MODULI OF NON-STANDARD NIKULIN SURFACES IN LOW GENUS

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ABSTRACT. Primitively polarized genus g Nikulin surfaces (S, M, H) are of two types, that we call standard and non-standard depending on whether the lattice embedding $\mathbb{Z}[H] \oplus_\perp \mathbf{N} \subset \operatorname{Pic} S$ is primitive. Here H is the genus g polarization and $\mathbf{N}$ is the Nikulin lattice. We concentrate on the non-standard case, which only occurs in odd genus. In particular, we study the birational geometry of the moduli space of non-standard Nikulin surfaces of genus g and prove its rationality for $g = 7, 11$ and the existence of a rational double cover of it when $g = 9$. Furthermore, if (S, M, H) is general in the above moduli space and $(C, M|_C)$ is a general Prym curve in $|H|$, we determine the dimension of the family of non-standard Nikulin surfaces of genus g containing $(C, M|_C)$ for $3 \leq g \leq 11$; this completes the study of the Prym-Nikulin map initiated in [KLV].

1. INTRODUCTION

A Nikulin surface is a $K3$ surface endowed with a non-trivial double cover branched along eight disjoint rational curves. Nikulin surfaces have attracted a lot of attention in recent time because of their relevance in the study of both the moduli [FV] and the syzygies [FK1, FK2] of Prym canonical curves. It is imperative to recall the lattice theoretical proof by Sarti and van Geemen [vGS] of the existence of exactly two types of polarized Nikulin surfaces, that we call *standard* and *non-standard* (cf. §2), the latter occurring only in odd genera. There are coarse moduli spaces $\mathcal{F}_g^{\mathbf{N},s}$ and $\mathcal{F}_g^{\mathbf{N},ns}$ parametrizing genus g primitively polarized Nikulin surfaces of standard and non-standard type, respectively; more precisely, a point of $\mathcal{F}_g^{\mathbf{N},s}$ (resp., $\mathcal{F}_g^{\mathbf{N},ns}$) represents a triple (S, M, H), where S is a standard (resp., non-standard) Nikulin surface, $H \in \operatorname{Pic} S$ is a genus g primitive polarization and the line bundle $M \in \operatorname{Pic} S$ defines the double cover branched along eight disjoint rational curves. Both $\mathcal{F}_g^{\mathbf{N},s}$ and $\mathcal{F}_g^{\mathbf{N},ns}$ are irreducible of dimension 11, cf. [Do, §3], [vGS, Prop. 2.3].

Up to now, only the moduli spaces $\mathcal{F}_g^{\mathbf{N},s}$ have been extensively studied, while non-standard Nikulin surfaces have not been adequately considered. This paper aims to (partially) fill this gap. We concentrate on the $\mathbb{P}^g$-bundle over $\mathcal{F}_g^{\mathbf{N},ns}$ parametrizing pairs $((S, M, H), C)$ such that $(S, M, H) \in \mathcal{F}_g^{\mathbf{N},ns}$ and $C \in |H|$. Let $\mathcal{P}_g^{\mathbf{N},ns}$ be the open set of pairs such that C is smooth and let $\mathcal{R}_g$ be the moduli space of Prym curves; we look at the diagram

$$(1)$$

$$
\begin{array}{ccc}
& \mathcal{P}_g^{\mathbf{N},ns} & \\
{\scriptstyle q_g^{\mathbf{N},ns}} \swarrow \quad {\scriptstyle \chi_g^{ns}}\downarrow \quad & & \searrow {\scriptstyle m_g^{\mathbf{N},ns}} \\
\mathcal{F}_g^{\mathbf{N},ns} \qquad & \mathcal{R}_g \longrightarrow & \mathcal{M}_g \,,
\end{array}
$$

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whose arrows can be described as follows: $q_g^{\mathbf{N},ns}$ and $m_g^{\mathbf{N},ns}$ are the obvious forgetful maps. Moreover, the *Prym-Nikulin map* χ_g^{ns} sends $((S,M,H),C)$ to the Prym curve $(C, M \otimes \mathcal{O}_C)$. In particular, $m_g^{\mathbf{N},ns}$ is just the composition of χ_g^{ns} and the forgetful map $\mathcal{R}_g \to \mathcal{M}_g$.

The main difference between the standard and non-standard case is that a general hyperplane section of a general polarized Nikulin surface of standard type is Brill-Noether general, while curves lying on non-standard Nikulin surfaces carry two unexpected theta-characteristics (cf. Prop. 2.3) that make them special in moduli. A first consequence is that the maps $m_g^{\mathbf{N},ns}$ and χ_g^{ns} can never be dominant. Furthermore, a heuristic count suggests that they cannot be generically finite for $g \leq 11$, cf. Remark 2.11. In [KLV] we proved that the map χ_g^{ns} is birational onto its image for (odd) genus $g \geq 13$, and the behaviour of the analogous map in the standard case was completely described. In this paper, we complete the picture by showing that:

Theorem 1.1. *The map χ_g^{ns} has generically*

- *9-dimensional fibers for $g = 3$;*
- *6-dimensional fibers for $g = 5$;*
- *4-dimensional fibers for $g = 7$;*
- *2-dimensional fibers for $g = 9$;*
- *1-dimensional fibers for $g = 11$.*

As already mentioned, hyperplane sections of non-standard Nikulin surfaces have some peculiar and compelling properties, that we now describe in more detail. A general genus g polarized non-standard Nikulin surface (S, M, H) carries two line bundles R, R' such that $H(-M) \simeq R \otimes R'$. The restrictions of R and R' to a general hyperplane section $C \in |H|$ are two theta-characteristics with positive dimensional spaces of global sections. For (odd) genus $g \geq 5$ both $h^0(\mathcal{O}_C(R)) \geq 2$ and $h^0(\mathcal{O}_C(R')) \geq 2$ and hence the theta divisor of the Jacobian of C has two singular points of given multiplicity. We precisely describe the images of $m_g^{\mathbf{N},ns}$ for $g = 3$ and 5, cf. Theorems 2.9 and 2.10:

– the image of $m_3^{\mathbf{N},ns}$ is the hyperelliptic locus in $\mathcal{M}_3$;
– the image of $m_5^{\mathbf{N},ns}$ coincides with the locus of curves in $\mathcal{M}_5$ possessing two autoresidual g_4^1; in particular, this locus is irreducible.

For $g \geq 7$ the situation becomes more intricate and the birational geometry of the moduli space $\mathcal{F}_g^{\mathbf{N},ns}$ is worth investigating. We prove:

Theorem 1.2. *The moduli space $\mathcal{F}_g^{\mathbf{N},ns}$ of non-standard Nikulin surfaces of genus g is:*

- *rational for $g = 7$ and $g = 11$;*
- *unirational with a rational double cover for $g = 9$.*

The proof of both Theorems 1.2 and 1.1 for $g \geq 7$ is given in §3-5 and relies on the description of nice projective models of non-standard Nikulin surfaces (S, M, H) in low genus. Set $r := h^0(R) - 1$ and $r' := h^0(R') - 1$. As already remarked by Garbagnati and Sarti in [GS], the line bundles R and R' enable to realize S as a subvariety of the intersection of the Segre variety $\mathbb{P}^{r'} \times \mathbb{P}^r \subset \mathbb{P}^{rr'+r+r'}$ with a linear space of dimension $g - 2$, namely, $\mathbb{P}(H^0(S, H(-M))^\vee)$. We are able to detect some geometric conditions that are also sufficient for such a subvariety of $(\mathbb{P}^{r'} \times \mathbb{P}^r) \cap \mathbb{P}^{g-2}$ to be a Nikulin surface of non-standard type.

For instance, a general non-standard Nikulin surface of genus 7 is a divisor of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, cf. [GS, §4.8]. Furthermore, a $K3$ surface in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2, 3)|$ is a Nikulin surface of non-standard type if and only if it contains two conics A_1 and A_2 that are contracted by the first projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and are mapped to the same plane conic by the second projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$.

Analogously, a general surface in $\mathcal{F}_9^{\mathbf{N}, ns}$ is a quadratic section of a Del Pezzo threefold $T := (\mathbb{P}^2 \times \mathbb{P}^2) \cap \mathbb{P}^7 \subset \mathbb{P}^8$, cf. [GS, §4.9]. Moreover, an element in $|\mathcal{O}_T(2, 2)|$ is a non-standard Nikulin surface if and only if it contains two sets of four lines that are contracted by the first and second projection, respectively.

As regards genus 11, a general surface S in $\mathcal{F}_{11}^{\mathbf{N}, ns}$ defines a divisor of type $(1, 2)$ in the threefold $T' := (\mathbb{P}^2 \times \mathbb{P}^3) \cap \mathbb{P}^9 \subset \mathbb{P}^{11}$. The projection $T' \to \mathbb{P}^3$ realizes T' as the blow-up of $\mathbb{P}^3$ along a rational normal cubic curve γ and we denote by P_γ the exceptional divisor. The surface S intersects P_γ along a rational quintic curve $\Gamma \subset T' \subset \mathbb{P}^9$ and in fact we show that the containment of Γ is a necessary and sufficient condition for a surface in $|\mathcal{O}_{T'}(1, 2)|$ to be a non-standard Nikulin surface of genus 11. The rationality results in Theorem 1.2 will follow from these characterizations.

