiness, their local geometry and their irreducibility; the last issue became known as the *Severi problem*. Nonemptiness has been established in many cases, as for instance $K3$ surfaces [MM, Ch1], abelian surfaces [KLM, KL], Enriques surfaces [CDGK]. As concerns their local geometry, Severi varieties behave well on rational surfaces and surfaces of Kodaira dimension 0, while on surfaces of general type wild unexpected phenomena occur, as highlighted in [CS, CC].

On the other hand, very little is known about the global geometry of Severi varieties even for surfaces of non-maximal Kodaira dimension. In particular, the Severi problem proves very challenging and has been solved in very few cases: for Hirzebruch surfaces by Tyomkin [Tyo], for Del Pezzo surfaces in the case of rational curves (that is, for maximal δ) by Testa [Tes], while partial results for blow-ups of the projective plane are due to Greuel-Lossen-Shustin [GLS]. In recent times, many papers focused on the case of toric surfaces [Bo, LT], and Zahariuc [Za] worked out the Severi problem for a general abelian surface with a polarization of any primitive type.

A vast literature is devoted to the case of $K3$ surfaces, motivated by the following well-known folklore conjecture.

Conjecture 1.1. *Let (Y, L) be a general polarized $K3$ surface of genus $g \geq 2$. Then, for any fixed $0 \leq \delta \leq g-1$, the Severi variety $V_\delta(Y, L)$ is irreducible.*

We recall that $\dim |L| = g$. The constraint $\delta \leq g-1$ is necessary because it is well-known that the linear system $|L|$ contains finitely many rational curves: since this number is computed by the Yau-Zaslow formula [Be] and is different from 1, the Severi variety $V_\delta(Y, L)$ is definitely reducible for $\delta = g$. Quite surprisingly, the above conjecture has remained open until now, despite numerous attempts. We will prove it for primitive linear systems as soon as $\delta \leq g-4$ and $g \geq 5$.

Theorem 1.2. *Let (Y, L) be a general primitively polarized $K3$ surface of genus $g \geq 2$. Then the following hold:*

(1) *for every $0 \leq \delta \leq g-1$, the closure of the Severi variety $\overline{V_\delta(Y, L)} \subset |L|$ is connected;*
(2) *if $g \geq 5$ and $0 \leq \delta \leq g-4$, the Severi variety $V_\delta(Y, L)$ is irreducible.*

Previous results in the literature due to Keilen [Kei], Kemeny [Kem], Ciliberto-Dedieu [CD2], Dedieu [De2] only concerned cases where δ is small with respect to the arithmetic genus g (roughly bounded by $g/4$) and it was clear that they cannot be further improved with similar proof techniques. A weaker form of the conjecture concerning the so-called *universal Severi variety* $\mathcal{V}_{g,\delta}$ was considered more approachable. Let $\mathcal{F}_g$ be the irreducible 19-dimensional moduli stack of genus g primitively polarized $K3$ surfaces. The stack $\mathcal{V}_{g,\delta}$ is smooth of pure dimension $19+g-\delta$ and admits a morphism $\phi_{g,\delta} : \mathcal{V}_{g,\delta} \to \mathcal{F}_g^\circ$ to a suitable open substack $\mathcal{F}_g^\circ$ of $\mathcal{F}_g$ whose fiber over a general point $(Y, L) \in \mathcal{F}_g$ equals the Severi variety $V_\delta(Y, L)$.

Conjecture 1.3. *For every $0 \leq \delta \leq g$, the universal Severi variety $\mathcal{V}_{g,\delta}$ is irreducible.*

This prediction makes perfect sense even for $\delta = g$, when it becomes a question on the monodromy of the finite morphism $\phi_{g,g}$. It is related to the non-existence of self-rational maps of degree > 1 on a general $K3$ surface in $\mathcal{F}_g$, which was predicted by Dedieu in [De1] and achieved by Chen in [Ch4]. Conjecture 1.3 was proved by Ciliberto-Dedieu [CD] for $2 \leq g \leq 11$ and $g \neq 10$, which is exactly the range where a general genus g curve lies on a $K3$ surface. We remark that, since the morphism $\phi_{g,\delta}$ is known to be smooth and dominant on all components of $\mathcal{V}_{g,\delta}$ for every δ [FKPS], Conjecture 1.1 implies Conjecture 1.3 for every $0 \leq \delta \leq g-1$. In particular, the following result comes straightforward from Theorem 1.2.

Corollary 1.4. *For every $g \geq 5$ and every $0 \leq \delta \leq g-4$ the universal Severi variety $\mathcal{V}_{g,\delta}$ is irreducible.*

The assumption $\delta \leq g-4$ in Theorem 1.2(2) and in Corollary 1.4 is due to proof technique, and is only used in the proof of Theorem 5.2. However, there is no evidence for the existence of counterexamples to Conjecture 1.1 in the remaining cases $g-3 \leq \delta \leq g-1$.

1.1. **Strategy and organization of the paper.** Theorem 1.2 is proved by degeneration to a so-called *Halphen surface* $\overline{S}_g \subset \mathbb{P}^g$, which has an elliptic singularity and is limit of primitively embedded $K3$ surfaces of genus g. These surfaces, introduced in [CD], appeared in the characterization of hyperplane sections of $K3$ surfaces accomplished by Arbarello-Bruno-Sernesi [ABS], and were first exploited in [ABFS] and then in [AB, FT]. We recall their construction. Let S be the blow-up of $\mathbb{P}^2$ at 9 general points and denote by $|L_g|$ the g-dimensional linear system on S parametrizing the strict transforms of plane curves of degree $3g$ having multiplicity g at the first 8 points that we have blown up and multiplicity $g-1$ at the last one; these are called *Du Val curves* of genus g after Du Val, who first considered them [Du]. The surface $\overline{S}_g$ is realized as the closure in $\mathbb{P}^g$ of the rational map $S \dashrightarrow \mathbb{P}^g$ defined by $|L_g|$. In particular, Severi varieties of nodal hyperplane curves on $\overline{S}_g$ are linked to Severi varieties $V_\delta(S, L_g)$ on S. A major advantage is that the surface S possesses polarizations L_g for every genus $g \geq 2$ and is thus the right environment where to perform some sort of induction. In Section 2, after recalling the main features of Halphen surfaces, we show that the results known for Severi varieties on a general $K3$ surface of genus g still hold true for $V_\delta(S, L_g)$. In particular, Chen's proof of the density of Severi varieties in any equigeneric locus on a general polarized $K3$ surface [Ch2, Ch3] works with essentially no change in the context of Halphen surfaces. Moreover, the irreducibility of $V_\delta(S, L_g)$ is easily obtained when δ is small with respect to g: this is the basis for our induction.

In §2.1, we show that for any fixed integer $k \geq 1$ the linear system $|L_g|$ sits as a linear space of codimension k inside of $|L_{g+k}|$. The main idea is to use this inclusion along with the irreducibility of $V_\delta(S, L_{g+k})$ for big enough k in order to deduce connectedness of $V_\delta(S, L_{g+k})$ for every $0 \leq \delta \leq g-1$. The main difficulty arises from the fact that the subspaces $|L_{g+k-j}| \subset |L_{g+k}|$ for $j \geq 2$ have excess intersection with the Severi varieties $\overline{V_\delta(S, L_{g+k})}$, as follows from the following equality:

$$|L_{g+k-1}| \cap \overline{V_\delta(S, L_{g+k})} = \bigcup_{h=0}^{\delta} \overline{V_{\delta-h}(S, L_{g+k-h-1})}.$$

To circumvent this problem, we need to consider *expanded degenerations* of the surface S along the divisor J introduced by Jun Li in [Li1, Li2]. We recall that an expanded degeneration of S is a semistable model of S

$$S[n]_0 := S \cup_J R \cup_J \ldots \cup_J R$$

obtained attaching to S a chain of $n \geq 0$ ruled surfaces $R := \mathbb{P}(\mathcal{O}_J \oplus N_{J/S})$ over J (cf. §3.1 for details). The theory of good degenerations of relative Hilbert schemes developed in [LW] is used to define an expanded linear system $|L_g|^{\exp}$, whose points parametrize stable curves that live in some expansion $S[n]_0$ of S and have no components in its singular locus or in the last copy of J. For every $0 \leq \delta \leq g$ we consider the *expanded Severi variety* $\overline{\mathcal{V}_\delta(S/J, L_g)}$, that is, the closure in $|L_g|^{\exp}$ of the Severi variety $V_\delta(S, L_g)$. We will also make use of the moduli stack $\mathcal{M}_{g-\delta}(S/J, L_g)$ of stable maps to some expansion of S introduced and studied in [Li1, Li2].

Having fixed $k >> 0$, we first show that $|L_{g+k}|^{\exp}$ admits a map $\tilde{\alpha}$ to a variety $\widetilde{|L_{g+k}|}$ obtained from $|L_{g+k}|$ blowing up the flag of subspaces $|L_0| \subset \ldots \subset |L_{g+k-2}|$. We then define a natural map $\Psi : \widetilde{|L_{g+k}|} \to \widetilde{\mathbb{P}^k}$, where $\widetilde{\mathbb{P}^k}$ is obtained from $\mathbb{P}^k$ again by blowing up a complete flag. It turns out that $|L_g|^{\exp}$ may be realized as a fiber of the composition $\Psi \circ \tilde{\alpha}$, and $\overline{\mathcal{V}_\delta(S/J, L_g)}$ as a fiber of the restriction ψ of $\Psi \circ \tilde{\alpha}$ to $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}$. We then show that ψ admits a section over an open subset of $\widetilde{\mathbb{P}^k}$ and, by a standard argument using Stein factorization and Zariski's Main Theorem, obtain the connectedness of $\overline{\mathcal{V}_\delta(S/J, L_g)}$ from the irreducibility of $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}$ (cf. Theorem 3.4).

Connectedness for positive dimensional Severi varieties on a general $K3$ surface is then obtained in Section 4. We consider a stable type II degeneration $Y_0 := S \cup_J S'$ constructed by appropriately gluing two surfaces S, S', which are both a blow-up of $\mathbb{P}^2$ at 9 general points as above, have isomorphic anticanonical divisor J and satisfy $N_{J/S} \simeq N^\vee_{J/S'}$. Let $\mathcal{Y} \to \mathbb{D}$ be a family of genus g polarized $K3$ surfaces degenerating to Y_0. The limit on Y_0 of a relative genus g linear system $|\mathcal{L}|^* \to \mathbb{D}^*$ is not unique (for instance, one of such limits contracts S' and maps S to $\overline{S}_g \subset \mathbb{P}^g$). Furthermore, in any such limit the deformations of a curve containing the singular locus of Y_0 encounter extra obstructions. To circumvent these problems, we apply the theory of good degenerations of relative Hilbert schemes developed in [LW] and obtain a good degeneration $|\mathcal{L}|^{exp} \to \mathbb{D}$ of the relative linear system $|\mathcal{L}|^* \to \mathbb{D}^*$. Points in the central fiber of $|\mathcal{L}|^{exp}$ parametrize curves living in some expanded degenerations

$$S \cup_J R \cup_J \ldots \cup_J R \cup_J S'$$

of Y_0. More precisely, the central fiber admits the following decomposition in a non-disjoint union of Cartier divisors:

$$\bigcup_{g_1+g_2=g} |L_{g_1}|^{\exp} \times |L'_{g_2}|^{\exp}. \tag{1.1}$$

Recalling that Severi varieties are functorially defined, we consider the closure in $|\mathcal{L}|^{exp}$ of the relative Severi variety $\mathcal{V}(\mathcal{Y}, \mathcal{L})^* \to \mathbb{D}^*$ and denote by $\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)}$ its central fiber. In Lemma 4.1 we prove that the latter coincides with the closure in (1.1) of the locus of curves with δ nodes outside of the singular locus of Y_0 (or of its expansions). In other words, $\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)}$

splits in the following non-disjoint union:

$$\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)} = \bigcup_{\substack{g_1+g_2=g\\ \delta_1+\delta_2=\delta}} \overline{\mathcal{V}_{\delta_1}(S/J, L_{g_1})} \times \overline{\mathcal{V}_{\delta_2}(S'/J, L'_{g_2})};$$

we mention that the analogous decomposition for the space of stable maps proved in [Li1, Li2] was already used in [MPT]. In Proposition 4.2 we exploit this decomposition and the connectedness of $\overline{\mathcal{V}_\delta(S/J, L_g)}$ obtained in the previous section to prove that $\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)}$ is connected as soon as $\delta \leq g-1$.

Part (1) of Theorem 1.2 is then the content of Theorem 4.3; a key point is that any two components of $\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)}$ can be connected through a sequence of components whose intersection is generically contained in the reduced locus of $\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)}$, as follows form Propositions 3.6 and 4.2.

Section 5 is then devoted to the proof of part (2). First of all, we show that, if $(S, L) \in \mathcal{F}_g$ is general and $0 \leq \delta \leq g-1$, two irreducible components of the Severi variety $\overline{V_\delta(S, L)}$ intersect in codimension 1, if they meet at all (cf. Propositions 5.1, 2.7 and Lemma 2.8). This is obtained by realizing $\overline{V_\delta(S, L)} \subset |L|$ as the image under a generically finite map of a degeneracy locus in $S^{[\delta]} \times |L|$ and using the fact that degeneracy loci of the expected dimension are Cohen-Macaulay. Knowing that $\overline{V_\delta(S, L)}$ is connected, in order to prove its irreducibility it is thus enough to show that the codimension 1 components of its singular locus cannot contain the intersection of two irreducible components. This holds true for $\overline{V_{\delta+1}(S, L)} \subset \mathrm{Sing}\overline{V_\delta(S, L)}$ as the relative normalization of $\overline{V_\delta(S, L)}$ along $\overline{V_{\delta+1}(S, L)}$ remains connected (which is also part of Theorem 4.3). Let W be any codimension 1 component of $\mathrm{Sing}\overline{V_\delta(S, L)}$ not contained in $\overline{V_{\delta+1}(S, L)}$. By deformation theory (cf. [CD2] for similar arguments), we show that a general point of W parametrizes either a curve whose singularities consist of (possibly non-transverse) smooth linear branches except at most for one cusp, or a curve whose normalization is hyperelliptic of genus $g-\delta$. In the former case, it turns out that $\overline{V_\delta(S, L)}$ is unibranched along W. The latter case can be excluded as soon as W has dimension ≥ 3, or equivalently, $\delta \leq g-4$, because curves in $|L|$ with hyperelliptic normalization of any fixed geometric genus ≥ 2 are known to move in dimension 2 (cf. [KLM, Rmk. 5.6]); this is the only part of the proof where the assumption $\delta \leq g-4$ is used.

1.2. **Preliminaries on Severi varieties on $K3$ surfaces.** We will here collect known properties of Severi varieties on $K3$ surfaces that are relevant for this paper and will be generalized to Halphen surfaces in Section 2. Standard deformation theory yields the following result (cf., e.g., [DS, §3–4]):

Proposition 1.5. *Let (Y, L) be a polarized $K3$ surface of genus g. For any fixed integer $0 \leq \delta \leq g$ the Severi variety $V_\delta(Y, L)$, if nonempty, is smooth of dimension $g-\delta$.*

Indeed, for any $C \in V_\delta(Y,L)$ the projective tangent space to $V_\delta(Y,L)$ at C coincides with $\mathbb{P}(H^0(Y, L \otimes I_N))$, where N is the scheme of nodes of C. Furthermore, the nodes of any such curve C can be smoothed independently. Therefore, the nonemptiness of $V_\delta(Y,L)$ for every δ reduces to the existence in the linear system $|L|$ of a nodal rational curve. This was achieved by Mori-Mukai for a general primitively polarized $K3$ surface, and was then generalized by Chen to non primitive polarizations.

Theorem 1.6 ([MM, Ch1]). *Let (Y,L) be a general $K3$ surface of genus g. For any fixed integer $0 \leq \delta \leq g$, the Severi variety $V_\delta(Y,L)$ is nonempty.*

For primitive polarizations, Chen obtained the following much stronger result:

Theorem 1.7 ([Ch2]). *Let (Y,L) be a general primitively polarized $K3$ surface of genus g. Then, all rational curves in the linear system $|L|$ are nodal.*

The above result is deeply linked to the natural question whether every curve in $|L|$ can be deformed to a nodal curve having the same geometric genus. A positive answer is again due to Chen and, defining the *equigeneric locus*

$$V^h(Y,L) := \{C \in |L| \text{ s.t. } C \text{ is integral of geometric genus } h\}$$

for every $0 \leq h \leq g$, it can be phrased in the following way.

Theorem 1.8 ([Ch3]). *Let (Y,L) be a general primitively polarized $K3$ surface of genus g. Then, for every $0 \leq \delta \leq g$, the Severi variety $V_\delta(Y,L)$ and the equigeneric locus $V^{g-\delta}(Y,L)$ have the same closure in $|L|$.*

Acknowledgements: We are especially grateful to Xi Chen and Adrian Zahariuc for pointing out a mistake in a previous version of this paper and for suggesting Proposition 2.10. We also thank Thomas Dedieu and Edoardo Sernesi for numerous valuable conversations on the topic. We were supported by the Italian PRIN-2017 "Moduli Theory and Birational Classification", PRIN-2020 "Curves, Ricci flat Varieties and their Interactions" and by GNSAGA.