Concerning the fibers of the moduli map χ_g^{ns}, the case of genus 7 has some special features. Let $C \subset (\mathbb{P}^r \times \mathbb{P}^{r'}) \cap \mathbb{P}^{g-2}$ be a general genus g Nikulin section in the non-standard case. In genus 9 a general quadratic section of the threefold T containing C is a non-standard Nikulin surface; the same holds in genus 11 if one considers in the threefold T' a general divisor of type $(1, 2)$ through C. The situation in genus 7 is divergent: a general $K3$ surface in the linear system $|\mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2}(2, 3)|$ is not a Nikulin surface. This difference depends on the fact that, contrary to what happens for $g = 9, 11$, in genus 7 the embedded curve $C \subset \mathbb{P}^{g-2}$ is not quadratically normal. As a relevant consequence, the image of χ_7^{ns} lies in the ramification locus of the Prym map $\mathcal{R}_7 \to \mathcal{A}_6$, cf. Remark 3.5 and [Be]. This suggests an interesting behaviour of Nikulin sections with respect to their Prym varieties. In the standard case this phenomenon was already pointed out in [FV], where the image of $\chi_6^{\mathbf{N}, s}$ is identified with the ramification locus of the Prym map $\mathcal{R}_6 \to \mathcal{A}_5$, but was still unknown in the non-standard case.

2. Nikulin surfaces of non-standard type and Segre varieties

We recall some basic definitions and properties.

Definition 2.1. A polarized *Nikulin* surface of genus $g \geq 2$ is a triple (S, M, H) such that S is a smooth $K3$ surface, $\mathcal{O}_S(M), H \in \operatorname{Pic} S$ and the following conditions are satisfied:

- S contains 8 mutually disjoint rational curves $N_1, \ldots, N_8$ such that

$$N_1 + \cdots + N_8 \sim 2M.$$

- H is nef, $H^2 = 2(g - 1)$ and $H \cdot M = 0$.

We say that (S, M, H) is *primitively polarized* if in addition H is primitive in $\operatorname{Pic} S$.

Definition 2.2. Let (S, M, H) be a Nikulin surface of genus g. Its *Nikulin lattice* $\mathbf{N} = \mathbf{N}(S, M)$ is the rank 8 sublattice of $\operatorname{Pic} S$ generated by $N_1, \ldots, N_8$ and M.

One also defines the rank 9 lattice

$$\Lambda = \Lambda(S, M, H) := \mathbb{Z}[H] \oplus_\perp \mathbf{N} \subset \operatorname{Pic} S.$$

If the embedding $\Lambda \subset \operatorname{Pic} S$ is primitive, we call (S, M, H) a *Nikulin surface of standard type*, else we call it a *Nikulin surface of non-standard type*.

There are coarse moduli spaces $\mathcal{F}_g^{\mathbf{N}, s}$ (respectively, $\mathcal{F}_g^{\mathbf{N}, ns}$) parametrizing polarized Nikulin surfaces of genus g of standard (resp., non-standard) type. Both $\mathcal{F}_g^{\mathbf{N}, s}$ and $\mathcal{F}_g^{\mathbf{N}, ns}$ are irreducible of dimension 11 and their very general members have Picard number nine, cf. [Do, §3], [vGS, Prop. 2.3]. By [vGS, Prop. 2.2], if (S, M, H) is a non-standard Nikulin surface of genus g, then g is odd and the embedding $\Lambda \subset \operatorname{Pic} S$ has index two. More precisely (cf. [GS, Prop. 2.1 and Cor. 2.1]), possibly after renumbering the curves N_i, there are $R, R' \in \operatorname{Pic} S$ such that

- $H - N_1 - N_2 - N_3 - N_4 \sim 2R$ and $H - N_5 - N_6 - N_7 - N_8 \sim 2R'$ if $g \equiv 1 \mod 4$;
- $H - N_1 - N_2 \sim 2R$ and $H - N_3 - \cdots - N_8 \sim 2R'$ if $g \equiv 3 \mod 4$.

Moreover, when $\operatorname{rk} \operatorname{Pic} S = 9$, then $\operatorname{Pic} S \simeq \mathbb{Z}[R] \oplus \mathbf{N}$ by [GS, Prop. 2.1 and Cor. 2.1]. We also need to define the line bundle $L := H - M$, which satisfies $L^2 = 2(g - 3)$ and $L \cdot N_i = 1$ for $i = 1, \ldots, 8$.

We henceforth concentrate on Nikulin surfaces of non-standard type.

First of all we show that hyperplane sections of non-standard Nikulin surfaces are rather special.

Proposition 2.3. *Let (S, M, H) be a general non-standard Nikulin surface of genus $g \equiv 1 \mod 4$ (respectively, $g \equiv 3 \mod 4$) and let L, R and R' be as above. Then*

(i) *R and R' are globally generated with $h^1(R) = h^1(R') = 0$ if $g \geq 5$;*

(ii) *$h^0(L) = g - 1$ and L is very ample if $g \geq 7$ and is ample and globally generated defining a degree two morphism onto $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ if $g = 5$;*

(iii) *if $g \geq 5$, then for any smooth curve C in $|H|$, the line bundles $\mathcal{O}_C(R)$ and $\mathcal{O}_C(R')$ are theta-characteristics satisfying $h^0(S, R) = h^0(C, \mathcal{O}_C(R)) = (g+3)/4$ (resp., $(g+5)/4$) and $h^0(S, R') = h^0(C, \mathcal{O}_C(R')) = (g+3)/4$ (resp., $(g+1)/4$).*

Proof. Since all properties are open in the moduli space, one may prove them for a non-standard Nikulin surface with $\operatorname{rk} \operatorname{Pic} S = 9$. Then (i) is proved in [GS, Prop. 3.5(2)] (recalling that a linear system on a $K3$ surface without base components is base point free) and (ii) in [GS, Prop. 3.2 and Lemma 3.1], using the classical numerical criteria of Saint-Donat [SD]. As $L^2 = 2(g - 3) > 0$, we have $h^1(L) = h^2(L) = 0$, whence $h^0(L) = g - 1$ by Riemann-Roch.

To prove (iii) we note that $R - H \sim -(R + N_1 + N_2 + N_3 + N_4)$ (resp., $-(R + N_1 + N_2)$). Thus, $h^0(R - H) = 0$. Moreover, the linear system $|R|$ contains irreducible members thanks to (i), and hence $|H - R|$ contains a divisor D that is the union of an irreducible element in $|R|$ and four rational irreducible tails. In particular, one has $h^0(\mathcal{O}_D) = 1$ and thus $h^1(R - H) = 0$. The standard restriction sequence yields $h^0(S, R) = h^0(C, \mathcal{O}_C(R)) = \frac{1}{2}R^2 + 2$ by Riemann-Roch and (i). The rest then follows from an easy computation and the same argument applies to R'. □

Remark 2.4. In the embedding $S \subset \mathbb{P}^{g-2}$ defined by $|L|$, any smooth C in $|H|$ is mapped to a Prym-canonical curve, as $L|_C \simeq \omega_C \otimes \mathcal{O}_C(M)$ satisfies $L|_C^{\otimes 2} \simeq \omega_C^{\otimes 2}$, and all $N_1, \ldots, N_8$ are mapped to lines.

From now on, we will set $r := h^0(S, R) - 1$ and $r' := h^0(S, R') - 1$. By Proposition 2.3, as soon as $g \geq 7$, the two linear systems $|R'|$ and $|R|$ (and their restrictions to C) define an embedding

$$(2) \qquad C \subset S \subset \mathbb{P}^{r'} \times \mathbb{P}^r \subset \mathbb{P}^{rr'+r+r'},$$

where the second inclusion is the Segre embedding.

Notation 2.5. We let $p : \mathbb{P}^{r'} \times \mathbb{P}^r \to \mathbb{P}^r$ and $p' : \mathbb{P}^{r'} \times \mathbb{P}^r \to \mathbb{P}^{r'}$ be the two projection maps. For any subvariety $X \subset \mathbb{P}^{r'} \times \mathbb{P}^r$, we denote by p'_X and p_X the restrictions to X of p' and p, respectively. In particular, p'_S and p_S are the maps defined by $|R'|$ and $|R|$, respectively.

We use the standard notation $\mathcal{O}_{\mathbb{P}^{r'} \times \mathbb{P}^r}(a, b) := p'^* \mathcal{O}_{\mathbb{P}^{r'}}(a) \otimes p^* \mathcal{O}_{\mathbb{P}^r}(b)$, and for any subvariety $X \subset \mathbb{P}^{r'} \times \mathbb{P}^r$, we set $\mathcal{O}_X(a, b) \simeq \mathcal{O}_{\mathbb{P}^{r'} \times \mathbb{P}^r}(a, b)|_X$ and refer to elements in the corresponding linear systems as *divisors of bidegree (a, b) on X*.

Definition 2.6. We say that a curve in $\mathbb{P}^{r'} \times \mathbb{P}^r$ is *vertical* if it is contracted by p' and *horizontal* if it is contracted by p.

Any line ℓ in $\mathbb{P}^{r'} \times \mathbb{P}^r$ is either vertical or horizontal. If $g \equiv 1 \mod 4$, then

- $N_1, \ldots, N_4$ are vertical, as $N_1 \cdot R' = \cdots = N_4 \cdot R' = 0$;
- $N_5, \ldots, N_8$ are horizontal, as $N_5 \cdot R = \cdots = N_8 \cdot R = 0$.

If instead $g \equiv 3 \mod 4$, then

- N_1, N_2 are vertical, as $N_1 \cdot R' = N_2 \cdot R' = 0$;
- $N_3, \ldots, N_8$ are horizontal, as $N_3 \cdot R = \cdots = N_8 \cdot R = 0$.

We will make use of the following:

Lemma 2.7. *Let (S, M, H) be a general non-standard Nikulin surface of genus g with $5 \leq g \leq 15$ and let L, R and R' be as above. Then the multiplication map*

$$\mu_{R,R'} : H^0(S, R) \otimes H^0(S, R') \longrightarrow H^0(S, L)$$

is surjective. Furthermore, it is isomorphic to the multiplication map

$$\mu_{\eta,\eta'} : H^0(C, \eta) \otimes H^0(C, \eta') \longrightarrow H^0(C, \eta \otimes \eta'),$$

where C is any smooth irreducible curve in $|H|$ and η and η' are the restrictions to C of the line bundles R and R', respectively.