2. Halphen surfaces and their Severi varieties

Let S be the blow-up of $\mathbb{P}^2$ at nine general points $p_1, \ldots, p_9$ and denote by $E_1, \ldots, E_9$ the exceptional curves of this blow-up. As the points p_i are general, there exists a unique plane cubic passing through the p_i's, whose strict transform on S we denote by J. Hence, J is the only anticanonical divisor on S and satisfies

$$J \sim -K_S \sim 3l - E_1 - \cdots - E_9,$$

where ℓ is the strict transform of a line in $\mathbb{P}^2$. For any fixed $g \geq 1$, let C be the strict transform on S of a so-called *Du Val curve of genus g*, that is, a plane curve of degree $3g$ having points of multiplicity g at $p_1, \dots, p_8$ and a point of multiplicity $g-1$ at p_9:

$$C \sim 3g\ell - gE_1 - \dots - gE_8 - (g-1)E_9.$$

Defining $L_g := \mathcal{O}_S(C) \in \mathrm{Pic}(S)$, the linear system $|L_g|$ has dimension g and its general element is a smooth irreducible curve of genus g. Since $C \cdot J = 1$, every irreducible curve $C \in |L_g|$ intersects J at the same point, that we denote by $p_{10}(g)$. It turns out that $p_{10}(g)$ is the only base point of $|L_g|$ (cf. [ABFS, Lem. 2.4]) and is uniquely determined by the condition

$$gp_1 + \ldots + gp_8 + (g-1)p_9 + p_{10}(g) \in |\mathcal{O}_J(3g\ell)|.$$

We will sometimes use the notation $L_0 := E_9$ and $p_{10}(0) = p_9$.

Let $\sigma : \tilde{S} \longrightarrow S$ be the blow-up of S at $p_{10}(g)$. We still denote by $E_1, \dots, E_9$ the inverse images under σ of the exceptional curves on S and by E_{10} the exceptional divisor of σ. Let $\tilde{J}$ be the strict transform of J and $\tilde{C}$ be the strict transform of a curve $C \in |L_g|$. The following relations hold on $\tilde{S}$:

$$\begin{aligned} -K_{\tilde{S}} &\sim \tilde{J} \sim 3\ell - E_1 - \dots - E_{10}, \\ \tilde{C} &\sim 3g\ell - gE_1 - \dots - gE_8 - (g-1)E_9 - E_{10}\,, \\ \tilde{C} \cdot \tilde{J} &= 0\,. \end{aligned} \tag{2.1}$$

The line bundle $\tilde{L}_g := \mathcal{O}_{\tilde{S}}(\tilde{C})$ is base-point-free [ABFS, Lem. 2.4] and thus defines a morphism from $\tilde{S}$ to a surface $\overline{S}_g \subset \mathbb{P}^g$ having trivial dualizing sheaf, canonical hyperplane sections and a single elliptic singularity o resulting from the contraction of $\tilde{J}$. As in [AB], we call such a surface $\overline{S}_g \subset \mathbb{P}^g$ a polarized Halphen surface of genus g. A general hyperplane section of $\overline{S}_g$ is a smooth irreducible curve of genus g [ABFS, Lem. 2.4], while a general hyperplane section of $\overline{S}_g$ passing through o has a cusp at o. The following result is due to Arbarello-Bruno-Sernesi:

Proposition 2.1 ([ABS], Cor. 10.5). *If the points $p_1, \dots, p_9$ are general, the surface $\overline{S}_g$ is the limit of smooth $K3$ surfaces in $\mathbb{P}^g$.*

Halphen surfaces $\overline{S}_g$ as above share some common behaviour with $K3$ surfaces of Picard rank 1. This depends on the following property, firstly exploited by Arbarello-Bruno-Farkas-Saccà [ABFS].

Lemma 2.2. *If the points $p_1, \dots, p_9$ are general, for any fixed integer $g \geq 1$ the only possible decompositions of L_g into two effective line bundles are of the form*

$$L_g \simeq \mathcal{O}_S(kJ) \otimes L_{g-k}$$

for some $0 \leq k \leq g-1$.

Proof. By choosing $p_1, \dots, p_9$ general, we may assume that S contains no (-2)-curves and that $h^0(S, \mathcal{O}_S(kJ)) = 1$ for every $k \geq 1$; in other words,

$p_1, \dots, p_9$ are chosen k-general in the sense of [AB, Def. 2.2] for every $k \geq 1$ (cf. also [CD]). Let $L_g \simeq N \otimes M$ be a decomposition into two effective line bundles $N, M \in \mathrm{Pic}(S)$. Since $c_1(L_g) \cdot J = 1$ and $(J)^2 = 0$, possibly by exchanging N and M we obtain $J \cdot c_1(N) = 0$ and $J \cdot c_1(M) = 1$. The statement thus follows by a theorem of Nagata ([ABFS, Prop. 2.3]), ensuring that under the genericity assumption the only effective divisors having vanishing intersection with J are the nonnegative multiples of J. □

The above result was used by Arbarello-Bruno-Farkas-Saccà in order to prove the following analogue of Lazarsfeld's Theorem.

Theorem 2.3 ([ABFS], Thm. 4.4)**.** *If the points $p_1, \dots, p_9$ are general, then a general curve $C \in |L_g|$ satisfies Petri's Theorem and all irreducible nodal curves in $|L_g|$ satisfy the Brill-Noether Theorem.*

We now investigate Severi varieties $V_\delta(S, L_g)$ and equigeneric loci $V^h(S, L_g)$ on S. We recall that the normalization $\tilde{V}^h(S, L_g)$ of $V^h(S, L_g)$ admits a universal family $\mathcal{C} \to \tilde{V}^h(S, L_g)$ together with a simultaneous resolution of singularities $\tilde{\mathcal{C}} \to \mathcal{C}$ (cf. [Tei, I, Thm. 1.3.2] and also [DS, Thm. 1.5]). This implies the existence of an étale cover $W \to \tilde{V}^h(S, L_g)$ along with a generically injective morphism $w : W \to M_h(S, L_g)$ to the coarse moduli space of genus h stable maps in $|L_g|$. The image of w consists of the irreducible components of $M_h(S, L_g)$ parametrizing stable maps which are smoothable, that is, can be deformed to a map from a nonsingular curve, birational to its image (cf. [Va]). We denote by $M_h(S, L_g)^{\mathrm{sm}}$ the closure in $M_h(S, L_g)$ of the image of w.

Viceversa, by [Ko1, I.6] the semi-normalization $\tilde{M}_h(S, L_g)^{\mathrm{sm}}$ of $M_h(S, L_g)^{\mathrm{sm}}$ admits a morphism

$$\mu : \tilde{M}_h(S, L_g)^{\mathrm{sm}} \to \overline{V^h(S, L_g)} \subset |L_g|, \tag{2.2}$$

that maps a stable map $f : C \to S$ to its image $f(C)$.

Proposition 2.4. *The following hold true:*

(i) *For every $0 \leq \delta \leq g$ the Severi variety $V_\delta(S, L_g)$ is nonempty and smooth of dimension $g - \delta$.*
(ii) *For every $0 \leq h \leq g$ the equigeneric locus $V^h(S, L_g)$ and $M_h(S, L_g)^{\mathrm{sm}}$ have pure dimension h.*
(iii) *For every $0 \leq h \leq g$ a general point C in any irreducible component of $V^h(S, L_g)$ is immersed[1]; equivalently, a general map f in any irreducible component of $M_h(S, L_g)^{\mathrm{sm}}$ is unramified. In particular, both $V^h(S, L_g)$ and $M_h(S, L_g)^{\mathrm{sm}}$ are generically reduced.*

Proof. We recall that the expected dimension $V_\delta(S, L_g)$ is $g - \delta$. The nonemptiness statement in (i) follows from [GLS, Thm. B]. By standard deformation theory, the projective tangent space to $V_\delta(S, L_g)$ at a point C is isomorphic

[1]A curve is called immersed if the differential of its normalization map is everywhere injective.

to $\mathbb{P}(H^0(S, L_g \otimes I_N))$, where N is the scheme of nodes of C. Hence, $V_\delta(S, L_g)$ is smooth at C of dimension $g-\delta$ if and only if $h^0(L_g \otimes I_N) = g+1-\delta$, or equivalently, $h^1(L_g \otimes I_N) = 0$. This vanishing can be easily deduced by the short exact sequence

$$0 \to \mathcal{O}_S \to L_g \otimes I_N \to \omega_C(p_{10}(g)) \otimes I_N \to 0,$$

using the isomorphism $\omega_C(p_{10}(g)) \otimes I_N \simeq \nu_* \omega_{\tilde{C}}(p)$, where $\nu : \tilde{C} \to C$ is the normalization map and $p = \nu^{-1}(p_{10}(g))$.

As concerns part (ii), let C be a general point in any irreducible component V of $V^h(S, L_g)$ and let $f : \tilde{C} \to S$ be the stable map defined as the composition of the normalization map $\nu : \tilde{C} \to C$ with the inclusion $C \subset S$. The discussion above the statement of this proposition yields that $\dim_{[C]} V = \dim_{[f]} M_h(S, L_g)$ and, by standard deformation theory, the latter is bounded below by $\chi(N_f)$, where N_f is the normal sheaf to f defined by the short exact sequence

$$0 \to T_{\tilde{C}} \to f^* T_S \to N_f \to 0.$$

It is then easy to check that $\chi(N_f) = \chi(\omega_{\tilde{C}}(p)) = h$ and thus $\dim V \geq h$.

In order to prove equality, we apply a result by Arbarello and Cornalba [AC, p. 26] as in [DS, proof of Thm. 2.8] getting

$$\dim V = \dim T_{[C]} V \leq h^0(\tilde{C}, \overline{N}_f) = h^0(\omega_{\tilde{C}}(p - R)) \leq h,$$

where $\overline{N}_f$ denotes the quotient of N_f by its torsion subsheaf, which coincides with the zero divisor $R \subset \tilde{C}$ of the differential of f. Since $\omega_{\tilde{C}}(p)$ is globally generated off p and $p_{10}(g) = f(p)$ is a smooth point of C, we conclude that $\dim V = h$ (thus getting (ii)) and $R = 0$. Hence, C is immersed and this yields (iii) because $T_{[f]} M_h(S, L_g) = h^0(N_f) = h$ and μ is an isomorphism locally around $[f]$. □

It is natural to ask whether the closure in $|L_g|$ of the Severi variety $V_\delta(S, L_g)$ coincides with that of the equigeneric locus $V^{g-\delta}(S, L_g)$, as it happens on a general $K3$ surface. The following result generalizes Theorem 1.8 to our setting.

Proposition 2.5. *If the points $p_1, \ldots, p_9$ are general, then for every $g \geq 1$ and $0 \leq \delta \leq g$ one has the equality*

$$\overline{V_\delta(S, L_g)} = \overline{V^{g-\delta}(S, L_g)}$$

in the linear system $|L_g|$.

Proof. We follow Chen's proof of the analogous result for a general genus g polarized $K3$ surface [Ch3, Cor. 1.2]. Let V be any irreducible component of the equigeneric locus $\overline{V^h(S, L_g)}$ with $0 \leq h \leq g$. In order to prove that a general point of V parametrizes a nodal curve, it is enough to show that V contains a component of $V^{h-1}(S, L_g)$ as soon as $h \geq 1$, and that all rational curves in $|L_g|$ are nodal. Both the statements were proved for a general genus g polarized $K3$ surface by Chen (in [Ch3, Thm. 1.1] and [Ch2, Thm. 1.1], respectively), by specialization to a so-called Bryan-Leung $K3$ surface, that is, a $K3$ surface X_0 admitting an elliptic fibration $\pi : X_0 \to \mathbb{P}^1$ with a section s and 24 nodal singular fibers. If f is a fiber, the line bundle $L_0 := \mathcal{O}_{X_0}(s + gf)$ is a genus g polarization on X_0 and every element in $|L_0|$ is completely reducible, that is, it is union of s and g fibers of π.

We now exhibit a limit of our surfaces S that appeared in [ABFS, §4.1] and is very similar to a Bryan-Leung $K3$ surface. By specializing the points $p_1, \dots, p_9 \in \mathbb{P}^2$ to the base locus of a general pencil of plane cubics, the surface S specializes to a rational elliptic surface $q : S_0 \to \mathbb{P}^1$; the fibers of q are the anticanonical divisors of S_0 and thus q admits precisely 12 nodal singular fibers. It is easy to verify that on S_0 the exceptional divisor E_9 becomes a section of q and every element in the linear system $|L_g|$ is the union of E_9 with g fibers of q. Chen's proof of [Ch3, Thm. 1.1] works in our setting with no change, yielding that on S_0 (and thus on a general S) every component of $\overline{V^h(S, L_g)}$ with $h \geq 1$ contains a component of $V^{h-1}(S, L_g)$. Also the proofs in [Ch2] still work if, instead of a family of $K3$ surfaces whose central fiber is a Bryan-Leung $K3$ surface, one considers a family of surfaces $\mathcal{S} \to \Delta$ whose general fibers are general S and whose central fiber is S_0. The only difference that is worth remarking concerns [Ch2, Prop. 2.1], whose proof becomes even simpler in our case because every vector of the space $H^1(T_{S_0})$ parametrizing first order deformation of S_0 can be realized as the Kodaira-Spencer class of a projective family $\mathcal{S}$. □

The following result is a generalization of [Kem, CD2], ensuring irreducibility of Severi varieties in $|L_g|$ when δ is small with respect to g.

Proposition 2.6. *If $\delta \leq \frac{1}{6}g - \frac{1}{12}$, then $V_\delta(S, L_g)$ is irreducible.*

Proof. Let $U_\delta \subset S^{[\delta]}$ be the open subset parametrizing 0-dimensional subschemes consisting of δ distinct points none of which lies on J. The nodes of any curve $C \in V_\delta(S, L_g)$ define a point in U_δ because they all lie outside of J as $C \cdot J = 1$. As in [Kem, App. A, proof of Thm. A.0.6], the Severi variety is an open subset of a projective bundle over U_δ as soon as $H^1(L_g \otimes I_z^2) = 0$ for all $z \in U_\delta$. We will show that $H^1(L_g \otimes I_w) = 0$ for every $w \in S^{[3\delta]}$ whose support is disjoint from J. By contradiction, if this is not the case, up to replacing w with a subscheme of length $d \leq 3\delta$, we may assume that $h^1(L_g \otimes I_w) = 1$ (use [BS, Lem. 1.2]) and $h^1(L_g \otimes I_{w'}) = 0$ for every proper subscheme w' of w (that is, w is L_g-stable in the sense of Tyurin [Tyu, Def. 1.2]). By [Tyu, Lem. 1.2] there exists a rank 2 vector bundle E fitting into an extension

$$0 \to \mathcal{O}_S \to E \to L_{g-1} \otimes I_w \to 0, \tag{2.3}$$

where we have used that $L_g \otimes K_S \simeq L_{g-1}$. Since $c_1(E) = c_1(L_{g-1})$ and $c_2(E) = d \leq 3\delta$, the Riemann-Roch formula yields

$$\chi(E \otimes E^\vee) = c_1(E)^2 - 4c_2(E) + 4\chi(\mathcal{O}_S) = 2g - 3 - 4d + 4 \geq 2g - 12\delta + 1 \geq 2,$$

and thus either $h^0(E \otimes E^\vee) \geq 2$ or $h^2(E \otimes E^\vee) = h^0(E \otimes E^\vee \otimes K_S) \geq 1$. In both cases, E is not $\mu_{L_{g-1}}$- stable and thus sits in a destabilizing short exact sequence

$$0 \to N \to E \to M \otimes I_\xi \to 0, \tag{2.4}$$

where $\xi \subset S$ is a 0-dimensional subscheme and $N, M \in \mathrm{Pic}(S)$ satisfy

$$\mu_{L_{g-1}}(N) \geq \mu_{L_{g-1}}(E) = \frac{2g-3}{2} \geq \mu_{L_{g-1}}(M).$$

In particular, one gets $h^0(N^\vee) = 0$. We will use short exact sequence (2.4) and Lemma 2.2 to reach a contradiction. As in [Kn, Lem 3.6], by tensoring (2.3) with $N^\vee$ and taking global sections, one obtains $h^0(M \otimes I_w) > 0$ and thus M possesses a global section vanishing along a divisor that contains w; in particular, M is effective and $M \not\simeq \mathcal{O}(kJ)$ for any $k \geq 0$. If (2.4) splits, the same holds true for N by inverting the roles of N and M. Since $c_1(L_{g-1}) = c_1(N) + c_1(M)$, this would contradict Lemma 2.2: we conclude that (2.4) does not split.

In the case where $h^0(E \otimes E^\vee) \geq 2$, by standard computations (cf., e.g., [AF, Lem. 3.4]) one concludes that $h^0(N \otimes M^\vee) > 0$ and thus N is effective, too. By Lemma 2.2, we get that $N \simeq \mathcal{O}(kJ)$ and $M \simeq L'_{g-k}$ for some $0 \leq k \leq g-1$ and this contradicts the inequalities on the slopes.

In order to arrive at the same conclusion in the case where $h^0(E \otimes E^\vee \otimes K_S) \geq 1$, we tensor (2.3) with K_S and then apply $\mathrm{Hom}(E, -)$ in order to get

$$h^0(E \otimes E^\vee \otimes K_S) \leq \dim \mathrm{Hom}(E, N \otimes K_S) + \dim \mathrm{Hom}(E, M \otimes K_S \otimes I_\xi).$$

By applying $\mathrm{Hom}(-, M \otimes K_S \otimes I_\xi)$ to (2.3) and using the fact that $\mathrm{Hom}(N, M \otimes K_S \otimes I_\xi) = 0$ as $\mu_{L_{g-1}}(N) > \mu_{L_{g-1}}(M \otimes K_S)$, one obtains that $\mathrm{Hom}(E, M \otimes K_S \otimes I_\xi) = 0$. Analogously, applying $\mathrm{Hom}(-, N \otimes K_S)$ to (2.3), one shows that $1 \leq \mathrm{Hom}(E, N \otimes K_S) \simeq H^0(M^\vee \otimes N \otimes K_S)$; hence, N is effective yielding the same contradiction as above. □

The following result controls the intersection of two irreducible components of $\overline{V_\delta(S, L_g)}$.