Proof. The properties are open in the moduli space, so we may assume that $\operatorname{rk} \operatorname{Pic} S = 9$. The surjectivity of $\mu_{R,R'}$ follows from the generalization by Mumford of a theorem of Castelnuovo, cf. [Mu, Thm. 2, p. 41] (recalling that the assumption on ampleness is unnecessary) once we check that $h^1(R - R') = 0$ and $h^2(R - 2R') = 0$.

We have $2(R - R') \sim -N_1 - \cdots - N_4 + N_5 + \cdots + N_8$ (resp., $-N_1 - N_2 + N_3 + \cdots + N_8$) if $g \equiv 1 \mod 4$ (resp., $g \equiv 3 \mod 4$). Hence, $h^0(R - R') = h^0(R' - R) = 0$. As $(R - R')^2 = -4$, one has $h^1(R - R') = 0$.

We next prove that $h^2(R - 2R') = h^0(2R' - R) = 0$. We treat the case $g \equiv 1 \mod 4$, leaving the other case to the reader, as it is very similar.

We have $2R' - R \sim R + N_1 + \cdots + N_4 - N_5 - \cdots - N_8$. Since $N_i \cdot (2R' - R) = -1$ for $i \in \{1, 2, 3, 4\}$, we have $h^0(2R' - R) = h^0(R - N_5 - \cdots - N_8)$. The assumption on g implies that $(R - N_5 - \cdots - N_8)^2 \leq -4$. Hence, if $R - N_5 - \cdots - N_8$ is effective, it has nonvanishing h^1, which by Ramanujam's vanishing theorem [Ra, Lemma 3] implies that it is not 1-connected. Hence, there is an effective nontrivial decomposition $R - N_5 - \cdots - N_8 \sim A + B$ such that $A \cdot B \leq 0$. Since $\mathrm{Pic}\, S \simeq \mathbb{Z}[R] \oplus \mathbf{N}$ by [GS, Prop. 2.1 and Cor. 2.1], we may write

$$A \sim \alpha R + \frac{1}{2} \sum_{i=1}^{8} \alpha_i N_i \ \text{ and } \ B \sim \beta R + \frac{1}{2} \sum_{i=1}^{8} \beta_i N_i,$$

for integers $\alpha, \beta, \alpha_i, \beta_i$ satisfying

(3) $\alpha + \beta = 1, \quad \alpha_i + \beta_i = 0$ if $i \in \{1, 2, 3, 4\}$ and $\alpha_i + \beta_i = -2$ if $i \in \{5, 6, 7, 8\}$.

Effectivity requires that $\alpha \geq 0$ and $\beta \geq 0$, so that we can without loss of generality assume $\alpha = 1$ and $\beta = 0$. Therefore, $B \sim \frac{1}{2} \sum_{i=1}^{8} \beta_i N_i$ and effectivity requires that all $\beta_i \geq 0$ and all β_i are even. Write $\beta_i = 2\gamma_i$ for integers $\gamma_i \geq 0$. Then $\alpha_i = -2\gamma_i$ if $i \in \{1, 2, 3, 4\}$ and $\alpha_i = -2(\gamma_i + 1)$ if $i \in \{5, 6, 7, 8\}$, so that $A \sim R - \sum_{i=1}^{4} \gamma_i N_i - \sum_{i=5}^{8} (\gamma_i + 1) N_i$. Therefore,

$$
\begin{aligned}
A \cdot B &= \left(R - \sum_{i=1}^{4} \gamma_i N_i - \sum_{i=5}^{8} (\gamma_i + 1) N_i \right) \cdot \sum_{i=1}^{8} \gamma_i N_i \\
&= \sum_{i=1}^{4} \gamma_i (2\gamma_i + 1) + 2 \sum_{i=5}^{8} \gamma_i (\gamma_i + 1).
\end{aligned}
$$

Since at least one of the γ_i is strictly positive, we see that we get $A \cdot B \geq 3$, a contradiction.

As concerns the second statement, it is enough to remark that the line bundles $L - H$, $R - H$ and $R' - H$ all have vanishing h^0 and h^1, which can be proved as in the last part of the proof of Proposition 2.3.   □

As a consequence, for $g \geq 7$ the embeddings $C \subset S \subset \mathbb{P}^{rr'+r+r'}$ in (2) factor through the embedding $S \subset \mathbb{P}^{g-2}$ defined by $|L|$, and hence:

(4) $C \subset S \subset (\mathbb{P}^{r'} \times \mathbb{P}^r) \cap \mathbb{P}^{g-2} \subset \mathbb{P}^{rr'+r+r'}.$

Remark 2.8. It is not a priori obvious that the intersection $(\mathbb{P}^{r'} \times \mathbb{P}^r) \cap \mathbb{P}^{g-2}$ is transversal. However, since $\mathcal{F}_g^{\mathbf{N}, ns}$ is irreducible, as soon as one shows the existence of a non-standard Nikulin surface of genus g in some transversal intersection $(\mathbb{P}^{r'} \times \mathbb{P}^r) \cap \mathbb{P}^{g-2}$, one gets the transversality statement for a general Nikulin surface in $\mathcal{F}_g^{\mathbf{N}, ns}$.

The next two results prove Theorem 1.1 in genera 3 and 5.

Theorem 2.9. *The image of $m_3^{\mathbf{N}, ns}$ coincides with the hyperelliptic locus in $\mathcal{M}_3$. In particular, a general fiber of $m_3^{\mathbf{N}, ns}$ has dimension 9.*

Proof. Let $(S, M, H) \in \mathcal{F}_3^{\mathbf{N}, ns}$ and $C \in |H|$ be general. The restriction of the line bundle $R \in \mathrm{Pic}\, S$ to C is a g_2^1; in particular, the canonical map of C is a double cover of a plane conic C_K branched along 8 points. Furthermore, the linear system $|H|$ on S defines a double cover $\varphi_H : S \to X \subset \mathbb{P}^3$ of a cone X in $\mathbb{P}^3$ branched along a plane conic C_2 that is the image of the unique curve in $|R'|$, and a sextic C_6 that is the image of an

irreducible curve in the linear system $|H + R'|$ (cf. [GS, 4.3]). Note that $C_2 \cdot C_K = 2$, $C_6 \cdot C_K = 6$ and that C_2 and C_6 meet at the six points in X that are images of the curves $N_3, \ldots, N_8$. Furthermore, φ_H factors through

$$(5) \qquad S \xrightarrow{c} \overline{S} \xrightarrow{\pi} \mathbb{F}_2 \xrightarrow{\phi} X,$$

where $\mathbb{F}_2$ is the second Hirzebruch surface (with a section C_0 such that $C_0^2 = -2$ and class fiber denoted by f), the map ϕ is induced by the linear system $|C_0 + 2f|$ on $\mathbb{F}_2$, the map π is a double cover branched along the inverse image of C_2 and C_6, while c is the contraction of $N_3, \ldots, N_8$. Note that $(\pi \circ c)^{-1}(C_0) = N_1 \cup N_2$, and $\phi^* C_2 \in |C_0 + 2f|$ while $\phi^* C_6 \in |3C_0 + 6f|$.

It is not difficult to show that the desingularization S of any double cover $\overline{S}$ of $\mathbb{F}_2$ branched along the union of a smooth irreducible curve $C_2 \in |C_0 + 2f|$ and a smooth irreducible curve $C_6 \in |3C_0 + 6f|$ is a Nikulin surface. Indeed, S is a $K3$ surface by, e.g., [Re, Thm. 2.2]; furthermore, S has eight disjoint rational curves, two of which mapping to the section C_0 (call them N_1 and N_2) and six arising as exceptional divisors of the desingularization of $\overline{S}$ (call them $N_3, \ldots, N_8$), which has six double points at the inverse images of $C_2 \cap C_6$. The line bundle $H \in \operatorname{Pic} S$ obtained as pullback of $C_0 + 2f$ is a genus 3 polarization. We denote by $R \in \operatorname{Pic} S$ the pullback of f, and by $R' \in \operatorname{Pic} S$ the line bundle with a section vanishing at the strict transform in S of the ramification curve $\pi^{-1}(C_2) \subset \overline{S}$. In particular, we have

$$(6) \qquad H - N_1 - N_2 \sim (\pi \circ c)^*(C_0 + 2f) - (\pi \circ c)^* C_0 \sim 2R.$$

Setting $M := H - R - R'$, one easily checks that $N_1 + \cdots + N_8 \sim 2M$ and hence (S, M, H) is a genus 3 Nikulin surface of non-standard type by (6); it depends on $\dim |C_0 + 2f| + \dim |3C_0 + 6f| - \dim \operatorname{Aut}(\mathbb{F}_2) = 3 + 15 - 7 = 11$ moduli.