Proposition 2.7. *Fix $g \geq 2$ and $0 \leq \delta \leq g-1$. Let V and W be two intersecting components of $\overline{V_\delta(S, L_g)}$. Then every irreducible component of $V \cap W$ not contained in $|L_{g-1}|$ has pure codimension* 1*; furthermore, any such component contains a closed point which is a reduced point of $\overline{V_\delta(S, L_g)}$ (that is, the local ring of $\overline{V_\delta(S, L_g)}$ at that point has no nilpotents*[2]*).*

Proof. Set $U := |L_g| \setminus |L_{g-1}| \subset |L_g|$ and consider the incidence variety

$$I := \overline{\left\{(C, z) \in U \times S^{[\delta]} \text{ s.t. } C \in |L_g \otimes I_z^2|\right\}} \subset |L_g| \times S^{[\delta]}. \tag{2.5}$$

[2]Since this is an open condition, the same will hold for a general point in any component of $V \cap W$, that is, for all closed points in a Zariski open subset of it.

We will express I as the degeneracy locus of a map of vector bundles on $|L_g|\times S^{[\delta]}$. Let $p: S\times S^{[\delta]}\longrightarrow S$ and $q: S\times S^{[\delta]}\longrightarrow S^{[\delta]}$ be the projections, and denote by $\Delta\subset S\times S^{[\delta]}$ the universal subscheme. Let $E:=q_*(p^*L_g)$ denote the vector bundle of rank $g+1$ on $S^{[\delta]}$ whose fiber over any point $z\in S^{[\delta]}$ equals $H^0(S,L_g)$. Let $F:=q_*(p^*L_g|_{2\Delta})$ be the vector bundle of rank 3δ on $S^{[\delta]}$ whose fiber over a point $z\in S^{[\delta]}$ is the vector space $H^0(S,L_g|_{2z})$, where $2z$ denotes the 0-dimensional subscheme of S defined by the ideal I_z^2. There is a natural map

$$\phi: E\longrightarrow F$$

of vector bundles on $S^{[\delta]}$. Note that $|L_g|\times S^{[\delta]}$ is isomorphic to the projective bundle

$$\pi:\mathbb{P}(E)\longrightarrow S^{[\delta]}.$$

Denoting by $\mathcal{U}\subset\pi^*E$ the universal subbundle, we consider the degeneracy locus $D(\widetilde{\phi})$ of the map

$$\widetilde{\phi}:\mathcal{U}\longrightarrow\pi^*F,$$

of vector bundles on $\mathbb{P}(E)\simeq|L_g|\times S^{[\delta]}$ obtained by composing $\pi^*\phi$ with the inclusion of $\mathcal{U}$ in π^*E. By construction, the incidence variety I is contained in $D(\widetilde{\phi})$ and, if $(C,z)\in D(\widetilde{\phi})\setminus I$, then $C\in|L_{g-1}|$. It can be easily checked that the expected dimension of $D(\widetilde{\phi})$ equals $g-\delta$. In order to show that $D(\widetilde{\phi})$ has the expected dimension along I, we consider the projection $t: D(\widetilde{\phi})\longrightarrow|L_g|$. If $(C,z)\in D(\widetilde{\phi})$, the curve C is singular along z. Hence, if $(C,z)\in I$ and C is reduced, then the δ-invariant of C is $\geq\delta$: this implies that $t(I)\subset\overline{V^{g-\delta}(S,L_g)}=\overline{V_\delta(S,L_g)}$, where the equality follows from Proposition 2.5. On the other hand, if $(C,z)\in D(\widetilde{\phi})\setminus I$, then $C\in|L_{g-1}|\subset|L_g|$. We conclude that I consists of the irreducible components of $D(\widetilde{\phi})$ whose image is not entirely contained in $|L_{g-1}|$. In particular, $t_I:=t|_I$ is birational and a general curve in $t(I)$ is integral. The following Lemma 2.8 yields that the locus in $I\setminus(t^{-1}|L_{g-1}|\cap I)$ where the fibers of t_I are not finite has dimension $\leq g-\delta-2$, and thus

$$g-\delta=\dim\overline{V_\delta(S,L_g)}\geq\dim I;$$

hence, I consists of irreducible components of $D(\widetilde{\phi})$ that dominate $\overline{V_\delta(S,L_g)}$ and has pure dimension $g-\delta$, as expected. In particular, I is locally Cohen-Macaulay (cf. [ACGH, II, Prop. 4.1]). Being birational to $\overline{V_\delta(S,L_g)}$, which is generically reduced by Proposition 2.4, we conclude that I is reduced because generic reducedness is equivalent to reducedness for locally Cohen-Macaulay schemes. Furthermore, every irreducible component I' of the intersection of two components of I has codimension 1 by Hartshorne's Connectedness Theorem (cf. [Ei, Thm. 18.12]). Let Z be a component of the intersection of two irreducible components of $\overline{V_\delta(S,L_g)}$ such that Z is not contained in $|L_{g-1}|$. Since any component I' of $t^{-1}(Z)$ has codimension 1 in I, the following Lemma 2.8 yields that a general fiber of the restriction of t to I' is finite and thus Z has codimension 1 in $\overline{V_\delta(S,L_g)}$. If a general curve in Z has δ-invariant precisely δ, then the restriction of t to $t^{-1}(Z)$ is birational. In particular, the open subset of I where t_I is an isomorphism intersects $t^{-1}(Z)$ and thus, using that I is reduced, we may conclude that $\overline{V_\delta(S,L_g)}$ is reduced at a general closed point of Z. If instead a general curve in Z has δ-invariant $> \delta$, by dimensional reasons Z is a component of $\overline{V_{\delta+1}(S,L_g)} = \overline{V^{g-\delta-1}(S,L_g)}$. We recall that $\overline{V_\delta(S,L_g)} = \overline{V^{g-\delta}(S,L_g)}$ is singular at the points of $\overline{V^{g-\delta-1}(S,L_g)}$ (cf. [DH]). More precisely, exactly as in the case of a $K3$ surface (cf. [CD, Prop. 6]), Proposition 2.4(i) implies that locally around a general $C \in V^{g-\delta-1}(S,L_g)$, the locus $\overline{V^{g-\delta}(S,L_g)}$ is the union of $\delta+1$ transversal sheets corresponding to the partial normalizations of C of arithmetic genus $g-\delta$. In particular, since $\overline{V^{g-\delta-1}(S,L_g)}$ is generically reduced by Proposition 2.4(iii), then every irreducible component of $\overline{V^{g-\delta-1}(S,L_g)}$ contains a closed point which is a reduced point of $\overline{V^{g-\delta}(S,L_g)}$(that is, the local ring of $\overline{V^{g-\delta}(S,L_g)}$ at a general closed point of $\overline{V^{g-\delta-1}(S,L_g)}$ has no nilpotents). □

Lemma 2.8. *Let $I \subset |L_g| \times S^{[\delta]}$ be the incidence variety defined in* (2.5) *and let $t_I : I \longrightarrow |L_g|$ be the first projection. Then, the locus in $I \setminus t_I^{-1}|L_{g-1}|$ where the fibers of t_I are not finite has dimension $\leq g-\delta-2$.*

Proof. Given $C \in \overline{V_\delta(S,L_g)}$, let us denote by $\nu : \tilde{C} \to C$ its normalization, by $A_C := Hom_{\mathcal{O}_C}(\nu_*\mathcal{O}_{\tilde{C}}, \mathcal{O}_C)$ its adjoint ideal, and by $E_C \subset C$ the subscheme defined by A_C. First of all, we show that, if $(C,z) \in I \setminus t_I^{-1}|L_{g-1}|$, then $z \subset E_C$. Indeed, by normalizing an affine curve contained in I passing through the point (C,z) and pulling back the universal families over I, one may construct a family of curves $f : \mathcal{C} \to B$ and a family of 0-dimensional subschemes $h : \mathcal{Z} \to B$ over a smooth 1-dimensional scheme B such that $\mathcal{Z} \subset \mathcal{C}$, a general fiber $C_b := f^{-1}(b)$ is a curve in $V_\delta(S,L_g)$ and $z_b := h^{-1}(b)$ is its scheme of nodes, while for a special point $b_0 \in B$ we have $C := f^{-1}(b_0)$ and $z := h^{-1}(b_0)$. Let $n : \mathcal{C}' \to \mathcal{C}$ be the normalization of $\mathcal{C}$ along $\mathcal{Z}$. For a general $b \in B$ the curve $C'_b := n^{-1}(f^{-1}(b))$ is the normalization of C_b, while $C' := n^{-1}(f^{-1}(b_0))$ is the partial normalization of C along z. In particular, z is the subscheme of C defined by the ideal $Hom_{\mathcal{O}_C}(n_*\mathcal{O}_{C'}, \mathcal{O}_C)$. The inclusion $z \subset E_C$ thus follows from the factorization of ν as $\tilde{C} \to C' \to C$.

For any $k \geq 1$, let $Z_{\delta,k} \subset \overline{V_\delta(S,L_g)}$ be the locus of irreducible curves $C \in \overline{V_\delta(S,L_g)}$ such that the subscheme $E_C \subset C$ defined by A_C contains a k-dimensional family of subschemes of length-δ; the above discussion yields that, if $\dim t_I^{-1}(t_I(C)) \geq k$, then $C \in Z_{\delta,k}$. We will show that $Z_{\delta,k} \subset \overline{V_{\delta+k+2}(S,L_g)}$ for all $0 \leq \delta \leq g-1$ and $k \geq 1$ and thus

$$\dim t_I^{-1}(Z_{\delta,k}) = \dim Z_{\delta,k} + k \leq \dim \overline{V_{\delta+k+2}(S,L_g)} + k = g-\delta-2, \tag{2.6}$$

that yields our statement .

We proceed by induction on k. The case $k=1$ amounts to showing that, if $C \in Z_{\delta,1}$, then the δ-invariant $\delta(C)$ of C (i.e., the length of E_C) is $\geq \delta+3$; this holds true because any subscheme ξ_t of length δ contained in E_C corresponds to a partial normalization $\nu_t : \hat{C}_t \to C$ with $p_a(\hat{C}_t) = p_a(C) - \delta$. If $\delta(C) = \delta$, then necessarily $\nu_t = \nu$, and thus $\xi_t = E_C$ is unique. Analogously, if $\delta(C) = \delta + 1$, then any such $\hat{C}_t$ is obtained from $\tilde{C}$ by creating either one node or one cusp at the finitely many points of $\tilde{C}$ mapping to the singular locus of C; hence, the partial normalizations ν_t (or, equivalently, the subschemes ξ_t) are finitely many in this case. The remaining case $\delta(C) = \delta + 2$ is treated in the same way using the fact that the only singularity having δ-invariant equal to 2 are tacnodes, ramphoid cusps and triple points of embedding dimension 3.

We now assume that the inclusion $Z_{\delta,h} \subset \overline{V_{\delta+h+2}(S, L_g)}$ holds for any $0 \le \delta \le g-1$ and $1 \le h \le k-1$, and prove it for $h = k \ge 2$. Fix a general $C \in Z_{\delta,k}$, that is, C has a k-dimensional family of length-δ subschemes contained in E_C. We will prove that C possesses a $(k-1)$-dimensional family of subschemes of length $\delta + 1$; this is enough to conclude because it implies that $C \in Z_{\delta+1,k-1} \subset \overline{V_{\delta+k+2}(S, L_g)}$, where the inclusion follows from the induction assumption. In order to pass from subschemes of length δ to subschemes of length $\delta+1$, we consider the nested Hilbert scheme $S^{[\delta,\delta+1]}$ parametrizing pairs $(\xi, \xi') \in S^{[\delta]} \times S^{[\delta+1]}$ such that $\xi \subset \xi'$. This is endowed with two natural morphisms

$$\begin{tikzcd} & S^{[\delta,\delta+1]} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ S^{[\delta]} \times S & & S^{[\delta+1]} \times S, \end{tikzcd}$$

mapping a pair (ξ, ξ') to (ξ, x) and (ξ', x), respectively, with $x \in S$ being the point where ξ and ξ' differ. As explained in [Le, p.12], the dimensions of the fibers of ϕ and ψ are related as follows: given $(\xi, \xi') \in S^{[\delta,\delta+1]}$, if $\phi^{-1}(\phi(\xi, \xi')) \simeq \mathbb{P}^{i-1}$, then $\psi^{-1}(\psi(\xi, \xi')) \simeq \mathbb{P}^{i'-2}$ for some integer i' satisfying $|i - i'| \le 1$. Let us consider the k-dimensional family

$$B := \{\xi \in S^{[\delta]} \mid \xi \subset E_C \subset C\},$$

the subscheme

$$W := \{(\xi, \xi') \in S^{[\delta,\delta+1]} \mid \xi \subset \xi' \subset E_C\} \subset \phi^{-1}(B \times \mathrm{Supp}(E_C)),$$

and set $B' := \psi(W) \subset \psi(\phi^{-1}(B \times \mathrm{Supp}(E_C)))$. We want to show that B' has dimension $\ge k-1$. Let $(\xi, \xi') \in \phi^{-1}(B \times \mathrm{Supp}(E_C))$ be general, and denote by $i-1$ the dimension of $\phi^{-1}(\phi(\xi, \xi'))$ and by $i'-2$ the dimension of $\psi^{-1}(\psi(\xi, \xi'))$. Since $i - i' \ge -1$, we obtain

$$\dim\, \psi(\phi^{-1}(B \times \mathrm{Supp}(E_C))) = \dim B + i - 1 - (i' - 2) \ge k.$$

In order to conclude that $\dim B' \ge k-1$, it is thus enough to show that the fiber at (ξ, ξ') of the restriction $\phi|_W$ has codimension at most 1 in the fiber $\phi^{-1}(\xi, x)$, where $(\xi, x) = \phi(\xi, \xi')$. We need to recall that $\phi^{-1}(\xi, x) \simeq$

$\mathbb{P}((I_\xi/m_x I_\xi)^\vee)$ where m_x is the maximal ideal of the point x; indeed, any pair $(\xi,\xi')\in\phi^{-1}(\xi,x)$ corresponds to a short exact sequence

$$0\longrightarrow I_{\xi'}\longrightarrow I_\xi\xrightarrow{\alpha}\mathcal{O}_x\longrightarrow 0,$$

(where $\mathcal{O}_x$ is the structure sheaf of x) and thus to a linear map $\alpha_x: I_\xi/m_xI_\xi\to\mathbb{C}$. Since $\xi\subset E_C$, we have an inclusion $\iota: I_{E_C/S}\to I_\xi$ and a linear map $\iota_x: I_{E_C/S}/m_xI_{E_C/S}\to I_\xi/m_xI_\xi$; the subscheme ξ' is contained in E_C precisely when the composition $\alpha_x\circ\iota_x$ is zero. Note that this is a linear condition and, in order to show that it is a codimension 1 condition on $\mathbb{P}((I_\xi/m_xI_\xi)^\vee)$, we prove that $I_{E_C/S}/m_xI_{E_C/S}$ is 1-dimensional, or equivalently, E_C is contained in at most one subscheme of S of length $\delta(C)+1$ differing from E_C only at x. Consider the standard short exact sequence of ideals

$$0\longrightarrow\mathcal{O}_S(-C)\xrightarrow{j} I_{E_C/S}\longrightarrow A_C\longrightarrow 0,$$

where j is the multiplication by the section of L_g defining C. We recall (cf., e.g., [DS, §3]) that the support of E_C is the singular locus of C and that, if C has multiplicity h at a point $x\in\mathrm{Supp}(E_C)$, then $C\in|L_g\otimes m_x^h|$. The computation of the adjoint ideal of a curve contained in a smooth surface in terms of an embedded resolutions of its singularities (cf., e.g., [CD, Lem. 3.7]) thus yields that $C\in|L_g\otimes m_xI_{E_C/S}|$; in other words, the image of j is contained in $m_xI_{E_C/S}$ and we have an isomorphism $I_{E_C/S}/m_xI_{E_C/S}\simeq A_C/m_xA_C$. It is thus enough to verify that $A_C/m_xA_C\simeq\mathbb{C}$, or equivalently, E_C is contained in at most one subscheme τ of C of length $\delta(C)+1$ differing from E_C only at x. Since $\tau\in\mathbb{P}((A_C/m_xA_C)^\vee)$, its unicity follows as soon as we show that we have finitely many choices for it. This holds true because any such τ corresponds to a rank 1 torsion free sheaf on C of the form $\nu_*\mathcal{O}_{\tilde{C}}(y)$ for one of the finitely many points y mapping to x. $\square$

2.1. **Severi varieties and excess intersections.** For every $g\geq 2$ and $0\leq p\leq g$, we consider the natural injection

$$i_{p,g}:|L_p|\hookrightarrow|L_g|$$

mapping a curve $C\in|L_p|$ to the divisor $C+(g-p)J\in|L_g|$.

Lemma 2.9. *The image $i_{p,g}(|L_p|)\subset|L_g|$ coincides with the codimension $g-p$ linear subspace $|L_g\otimes I_x^{g-p}|$, where $x\in J$ is a general point. In particular, identifying $|L_p|$ with its image in $|L_g|$ under the map $i_{p,g}$ for every $0\leq p\leq g$, we have the following chain of inclusions in $|L_g|$:*

$$\{E_9\}=|L_0|\subset|L_1|\subset\cdots\subset|L_{g-1}|\subset|L_g| \tag{2.7}$$

Proof. The inclusion $i_{p,g}(|L_p|)\subset|L_g\otimes I_x^{g-p}|$ is obvious. In order to prove equality, it is enough to show that $h^0(L_g\otimes I_x^{g-p})=p+1$. We proceed by induction on $g-p$. The case $g-p=1$ is trivial and the induction step follows from the short exact sequences

$$0\longrightarrow L_{g-1}\otimes I_x^{g-p-1}\longrightarrow L_g\otimes I_x^{g-p}\longrightarrow L_g\otimes I_x^{g-p}|_J\longrightarrow 0,$$

along with the isomorphism $L_g \otimes I_x^{g-p}|_J \simeq \mathcal{O}_J(p_{10}(g) - (g-p)x)$. $\square$

It is natural to investigate the intersection of a Severi variety $\overline{V_\delta(S, L_g)}$ with the subspaces $|L_p|$ in the flag (2.7). The following proposition yields that this intersection is not dimensionally proper as soon as $p \leq g-2$. Even though we will not use this result in the rest of the paper, we include it as it motivates the necessity of introducing expanded Severi varieties in the next section.

Proposition 2.10. *For every $g \geq 2$ and $0 \leq \delta \leq g-1$, the following equality holds in $|L_g|$:*

$$|L_{g-1}| \cap \overline{V_\delta(S, L_g)} = \bigcup_{h=0}^{\delta} \overline{V_{\delta-h}(S, L_{g-h-1})}. \tag{2.8}$$

Proof. First of all, we verify the inclusion $\supset$ in (2.8) by showing that, if $0 \leq h \leq \delta$ and C is general in any irreducible component of $\overline{V_{\delta-h}(S, L_{g-h-1})}$, the curve $X = C + (h+1)J \in |L_g|$ can be deformed to an irreducible curve in $|L_g|$ of geometric genus $g - \delta$ (which thus lies in $\overline{V_\delta(S, L_g)}$ by Proposition 2.5). The curve X is the image of a stable map $f : \tilde{C} \cup_p \tilde{J} \to S$ of genus $g-\delta$, where $\tilde{J}$ is a smooth elliptic degree $h+1$ cover of J, the curve $\tilde{C}$ is the normalization of C and has genus $g-\delta-1$, and the gluing point p is mapped to $C \cap J = \{p_{10}(g-h-1)\}$. We denote by $f_C := f|_{\tilde{C}} : \tilde{C} \to C \subset S$ and by $f_J := f|_{\tilde{J}} : \tilde{J} \to J \subset S$ the restrictions of f to $\tilde{C}$ and $\tilde{J}$, respectively. As f_J is étale and C is nodal, both f_J and f_C are unramified and the same holds true for the map f since C and J intersect transversally at $p_{10}(g-h-1)$. The normal sheaf N_f sits in the following short exact sequence:

$$0 \longrightarrow N_f(-p)|_{\tilde{J}} \longrightarrow N_f \longrightarrow N_f|_{\tilde{C}} \longrightarrow 0.$$

By [GHS, Lem. 2.5] we have isomorphisms

$$N_f|_{\tilde{C}} \simeq N_{f_C}(p) \simeq \omega_{\tilde{C}}(2p),$$

and analogously

$$N_f(-p)|_{\tilde{J}} \simeq N_{f_J} \simeq f_J^* \mathcal{O}_J(J).$$

Since the line bundle $f_J^* \mathcal{O}_J(J)$ is non-trivial of degree 0, we obtain that $h^0(N_f) = h^0(\omega_{\tilde{C}}(2p)) = g - \delta$ and $h^1(N_f) = 0$, and thus f defines a smooth point of a $(g-\delta)$-dimensional component of $M_{g-\delta}(S, L_g)$. However, f_C is an unramified stable map of genus $g-1-\delta$ and thus $\dim_{[f_C]} M_{g-1-\delta}(S, L_{g-h-1}) = g-1-\delta$. Analogously, the map f_J is rigid in $M_1(S, (h+1)J)$. Hence, a general deformation of f parametrizes a stable map from an integral (and smooth by dimensional arguments) curve of genus $g - \delta$. This proves that f is smoothable and thus the existence of the morphism (2.2) yields that $X \in \overline{V_\delta(S, L_g)}$.

It remains to verify the inclusion $\subset$ in (2.8). Any irreducible component V of $|L_{g-1}| \cap \overline{V_\delta(S, L_g)}$ satisfies $\dim V = g - 1 - \delta$ because no component of $\overline{V_\delta(S, L_g)}$ is contained in $|L_{g-1}|$ by Proposition 2.4 and $|L_{g-1}|$ is a hyperplane in $|L_g|$. Assume now that a general element X of V parametrizes a curve in $|L_{g-h-1}| \setminus |L_{g-h-2}|$ for some $h \geq 0$, that is, $X = C + (h+1)J$ with C irreducible. We need to show that $h \leq \delta$ and $C \in V_{\delta-h}(S, L_{g-h-1})$; by dimensional reasons and Proposition 2.5, it is enough to check that C has geometric genus $\leq g-1-\delta$. This trivially follows because $X = C+(h+1)J \in \overline{V_\delta(S, L_g)} = \overline{V^{g-\delta}(S, L_g)}$ and J is an elliptic curve. $\square$

3. Connectedness of expanded Severi varieties on Halphen surfaces

3.1. **Jun Li's expanded degenerations and expanded Severi varieties.** We briefly recall the theory of expanded degenerations developed by Jun Li [Li1, Li2]. An expanded degeneration of S along J is a semistable model of S

$$S[n]_0 := S \cup_J R \cup_J \cdots \cup_J R, \quad R := \mathbb{P}(\mathcal{O}_J \oplus N_{J/S}),$$

which is the union of S with a length-n tree of ruled surfaces R as above for some $n \geq 0$. More precisely, denoting by J_0 and J_∞ the two distinguished sections on R such that $N_{J_0/R} \simeq N^\vee_{J/S}$ and $N_{J_\infty/R} \simeq N_{J/S}$, the above expansion $S[n]_0$ is obtained by gluing the first copy of R with S along J_0, while two adjacent copies of R are glued identifying the J_∞ on the left surface with the J_0 on the right one. The section J_∞ on the latter copy of R is referred to as the *relative divisor* J.

We apply Li and Wu's construction [LW] (cf. also [Li3] for a nice survey) of stacks of relative ideal sheaves with fixed Hilbert polynomial. This provides a moduli stack $|L_g|^{\exp}$, which we call *expanded linear system*, that parametrizes equivalence classes of connected curves X living in some expanded degeneration $S[n]_0$ of S such that: the image of X under the projection $S[n]_0 \to S$ lies in $|L_g|$, the curve X is normal to the singular locus of $S[n]_0$ and to the relative divisor J, and its automorphism group is finite. Since $J \cdot L_g = 1$ on S, curves in $|L_g|^{\exp}$ can be easily described. The condition of X being normal to the singular locus $\mathrm{Sing} S[n]_0$ and to the relative divisor J reduces to the requirement that X has no component contained in these loci and intersects each component of $\mathrm{Sing} S[n]_0$ at a node that connects two irreducible components of X living in two adjacent components of $S[n]_0$. We decompose $X \subset S[n]_0$ in subcurves as follows:

$$X = X_0 \cup X_1 \cup \cdots \cup X_n, \tag{3.1}$$

where $X_0 \subset S$, while X_i is contained in the i-th copy of R if $i \geq 1$, and two adjacent X_i share a node along the singular locus of $S[n]_0$. Since its automorphism group is finite, X contains no fiber of a ruled surface R. Denoting by g_i the arithmetic genus of X_i, the condition that the image of X in S lies in $|L_g|$ yields $\sum_{i=0}^n g_i = g$. The curve $X_0 \in |L_{g_0}|$ while, denoting by f the numerical class of a fiber of $R \to J$, for $i \geq 1$ the numerical class of $X_i \subset R$ is $g_i J_0 + f$ (and thus $g_i > 0$ as X_i is not a fiber). Note that $(g_i J_0 + f) \cdot J_0 = 1$ and thus a reducible curve with this numerical class necessarily contains either J_0 or J_∞. We thus conclude that X_i is irreducible. The linear equivalence class of X_i is determined by the gluing condition as follows. Setting $\underline{g} = (g_0, \ldots, g_n)$ and $x_1(\underline{g}) := p_{10}(g_0 + g_1)$, we have $f(C_1) \in |N_1(\underline{g})|$ with $N_1(\underline{g}) := g_1 J_0 + f_{x_1(\underline{g})}$. As in [FT, §2], one verifies that $N_1(\underline{g})$ has two base points, namely, $p_{10}(g_0) \in J_0$ and $x_1(\underline{g}) \in J_\infty$. Analogously, setting $x_i(\underline{g}) := p_{10}(g_0 + \cdots + g_i)$ for every $1 \le i \le n$, we have $f(C_i) \in |N_i(\underline{g})|$ where the line bundle $N_i(\underline{g}) := g_i J_0 + f_{x_i(\underline{g})}$ has base points $x_{i-1}(\underline{g}) \in J_0$ and $x_i(\underline{g}) \in J_\infty$. In particular, the evaluation map

$$|L_g|^{\exp} \to J$$

at the relative divisor always takes the value $p_{10}(g)$.

The multiplicative group $\mathbb{C}^*$ acts fiberwise on R preserving the sections J_0 and J_∞; this induces an action of $(\mathbb{C}^*)^n$ on $S[n]_0$ for every $n \ge 1$. Two curves define the same point of $|L_g|^{\exp}$ if they live in the same $S[n]_0$ and lie in the same orbit under the action of $(\mathbb{C}^*)^n$. Summing up, thanks to the decomposition (3.1), a point $[X] \in |L_g|^{\exp}$ representing a curve in $S[n]_0$ defines points of the following moduli stacks

$$X_0 \in |L_{g_0}|\ ,\ [X_i] \in |N_i(\underline{g})|/\mathbb{C}^* \text{ for } i \ge 1.$$

Since $(g_i J_0 + f) \cdot f = g_i \ge 1$, any curve with numerical class $g_i J_0 + f$ is a degree g_i cover of J; in particular, the linear system $|N_i(\underline{g})|$ contains no rational curves. We will later use the following result concerning equigeneric loci in the linear systems $|N_i(\underline{g})|$ on R.

Proposition 3.1. *Let R and $N_i(\underline{g}) := g_i J_0 + f_{x_i(\underline{g})} \in \mathrm{Pic}(R)$ be defined as above. Then for every integer $1 \le h_i \le g_i$ the following hold:*

- (i) *Both the equigeneric locus $V^{h_i}(R, N_i(\underline{g})) \subset |N_i(\underline{g})|$ and the moduli stack of smoothable stable maps $\mathcal{M}_{h_i}(R, N_i(\underline{g}))^{\mathrm{sm}}$ with image in the linear system $|N_i(\underline{g})|$ have pure dimension h_i and are generically reduced.*
- (ii) *Let V and W be two intersecting components of $\overline{V^{h_i}(R, N_i(\underline{g}))}$ and let Z be an irreducible component of $V \cap W$ whose general point parametrizes a curve containing neither J_0 nor J_∞; then Z has pure codimension 1 in $\overline{V^{h_i}(R, N_i(\underline{g}))}$ and a general point of Z is a reduced point of $\overline{V^{h_i}(R, N_i(\underline{g}))}$.*

Proof. Let $\eta \in \mathrm{Pic}^0(J)$ be the line bundle such that $R = \mathbb{P}(\mathcal{O}_J \oplus \eta)$ and denote by $\phi : R \to J$ the natural projection; we have $J_\infty \equiv J_0 - \phi^*(\eta)$. We recall from [FT] that curves in $|N_i(\underline{g})|$ have arithmetic genus g_i and $\dim |N_i(\underline{g})| = g_i$. According to our notation, the moduli stack $\mathcal{M}_{h_i}(R, N_i(\underline{g}))$ parametrizes maps f such that $f(C)$ lies in the linear system $|N_i(\underline{g})|$; this is a closed substack of the moduli stack $\mathcal{M}_{h_i}(R, g_i J_0 + f)$ where only the numerical class of $f(C)$ is fixed. Let $\mathcal{M}_{h_i}(R, N_i(\underline{g}))^{sm}$ and $\mathcal{M}_{h_i}(R, g_i J_0 + f)^{sm}$ be the closed substacks parametrizing smoothable maps. The deformations of a map $[f] \in \mathcal{M}_{h_i}(R, g_i J_0 + f)^{sm}$ are governed by the normal sheaf N_f.

As in the proof of Proposition 2.4(ii)-(iii), one shows that a general $[f]$ in any irreducible component of $\mathcal{M}_{h_i}(R, g_iJ_0 + f)^{sm}$ is unramified and thus $\mathcal{M}_{h_i}(R, g_iJ_0 + f)^{sm}$ is generically reduced and has pure dimension $h_i + 1$. Being a fiber of the evaluation map $\mathcal{M}_{h_i}(R, g_iJ_0 + f)^{sm} \to J_0$, the stack $\mathcal{M}_{h_i}(R, N_i(\underline{g}))^{sm}$ is generically reduced and of pure dimension h_i. The same holds true for the equigeneric locus $\overline{V^{h_i}(R, N_i(\underline{g}))}$ thanks to the existence of a birational map $\tilde{\mu} : \tilde{\mathcal{M}}_{h_i}(R, g_iJ_0 + f)^{sm} \to \overline{V^h(R, N_i(\underline{g}))}$ from the seminormalization $\tilde{\mathcal{M}}_{h_i}(R, g_iJ_0 + f)^{sm}$ of $\mathcal{M}_{h_i}(R, g_iJ_0 + f)^{sm}$. This proves (i).

To obtain (ii), one proceeds exactly as in the proofs of Proposition 2.7 and Lemma 2.10, where the density of the Severi variety in the equigeneric locus (whose validity on R is unknown) was never used. The proofs work the same way. In particular, in the proof of Proposition 2.7 the fact that a general point in $t(I)$ parametrizes an integral curve still holds true; indeed, the equality $N_i(\underline{g}) \cdot J_0 = 1$ implies that all curves in the linear system $|N_i(\underline{g})|$ are integral except for those lying in the two hyperplanes of $|N_i(\underline{g})|$ defined by the linear subsystem $J_0 + |(g_i - 1)J_0 + f_{x_i(\underline{g})}|$ and $J_\infty + |(g_i - 1)J_0 + f_{y_i(\underline{g})}|$, where $y_i(\underline{g}) := p_{10}(g_0 + \cdots + g_i - 1)$. □

Coming back to $|L_g|^{\exp}$, this is a proper and separated Deligne-Mumford stack [Li3, Thm. 3.36]. We briefly recall why it is DM. By Li's construction [Li4, §2], there is a scheme

$$S[n] \to \mathbb{A}^n$$

(obtained from $S \times \mathbb{A}^n$ via a sequence of blow-ups) combining all possible expansions $S[k]_0$ for $0 \leq k \leq n$; in particular, a fiber over a general $t \in \mathbb{A}^n$ is isomorphic to S, the central fiber over $0 \in \mathbb{A}^n$ is the n-th expansion $S[n]_0$, while the fibers over any coordinate $(n - k)$-dimensional plane in $\mathbb{A}^n$ are isomorphic to $S[k]_0$. The natural action of $(\mathbb{C}^*)^n$ on $\mathbb{A}^n$ lifts to an action on $S[n]$ so that its restriction to $S[n]_0$ is trivial on S, while the i-th copy of $\mathbb{C}^*$ acts on the i-th copy of R fiberwise so that J_0 and J_∞ are fixed. By its construction, $S[n]$ is endowed with a projection

$$\beta_n : S[n] \to S$$

and we consider the line bundle $L_g[n] := \beta_n^* L_g$. Let $|L_g[n]|^{Li}$ be the closed DM substack of the linear system $|L_g[n]|$ parametrizing curves that live in some expanded degeneration $S[k]_0$ with $k \leq n$ and satisfy the same conditions required to define a point in $|L_g|^{\exp}$. Note that $|L_g[n]|^{Li}$ admits a $(\mathbb{C}^*)^n$ action that is induced by the one on $S[n]$ and has finite stabilizers. Since the linear systems $|N_i(\underline{g})|$ contain no rational curves, in the decomposition (3.1) for a curve in $|L_g|^{\exp}$ one has $n \leq g$; in other words, there is a surjective map $|L_g[g]|^{Li} \to |L_g|^{\exp}$. The fact that $|L_g|^{\exp}$ is DM thus follows from the properties of the induced map

$$|L_g[g]|^{Li}/(\mathbb{C}^*)^g \to |L_g|^{\exp},$$

which is surjective, finite and étale.

We will also occasionally make use of the moduli stack $\mathcal{M}_{g-\delta}(S/J, L_g)$ of *stable relative maps to expanded degenerations* of (S, J) with multiplicity 1 along the relative divisor J; its introduction by Jun Li [Li1, Li2] is prior to the constructions in [LW] recalled above. Remembering that $J \cdot L_g = 1$ on S, a stable relative map of genus $g - \delta$ to the expansion $S[n]_0$ is a map $f : C \to S[n]_0$ from a connected prestable curve of genus $g - \delta$ such that the image $f(C)$ defines a point of the expanded linear system $|L_g|^{\mathrm{exp}}$, no component of C is mapped entirely to the singular locus of $S[n]_0$ or to the relative divisor J and the automorphism group of f is finite. By (3.1), any such map can be thus decomposed as

$$f = f_0 \cup \cdots \cup f_n : C = C_0 \cup C_1 \cup \cdots \cup C_n \to S[n]_0, \tag{3.2}$$

where

$$[f_0] \in \mathcal{M}_{h_0}(S, L_{g_0}) \ , \ [f_i] \in \mathcal{M}_{h_i}(R, N_i(\underline{g}))/\mathbb{C}^* \text{ for } i \geq 1$$

for some integers $g_i \geq h_i \geq 0$ satisfying $\sum_{i=0}^n g_i = g$, $\sum_{i=0}^n h_i = g - \delta$ and $h_i > 0$ if $i \neq 0$.

For every $0 \leq \delta \leq g$, we define the *expanded Severi variety* $\overline{\mathcal{V}_\delta(S/J, L_g)}$ to be the closure in $|L_g|^{\mathrm{exp}}$ of the Severi variety $V_\delta(S, L_g)$. Closed points of $\overline{\mathcal{V}_\delta(S/J, L_g)}$ thus parametrize curves $X = X_0 \cup X_1 \cup \ldots \cup X_n$ in $|L_g|^{\mathrm{exp}}$ whose normalization outside of the nodes $n_i := X_i \cap X_{i+1}$ is a nodal connected curve of arithmetic genus $\leq g - \delta$. Denoting by $\mathcal{M}_{g-\delta}(S/J, L_g)^{sm}$ the substack of $\mathcal{M}_{g-\delta}(S/J, L_g)$ parametrizing smoothable maps and by $\tilde{\mathcal{M}}_{g-\delta}(S/J, L_g)^{sm}$ its semi-normalization, the stack $\overline{\mathcal{V}_\delta(S/J, L_g)}$ can be alternatively described as the image of the natural map

$$\tilde{\mu} : \tilde{\mathcal{M}}_{g-\delta}(S/J, L_g)^{sm} \to |L_g|^{\mathrm{exp}}$$

sending a stable map to its image. Our next goal is to prove connectedness of $\overline{\mathcal{V}_\delta(S/J, L_g)}$ for $0 \leq \delta < g$.