We use this in order to prove that a general hyperelliptic curve of genus 3 lies on a Nikulin surface of non-standard type. Let C be a general hyperelliptic curve of genus 3 and let $C_K \subset \mathbb{P}^2$ be the canonical image of C, which is a smooth plane conic. We denote by $x_1, \ldots, x_8$ the image in C_K of the eight Weierstrass points on C and by X the cone in $\mathbb{P}^3$ over C_K. The desingularization of X is then isomorphic to $\mathbb{F}_2$. By abuse of notation, we still denote by the same name the inverse images in $\mathbb{F}_2$ of the curve C_K and the points $x_1, \ldots, x_8$. It is then enough to remark that both the linear systems $|(C_0 + 2f) \otimes \mathcal{I}_{x_1 + x_2}|$ and $|(3C_0 + 6f) \otimes \mathcal{I}_{x_3 + \cdots + x_8}|$ are nonempty and contain smooth members. $\qquad \square$

Theorem 2.10. *The map $m_5^{\mathbf{N}, ns}$ has generically 6-dimensional fibers and its image coincides with the locus of curves in $\mathcal{M}_5$ possessing two autoresidual g_4^1. In particular, this locus is irreducible of dimension 10.*

Proof. By [GS, 4.6(b)], the nodal model of a Nikulin surface S of non-standard type and genus 5 is the complete intersection in $\mathbb{P}^5$ of three quadrics Q_1, Q_2, Q_3 such that Q_3 is smooth, while Q_1 and Q_2 have rank 3 and disjoint singular loci.

Vice versa, we are going to show that the minimal desingularization of any complete intersection $\overline{S} = Q_1 \cap Q_2 \cap Q_3$ of three quadrics in $\mathbb{P}^5$ with the above properties is automatically a Nikulin surface of non-standard type and genus 5. For $i = 1, 2$, let π_i be the plane vertex of Q_i. The plane π_1 (respectively, π_2) intersects S at four nodes $P_1, \ldots, P_4$ (resp., $P_5, \ldots P_8$). Let $q : S \to \overline{S}$ be the minimal desingularization of $\overline{S}$ and let $N_i := q^{-1}(P_i)$ for $1 \leq i \leq 8$. The line bundle $H := q^*(\mathcal{O}_{\overline{S}}(1))$ is a genus 5

polarization on S. Up to a change of coordinates, the quadrics Q_1 and Q_2 have defining equations $z_0 z_1 - z_2^2 = 0$ and $z_3 z_4 - z_5^2 = 0$, respectively; hence, $\pi_1 : z_0 = z_1 = z_2 = 0$ and $\pi_2 : z_3 = z_4 = z_5 = 0$. The hyperplanes $z_0 = 0$ and $z_1 = 0$ generate a pencil of hyperplanes in $\mathbb{P}^5$ all passing through the points $P_1, \ldots, P_4$ and cutting out on $\overline{S}$ a curve with multiplicity two; therefore, there exists a line bundle $R \in \operatorname{Pic} S$ such that $2R \sim H - N_1 - N_2 - N_3 - N_4$. Analogously, one shows the existence of a line bundle $R' \in \operatorname{Pic} S$ such that $2R' \sim H - N_5 - N_6 - N_7 - N_8$. Hence, S is a Nikulin surface of non-standard type.

We are now ready to detect the image of $m_5^{\mathbf{N},ns}$. First of all, it is straightforward that the line bundles R and R' on a genus 5 Nikulin surface of non-standard type cut out two autoresidual g_4^1 on a general hyperplane section. The other way around, let us consider a genus 5 curve C possessing two autoresidual g_4^1; these determine two rank-3 quadrics q_1 and q_2 in $\mathbb{P}^4$ containing the canonical image of C, cf. [ACGH, p. 208]. Since any component of the locus in $\mathcal{M}_5$ of curves with two autoresidual g_4^1 has dimension at least 10, we can assume C not to be bielliptic; this ensures that the singular lines of q_1 and q_2 do not intersect, cf. [ACGH, ch. VI, F]. Fix an embedding $\mathbb{P}^4 \subset \mathbb{P}^5$ and let Q_1 (respectively, Q_2) be the cone over q_1 (resp. q_2) with vertex a point P_1 (resp., P_2) in $\mathbb{P}^5 \setminus \mathbb{P}^4$; then, both Q_1 and Q_2 are quadrics of rank 3 and one can choose the points P_1 and P_2 so that their singular loci are disjoint. It is easy to check that $h^0(\mathbb{P}^5, \mathcal{I}_{C/\mathbb{P}^5}(2)) = 9$ and a general quadric Q_3 containing C is smooth since C cannot be trigonal (cf. [ACGH, ch. VI, F]); therefore, the surface $\overline{S} = Q_1 \cap Q_2 \cap Q_3$ is the nodal model of a Nikulin surface of non-standard type. Furthermore, the fiber of $m_5^{\mathbf{N},ns}$ over $[C]$ is parametrized by $\mathbb{P}(H^0(\mathbb{P}^5, \mathcal{I}_{C/\mathbb{P}^5}(2))/\langle Q_1, Q_2 \rangle) = \mathbb{P}^6$. $\qquad\square$

The rest of the paper will focus on the cases $g = 7, 9, 11$.

Remark 2.11. The following heuristic count shows that the expected dimension of a general fiber of χ_g^{ns} for $g = 7, 9, 11$ is the one obtained in Theorem 1.1.

When $g = 7$, a general hyperplane section C carries two theta-characteristics with a space of global sections of dimension 3 and 2, respectively, by Proposition 2.3. The moduli spaces of such curves have codimensions 3 and 1, respectively, in $\mathcal{M}_g$ or $\mathcal{R}_g$, by [Te], thus one expects the target of χ_7^{ns} to have dimension $18 - 3 - 1 = 14$ and the fibers to have dimension $11 + 7 - 14 = 4$.

When $g = 9$, a general hyperplane section C carries two theta-characteristics with a 3-dimensional space of global sections, by Proposition 2.3. The moduli spaces of such curves have codimension 3 in $\mathcal{M}_g$ or $\mathcal{R}_g$, by [Te], thus one expects the target of χ_9^{ns} to have dimension $24 - 3 - 3 = 18$ and the fibers to have dimension $11 + 9 - 18 = 2$.

When $g = 11$, a general hyperplane section C carries two theta-characteristics with 4 and 3 sections, respectively, by Proposition 2.3. The moduli spaces of such curves have codimensions 6 and 3, respectively, in $\mathcal{M}_g$ or $\mathcal{R}_g$, by [Te], thus one expects the target of χ_{11}^{ns} to have dimension $30 - 6 - 3 = 21$ and the fibers to have dimension $11 + 11 - 21 = 1$.

3. The case of genus 7

Let (S, M, H) be a general primitively polarized Nikulin surface of non-standard type of genus 7. Let $L = H - M$ and

$$R \sim \frac{1}{2}\left(H - N_1 - N_2\right) \ \text{ and } \ R' \sim L - R \sim \frac{1}{2}\left(H - N_3 - \cdots - N_8\right)$$

be as in §2. By Proposition 2.3, the line bundle L defines an embedding $S \subset \mathbb{P}^5$ and the embeddings in (4) are as follows:

$$S \subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5.$$

Here $|R| = |\mathcal{O}_S(0,1)|$ is a net of genus 2 curves of degree $R \cdot L = 5$ and $|R'| = |\mathcal{O}_S(1,0)|$ is a pencil of elliptic curves of degree $R' \cdot L = 3$. By the adjunction formula, $S \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$, cf. [GS, §4.8]. We want to identify the locus in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ parametrizing Nikulin surfaces of non-standard type. Since $R' \cdot N_1 = R' \cdot N_2 = 0$, two elements of $|R'|$ split as $N_1 + A_1$ and $N_2 + A_2$. In particular A_1, A_2 are two disjoint conics in the embedding $S \subset \mathbb{P}^5$, mapped into conics in $\mathbb{P}^2$ by p, as $R \cdot A_1 = R \cdot A_2 = 2$. Furthermore, one can prove that A_1 and A_2 are irreducible by specializing to the case where $\operatorname{rk} \operatorname{Pic} S = 9$ and proceeding as in [GS, proof of Prop. 3.5(2)].

Lemma 3.1. *We have* $p(A_1) = p(A_2)$.

Proof. Since $N_j \cdot R' = 1$ and $N_j \cdot N_1 = N_j \cdot N_2 = 0$ for $j \geq 3$, we have $N_j \cdot A_1 = N_j \cdot A_2 = 1$ for $j \geq 3$. It is then enough to note that the six points $z_j := p(N_j)$, $j = 3, \dots, 8$, are distinct and belong to both the conics $p(A_1)$ and $p(A_2)$. $\square$

We call A_1 and A_2 the *vertical conics of S*.

Using the fact that $R' \sim N_1 + A_1 \sim N_2 + A_2 \sim \frac{1}{2}(H - N_3 - \cdots - N_8)$, we obtain $2R' \sim (N_1 + A_1) + (N_2 + A_2) \sim H - N_3 - \cdots - N_8$, whence

$$(7) \qquad\qquad H \sim N_1 + \cdots + N_8 + A_1 + A_2.$$

3.1. Rationality of $\mathcal{F}_7^{\mathbf{N},ns}$.
Fix any smooth conic $A \subset \mathbb{P}^2$ and two disjoint vertical conics $A_1, A_2 \subset \mathbb{P}^1 \times \mathbb{P}^2$ such that $p(A_1) = p(A_2) = A$. The surface $\mathbb{P}^1 \times A$ is of bidegree $(0,2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Consider the inclusion

$$|\mathcal{I}_{\mathbb{P}^1 \times A / \mathbb{P}^1 \times \mathbb{P}^2}(2,3)| \subset |\mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times \mathbb{P}^2}(2,3)|.$$

Proposition 3.2. *A general member of* $|\mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ *is smooth and every smooth* $S \in |\mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ *is a non-standard Nikulin surface of genus 7 polarized by* $\mathcal{O}_S(2,2)(-A_1 - A_2)$.

Moreover, $\dim |\mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times \mathbb{P}^2}(2,3)| = 15$ *and* $\dim |\mathcal{I}_{\mathbb{P}^1 \times A / \mathbb{P}^1 \times \mathbb{P}^2}(2,3)| = 8$.

Proof. The standard exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathbb{P}^1 \times A / \mathbb{P}^1 \times \mathbb{P}^2}(2,3) \longrightarrow \mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times \mathbb{P}^2}(2,3) \longrightarrow \mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times A}(2,3) \longrightarrow 0$$

along with the isomorphisms $\mathcal{I}_{\mathbb{P}^1 \times A / \mathbb{P}^1 \times \mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0,-2)$ and $\mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times A}(2,3) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2,0) \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,6) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,6)$ proves the dimensional statements, the global generation of $\mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times \mathbb{P}^2}(2,3)$ and the surjectivity of the restriction map of linear systems

$$\rho : |\mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times \mathbb{P}^2}(2,3)| \longrightarrow (A_1 \cup A_2) + |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,6)|.$$

Hence, a general $S \in |\mathcal{I}_{A_1 \cup A_2 / \mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ is smooth and

$$S \cdot (\mathbb{P}^1 \times A) = A_1 + A_2 + N_3 + \cdots + N_8 \in |\mathcal{O}_S(0,2)|,$$

with $N_3, \dots, N_8$ disjoint horizontal lines. At the same time, $|\mathcal{O}_S(1,0)|$ is a pencil of elliptic curves of degree 3 on S such that $\mathcal{O}_S(1,0) \cdot A_i = 0$ for $i = 1,2$, and hence contains two elements of the form $N_i + A_i$ with N_i a line for $i = 1,2$. Furthermore, N_1 and N_2 are mutually disjoint, as well as disjoint from the other N_j for $j = 3, \dots, 8$.

Note that the divisor $N_1 + \cdots + N_8 \in |\mathcal{O}_S(2,2)(-2A_1 - 2A_2)|$ and thus is 2-divisible in $\operatorname{Pic} S$. It is now straightforward that S satisfies the desired properties; in particular, (7) implies that S is of non-standard type. $\qquad\square$

Two smooth elements in $|\mathcal{I}_{A_1 \cup A_2/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ are isomorphic if and only if they are in the same orbit under the action of the stabilizer G of $A_1 \cup A_2$ in $\operatorname{Aut}(\mathbb{P}^1 \times A)$. The group G is 4-dimensional, since it is the product of the stabilizer of two points in $\mathbb{P}^1$ and of the group $\operatorname{Aut} A$. Hence the quotient $|\mathcal{I}_{A_1 \cup A_2/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|/G$ is 11-dimensional and we have a birational map

$$|\mathcal{I}_{A_1 \cup A_2/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|/G \dashrightarrow \mathcal{F}_7^{\mathbf{N},ns}.$$

Theorem 3.3. *The moduli space $\mathcal{F}_7^{\mathbf{N},ns}$ is rational.*

Proof. The blow-up of $\mathbb{P}^{15} := |\mathcal{I}_{A_1 \cup A_2/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ along $\mathbb{P}^8 := |\mathcal{I}_{\mathbb{P}^1 \times A/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ is a $\mathbb{P}^9$-bundle $\pi : \mathbb{P} \to \mathbb{P}^6$. Let $o \in \mathbb{P}^6$, then $\pi^{-1}(o)$ is a 9-dimensional linear system generated by $\mathbb{P}^8$ and by an element $S \in \mathbb{P}^{15}$ not containing $\mathbb{P}^1 \times A$. It is useful to remark that then the base locus of $\pi^{-1}(o)$ is $S \cdot (\mathbb{P}^1 \times A) = A_1 + A_2 + N_{o1} + \cdots + N_{o6}$, where the last six summands are the 'horizontal' lines in the surface S. Let $p'_A : \mathbb{P}^1 \times A \to A$ be the projection map. Since $N_{o1} + \cdots + N_{o6} \in |\mathcal{O}_A(3)|$, this yields an immediate identification

$$\mathbb{P}^6 := |\mathcal{O}_A(3)| = |\mathcal{O}_{\mathbb{P}^1}(6)|,$$

under the linear isomorphism sending o to $n := (p'_A)_*(N_{o1} + \cdots + N_{o6})$. Now it is clear that G acts linearly on $\mathbb{P}$ and on $\mathbb{P}^6$. Furthermore, by Castelnuovo's criterion, $\mathbb{P}^6/G$ is a unirational surface, hence it is rational. To complete the proof it suffices to show that $\mathbb{P}/G$ is a $\mathbb{P}^9$-bundle over a nonempty open set of $\mathbb{P}/G$. Let $U \subset \mathbb{P}^6$ be the open set of the degree six divisors $n \in |\mathcal{O}_A(3)|$ such that the stabilizer of n in $\operatorname{Aut} A$ is trivial; this is nonempty since there are no non-trivial automorphisms of $\mathbb{P}^1$ mapping a set of 6 general points to itself. This immediately implies that, whenever $o \in U$, the stabilizer of $\pi^{-1}(o)$ in G is trivial: otherwise n would be invariant under the action of some non trivial $\gamma \in G$. Let $\mathbb{P}_U$ be the restriction of $\mathbb{P}$ to U. Since the stabilizer of $\pi^{-1}(o)$ is trivial along U, it follows from Kempf's descent lemma, cfr. [DN], that $\mathbb{P}_U$ descends to a $\mathbb{P}^9$-bundle $\mathbb{P}_U/G$ over U/G. This implies the statement. $\qquad\square$

3.2. **The fibre of the Prym-Nikulin map χ_7^{ns}.** We start with a general point (S, M, H) in $\mathcal{F}_7^{\mathbf{N},ns}$ and a general smooth $C \in |H|$. We still denote by A_1 and A_2 the two vertical conics of S.

Lemma 3.4. *We have*
  (i) $h^0(\mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2}(2,2)) = h^1(\mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2}(2,2)) = 1$,
  (ii) *C is not quadratically normal in $\mathbb{P}^5$,*
  (iii) $h^0(\mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)) = 6$,
  (iv) $h^0(\mathcal{I}_{C \cup A_1 \cup A_2/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)) = 4$.

Proof. Item (i) follows from the exact sequence

$$0 \longrightarrow \mathcal{I}_{S/\mathbb{P}^1 \times \mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(-2,-3) \longrightarrow \mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2} \longrightarrow \mathcal{I}_{C/S} \simeq \mathcal{O}_S(-H) \longrightarrow 0$$

tensored by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,2)$ and the isomorphisms

$$\mathcal{O}_S(2L - H) \simeq \mathcal{O}_S(H - 2M) \simeq \mathcal{O}_S(A_1 + A_2),$$

cf. (7). Item (ii) is an immediate consequence of (i).

Item (iii) follows from the above sequence tensored by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,3)$ and the equality $h^0(S, R + A_1 + A_2) = 5$. Item (iv) follows similarly. $\qquad\square$

Remark 3.5. Lemma 3.4(ii) is of particular interest. Indeed, it implies that the image of the moduli map $\chi_7^{ns} : \mathcal{P}_7^{\mathbf{N},ns} \to \mathcal{R}_7$ lies in the ramification locus of the Prym map $\mathcal{R}_7 \to \mathcal{A}_6$, cf. [Be].

Theorem 1.1 in genus 7 follows by detecting the locus $\mathcal{D}_C$ in $|\mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ that parametrizes Nikulin surfaces of non-standard type.

Theorem 3.6. *The fibre of* $\chi_7^{ns} : \mathcal{F}_7^{\mathbf{N},ns} \to \mathcal{R}_7$ *over* C *is* 4-*dimensional.*

Proof. We consider the 5-dimensional linear system $|\mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$, cf. Lemma 3.4(iii), along with its linear subsystem $|\mathcal{I}_{C \cup A_1 \cup A_2/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)| \subset |\mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$, which has dimension 3 and parametrizes Nikulin surfaces of non-standard type by Lemma 3.4(iv) and Proposition 3.2.

We are going to show the existence of a one-dimensional family of such linear subsystems, the union of which is a hypersurface $\mathcal{D}_C$ in $|\mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ parametrizing Nikulin surfaces of non-standard type.

Lemma 3.4(i) yields that $C \subset Y \subset \mathbb{P}^1 \times \mathbb{P}^2$, where Y is integral of bidegree $(2,2)$. The linear system $|\mathcal{O}_Y(1,0)|$ is a ruling of conics on Y, and A_1, A_2 are in this ruling, since $C \subset Y$ and $A_j \cdot C = 6$. For each $x \in \mathbb{P}^1$ we denote by A_x the conic over the point x. Consider the map

$$p_* : |\mathcal{O}_Y(1,0)| \longrightarrow |\mathcal{O}_{\mathbb{P}^2}(2)|,$$

sending A_x to $p_* A_x$. Since $p_Y : Y \to \mathbb{P}^2$ has degree two, the map p_* has degree one or two. As $p_* A_1 = p_* A_2 = A$, it has degree two. Hence there exists an involution $\iota : \mathbb{P}^1 \to \mathbb{P}^1$ such that $p_* A_x = p_* A_{\iota(x)}$. Thus we have a fibration

$$\mathcal{D}_C \longrightarrow \mathbb{P}^1,$$

sending a surface S to the pair of conjugated points defined by its vertical conics; in other words, the base $\mathbb{P}^1$ is the quotient of $|\mathcal{O}_Y(1,0)|$ by the involution ι and the fiber over a point $\langle x, \iota(x) \rangle \in \mathbb{P}^1$ is the 3-dimensional linear subsystem $|\mathcal{I}_{A_x \cup A_{\iota(x)} \cup C/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)| \subset |\mathcal{I}_{C/\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$. Hence $\mathcal{D}_C$ is 4-dimensional.