3.2. **A sequence of blow-ups.** We fix $k >> 0$ so that $\overline{V_\delta(S, L_{g+k})}$ is irreducible by Proposition 2.6; the same holds true for the expanded linear system $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}$. To prove connectedness of $\overline{\mathcal{V}_\delta(S/J, L_g)}$, we will realize it as a fiber of a morphism $\overline{\mathcal{V}_\delta(S/J, L_{g+k})} \to \widetilde{\mathbb{P}^k}$ admitting a section, where $\widetilde{\mathbb{P}^k}$ is obtained from $\mathbb{P}^k$ via a sequence of blow-ups.

As a first step, we perform a sequence of blow-ups of $|L_{g+k}|$ and in §3.3 we will relate it to the expanded linear system $|L_{g+k}|^{\mathrm{exp}}$.

Inside $|L_{g+k}|$ we consider the chain of inclusions provided by Lemma 2.9

$$\{E_9\} = |L_0| \subset |L_1| \subset \cdots \subset |L_{g+k-1}| \subset |L_{g+k}|.$$

We start by blowing up $|L_{g+k}|$ along $|L_0|$ and denote by E_0 the exceptional divisor. We then blow up the strict transform of $|L_1|$ and denote by E_1 the exceptional divisor, and so on until we finally blow up the strict transform of $|L_{g+k-2}|$ and get the last exceptional divisors E_{g+k-2}. Let

$$\pi : \widetilde{|L_{g+k}|} \longrightarrow |L_{g+k}|$$

be the composition of these $g+k-1$ blow-ups [3].

Let $q : |L_{g+k}| \dashrightarrow \mathbb{P}^k$ be the projection of $|L_{g+k}|$ from $|L_{g-1}|$, and let $\tilde{q} : Bl_{|L_{g-1}|}|L_{g+k}| \to \mathbb{P}^k$ be its minimal resolution. Since $\pi^{-1}|L_{g-1}| = E_0 + \cdots + E_{g-1}$ is a Cartier divisor on $\widetilde{|L_{g+k}|}$, the universal property of blow-ups [St, 71.17] implies that π factors trough a map $\tilde{\pi} : \widetilde{|L_{g+k}|} \to Bl_{|L_{g-1}|}|L_{g+k}|$. We consider the composition $p := \tilde{q} \circ \tilde{\pi} : \widetilde{|L_{g+k}|} \to \mathbb{P}^k$.

For $0 \leq j \leq k-2$ consider the j-dimensional projective subspace $W_j = \overline{q(|L_{g+j}|)} \subset \mathbb{P}^k$, and blow-up $\mathbb{P}^k$ first at the point W_0, then at the strict transform of the line W_1 and so on, until finally at the strict transform of W_{k-2}. We denote by $b : \widetilde{\mathbb{P}^k} \to \mathbb{P}^k$ the composition of these $k-1$ blow-ups. Since $p^{-1}(W_0) = E_g$ is a Cartier divisor on $\widetilde{|L_{g+k}|}$, then again by the universal property of blow-ups p factors through a map $p_0 : \widetilde{|L_{g+k}|} \to Bl_{W_0}\mathbb{P}^k$. The inverse image under p_0 of the strict transform of W_1 in $Bl_{W_0}\mathbb{P}^k$ is again a Cartier divisor as it coincides with E_{g+1}, and thus p_0 factors through the blow-up of $Bl_{W_0}\mathbb{P}^k$ along the strict transform of W_1. By the same argument, after $k-1$ steps we obtain that p factors through a map

$$\Psi : \widetilde{|L_{g+k}|} \to \widetilde{\mathbb{P}^k}. \tag{3.3}$$

We will need the following results concerning Ψ.

Lemma 3.2. *Let $\widetilde{H_1}$ denote the strict transform of $|L_{g+k-1}|$ in $\widetilde{|L_{g+k}|}$. Then, the following hold:*

(i) *the intersection $\widetilde{H_1} \cap \left(\bigcap_{i=0}^{k-2} E_{g+i}\right)$ is a fiber of Ψ;*

(ii) *if $X = X_0 + (k+1)J \in |L_{g-1}| \setminus |L_{g-2}| \subset |L_{g+k}|$, then the fiber $\pi^{-1}(X)$ defines a section of $\Psi : \widetilde{|L_{g+k}|} \to \widetilde{\mathbb{P}^k}$.*

Proof. Denote by $e_0, \ldots, e_{k-2}$ the exceptional divisors of $b : \widetilde{\mathbb{P}^k} \to \mathbb{P}^k$, by numbering them so that $b(e_j)$ has dimension j. By construction, we have $E_{g+j} = \Psi^*(e_j)$ for every $0 \leq j \leq k-2$. Furthermore, on $\widetilde{\mathbb{P}^k}$ there exists a divisor $\widetilde{D_1} \in |b^*\mathcal{O}_{\mathbb{P}^k}(1) - \sum_{j=0}^{k-2} e_j|$ such that $\widetilde{H_1} = \Psi^*\widetilde{D}_1$. It then follows that the intersection

$$\widetilde{H_1} \cap \left(\bigcap_{i=0}^{k-2} E_{g+i}\right)$$

is the inverse image under Ψ of the point $\xi \in \widetilde{D_1} \cap e_0 \cap \ldots \cap e_{k-2}$ determined by $\mathbb{P}(N_{D_1/\mathbb{P}^k, W_0})$, where $D_1 = b(\widetilde{D}_1) = \pi(|L_{g+k-1}|)$. This gives (i).

[3] the attentive reader may note that for our purposes it would be enough to perform only $k+1$ blow-ups, starting from that of $|L_{g-1}|$; however, we consider the choice of blowing up the entire flag more natural.

Let $X \in |L_{g-1}| \setminus |L_{g-2}|$ be as in (ii). By construction, the fiber $\pi^{-1}(X)$ is a $\mathbb{P}^k = \mathbb{P}(N_{|L_{g-1}|/|L_{g+k}|,X})$ blown-up at the point $\mathbb{P}(N_{|L_{g-1}|/|L_g|,X})$ (the exceptional divisor being identified with $\mathbb{P}(N_{|L_g|/|L_{g+k}|,X})$) and then at the strict transform of the line $\mathbb{P}(N_{|L_{g-1}|/|L_{g+1}|,X})$ (with exceptional divisor given by a $\mathbb{P}(N_{|L_{g+1}|/|L_{g+k}|,X}) = \mathbb{P}^{k-2}$-bundle) and so on until, finally, at the strict transform of $\mathbb{P}(N_{|L_{g-1}|/|L_{g+k-2}|,X})$ (the exceptional divisor over it being a $\mathbb{P}(N_{|L_{g+k-2}|/|L_{g+k}|,X}) = \mathbb{P}^1$-bundle). Hence, Ψ maps $\pi^{-1}(X)$ isomorphically onto $\widetilde{\mathbb{P}^k}$, that is, $\pi^{-1}(X)$ defines a section of $\Psi : \widetilde{|L_{g+k}|} \to \widetilde{\mathbb{P}^k}$. □

3.3. **Connectedness results.** We will now use the map Ψ in (3.3) to obtain a map from the expanded linear system $|L_{g+k}|^{\exp}$ defined in §3.1 to $\widetilde{\mathbb{P}^k}$, a fiber of which will be isomorphic to $|L_g|^{\exp}$.

Proposition 3.3. *The natural projection $\alpha : |L_{g+k}|^{\exp} \to |L_{g+k}|$ factors through a map $\tilde{\alpha} : |L_{g+k}|^{\exp} \to \widetilde{|L_{g+k}|}$. Furthermore, the composition*

$$\Psi \circ \tilde{\alpha} : |L_{g+k}|^{\exp} \to \widetilde{\mathbb{P}^k}$$

has a fiber that is isomorphic to $|L_g|^{\exp}$ and parametrizes curves $X = X_0 \cup X_1 \cup \ldots \cup X_k$ living in some expanded degeneration $S[n]_0$ with $n \geq k$ so that $[X_0] \in |L_g|^{\exp}$ and $X_i \simeq J \in |J_0 + f_{p_{10}(g+i)}|$ for $1 \leq i \leq k$ [4].

Proof. We exploit the family $S[1] \to \mathbb{A}^1$, whose general fibers S_t are isomorphic to S and whose central fiber is $S[1]_0 = S \cup R$. Let $\beta_1 : S[1] \to S$ be the projection and set $L_{g+k}[1] := \beta_1^* L_{g+k}$. We exploit the theory of good degenerations of relative Hilbert schemes introduced and studied in [LW]. Let

$$e : |L_{g+k}[1]|^{\exp} \to \mathbb{A}^1$$

be the moduli stack of the good degeneration of the relative linear system $|L_{g+k}| \times (A^1 \setminus \{0\}) \to A^1 \setminus \{0\}$. Over a point $t \in \setminus\{0\}$, the fiber of e is the linear system $|L_{g+k}|$, while the fiber over 0 is the stack $|L_{g+k}[1]|_0^{\exp}$, parametrizing equivalence classes of curves $X = X_0 \cup X_1 \cup \ldots \cup X_n \cup X_0'$ in some expanded degeneration of $S[1]_0$ (or equivalently, in some expanded degenerations $S[n]_0$ of S with $n \geq 1$) whose image in S lies in $|L_{g+k}|$. By [Li3, Thm. 3.37] (whose analogue for stable maps was proved in [Li1, Li2] and is clearly exposed in [Li4, Lem. 16]), the stack $|L_{g+k}[1]|_0^{\exp}$ admits the following decomposition:

$$|L_{g+k}[1]|_0^{\exp} = \bigcup_{0 \leq g_0 \leq g+k-1} |L_{g_0}|^{\exp} \times |(g+k-g_0)J_0 + f_{p_{10}(g+k)}|^{\exp}, \tag{3.4}$$

[4] we will say that such a curve X has a tail of at least k copies of J

where $|(g+k-g_0)J_0+f_{p_{10}(g+k)}|^{\exp}$ stands for the expansion of the linear system $|(g+k-g_0)J_0+f_{p_{10}(g+k)}|$ relative to the divisor $J_0 \subset R$. In the above decomposition, each factor appears with multiplicity one[5] and defines a Cartier divisor in $|L_{g+k}[1]|^{\exp}$ that we denote by $\mathcal{D}'_{g_0}$. As it happens for $|L_{g+k}|^{\exp}$, points of $|L_{g+k}[1]|^{\exp}$ parametrize curves living in some expansion $S[n]_0$ with $n \leq g+k$ and thus there is a surjection $|L_{g+k}[g+k]|^{Li} \to |L_{g+k}[1]|^{\exp}$; the divisor $\mathcal{D}'_{g_0}$ is the image of a Cartier divisor on $|L_{g+k}[g+k]|^{Li}$, which is $(\mathbb{C}^*)^{g+k}$-equivariant. Its image under the étale map

$$|L_{g+k}[g+k]|^{Li}/(\mathbb{C}^*)^{g+k} \to |L_{g+k}|^{\exp}$$

thus defines a Cartier divisor on $|L_{g+k}|^{\exp}$, that we denote by $\mathcal{D}_{g_0}$. By the definition of $\mathcal{D}'_{g_0}$, if $[X] \in \mathcal{D}_{g_0}$ then $\alpha(X) \in |L_{g_0}| \subset |L_{g+k}|$. Therefore, by Lemma 2.9, for any fixed $0 \leq g' < g+k$ one has the equality

$$\alpha^{-1}(|L_{g'}|) = \bigcup_{0 \leq g_0 \leq g'} \mathcal{D}_{g_0}, \tag{3.5}$$

implying that $\alpha^{-1}(|L_{g'}|)$ is a Cartier divisor on $|L_{g+k}|^{\exp}$. The case $g'=0$ yields, by the universal property of blow-ups [St, 70.17], a factorization of α through the blow-up of $|L_{g+k}|$ along $|L_0|$:

$$\begin{tikzcd} |L_{g+k}|^{\exp} \arrow[r, "\alpha_0"] \arrow[rr, bend right, "\alpha"'] & Bl_{|L_0|}|L_{g+k}| \arrow[r, "b_0"] & |L_{g+k}|. \end{tikzcd}$$

Denoting by $|L_1|^t$ the strict transform of $|L_1|$ in $Bl_{|L_0|}|L_{g+k}|$, we have

$$\alpha^{-1}(|L_1|) = \alpha_0^{-1}(|L_1|^t + E_0) = \alpha_0^{-1}(|L_1|^t) + \alpha^{-1}(|L_0|),$$

and thus conclude that $\alpha_0^{-1}(|L_1|^t)$ is a Cartier divisor as it is difference of two Cartier divisors. Therefore, α_0 factors through the blow-up of $Bl_{|L_0|}|L_{g+k}|$ along $|L_1|^t$. By the same argument, after $g+k-1$ steps one obtains the desired factorization of α through a map $\tilde{\alpha} : |L_{g+k}|^{\exp} \to \widetilde{|L_{g+k}|}$. By construction, one has

$$\tilde{\alpha}^{-1}(E_0) = \alpha^{-1}|L_0| = D_0$$

and then

$$\tilde{\alpha}^{-1}(E_1) = \alpha^{-1}|L_1| - \tilde{\alpha}^{-1}(E_0) = \mathcal{D}_1.$$

Analogously, for every $1 \leq i \leq g+k-2$ one has

$$\tilde{\alpha}^{-1}(E_i) = \alpha^{-1}|L_i| - \sum_{j=0}^{i-1} \tilde{\alpha}^{-1}(E_j) = \mathcal{D}_i.$$

[5]this follows from the fact that $L_g \cdot J = 1$ and thus any curve X in $|L_{g+k}[1]|_0^{\exp}$ intersects each component of the singular locus of $S[n]_0$ at a node that connects two irreducible components of X living in two adjacent components of $S[n]_0$.

Finally, one computes that

$$\tilde{\alpha}^{-1}(\widetilde{H}_1) = \alpha^{-1}|L_{g+k-1}| - \sum_{j=0}^{g+k-2} \tilde{\alpha}^{-1}(E_j) = \mathcal{D}_{g+k-1}. \tag{3.6}$$

We recall that points of $\mathcal{D}_i$ represent curves $X = X_0 \cup X_1$ such that $X_0 \in |L_i|^{\exp}$ and $X_1 \in |(g+k-i)J_0 + f_{p_{10}(g+k)}|^{\exp}$. Hence, points in the intersection $\tilde{\alpha}^{-1}(\widetilde{H}_1) \cap \tilde{\alpha}^{-1}(E_{g+k-2}) = \tilde{\alpha}^{-1}(\widetilde{H}_1 \cap E_{g+k-2}) = \mathcal{D}_{g+k-1} \cap \mathcal{D}_{g+k-2}$ parametrize curves $X_0 \cup X_1 \cup X_2$ such that $X_0 \in |L_{g+k-2}|^{\exp}$, $X_1 \simeq J \in |J_0 + f_{p_{10}(g+k-1)}|$ and $X_2 \simeq J \in |J_0 + f_{p_{10}(g+k)}|$; since X_1 and X_2 are sections of $R \to J$, they are both isomorphic to J and thus X has a tail of at least 2 copies of J. By further intersecting with $\tilde{\alpha}^{-1}(E_{g+k-3}) = \mathcal{D}_{g+k-3}$, we select curves with a tail of at least 3 copies of J, and so on.

In conclusion, the intersection

$$\tilde{\alpha}^{-1}\left(\widetilde{H_1} \cap \left(\bigcap_{i=g}^{g+k-2} E_i\right)\right) = \bigcap_{i=g}^{g+k-1} \mathcal{D}_i \tag{3.7}$$

parametrizes curves $X = X_0 \cup X_1 \cup \ldots \cup X_k$ living in some expanded degeneration $S[m+k]_0 = S[m]_0 \cup R \cup \ldots \cup R$ with $m \geq 0$ so that $[X_0] \in |L_g|^{\exp}$ and $X_i \simeq J \in |J_0 + f_{p_{10}(g+i)}|$ for $1 \leq i \leq k$. For $1 \leq i \leq k$ the class of X_i under the action of $\mathbb{C}^*$ is uniquely determined. In particular, the intersection (3.7), which is a fiber of $\Psi \circ \tilde{\alpha}$ by Lemma 3.2, is isomorphic to $|L_g|^{\exp}$. $\square$

Theorem 3.4. *For every $g \geq 2$ and $0 \leq \delta \leq g-1$, the stack $\overline{\mathcal{V}_\delta(S/J, L_g)}$ is connected. The same holds true for the relative normalization $\overline{\mathcal{V}_\delta(S/J, L_g)}^n$ of $\overline{\mathcal{V}_\delta(S/J, L_g)}$ along $\overline{\mathcal{V}_{\delta+1}(S/J, L_g)}$.*

Proof. Fix $k >> 0$ so that $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}$ is irreducible, and consider the restriction

$$\psi := \Psi \circ \tilde{\alpha}|_{\overline{\mathcal{V}_\delta(S/J, L_{g+k})}} : \overline{\mathcal{V}_\delta(S/J, L_{g+k})} \to \widetilde{\mathbb{P}^k}$$

of $\Psi \circ \tilde{\alpha}$ to the expanded Severi variety $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}$. Let $\xi \in \widetilde{\mathbb{P}^k}$ be the point such that the fiber $(\Psi \circ \tilde{\alpha})^{-1}(\xi)$ parametrizes curves $X = X_0 \cup J \cup \ldots \cup J$ with $X_0 \in |L_g|^{\exp}$ and a tail of k copies of J as in Proposition 3.3. First of all, we show that $\psi^{-1}(\xi)$ parametrizes those curves $X = X_0 \cup J \cup \ldots \cup J$ such that $X_0 \in \overline{\mathcal{V}_\delta(S/J, L_g)}$.