It remains to show that the moduli map $m_C : \mathcal{D}_C \dashrightarrow \mathcal{F}_7^{\mathbf{N},ns}$ is generically finite. This easily follows since there are finitely many automorphism of $\mathbb{P}^1 \times \mathbb{P}^2$ fixing C; indeed, any of them different from the identity would induce a non-trivial automorphism of C itself. $\qquad\square$

4. The case of genus 9

Let (S, M, H) be a general primitively polarized Nikulin surface of non-standard type of genus 9. Let $L = H - M$ and

$$R \sim \frac{1}{2}(H - N_1 - N_2 - N_3 - N_4) \quad \text{and} \quad R' \sim L - R \sim \frac{1}{2}(H - N_5 - N_6 - N_7 - N_8)$$

be as in §2. We have $R^2 = R'^2 = 2$ and $R \cdot R' = 4$. By Proposition 2.3, the line bundle L defines an embedding $S \subset \mathbb{P}^7$ and $|R|$ and $|R'|$ are base point free linear systems whose

general member is a smooth, irreducible curve of genus 2. As in (4), the embeddings $C \subset S \subset \mathbb{P}^7$ thus factor as

$$S \subset \left(\mathbb{P}^2 \times \mathbb{P}^2\right) \cap \mathbb{P}^7 \subset \mathbb{P}^8.$$

We may assume that the intersection

$$T := \left(\mathbb{P}^2 \times \mathbb{P}^2\right) \cap \mathbb{P}^7$$

is transversal (cf. Remark 2.8 and Proposition 4.3 below) and hence a sextic Del Pezzo threefold. Since $\omega_T \simeq \mathcal{O}_T(-2, -2)$, we have, by adjunction, cf. [GS, §4.9]:

Lemma 4.1. *The surface S is the complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$ of a hyperplane section and of a quadratic section defined by a quadric Q:*

$$S = Q \cap \mathbb{P}^7 \cap \left(\mathbb{P}^2 \times \mathbb{P}^2\right) = Q \cap T \subset \mathbb{P}^8.$$

The first and second projections $p'_S : S \to \mathbb{P}^2$ and $p_S : S \to \mathbb{P}^2$ are double coverings of $\mathbb{P}^2$, contracting the set of lines $\{N_1, \ldots, N_4\}$ and $\{N_5, \ldots, N_8\}$, respectively.

The line bundle

(8) $$E := H - N_1 - \cdots - N_8.$$

plays a crucial role.

Lemma 4.2. *The linear system $|E|$ is an elliptic pencil on S. Furthermore, for any $F \in |E|$, we have:*

  (i) *The maps $p'_F : F \to \mathbb{P}^2$ and $p_F : F \to \mathbb{P}^2$ are double coverings onto smooth conics A' and A, respectively;*
  (ii) *$F = (A' \times A) \cap \mathbb{P}^7 \subset \left(\mathbb{P}^2 \times \mathbb{P}^2\right) \cap \mathbb{P}^7 = T$.*
  (iii) *The two surfaces $Y' := (A' \times \mathbb{P}^2) \cap \mathbb{P}^7$ and $Y := \left(\mathbb{P}^2 \times A\right) \cap \mathbb{P}^7$ are minimal sextic scrolls (isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) embedded in T such that $F = Y' \cap Y$ and F is anticanonical in Y' and Y. Moreover, $N_1 \cup \cdots \cup N_4 \subset Y'$ and $N_5 \cup \cdots \cup N_8 \subset Y$.*

Proof. Using the fact that $\operatorname{rk} \operatorname{Pic} S = 9$, it is easy to check that E is nef and primitive, whence an elliptic pencil. Let $F \in |E|$. As p_S has degree two, p_F is either birational or of degree two onto its image. In the former case the image would be a quartic curve, as $R \cdot E = 4$; however, p contracts N_i, $i = 1, 2, 3, 4$, and $N_i \cdot E = 2$, so the quartic would have four singular points, a contradiction. The same works for p'_F. Hence, (i) is proved.

Letting $A = p(F)$ and $A' = p'(F)$, we have

$$F \subset \left(A \times A'\right) \subset \left(\mathbb{P}^2 \times \mathbb{P}^2\right) \cap \mathbb{P}^7 = T$$

Moreover, $A \times A'$ is the 2-Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)|$. Hence F is a hyperplane section of it, proving (ii). Property (iii) easily follows since the projection $p'_Y : Y' \to A'$ realizes Y' as the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ over $A' \simeq \mathbb{P}^1$, and similarly for Y. $\square$

4.1. A rational parametrization of a double cover of $\mathcal{F}_9^{\mathbf{N}, ns}$.
Let us fix a Del Pezzo threefold $T := \left(\mathbb{P}^2 \times \mathbb{P}^2\right) \cap \mathbb{P}^7 \subset \mathbb{P}^8$. Since T is smooth, the restriction map $\operatorname{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) \to \operatorname{Pic} T$ is an isomorphism by the Lefschetz Theorem, whence T contains no plane. In particular, both projections $p'_T : T \to \mathbb{P}^2$ and $p_T : T \to \mathbb{P}^2$ realize T as a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. We fix four vertical lines $N_1, \ldots, N_4$ and four horizontal lines $N_5, \ldots, N_8$ in T such that the points $p'(N_1), \ldots, p'(N_4)$ are in general position, and the same for $p(N_5), \ldots p(N_8)$.

Proposition 4.3. *A general member of* $|\mathcal{I}_{N_1\cup\cdots\cup N_8/T}(2,2)|$ *is smooth and every smooth* $S \in |\mathcal{I}_{N_1\cup\cdots\cup N_8/T}(2,2)|$ *is a non-standard Nikulin surface of genus* 9 *polarized by* $\mathcal{O}_S(2,0)(N_5+\cdots+N_8)$.

Moreover, $\dim |\mathcal{I}_{N_1\cup\cdots\cup N_8/T}(2,2)| = 3$.

Proof. Set
$$b' := \{p'(N_1),\ldots,p'(N_4)\} \ \text{ and } \ b := \{p(N_5),\ldots p(N_8)\}.$$
and let A' (respectively, A) be any smooth conic passing through b' (resp., b). Define the following surfaces contained in T:

$$(9) \qquad Y' := (A' \times \mathbb{P}^2)\cap \mathbb{P}^7 \in |\mathcal{O}_T(2,0)| \ \text{ and } \ Y := \left(\mathbb{P}^2 \times A\right)\cap\mathbb{P}^7 \in |\mathcal{O}_T(0,2)|,$$

which are minimal sextic scrolls isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$. One easily verifies that $F := Y'\cap Y$ is anticanonical in both Y' and Y and that $N_1\cup\cdots\cup N_4 \subset Y'$ and $N_5\cup\cdots\cup N_8 \subset Y$. More precisely,

$$\begin{aligned}
N_1+\cdots+N_4 &\in |\mathcal{O}_{Y'}(2,0)| \simeq |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,0)| \\
N_5+\cdots+N_8 &\in |\mathcal{O}_Y(0,2)| \simeq |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,4)|.
\end{aligned}$$

We have $\mathcal{I}_{F/Y'\cup Y} \simeq \mathcal{I}_{F/Y'} \oplus \mathcal{I}_{F/Y}$. Tensoring by $\mathcal{O}_{Y\cup Y'}(2,2)$ and using the fact that $F \in |\mathcal{O}_{Y'}(0,2)|$ and $F \in |\mathcal{O}_Y(2,0)|$ by (9), we get

$$(10) \qquad \mathcal{I}_{F/Y'\cup Y}(2,2) \simeq \mathcal{O}_{Y'}(2,0) \oplus \mathcal{O}_Y(0,2) \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,4)$$

We also have a short exact sequence

$$(11) \qquad 0 \longrightarrow \mathcal{I}_{Y'\cup Y/T}(2,2) \simeq \mathcal{O}_T \longrightarrow \mathcal{I}_{F/T}(2,2) \longrightarrow \mathcal{I}_{F/Y'\cup Y}(2,2) \longrightarrow 0,$$

where the isomorphism follows as $Y'\cup Y \in |\mathcal{O}_T(2,2)|$ by (9). From (10) and (11) we get that $\mathcal{I}_{F/T}(2,2)$ is globally generated and the restriction map of linear systems

$$|\mathcal{I}_{F/T}(2,2)| \longrightarrow (F + |\mathcal{O}_{Y'}(2,0)|) \times (F + |\mathcal{O}_Y(0,2)|)$$

is surjective. Hence, there is a smooth $S \in |\mathcal{I}_{F/T}(2,2)|$ containing $N_1 \cup \cdots \cup N_8$, and

$$(12) \qquad\qquad\qquad S\cdot Y' \ = \ N_1+\cdots+N_4 + F \in |\mathcal{O}_S(2,0)|$$

$$(13) \qquad\qquad\qquad S\cdot Y \ = \ N_5+\cdots+N_8 + F \in |\mathcal{O}_S(0,2)|.$$

In particular, the divisor

$$N_1+\cdots+N_8 \in |\mathcal{O}_S(2,2)(-2F)|$$

is 2-divisible in $\operatorname{Pic} S$. It is then easy to see that S is a non-standard Nikulin surface of genus 9 polarized by $\mathcal{O}_S(2,0)(N_5+\cdots+N_8)$.