Take a general curve $X = X_0 \cup J \cup \ldots \cup J$ with a tail of k copies of J that lies in $\psi^{-1}(\xi)$. Defining a point in $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}$, the curve X is the limit of a family of integral curves in $|L_{g+k}|$ possessing δ nodes; the limit of any of these nodes is non-disconnecting[6], and thus lies in X_0. The normalization of X outside of its disconnecting nodes thus has arithmetic genus $\leq g+k-\delta$.

[6]A node x of a connected curve X is said to be disconnecting if the normalization of X at x is not connected. In a family of curves, a disconnecting node of the central fiber cannot be the limit of non-disconnecting nodes on general fibers. Indeed, otherwise, the normalization would produce a flat family of curves whose general fiber is connected and

By dimensional reasons using that every component of $\psi^{-1}(\xi)$ has dimension $\geq g-\delta$ and Propositions 2.4 and 3.1, we conclude that X_0 is irreducible and $X_0 \in V_\delta(S, L_g) \subset \overline{\mathcal{V}_\delta(S/J, L_g)}$.

Viceversa, we need to show that if $X_0 \in V_\delta(S, L_g)$, the curve $X = X_0 \cup J \cup \ldots \cup J \subset S[k]_0$ obtained by attaching to X_0 a chain of k copies of J, each defining the only point of $|J_0 + f_{p_{10}(g+i)}|/\mathbb{C}^*$ in the i-th copy of R, lies in $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}$. Let $f_0 : C_0 \to S$ denote the stable map in $\mathcal{M}_{g-\delta}(S, L_g)^{sm}$ whose image is X_0 and take $[f_i] \in \mathcal{M}_1(R, J_0 + f_{p_{10}(g+i)})/\mathbb{C}^*$ for $1 \leq i \leq k$; the map $f = f_0 \cup f_1 \cup \ldots \cup f_k$ is unramified and thus, by dimensional reasons using Propositions 2.4 and 3.1, one checks that $[f] \in \mathcal{M}_{g+k-\delta}(S/J, L_{g+k})^{sm}$. Since X coincides with the image of f, we conclude that $[X] \in \overline{\mathcal{V}_\delta(S/J, L_{g+k})}$. We have thus proved that $\overline{\mathcal{V}_\delta(S/J, L_g)}$ is isomorphic to the fiber $\psi^{-1}(\xi)$.

We claim that ψ admits a section $s_V : V \to \overline{\mathcal{V}_\delta(S/J, L_{g+k})}$ over an open subset $V \subset \widetilde{\mathbb{P}^k}$. Assuming this, we consider the Stein factorization of ψ:

$$\begin{tikzcd} \overline{\mathcal{V}_\delta(S/J, L_{g+k})} \arrow[r, "\psi'"] \arrow[rr, bend right, "\psi"'] & B \arrow[r, "b"] & \widetilde{\mathbb{P}^k}, \end{tikzcd}$$

where b is finite and ψ' has connected fibers. The composition $t_V := \psi' \circ s_V$ is a section of b over V. Denote by $B' \subset B$ the closure of the image of t_V; since B and $\widetilde{\mathbb{P}^k}$ have the same dimension and B is irreducible, we conclude that $B' = B$. Therefore, b is a finite birational morphism and, since $\widetilde{\mathbb{P}^k}$ is normal, Zariski's Main Theorem implies that b is an isomorphism. We conclude that the fibers of ψ are isomorphic to those of ψ', which are all connected; in particular, $\overline{\mathcal{V}_\delta(S/J, L_g)}$ is connected.

It only remains to prove our claim. Take $X_0 \in V_\delta(S, L_{g-1})$ with $X_0 \notin |L_{g-2}|$ and set $X := X_0 + (k+1)J \in |L_{g-1}| \subset |L_{g+k}|$. Recalling that $\alpha = \pi \circ \tilde{\alpha}$ and that $\pi^{-1}(X)$ defines a section of Ψ by Lemma 3.2(ii), to construct a section of ψ over an open subset $V \subset \widetilde{\mathbb{P}^k}$ it is enough to show that $\alpha^{-1}(X) = \tilde{\alpha}^{-1}(\pi^{-1}(X))$ is contained in $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}$ and that the restriction of $\tilde{\alpha}$ to $\alpha^{-1}(X)$ is birational.

By (3.4) and (3.5), the fiber $\alpha^{-1}(X)$ is contained in the divisor $\mathcal{D}_{g-1}$ and is isomorphic to $|(k+1)J_0 + f_{p_{10}(g)}|^{\exp}/\mathbb{C}^*$. Let $f_0 : C_0 \to X_0 \subset S$ denote the stable map in $\mathcal{M}_{g-1-\delta}(S, L_{g-1})^{sm}$ obtained by composing the normalization map of X_0 with the inclusion $X_0 \subset S$. By dimensional reasons using Propositions 2.4 and 3.1, all stable maps $f = f_0 \cup f_1$ with $[f_1] \in \mathcal{M}_{k+1}(R/J_0, (k+1)J_0 + f_{p_{10}(g)})^{sm}/\mathbb{C}^*$ are smoothable. As a consequence, all curves $X_0 \cup X_1$ with $[X_1] \in |(k+1)J_0 + f_{p_{10}(g)}|^{\exp}/\mathbb{C}^*$ define

whose central fiber is not, in contradiction with lower semicontinuity of the number of connected components.

points of $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}$ and thus

$$\overline{\mathcal{V}_\delta(S/J, L_{g+k})} \supset \tilde{\alpha}^{-1}(\pi^{-1}(X)) = \alpha^{-1}(X) \simeq |(k+1)J_0 + f_{p_{10}(g)}|^{\exp}/\mathbb{C}^*.$$

In particular, $\tilde{\alpha}^{-1}(\pi^{-1}(X))$ and $\pi^{-1}(X)$ have both dimension k and hence a general fiber of the restriction $\tilde{\alpha}|_{\pi^{-1}(X)}$ is finite. However, $\tilde{\alpha}$ is birational and $\widetilde{|L_{g+k}|}$ is smooth, so that all fibers of $\tilde{\alpha}$ are connected by Zariski's Main Theorem. We thus conclude that the restriction of $\tilde{\alpha}$ to $\pi^{-1}(X)$ is birational, as desired.

Let now $\psi^n : \overline{\mathcal{V}_\delta(S/J, L_{g+k})}^n \to \widetilde{\mathbb{P}^k}$ denote the composition of the relative normalization $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}^n \to \overline{\mathcal{V}_\delta(S/J, L_{g+k})}$ along $\overline{\mathcal{V}_{\delta+1}(S/J, L_{g+k})}$ with the map ψ. The fiber $(\psi^n)^{-1}(\xi)$ is isomorphic to the relative normalization $\overline{\mathcal{V}_\delta(S/J, L_g)}^n$ of $\overline{\mathcal{V}_\delta(S/J, L_g)}$ along $\overline{\mathcal{V}_{\delta+1}(S/J, L_g)}$; therefore, in order to prove that $\overline{\mathcal{V}_\delta(S/J, L_g)}^n$ remains connected it is enough to show that, up to restricting V, the section $s_V : V \to \overline{\mathcal{V}_\delta(S/J, L_{g+k})}$ constructed above lifts to a section $s_V^n : V \to \overline{\mathcal{V}_\delta(S/J, L_{g+k})}^n$. This holds true because a general element of $\alpha^{-1}(X)$ does not lie in $\overline{\mathcal{V}_{\delta+1}(S/J, L_g)}$, and thus $\alpha^{-1}(X)$ is birational to its inverse image in $\overline{\mathcal{V}_\delta(S/J, L_{g+k})}^n$. □

Since the image of the expanded Severi variety $\overline{\mathcal{V}_\delta(S/J, L_g)}$ under the projection $|L_g|^{\exp} \to |L_g|$ coincides with $\overline{V_\delta(S, L_g)}$, we obtain the following corollary of independent interest.

Corollary 3.5. *For every $g \geq 2$ and $0 \leq \delta \leq g-1$, the closure of the Severi variety $\overline{V_\delta(S, L_g)} \subset |L_g|$ is connected.*

In the next section, we will make use of the following result.

Proposition 3.6. *Fix $g \geq 2$ and $0 \leq \delta \leq g-1$. Let $\mathcal{V}$ and $\mathcal{W}$ be two intersecting irreducible component of $\overline{\mathcal{V}_\delta(S/J, L_g)}^n$ and let $\mathcal{Z}$ be a component of their intersection; then a general closed point of $\mathcal{Z}$ is a reduced point of $\overline{\mathcal{V}_\delta(S/J, L_g)}^n$.*

Proof. Take $\mathcal{Z}$ as in the statement. Since normal singularities are unibranched (cf., e.g., [Ko2]), then $\mathcal{Z}$ is not contained in the inverse image of $\overline{\mathcal{V}_{\delta+1}(S/J, L_g)}$ under the normalization map $\overline{\mathcal{V}_\delta(S/J, L_g)}^n \to \overline{\mathcal{V}_\delta(S/J, L_g)}$.

We first assume that a general point ζ of $\mathcal{Z}$ parametrizes an irreducible curve, so that locally around ζ the morphism

$$\overline{\mathcal{V}_\delta(S/J, L_g)} \to \overline{V_\delta(S, L_g)}$$

(obtained as restriction of $|L_g|^{\exp} \to |L_g|$) is an isomorphism; the result then follows from Proposition 2.7.

We now treat the case where a general point ζ of $\mathcal{Z}$ parametrizes a curve $X = X_0 \cup \ldots \cup X_n \in S[n]_0$ for some $n \geq 1$. More precisely, there exist $\underline{h} = (h_0, \ldots, h_n) \in \mathbb{Z}^{n+1}$ and $\underline{g} = (g_0, \ldots, g_n) \in \mathbb{Z}^{n+1}$ with $\sum_{i=0}^n h_i = g - \delta$,

$\sum_{i=0}^{n} g_i = g$, $0 \leq h_0 \leq g_0$ and $1 \leq h_i \leq g_i$ for $i > 0$, such that $\mathcal{Z}$ is contained in the substack $\mathcal{V}(\underline{h}, \underline{g})$ of $\overline{\mathcal{V}_\delta(S/J, L_g)}$ parametrizing curves

$$X = X_0 \cup X_1 \cup \ldots \cup X_n \subset S \cup R \cup \ldots \cup R = S[n]_0$$

such that X_i has arithmetic genus g_i and geometric genus h_i. The closure $\overline{\mathcal{V}(\underline{h}, \underline{g})}$ of $\mathcal{V}(\underline{h}, \underline{g})$ in $|L_g|^{\exp}$ fills up some components of the intersection of $\overline{\mathcal{V}_\delta(S/J, L_g)}$ with the n Cartier divisors $\mathcal{D}_{g_0}, \mathcal{D}_{g_0+g_1}, \ldots, \mathcal{D}_{g_0+\cdots+g_{n-1}}$ defined in the proof of Proposition 3.3. More precisely, $\mathcal{V}(\underline{h}, \underline{g})$ is isomorphic to the open substack

$$U \subset \overline{V^{h_0}(S, L_{g_0})} \times \left[\overline{V^{h_1}(R, N_1(\underline{g}))}/\mathbb{C}^*\right] \times \cdots \times \left[\overline{V^{h_n}(R, N_n(\underline{g}))}/\mathbb{C}^*\right]$$

parametrizing curves with no components in the singular locus of $S[n]_0$. Propositions 2.4 and 3.1 then imply that $\mathcal{V}(\underline{h}, \underline{g})$ is a complete intersection in $\overline{\mathcal{V}_\delta(S/J, L_g)}$ and has codimension n therein. In particular, if $\mathcal{V}(\underline{h}, \underline{g})$ is reduced at a general point of $\mathcal{Z}$, then[7] the same holds true for $\overline{\mathcal{V}_\delta(S/J, L_g)}$. Again by Propositions 2.4 and 3.1, $\mathcal{V}(\underline{h}, \underline{g})$ is generically reduced and we thus conclude that $\overline{\mathcal{V}_\delta(S/J, L_g)}$ is reduced at a general point of $\mathcal{Z}$ if $\mathrm{codim}\mathcal{Z} = n$ (that is, when $\mathcal{Z}$ fills up a component of $\overline{\mathcal{V}(\underline{h}, \underline{g})}$). If instead $\mathrm{codim}\mathcal{Z} > n$, then $\mathcal{V}' := \mathcal{V} \cap \mathcal{V}(\underline{h}, \underline{g})$ and $\mathcal{W}' := \mathcal{W} \cap \mathcal{V}(\underline{h}, \underline{g})$ are unions of components of $\mathcal{V}(\underline{h}, \underline{g})$ and $\mathcal{Z}$ is a component of the intersection $\mathcal{V}' \cap \mathcal{W}'$. Hence, $\mathcal{V}'$ and $\mathcal{W}'$ can be identified with open substacks of two products $\mathcal{V}'_0 \times [\mathcal{V}'_1/\mathbb{C}^*] \times \cdots [\mathcal{V}'_n/\mathbb{C}^*]$ and $\mathcal{W}'_0 \times [\mathcal{W}'_1/\mathbb{C}^*] \times \cdots [\mathcal{W}'_n/\mathbb{C}^*]$, where $\mathcal{V}'_0, \mathcal{W}'_0$ are components of $\overline{V^{h_0}(S, L_{g_0})}$, while $\mathcal{V}'_i, \mathcal{W}'_i$ are components of $\overline{V^{h_i}(R, N_i(\underline{g}))}$ for $i \geq 1$. Hence, $\mathcal{Z}$ can be identified with an open subset of an irreducible component of

$$(\mathcal{V}'_0 \cap \mathcal{W}'_0) \times [\mathcal{V}'_1 \cap \mathcal{W}'_1/\mathbb{C}^*] \times \cdots [\mathcal{V}'_n \cap \mathcal{W}'_n/\mathbb{C}^*];$$

again Propositions 2.7 and 3.1 (and the fact that a categorical quotient of a generically reduced object is still generically reduced by the universal property) imply that $\mathcal{V}(\underline{h}, \underline{g})$ is reduced at a general closed point of $\mathcal{Z}$, and thus the same holds true for $\overline{\mathcal{V}_\delta(S/J, L_g)}$. □

4. Connectedness on a general $K3$ surface

In this section we will show how Theorem 3.4 implies connectedness of positive dimensional Severi varieties on a general polarized $K3$ surface.

Let S and S' be two surfaces both obtained as blow-ups of $\mathbb{P}^2$ at two 9-uples of general points, $p_1, \ldots, p_9$ and $p'_1, \ldots, p'_9$, respectively. We assume that the anticanonical divisors on S and S' are represented by the same

[7] it is enough to use that, given a commutative ring R, a prime ideal $p \subset R$ and an element $x \in p$ which is not a zero divisor of R, if the quotient ring $(R/(x))_{p/(x)} \simeq R_p/xR_p$ has no nilpotents, then the same holds true for R_p. This can be proved as follows: if $r \in R_p$ is nilpotent, then $r = xy \in xR_p$ and y itself is nilpotent; this shows that the Nilradical $\mathcal{N}(R_p)$ of R_p satisfies $\mathcal{N}(R_p) = x\mathcal{N}(R_p)$ and thus $\mathcal{N}(R_p) = (0)$ by Nakayama's Lemma.

elliptic curve J and that the relation $N_{J/S} \simeq N^{\vee}_{J/S'}$ holds. We glue S and S' along J so that $p'_9 = p_{10}(g)$. Since $N_{J/S} \simeq \mathcal{O}_J(p_{10}(h) - p_{10}(h-1))$ for every $h \geq 0$, and the same holds for S', our assumptions yield $p_{10}(h) = p_{10}(g-h)'$ for every $0 \leq h \leq g$. In particular, all the pairs $(L_h, L'_{g-h}) \in \mathrm{Pic}(S) \times \mathrm{Pic}(S')$ define equivalent polarizations on $Y_0 := S \cup_J S'$, in the sense that they differ only by the twist for a multiple of $(N_{J/S}, N_{J/S'})$; we set $L := [(L_h, L'_{g-h})]$. The surface Y_0 is a stable $K3$ surface of type II and thus occurs as the central fiber of a flat map

$$\chi : \mathcal{Y} \to \mathbb{D}$$

over a disc whose general fiber Y_t is a smooth $K3$ surface of genus g (cf. [Fr, Prop. 2.5, Thm. 5.10]). Furthermore, for every $0 \leq h \leq g$ the family $\mathcal{Y}$ comes equipped with a relative line bundle $\mathcal{L}(h)$ restricting to the genus g polarization L_t on a general fiber Y_t and to the polarization (L_h, L'_{g-h}) on Y_0. Since the line bundles $\mathcal{L}(h)$ only differ by a twist for a multiple of some component of the central fiber, they are all equivalent and we call $\mathcal{L}$ their class. We remark that this degeneration in the particular case where $S = S'$ and the points $p_1, \ldots, p_9 \in \mathbb{P}^2$ are the base locus of a general pencil of plane cubics is the one used in [MPT, §4].

We denote by

$$\chi_\delta : \mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L}) \to \mathbb{D}$$

the moduli stack of connected stable maps to expanded degenerations of χ constructed in [Li1, Li2]. Over a point $t \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}$, the fiber of χ_δ is simply the moduli stack $\mathcal{M}_{g-\delta}(Y_t, L_t)$ of ordinary stable maps on Y_t, while the fiber over 0 is the stack $\mathcal{M}_{g-\delta}(Y_0^{\exp}, L)$ parametrizing stable maps (in the sense of J. Li) to some expanded target degeneration of Y_0 of the form:

$$Y_0[n]_0 := S \cup_J R \cup_J \ldots \cup_J R \cup_J S'$$

for some $n \geq 0$. As already used in [MPT], a stable map to such an expanded degeneration can be split in a non-unique way into relative stable maps to (S, J) and (S', J). In particular, $\mathcal{M}_{g-\delta}(Y_0^{\exp}, L)$ can be written as a non-disjoint union

$$\mathcal{M}_{g-\delta}(Y_0^{\exp}, L) = \bigcup_{\substack{g_1+g_2=g \\ h_1+h_2=g-\delta}} \mathcal{M}_{h_1}(S/J, L_{g_1}) \times \mathcal{M}_{h_2}(S'/J, L'_{g_2}), \tag{4.1}$$

where each factor in the above decomposition can be realized as a Cartier divisor on $\mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L})$. Let

$$\mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L})^{sm} \to \mathbb{D}$$

be the substack of $\mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L})$ whose fiber over $t \in \mathbb{D}^*$ is the substack $\mathcal{M}_{g-\delta}(Y_t, L_t)^{sm}$ of $\mathcal{M}_{g-\delta}(Y_t, L_t)$ parametrizing smoothable stable maps, and set $\mathcal{M}_{g-\delta}(Y_0^{\exp}, L)^{sm} := \mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L})^{sm} \times_{\mathbb{D}} 0$.

Similarly, we denote by

$$e : |\mathcal{L}|^{\exp} \to \mathbb{D}$$

the good degeneration of the relative linear system $|\mathcal{L}|^* \to \mathbb{D}^*$, which is a particular case of good degenerations of relative Hilbert schemes introduced and studied in [LW]. The space $|\mathcal{L}|^{\exp}$ is a Deligne-Mumford stack, separated, proper over $\mathbb{D}$ and of finite type. A fiber of e over a general $t \in \mathbb{D}$ is isomorphic to the linear system $|L_t|$ on Y_t. On the other hand, points of the central fiber parametrize equivalence classes of curves $X = X_0 \cup X_1 \cup \ldots \cup X_n \cup X_0'$ in some expanded target degenerations $Y_0[n]_0$ of Y_0 with no components in the singular locus of $Y_0[n]_0$. More precisely, by [LW, Thm. 5.27] the central fiber can be decomposed in the following non-disjoint union of Cartier divisors

$$\bigcup_{g_1+g_2=g} |L_{g_1}|^{\exp} \times |L'_{g_2}|^{\exp}. \tag{4.2}$$

Since Severi varieties may be defined functorially, for any fixed $0 \leq \delta \leq g$ there is a χ-relative Severi variety $s_\delta : \mathcal{V}_\delta(\mathcal{Y}, \mathcal{L})^* \to \mathbb{D}^*$ such that the fiber over $t \in \mathbb{D}^*$ is the Severi variety $V_\delta(Y_t, L_t)$. We denote by $\overline{\mathcal{V}_\delta(\mathcal{Y}^{\exp}, \mathcal{L})}$ the closure of $\mathcal{V}_\delta(\mathcal{Y}, \mathcal{L})^*$ in $|\mathcal{L}|^{\exp}$ and by

$$\overline{s}_\delta : \overline{\mathcal{V}_\delta(\mathcal{Y}^{\exp}, \mathcal{L})} \to \mathbb{D}$$

the natural morphism. The stack $\overline{\mathcal{V}_\delta(\mathcal{Y}^{\exp}, \mathcal{L})}$ can be alternatively realized as the image of the natural map from the semi-normalization of $\mathcal{M}_{g-\delta}(\mathcal{Y}^{\exp}, \mathcal{L})^{sm}$ to $|\mathcal{L}|^{\exp}$. The analogue of the decomposition (4.1) for the central fiber $\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)} := \overline{\mathcal{V}_\delta(\mathcal{Y}^{\exp}, \mathcal{L})} \times_{\mathbb{D}} 0$ of $\overline{s}_\delta$ is then stated in the following result.

Lemma 4.1. *The stack $\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)}$ decomposes in the following non-disjoint union:*

$$\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)} = \bigcup_{\substack{g_1+g_2=g \\ \delta_1+\delta_2=\delta}} \overline{\mathcal{V}_{\delta_1}(S/J, L_{g_1})} \times \overline{\mathcal{V}_{\delta_2}(S'/J, L'_{g_2})}. \tag{4.3}$$

Each factor in the decomposition (4.3) *appears with multiplicity* 1 *and defines a Cartier divisor in $\overline{\mathcal{V}_\delta(\mathcal{Y}^{\exp}, \mathcal{L})}$.*

Proof. As concerns the inclusion $\supset$, we recall that a general point in any irreducible component of $\overline{\mathcal{V}_{\delta_1}(S/J, L_{g_1})}$ (respectively, $\overline{\mathcal{V}_{\delta_2}(S'/J, L'_{g_2})}$) parametrizes an irreducible curve $C \in V_{\delta_1}(S, L_{g_1})$ (respectively, $C' \in V_{\delta_2}(S', L'_{g_2})$). By gluing C and C' along their intersection point $p_{10}(g_1) = p_{10}(g_2)'$ with J, one obtains a curve $X = C \cup C' \subset Y_0 = S \cup S'$ with $\delta_1 + \delta_2$ nodes outside of X; there is no obstruction to deforming such an X outside of the central fiber of $\overline{s}_\delta$ and this proves $\supset$.

We now prove the opposite inclusion $\subset$. Let $\mathcal{V}$ be a component of the central fiber $\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)}$ of $\overline{s}_\delta$. Then $\mathcal{V}$ has dimension $g - \delta$ by upper semicontinuity along with the fact that $\mathbb{D}$ is 1-dimensional. A general point of $\mathcal{V}$ parametrizes a curve $X \subset Y_0[n]_0$ for some $n \geq 0$ such that $X \in |\mathcal{L}|^{\exp}$ and the normalization of X outside of the singular locus of $Y_0[n]_0$ has arithmetic genus $h \leq g - \delta$. Propositions 2.4 and 3.1 then yield $n = 0$ and

$h = g - \delta$, so that $X = C \cup C'$ with $C \in \overline{V^{g_1-\delta_1}(S, L_{g_1})} = \overline{V_{\delta_1}(S, L_{g_1})}$ and $C' \in \overline{V^{g_2-\delta_2}(S', L_{g_2})} = \overline{V_{\delta_2}(S', L'_{g_2})}$ for some integers $0 \leq \delta_1 \leq g_1$, $0 \leq \delta_2 \leq g_2$ such that $g_1 + g_2 = g$ and $\delta_1 + \delta_2 = \delta$. This proves $\subset$.

The last part of the statement is a consequence of the same property for the decompositions (4.1) and (4.2), that follow from [Li2] and [LW, Thm. 5.27] since $L_h \cdot J = 1$ for every $h \geq 0$. □

Proposition 4.2. *If $0 \leq \delta \leq g-1$, every component $\mathcal{V}$ of $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}$ can be connected to some component $\mathcal{V}_r$ of $|L_0| \times \overline{\mathcal{V}_\delta(S'/J, L'_g)}$ through a sequence of irreducible components*

$$\mathcal{V} = \mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_r$$

such that for all $0 \leq i \leq r-1$ the following hold:

(i) *the intersection $\mathcal{V}_i \cap \mathcal{V}_{i+1}$ has codimension 1;*

(ii) *a general point of $\mathcal{V}_i \cap \mathcal{V}_{i+1}$ parametrizes a curve*

$$X = X_0 \cup X_1 \cup X_0' \subset Y_0[1]_0$$

such that the components $X_0 \subset S$ and $X_0' \subset S'$ are nodal, the component $X_1 \subset R$ is immersed, the normalization of X outside of its intersection points with the singular locus of $Y_0[1]_0$ has arithmetic genus precisely $g - \delta$.

In particular, $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}$ is reduced at a general closed point of any irreducible component of $\mathcal{V}_i \cap \mathcal{V}_{i+1}$.

Proof. By Lemma 4.1, there exist integers $0 \leq \delta_1 \leq g_1$, $0 \leq \delta_2 \leq g_2$ such that $g_1 + g_2 = g$ and $\delta_1 + \delta_2 = \delta$ and $\mathcal{V}_0 := \mathcal{V} = \mathcal{W}_0 \times \mathcal{W}_0'$ for some irreducible components $\mathcal{W}_0$ of $\overline{\mathcal{V}_{\delta_1}(S/J, L_{g_1})}$ and $\mathcal{W}_0'$ of $\overline{\mathcal{V}_{\delta_2}(S'/J, L'_{g_2})}$.

If $g_1 - \delta_1 \geq 1$, then $\mathcal{W}_0$ contains in codimension 1 points that parametrize curves $C = C_0 \cup C_1 \subset S[1]_0$ with $C_0 \in V_{\delta_1}(S, L_{g_1-1})$ and $C_1 \simeq J \in |J_0 + f_{p_{10}(g_1)}|$. For any $C_0' \in V_{\delta_2}(S', L'_{g_2})$ the nodal curve $C := C_0 \cup C_1 \cup C_0'$ also defines a point of $\overline{\mathcal{V}_{\delta_1}(S/J, L_{g_1-1})} \times \overline{\mathcal{V}_{\delta_2}(S'/J, L'_{g_2+1})}$ and this proves that $\mathcal{V}_0 := \mathcal{W}_0 \times \mathcal{W}_0'$ can be connected to a component

$$\mathcal{V}_1 = \mathcal{W}_1 \times \mathcal{W}_1' \subset \overline{\mathcal{V}_{\delta_1}(S/J, L_{g_1-1})} \times \overline{\mathcal{V}_{\delta_2}(S'/J, L'_{g_2+1})}$$

so that the intersection $\mathcal{V}_0 \cap \mathcal{V}_1$ satisfies conditions (i)-(ii) in the statement. By repeating the same argument $g_1 - \delta_1$ times, we find a sequence

$$\mathcal{V} = \mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_{g_1-\delta_1}$$

of irreducible components of $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}$ such that $\mathcal{V}_i = \mathcal{W}_i \times \mathcal{W}_i'$ is an irreducible component of $\overline{\mathcal{V}_{\delta_1}(S/J, L_{g_1-i})} \times \overline{\mathcal{V}_{\delta_2}(S'/J, L'_{g_2+i})}$ and that conditions (i)-(ii) in the statement hold for the intersection $\mathcal{V}_i \cap \mathcal{V}_{i+1}$.

We now start from

$$\mathcal{V}_{g_1-\delta_1} = \mathcal{W}_{g_1-\delta_1} \times \mathcal{W}'_{g_1-\delta_1} \subset \overline{\mathcal{V}_{\delta_1}(S/J, L_{\delta_1})} \times \overline{\mathcal{V}_{\delta_2}(S'/J, L'_{g-\delta_1})}$$

and notice that $\mathcal{W}'_{g_1-\delta_1} \subset \overline{\mathcal{V}_{\delta_2}(S'/J, L'_{g-\delta_1})}$ contains in codimension 1 points that parametrize curves $D = \tilde{J} \cup D'_0 \subset S[1]'_0$, where $D'_0 \in |L'_{g-\delta-1}|$ and $\tilde{J}$ is an irreducible elliptic curve such that $\tilde{J} \in V^1(R, (\delta_2+1)J_0 + f_{p_{10}(\delta+1)})$. For every rational curve $D_0 \in V_{\delta_1}(S, L_{\delta_1})$ the curve $D := D_0 \cup \tilde{J} \cup D'_0$ also defines a point of $\overline{\mathcal{V}_\delta(S/J, L_{\delta+1})} \times |L'_{g-\delta-1}|^{\mathrm{exp}}$; this proves that $\mathcal{V}_{g_1-\delta_1}$ can be connected to a component

$$\mathcal{V}_{g_1-\delta_1+1} := \mathcal{W}_{g_1-\delta_1+1} \times |L'_{g-\delta-1}|^{\mathrm{exp}} \subset \overline{\mathcal{V}_\delta(S/J, L_{\delta+1})} \times |L'_{g-\delta-1}|^{\mathrm{exp}}$$

so that (i) and (ii) hold.

Finally, we use the fact that the component $\mathcal{W}_{g_1-\delta_1+1} \subset \overline{\mathcal{V}_\delta(S/J, L_{\delta+1})}$ contains in codimension 1 curves of the form $E_9 + \overline{J}$, where E_9 is the ninth exceptional divisor on S (and thus the only curve in the linear system $|L_0|$) and $\overline{J}$ is an irreducible curve such that $\overline{J} \in V^1(R, (\delta+1)J_0 + f_{p_{10}(\delta+1)})$. For any curve $F'_0 \in |L'_{g-\delta-1}|$, the divisor $E_9 + \overline{J} + F'_0 \in Y_0[1]_0$ also defines a point of $|L_0| \times \overline{\mathcal{V}_\delta(S'/J, L'_g)}$. This proves that $\mathcal{V}_{g_1-\delta_1+1}$ is connected to a component $\mathcal{V}_{g_1-\delta_1+2}$ of $|L_0| \times \overline{\mathcal{V}_\delta(S'/J, L'_g)}$ so that (i)-(ii) hold for $\mathcal{V}_{g_1-\delta_1+1} \cap \mathcal{V}_{g_1-\delta_1+2}$.

It only remains to prove that conditions (i)-(ii) imply $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}$ is reduced at a general point the intersection $\mathcal{V}_i \cap \mathcal{V}_{i+1}$. Conditions (i)-(ii) together with Propositions 2.4 and 3.1 yield that every component of $\mathcal{V}_i \cap \mathcal{V}_{i+1}$ is birational to an open substack of

$$\overline{V_{\delta_0}(S, L_{g_0})} \times \left[\overline{V^{h_1}(R, g_1 J_0 + f_{p_{10}(g_0+g_1)})}/\mathbb{C}^*\right] \times \overline{V_{\delta'_0}(S', L'_{g'_0})}$$

for some integers $0 \le \delta_0 \le g_0$, $0 \le \delta'_0 \le g'_0$, $1 \le h_1 \le g_1$ such that $g_0 + g_1 + g'_0 = g$ and $\delta_0 + g_1 - h_1 + \delta'_0 = \delta$. Reducedness of $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}$ at general points of $\mathcal{V}_i \cap \mathcal{V}_{i+1}$ thus follows from Propositions 2.4 and 3.1 (using the same argument explained in footnote [7]). □

Theorem 4.3. *Let (Y, L) be a general primitively polarized $K3$ surface of genus $g \ge 2$ and fix $0 \le \delta \le g-1$. Then the closure of the Severi variety $\overline{V_\delta(Y, L)} \subset |L|$ is connected and the same holds true for the relative normalization $\overline{V_\delta(Y, L)}^n$ of $\overline{V_\delta(Y, L)}$ along $\overline{V_{\delta+1}(Y, L)}$.*

Proof. Let $\overline{s}_\delta : \overline{\mathcal{V}_\delta(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})} \to \mathbb{D}$ be the good degeneration of the relative Severi variety to the family $\chi : \mathcal{Y} \to \mathbb{D}$ as at the beginning of this section. We denote by $s_\delta^n : \overline{\mathcal{V}_\delta(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})}^n \to \mathbb{D}$ the relative normalization of $\overline{\mathcal{V}_\delta(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})}$ along $\overline{\mathcal{V}_{\delta+1}(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})}$. In particular, a general fiber of s_δ^n is the normalization of $\overline{V_\delta(Y_t, L_t)}$ along $\overline{V_{\delta+1}(Y_t, L_t)}$, while the central fiber is the normalization $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}^n$ of $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}$ along $\overline{\mathcal{V}_{\delta+1}(Y_0^{\mathrm{exp}}, L)}$. We need to show that a general fiber of s_δ^n is connected.

First of all, we note that the central fiber $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}$ of $\overline{s}_\delta$ is generically reduced. Indeed, this follows directly from Lemma 4.1 and Proposition 2.4(iii). Obviously, the same holds true for the central fiber of s_δ^n, thus implying that two components of a general fiber of s^n_δ remain distinct also on the central fiber.

By Proposition 4.2, every component of the central fiber $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}$ of $\overline{s}_\delta$ can be connected to $|L_0| \times \overline{\mathcal{V}_\delta(S'/J, L'_g)}$ through a sequence of irreducible components $\mathcal{V} = \mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_r$ with $\mathcal{V}_r \subset |L_0| \times \overline{\mathcal{V}_\delta(S'/J, L'_g)}$ such that for all $0 \leq i \leq r-1$ the stack $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}$ is reduced at a general point of any component of $\mathcal{V}_i \cap \mathcal{V}_{i+1}$. Again by Proposition 4.2, no component of the latter intersection is contained in $\overline{\mathcal{V}_{\delta+1}(Y_0^{\mathrm{exp}}, L)}$ and we conclude that also every component $\mathcal{V}^n$ of the normalization $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}^n$ can be connected to $|L_0| \times \overline{\mathcal{V}_\delta(S'/J, L'_g)}^n$ through a sequence of irreducible components $\mathcal{V}^n = \mathcal{V}^n_0, \mathcal{V}^n_1, \ldots, \mathcal{V}^n_r$ with $\mathcal{V}^n_r \subset |L_0| \times \overline{\mathcal{V}_\delta(S'/J, L'_g)}^n$ such that $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}^n$ is reduced at a general closed point of any component of the intersection $\mathcal{V}^n_i \cap \mathcal{V}^n_{i+1}$. Since $|L_0|$ is a point, Proposition 3.6 implies that $|L_0| \times \overline{\mathcal{V}_\delta(S'/J, L'_g)}^n$ is connected and that $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}^n$ is reduced at general closed points of the intersection of two irreducible components of $|L_0| \times \overline{\mathcal{V}_\delta(S'/J, L'_g)}^n$.