Finally, the sequence

$$0 \longrightarrow \mathcal{I}_{S/T}(2,2) \simeq \mathcal{O}_T \longrightarrow \mathcal{I}_{N_1\cup\cdots\cup N_8/T}(2,2) \longrightarrow \mathcal{I}_{N_1\cup\cdots\cup N_8/S}(2,2) \simeq \mathcal{O}_S(2F) \longrightarrow 0,$$

yields $h^0(\mathcal{I}_{N_1\cup\cdots\cup N_8/T}(2,2)) = 4$. $\qquad\qquad\square$

We obtain a nice parametrization of the moduli space $\mathcal{F}_9^{\mathbf{N},ns}$. We fix four vertical lines $N_1,\ldots,N_4$ in T, and observe that in the space of the Segre embedding one has

$$\langle N_1\cup\cdots\cup N_4\rangle = \mathbb{P}^7$$

since $N_1,\ldots,N_4$ are contained in a minimal sextic scroll $Y' \simeq \mathbb{P}^1\times\mathbb{P}^1 \subset \mathbb{P}^7$ defined as in the previous proof. It is clear that, up to the action of $\operatorname{Aut} T$, we can choose this set of four lines up to the ordering of its elements. Since these four lines are spanning

$\langle T \rangle = \mathbb{P}^7$ and the automorphisms of T are the automorphisms of $\mathbb{P}^2 \times \mathbb{P}^2$ fixing this $\mathbb{P}^7$, the stabilizer of $N_1 \cup N_2 \cup N_3 \cup N_4$ in $\operatorname{Aut} T$ coincides with the stabilizer in $\operatorname{Aut}(\mathbb{P}^2 \times \mathbb{P}^2)$ of the same set. Recall that

$$\operatorname{Aut}(\mathbb{P}^2 \times \mathbb{P}^2) \simeq \operatorname{PGL}(3) \times \operatorname{PGL}(3) \times \mathbb{Z}/2\mathbb{Z},$$

where the $\mathbb{Z}/2\mathbb{Z}$-factor is due to the involution interchanging the two factors of $\mathbb{P}^2 \times \mathbb{P}^2$. For $i = 1, \ldots, 4$ we have $N_i = \{o_i\} \times \ell_i$, where $o_i = p'(N_i)$ is a point and $\ell_i = p(N_i)$ is a line. The stabilizer of $N_1 \cup \cdots \cup N_4$ acts on the set of pairs $\{(o_1, \ell_1), \ldots, (o_4, \ell_4)\}$. Hence the stabilizer is the diagonal embedding $S_4 \subset S_4 \times S_4$. The action is the diagonal action: $\alpha(o_i, \ell_i) = (\alpha(o_i), \alpha(\ell_i))$. We define $N_{1\ldots4} := \{N_1, \ldots, N_4\}$ and choose a general set $N_{5\ldots8} := \{N_5, \ldots, N_8\}$ of four horizontal lines, or equivalently, four points in $p(T) = \mathbb{P}^2$. Then the moduli space of pairs $(N_{1\ldots4}, N_{5\ldots8})$ is precisely the quotient

$$(\mathbb{P}^2)^4 / S_4,$$

where $S_4 \subset \operatorname{Aut} T$ is the previous group of automorphisms. Hence it acts as above: $\alpha(o, \ell) = (\alpha(o), \alpha(\ell))$ and $\alpha(\ell, o) = (\alpha(\ell), \alpha(o))$. Thus we have:

Theorem 4.4. *The quotient $(\mathbb{P}^2)^4 / S_4$ is the 4-symmetric product of $\mathbb{P}^2$ and hence is rational.*

For a general pair (N_{1234}, N_{5678}), with $N_{1\ldots4}$ fixed, the linear system $|\mathcal{I}_{N_1 \cup \ldots \cup N_8/T}(2,2)|$ defines a $\mathbb{P}^3$-bundle over $(\mathbb{P}^2)^4$. This bundle descends to $(\mathbb{P}^2)^4 / S_4$, thus implying the following:

Theorem 4.5. *The moduli space of fourtuples (S, M, H, N_{1234}) is rational and a double cover of $\mathcal{F}_9^{\mathbf{N}, ns}$.*

4.2. The fibre of the Prym-Nikulin map χ_9^{ns}. Let both $(S, M, H) \in \mathcal{F}_9^{\mathbf{N}, ns}$ and $C \in |H|$ be general. Let E be as in (8) and recall Lemma 4.2. The genus 9 case of Theorem 1.1 is a consequence of the next two results.

Lemma 4.6. *We have*

$$\dim |\mathcal{I}_{C/T}(2,2)| = 2.$$

In particular, C is quadratically normal.

Proof. Fix any $F \in |E|$. Since $2L \sim C + F$ and T is projectively normal, the curve $C \cup F$ is the complete intersection in T of two quadratic sections. Therefore, we have

(14) $h^0(\mathcal{I}_{C \cup F/T}(2,2)) = 2$ and $h^1(\mathcal{I}_{C \cup F/T}(2,2)) = h^2(\mathcal{I}_{C \cup F/T}(2,2)) = 0.$

We consider the standard exact sequence

(15) $0 \longrightarrow \mathcal{I}_{C \cup F/T}(2,2) \longrightarrow \mathcal{I}_{C/T}(2,2) \oplus \mathcal{I}_{F/T}(2,2) \longrightarrow \mathcal{I}_{C \cap F/T}(2,2) \longrightarrow 0.$

Taking cohomology in (10) and (11) yields

(16) $h^0(\mathcal{I}_{F/T}(2,2)) = 11$ and $h^1(\mathcal{I}_{F/T}(2,2)) = h^2(\mathcal{I}_{F/T}(2,2)) = 0.$

This, together with the sequence

$$0 \longrightarrow \mathcal{I}_{F/T}(2,2) \longrightarrow \mathcal{I}_{F \cap C/T}(2,2) \longrightarrow \mathcal{I}_{F \cap C/F}(2,2) \simeq \mathcal{O}_F(2L - C) \longrightarrow 0,$$

and the fact that $2L - C \sim F$ and $\mathcal{O}_F(F) \simeq \mathcal{O}_F$, yields

(17) $h^0(\mathcal{I}_{F \cap C/T}(2,2)) = h^0(\mathcal{I}_{F/T}(2,2)) + h^0(\mathcal{O}_F) = 12.$

Thus, the cohomology of (15) together with (14), (16) and (17) yields $h^0(\mathcal{I}_{C/T}(2,2)) = 3$.

The fact that C is quadratically normal is easily checked. $\qquad\square$

Proposition 4.7. *A general* $S' \in |\mathcal{I}_{C/T}(2,2)|$ *defines a point of* $\mathcal{F}_9^{\mathbf{N},ns}$, *and the moduli map* $|\mathcal{I}_{C/T}(2,2)| \dashrightarrow \mathcal{F}_9^{\mathbf{N},ns}$ *is generically injective.*

Proof. As $S \cdot S' \sim 2L$ on S, we have

$$(18) \qquad\qquad S' \cdot S = F + C \in |\mathcal{O}_{S'}(2,2)|$$

for some $F \in |E|$. Let Y' and Y be as in Lemma 4.2(iii).

Using the fact that F is anticanonical on Y', it is not difficult to show that

$$S' \cdot Y' = N_1' + \cdots + N_4' + F \in |\mathcal{O}_{Y'}(2,2)| \simeq |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(6,2)|,$$

with $N_1', \ldots, N_4'$ four disjoint lines in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)|$. Similarly, one shows that

$$S' \cdot Y = N_5' + \cdots + N_8' + F \in |\mathcal{O}_Y(2,2)| \simeq |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,6)|,$$

with $N_5', \ldots, N_8'$ four disjoint lines in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)|$. Hence S' is a non-standard Nikulin surface of genus 9 by Proposition 4.3.

We now show that the moduli map $m_C : |\mathcal{I}_{C/T}(2,2)| \dashrightarrow \mathcal{F}_9^{\mathbf{N},ns}$ is generically injective. Assume that $m_C(S') = m_C(S'')$, for distinct $S', S'' \in |\mathcal{I}_{C/T}(2,2)|$. Then there exists $\alpha \in \mathrm{Aut}(T)$ such that $\alpha(S') = S''$. In particular, such an α would fix C and thus induce a non-trivial automorphism of C. This is a contradiction because the image of $m_9^{\mathbf{N},ns}$ has dimension at least $20 - 2 = 18$, while the maximal dimension of a component of the locus in $\mathcal{M}_9$ of curves with a non-trivial automorphism is $2g - 1 = 17$, cf. [Co]. $\qquad\square$

5. The case of genus 11

Let (S, M, H) be a general primitively polarized Nikulin surface of non-standard type of genus 11. Let $L = H - M$, then we have as in §2

$$R \sim \frac{1}{2}(H - N_1 - N_2) \quad \text{and} \quad R' \sim L - R \sim \frac{1}{2}(H - N_3 - \cdots - N_8).$$

By Proposition 2.3, the line bundle L defines an embedding $S \subset \mathbb{P}^9$. Moreover $|R|$ and $|R'|$ are base point free linear systems, respectively of dimensions 3 and 2, such that $R^2 = 4$, $R'^2 = 2$ and $R \cdot R' = 5$. The embedding $S \subset \mathbb{P}^9$ factors as follows

$$S \subset (\mathbb{P}^2 \times \mathbb{P}^3) \cap \mathbb{P}^9 \subset \mathbb{P}^{11},$$

where the inclusion $\mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}$ is the Segre embedding and $\mathbb{P}^9$ is linearly embedded. We may assume (cf. Remark 2.8 and Proposition 5.5 below) that the intersection

$$T := (\mathbb{P}^2 \times \mathbb{P}^3) \cap \mathbb{P}^9$$

is transversal, so that T is a smooth threefold with $K_T \sim \mathcal{O}_T(-1,-2)$. Hence, by the adjunction formula, S is a divisor of type $(1,2)$ in T and we can conclude as follows.

Lemma 5.1. *The surface S belongs to* $|-K_T|$ *and is a complete intersection in* $\mathbb{P}^2 \times \mathbb{P}^3$ *of three divisors, respectively of type* $(1,1)$, $(1,1)$ *and* $(1,2)$.