We use this in order to conclude that a general fiber of s^n_δ is connected, too. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two connected components of $\overline{\mathcal{V}_\delta(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})}^n$ and, by contradiction, assume that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is contained in the central fiber of s^n_δ; in particular, $\mathcal{Z}_1 \cap \mathcal{Z}_2$ has codimension ≥ 2 in $\overline{\mathcal{V}_\delta(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})}^n$. By the above discussion, we may also assume that $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}^n$ is reduced at general closed points of $\mathcal{Z}_1 \cap \mathcal{Z}_2$. Let $\tau : \mathcal{V}_\delta(\widetilde{\mathcal{Y}^{\mathrm{exp}}}, \mathcal{L}) \to \overline{\mathcal{V}_\delta(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})}^n$ be the normalization of $\overline{\mathcal{V}_\delta(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})}^n$. By Serre's criterion for normality[8] [Ma, Thm. 23.8], $\mathcal{V}_\delta(\widetilde{\mathcal{Y}^{\mathrm{exp}}}, \mathcal{L})$ satisfies Serre's condition (S_2) and thus the central fiber of $\tau \circ s^n_\delta$, being a Cartier divisor, satisfies Serre's condition (S_1) (that is, it has no embedded points). Since generic reducedness along with condition (S_1) is equivalent to reducedness [Ma, p. 183] and the central fiber of s^n_δ is generically reduced, we conclude that the central fiber of $\tau \circ s^n_\delta$ is reduced. By [St, 76.36.8], the number of its connected components is thus the same as the number of connected components of a general fiber of $\tau \circ s^n_\delta$. This implies that the normalization τ has separated $\mathcal{Z}_1$ and $\mathcal{Z}_2$ and thus $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is contained in the non-normal locus of $\overline{\mathcal{V}_\delta(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})}^n$. Since $\mathcal{Z}_1 \cap \mathcal{Z}_2$ has codimension ≥ 2, again by Serre's criterion for normality we conclude that $\overline{\mathcal{V}_\delta(\mathcal{Y}^{\mathrm{exp}}, \mathcal{L})}^n$ is not (S_2) at a general point of $\mathcal{Z}_1 \cap \mathcal{Z}_2$. Hence[9], the central fiber $\overline{\mathcal{V}_\delta(Y_0^{\mathrm{exp}}, L)}^n$ of s^n_δ, which is a Cartier divisor containing $\mathcal{Z}_1 \cap \mathcal{Z}_2$, is not (S_1) at a general point of $\mathcal{Z}_1 \cap \mathcal{Z}_2$. Since this contradicts the assumption of

[8]because normality and reducedness of an algebraic stack $\mathbb{T}$ can be checked on an atlas T, we will say that $\mathbb{T}$ is reduced/ normal/ satisfies Serre's condition (S_k) if T does.

[9]it is enough to use that, if R is a commutative ring and $x \in R$ is not a zero divisor, then for every prime ideal $p \subset R$ containing x, one has $(R/(x))_{p/(x)} \simeq R_p/xR_p$ and $\mathrm{depth} R_p/xR_p = \mathrm{depth} R_p - 1$.

$\overline{\mathcal{V}_\delta(Y_0^{\exp}, L)}^n$ being reduced at a general point of $\mathcal{Z}_1 \cap \mathcal{Z}_2$, we conclude that a general fiber of s_δ^n is connected. □

5. Irreducibility on a general $K3$ surface

In this section, we will focus on the Severi problem for general polarized $K3$ surfaces. As we have already proved that Severi varieties of positive dimension are connected, the irreducibility problem can be approached by investigating how two irreducible components may intersect.

Proposition 5.1. *Let (Y, L) be a general primitively polarized $K3$ surface of genus $g \geq 2$ and fix $0 \leq \delta \leq g-1$. The intersection of two irreducible components of $\overline{V_\delta(Y, L)}$, if nonempty, has pure codimension 1.*

Proof. Since (Y, L) is general, we may assume that all curves in $|L|$ are integral and that $\overline{V^\delta(Y, L)} = \overline{V_{g-\delta}(Y, L)}$. The proof proceeds as the one of Proposition 2.7 and is actually easier because all curves in $|L|$ are integral. In this case $I = D(\widetilde{\phi})$ and the same proof as that of Lemma 2.8 implies that the locus in I where the fibers of the projection $t : I \longrightarrow |L|$ have positive dimension has dimension $\leq g - \delta - 2$. □

Theorem 5.2. *Let (Y, L) be a general primitively polarized $K3$ surface of genus $g \geq 4$ and fix $0 \leq \delta \leq g-4$. Then, the Severi variety $V_\delta(Y, L)$ is irreducible.*

Proof. Since (Y, L) is general, we may assume that all curves in $|L|$ are integral. By Theorem 4.3, $\overline{V_\delta(Y, L)}$ and its relative normalization $\overline{V_\delta(Y, L)}^n$ along $\overline{V_{\delta+1}(Y, L)}$ are connected. If reducible, $\overline{V_\delta(Y, L)}$ contains two irreducible components V, W whose intersection $V \cap W$ is nonempty and thus of codimension 1 by Proposition 5.1. Since $\overline{V_\delta(Y, L)}^n$ is still connected, we may further assume that $V \cap W$ is not contained in $\overline{V_{\delta+1}(Y, L)}$. It is therefore enough to show that no codimension 1 component of the singular locus of $\overline{V_\delta(Y, L)}$ may contain such an intersection.

Let Z be a component of $\operatorname{Sing}\overline{V_\delta(Y, L)}$ such that Z is not contained in $\overline{V_{\delta+1}(Y, L)}$ and Z has codimension 1. Let $C \in Z$ be a general point and denote by f the composition of the normalization map $\nu : \tilde{C} \to C$ with the inclusion of C in Y. Since Z has dimension $g - \delta - 1$, Theorem 1.8 and Proposition 1.5 imply that $\tilde{C}$ is a smooth irreducible curve of genus either $g - \delta$ or $g - \delta - 1$, and the latter case does not occur because Z is not contained in $\overline{V_{\delta+1}(Y, L)}$.

Hence, $\tilde{C}$ has genus $g - \delta$ and, by generality, we can assume that all points in a dense open subset of Z parametrize curves with the same singularities as C. We may thus apply [AC, p. 26] as in the proof of Proposition 2.4 to obtain

$$g - \delta - 1 = \dim Z \leq h^0(\overline{N}_f), \tag{5.1}$$

where $\overline{N}_f \simeq \omega_{\tilde{C}}(-R)$ with R being the ramification divisor of f. Inequality (5.1) then yields $\deg R \leq 2$.

If $\deg R = 0$, then f is unramified and $N_f = \overline{N}_f = \omega_{\tilde{C}}$, If $\deg R = 1$, then C has only one ordinary cusp and, denoting by Q the point of $\tilde{C}$ mapping to it, we have $N_f = \omega_{\tilde{C}}(-Q) \oplus \mathcal{O}_Q$. In both cases one has $h^0(N_f) = g - \delta$ and thus f defines a smooth point of the moduli space of genus $g-\delta$ stable maps $M_{g-\delta}(Y, L)$, and more precisely of the locus $M_{g-\delta}(Y, L)^{\mathrm{sm}}$ parametrizing smoothable stable maps. Let μ be the morphism from the semi-normalization of $M_{g-\delta}(Y, L)^{\mathrm{sm}}$ to $\overline{V^{g-\delta}(Y, L)} = \overline{V_\delta(Y, L)}$. Locally around f the morphism μ is injective: indeed, the inverse image under μ of an irreducible curve of geometric genus exactly $g-\delta$ is the only point defined by the composition of its normalization map with its inclusion in S. Therefore, the smoothness of $M_{g-\delta}(Y, L)^{\mathrm{sm}}$ at f yields that $\overline{V_\delta(S, L)}$, if singular at C, has a unibranched singularity there. In particular, W cannot lie in the intersection of two irreducible components of $\overline{V_\delta(S, L)}$.

We now treat the remaining case $\deg R = 2$, where (5.1) implies that $\tilde{C}$ is hyperelliptic and R is a divisor in the g^1_2. By [KLM, Rmk. 5.6], curves in $|L|$ with hyperelliptic normalization of any fixed geometric genus ≥ 2 move in dimension 2 and hence $g - \delta - 1 = \dim W \leq 2$ yielding a contradiction as soon as $\delta \leq g - 4$. □

References

[AB] E. Arbarello, A. Bruno, *Rank two vector bundles on polarised Halphen surfaces and the Gauss-Wahl map for du Val curves*, J. Éc. Polytech. Math **4** (2017), 257–285.

[AC] E. Arbarello, M. Cornalba, *Su una congettura di Petri*, Comment. Math. Helv. **56** (1981), 1–38.

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, *Geometry of algebraic curves*, Grundl. Math. Wiss. 267, Springer Verlag, 1985.

[AF] M. Aprodu, G. Farkas, *The Green Conjecture for smooth curves lying on arbitrary $K3$ surfaces*, Compositio Math. **147** (2011), 839–851.

[ABFS] E. Arbarello, A. Bruno, G. Farkas, G. Saccà, *Explicit Brill-Noether-Petri general curves*, Comment. Math. Helv. **91** (2016), 477–491.

[ABS] E. Arbarello, A. Bruno, E. Sernesi, *On hyperplane sections on K3 surfaces*, Alg. Geom. **4** (2017), 562–596.

[Be] A. Beauville, *Counting rational curves on $K3$ surfaces*, Duke Math. J. **97** (1999), 99–108.

[BS] M. Beltrametti, A. J. Sommese, *Zero cycles and k-th order embeddings of smooth projetive surfaces*, in *Projetive surfaces and their classification*, Cortona proceedings Symp. Math. INDAM **32**, pp. 33–48, New York, London: Academic press 1988

[BHT] F. Bogomolov, B. Hassett, Y. Tschinkel, *Constructing rational curves on $K3$ surfaces*, Duke Math. J. **157** (2011), 535–550.

[Bo] D. Bourqui, *Algebraic points, non-anticanonical heights and the Severi problem on toric varieties*, Proc. Lond. Math. Soc. **113** (2016), 474–514.

[CD] S. Cantat, I. Dolgachev, *Rational surfaces with a large group of automorphisms*, J. Amer. Math. Soc. **25** (2012), 863–905.

[Ch1] X. Chen, *Rational curves on $K3$ surfaces*, J. Algeb. Geom. **8** (1999), 245–278.

[Ch2] X. Chen, *A simple proof that rational curves on $K3$ surfaces are nodal*, Math. Ann. **324** (2002), 71–104.

[Ch3] X. Chen, *Nodal curves on $K3$ surfaces*, New York J. Math.**25** (2019), 168–173.

[Ch4] X. Chen, *Self rational maps of $K3$ surfaces*, `arXiv: 1008.1619v1`.

[CC] L. Chiantini, C. Ciliberto, On the Severi varieties of surfaces in $\mathbb{P}^3$, J. Algeb. Geom. **8** (1999), 67–83.

[CS] L. Chiantini, E. Sernesi, *Nodal curves on surfaces of general type*, Math. Ann. **307** (1997), 41–56.

[CD1] C. Ciliberto, T. Dedieu, *On universal Severi varieties of low genus $K3$ surfaces*, Math. Z. **271** (2012), 953–960.

[CD2] C. Ciliberto, T. Dedieu, *On the irreducibility of Severi varieties on K3 surfaces*, Proc. Amer. Math. Soc. **147** (2019), 4233–4244.

[CDGK] C. Ciliberto, T. Dedieu, C. Galati, A. L. Knutsen, *Nonemptiness of Severi varieties on Enriques surfaces*, `arXiv:2109.10735`

[CK] C. Ciliberto, A. L. Knutsen, *On k-gonal loci in Severi varieties on general $K3$ surfaces*, J. Math. Pures Appl. **101** (2014), 473–494.

[De1] T. Dedieu, *Severi varieties and self-rational maps of K3 surfaces*, Internat. J. Math. **20** (2009), 1455–1477.

[De2] T. Dedieu, *Comment on: On the irreducibility of the Severi variety of nodal curves in a smooth surface, by E. Ballico*, Arch. Math. **114**, 171–174 (2020).

[DS] T. Dedieu, E. Sernesi, *Equigeneric and equisingular families of curves on surfaces*, Publ. Mat. **61** (2017), 175–212.

[DH] S. Diaz, J. Harris, *Ideals associated to deformations of singular plane curves*, Trans. Amer. Math. Soc. **309** (1988), 433–468.

[Du] P. Du Val, *On rational surfaces whose prime sections are canonical curves*. Proc. London Math. Soc. **35** (1933), 1–13.

[Ei] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer, New York, 1995.

[EH] D. Eisenbud, J. Harris, *The Geometry of Schemes*, Graduate Texts in Mathematics 197, Springer, Berlin, 2000.

[FT] G. Farkas, N. Tarasca, *Du Val curves and the pointed Brill-Noether Theorem*, Sel. Math. **23** (2017), 2243–2259.

[FKPS] F. Flamini, A. L. Knutsen, G. Pacienza, E. Sernesi, *Nodal curves with general moduli on K3 surfaces*, Comm. Algebra **36** (2008), 3955–3971.

[Fr] R. Friedman, *Global smoothings of varieties with normal crossings*, Annals of Math. **118** (1983), 75–114.

[FL] W. Fulton, R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, in Algebraic Geometry, Proceedings of the Midwest Algebraic Geometry Conference, Chicago 1980, Lecture Notes in Math. **862**, Springer-Verlag, Berlin - Heidelberg - New York (1981), 26–92.

[GHS] T. Graber, J. Harris, J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), 57–67.

[GLS] G. M. Greuel, C. Lossen, E. Shustin, *Geometry of families of nodal curves on the blown-up projective plane*, Trans. Amer. Math. Soc. **350** (1998), 251–274.

[Ha] J. Harris, *On the Severi problem*, Invent. Math. **84** (1986), 445–461.

[HK] D. Huybrechts, M. Kemeny, *Stable maps and Chow groups*, Doc. Math. **18** (2013), 507–517.

[Kei] T. Keilen, *Irreducibility of equisingular families of curves*, Trans. Amer. Math. Soc. **355** (2003), 3485–3512.

[Kem] M. Kemeny, *The universal Severi variety of rational curves on $K3$ surfaces*, Bull. London Math. Soc. **45** (2013), 159–174.

[KL] A. L. Knutsen, M. Lelli-Chiesa, *Genus two curves on abelian surfaces*, to appear in Ann. Sci. Éc. Norm. Supér.

[KLM] A. L. Knutsen, M. Lelli-Chiesa, G. Mongardi, *Severi Varieties and Brill-Noether theory of curves on abelian surfaces*, J. Reine Angew. Math **749** (2019), 161–200.

[Ko1] J. Kollár, *Rational Curves on Algebraic Varieties*, Springer-Verlag, Germany, 1996.

[Ko2] J. Kollár, *Variants of normality for Noetherian schemes*, Pure Appl. Math. Q. **12** (2016), 1–31.

[KoL] J. Kollár, A. Landesman *A Fulton-Hansen theorem for almost homogeneous spaces*, Bollettino Unione Mat. Ital. **15** (2022), 253-268.

[Kn] A. L. Knutsen, *On kth-order embeddings of $K3$ surfaces and Enriques surfaces*, Manuscripta Math.**104** (2001), 211–237.

[Le] M. Lehn *Lectures on Hilbert schemes*, CRM proceedings, Centre de Recherches Mathématiques, Montreal (2003).

[LT] L. Lang, I. Tyomkin, *A note on the Severi problem for toric surfaces*, `arXiv:2007.11550`.

[Li1] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Diff. Geom. **57** (2001), 509–578.

[Li2] J. Li, *A degeneration formula for Gromov-Witten invariants*, J. Diff. Geom. **60** (2002), 199–293.

[Li3] J. Li, *Good degenerations of moduli spaces*, Handbook of moduli. Vol. II, 299–351, Adv. Lect. Math., (ALM) 25, International Press and Higher Education Press, Beijing, 2013.

[Li4] J. Li, *Lecture notes on relative GW invariants*, Intersection theory and moduli, ICTP Lect. Notes, XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, 41–96.

[LW] J. Li, B. Wu, *Good degeneration of Quot-schemes and coherent systems*, Comm. Anal. Geom. **23** (2015), 841–921.

[Ma] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Adv. Math. 8, Cambridge University Press, Cambridge, 1989.

[MPT] D. Maulik, R. Pandharipande, R. Thomas, *Curves on $K3$ surfaces and modular forms*, J. Topol. **3** (2010), 937–996.

[MM] S. Mori, S. Mukai, *The uniruledness of the moduli space of curves of genus* 11, Algebraic Geometry, Proc. Tokyo/Kyoto, 334–353, Lecture Notes in Math. **1016**, Springer, Berlin, (1983).

[Se] F. Severi, *Vorlesungen über Algebraische Geometrie*, Johnson Pub. (reprinted, 1968; 1st. ed., Leipzig 1921).

[St] Stacks Project Authors, *Stacks Project*, `https://stacks.math.columbia.edu` (2018).

[Tes] D. Testa, The irreducibility of the spaces of rational curves on del Pezzo surfaces, J. Algeb. Geom. **18** (2009), 37–61.

[Tei] B. Teissier *Résolution simultanée I, II, in Séminaire sur les singularités des surfaces, Cent. Math. Éc. Polytech., Palaiseau 1976–77*, Lecture Notes in Mathematics **777** (1980), Springer-Verlag, Berlin, 71–146.

[Tyo] I. Tyomkin, *On Severi varieties on Hirzebruch surfaces*, Int. Math. Res. Not. **23** (2007), 31 pp.

[Tyu] A. N. Tyurin, *Cycles, curves and vector bundles on an algebraic surface*, Duke Math. J. **54** (1987), 1–26 .

[Va] R. Vakil, *A Tool for Stable Reduction of Curves on Surfaces*, Advances in Algebraic Geometry Motivated by Physics, 145–154, Amer. Math. Soc., 2001.

[Za] A. Zahariuc, *The Severi problem for abelian surfaces in the primitive case*, J. Math. Pures Appl., in press.

Andrea Bruno: Dipartimento di Matematica e Fisica, Università Roma Tre
Largo San Leonardo Murialdo 1-00146 Roma, Italy
Email address: andrea.bruno@uniroma3.it

Margherita Lelli-Chiesa: Dipartimento di Matematica e Fisica, Università Roma Tre
Largo San Leonardo Murialdo 1-00146 Roma, Italy
Email address: margherita.lellichiesa@uniroma3.it