Let $(x,y) := (x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2 : y_3)$ be coordinates on $\mathbb{P}^2 \times \mathbb{P}^3$. The equations of S in $\mathbb{P}^2 \times \mathbb{P}^3$ can be written as

$$a_0 x_0 + a_1 x_1 + a_2 x_2 = b_0 x_0 + b_1 x_1 + b_2 x_2 = c_0 x_0 + c_1 x_1 + c_2 x_2 = 0,$$

where for $i = 0, 1, 2$ the coefficients a_i and b_i are linear forms while the c_i are quadratic forms in $(y_0 : y_1 : y_2 : y_3)$. The equations of T are

$$a_0 x_0 + a_1 x_1 + a_2 x_2 = b_0 x_0 + b_1 x_1 + b_2 x_2 = 0.$$

The morphism $p_T : T \to \mathbb{P}^3$ is birational and its inverse is described by

$$(y) \mapsto (a_1 b_2 - a_2 b_1, a_2 b_0 - a_0 b_2, a_0 b_1 - a_1 b_0) \times (y_0 : y_1 : y_2 : y_3).$$

Equivalently, p_T is the blow-up of the scheme γ defined by the 2×2 minors of

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}.$$

Since T is smooth, γ is a smooth (rational normal cubic) curve. Let $P_\gamma := p_T^{-1}(\gamma)$ be the exceptional divisor of p_T.

Lemma 5.2. *We have $P_\gamma \in |\mathcal{O}_T(-1, 2)|$ and $P_\gamma \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Under this identification, $\mathcal{O}_{P_\gamma}(0, 1) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 3)$ and $\mathcal{O}_{P_\gamma}(1, 0) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$.*

Proof. We have

$$\mathcal{O}_T(P_\gamma) \simeq \omega_T \otimes p_T^*(\omega_{\mathbb{P}^3}^\vee) \simeq \mathcal{O}_T(-1, -2) \otimes \mathcal{O}_T(0, 4) \simeq \mathcal{O}_T(-1, 2).$$

As is well known, $\mathcal{N}_{\gamma/\mathbb{P}^3} \simeq \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$, whence $P_\gamma \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Since $\gamma \subset \mathbb{P}^3$ is a curve of degree 3, it follows that $\mathcal{O}_{P_\gamma}(0, 1) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 3)$. Finally, we have

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) \simeq \omega_{P_\gamma} \simeq \mathcal{O}_{P_\gamma}(K_T + P_\gamma) \simeq \mathcal{O}_{P_\gamma}(-2, 0),$$

whence $\mathcal{O}_{P_\gamma}(1, 0) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$. $\square$

Lemma 5.3. *We have*

$$S \cdot P_\gamma = \Gamma + N_3 + \cdots + N_8,$$

where Γ is a smooth element of $|\mathcal{O}_{P_\gamma}(1, 0)| = |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)|$. In particular, p'_Γ is a two to one map onto a line.
Moreover, Γ has the following properties:

(i) $\Gamma \cdot N_3 = \cdots = \Gamma \cdot N_8 = 1$ *and* $\Gamma \cdot N_1 = \Gamma \cdot N_2 = 2$.
(ii) $\Gamma + N_1 + N_2 \sim R'$.

Proof. We know that $N_3, \ldots, N_8$ are contracted by p_S, whence they are six disjoint fibres of $p_{P_\gamma} : P_\gamma \to \gamma$. On the other hand, $S \in |\mathcal{O}_T(1, 2)|$, hence its restriction to P_γ belongs to $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 7)|$ by Lemma 5.2. This implies that $\Gamma \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)| = |\mathcal{O}_{P_\gamma}(1, 0)|$, and it immediately follows that p' maps Γ two to one onto a line. If Γ is not smooth, then it contains a fibre N_9 of p_{P_γ}. But then one can check (on S) that N_9 is orthogonal to $R, N_1, \ldots, N_8$. Hence $\operatorname{Pic} S$ has rank ≥ 10, against the generality of S. The properties (i) and (ii) are easy to check. $\square$

Consider the line $\ell := p'(\Gamma)$ and the surface

(19) $$P_\ell := p'^{-1}(\ell) \cap T \in |\mathcal{O}_T(1, 0)|.$$

Let $l_0 x_0 + l_1 x_1 + l_2 x_2 = 0$ be the equation of ℓ, with $l_0, l_1, l_2 \in \mathbb{C}$. Then P_ℓ is defined by

$$l_0 x_0 + l_1 x_1 + l_2 x_2 = a_0 x_0 + a_1 x_1 + a_2 x_2 = b_0 x_0 + b_1 x_1 + b_2 x_2 = 0.$$

The surface P_ℓ is a $\mathbb{P}^1$-bundle over ℓ and $p(P_\ell) \subset \mathbb{P}^3$ is a quadric through γ defined by the equation

$$\det \begin{pmatrix} l_0 & l_1 & l_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} = 0.$$

Lemma 5.4. *One has*

$$S \cdot P_\ell = \Gamma + N_1 + N_2.$$

Moreover, $p(P_\ell)$ is smooth and $P_\ell \simeq \mathbb{P}^1 \times \mathbb{P}^1$, with $\mathcal{O}_{P_\ell}(1,1) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1)$.

Proof. The first assertion follows from Lemma 5.3(ii) and (19). Next assume $p(P_\ell)$ is singular. Then it is a rank 3 cone of vertex $e = p(N_1) \cap p(N_2)$, and $e \in \gamma$. But then the curve $p_T^{-1}(e)$ is contained in $S \cap P_\ell$ as a proper component of Γ, against the irreducibility of Γ. Finally, since $p(P_\ell)$ is a smooth quadric, we have $\mathcal{O}_{P_\ell}(0,1) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$. Hence, the isomorphism $\mathcal{O}_{P_\ell}(1,1) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1)$ follows. $\square$

In the considerations so far, γ, T and P_γ are fixed and independent of S, whereas Γ depends on S and determines the line $\ell \subset \mathbb{P}^2$ and thus the surface P_ℓ. Actually, ℓ alone determines both P_ℓ and Γ, as $P_\ell = p'^{-1}_T(\ell)$ and $\Gamma = P_\ell \cap P_\gamma$. In order to parametrize all Nikulin surfaces we will indeed let $\ell \subset \mathbb{P}^2$ vary.

5.1. **Rationality of $\mathcal{F}_{11}^{\mathbf{N},ns}$.** Fix any smooth rational normal cubic curve $\gamma \subset \mathbb{P}^3$ and let $p_T : T \to \mathbb{P}^3$ be the blow-up along γ with exceptional divisor P_γ. Then $T \subset \mathbb{P}^2 \times \mathbb{P}^3$ and we denote as before by $p'_T : T \to \mathbb{P}^2$ the first projection. Any line $\ell \subset \mathbb{P}^2$ determines a surface $P_\ell := p'^{-1}(\ell) \cap T \in |\mathcal{O}_T(1,0)|$ and a curve $\Gamma_\ell := P_\ell \cap P_\gamma \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)|$, which is smooth for general ℓ.

Proposition 5.5. *Let ℓ be general. Then a general member of $|\mathcal{I}_{\Gamma_\ell/T}(1,2)|$ is smooth and every smooth $S \in |\mathcal{I}_{\Gamma_\ell/T}(1,2)|$ is a non-standard Nikulin surface of genus 11 polarized by $\mathcal{O}_S(1,2)(-\Gamma_\ell)$.*
  Moreover, $\dim |\mathcal{I}_{\Gamma_\ell/T}(1,2)| = 12$.

Proof. Consider the exact sequences of ideal sheaves

$$(20) \qquad 0 \longrightarrow \mathcal{I}_{P_\gamma/T}(1,2) \longrightarrow \mathcal{I}_{\Gamma_\ell/T}(1,2) \longrightarrow \mathcal{O}_{P_\gamma}(1,2)(-\Gamma_\ell) \longrightarrow 0$$

and

$$(21) \qquad 0 \longrightarrow \mathcal{I}_{P_\ell/T}(1,2) \longrightarrow \mathcal{I}_{\Gamma_\ell/T}(1,2) \longrightarrow \mathcal{O}_{P_\ell}(1,2)(-\Gamma_\ell) \longrightarrow 0.$$

By (19) and Lemma 5.2 we have

$$(22) \qquad \mathcal{I}_{P_\gamma/T}(1,2) \simeq \mathcal{O}_T(2,0) \text{ and } \mathcal{I}_{P_\ell/T}(1,2) \simeq \mathcal{O}_T(0,2),$$

and by Lemmas 5.2, 5.3 and 5.4 we have

$$(23) \qquad \mathcal{O}_{P_\gamma}(1,2)(-\Gamma_\ell) \quad \simeq \quad \mathcal{O}_{P_\gamma}(0,2) \quad \simeq \quad \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,6)$$
$$(24) \qquad \mathcal{O}_{P_\ell}(1,2)(-\Gamma_\ell) \quad \simeq \quad \mathcal{O}_{P_\ell}(2,0) \quad \simeq \quad \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,0).$$

Thus, either of (20) and (21) shows that $\mathcal{I}_{\Gamma_\ell/T}(1,2)$ is globally generated. In particular, a general $S \in |\mathcal{I}_{\Gamma_\ell/T}(1,2)|$ is smooth and hence a $K3$ surface by adjunction.

From (20)-(24) one obtains that $h^0(\mathcal{I}_{\Gamma_\ell/T}(1,2)) = 13$ and that the restriction maps

$$\rho_\gamma : |\mathcal{I}_{\Gamma_\ell/T}(1,2)| \longrightarrow \Gamma_\ell + |\mathcal{O}_{P_\gamma}(0,2)| = \Gamma_\ell + |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,6)|$$
$$\rho_\ell : |\mathcal{I}_{\Gamma_\ell/T}(1,2)| \longrightarrow \Gamma_\ell + |\mathcal{O}_{P_\ell}(2,0)| = \Gamma_\ell + |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,0)|$$

are surjective. A general member of $|\mathcal{O}_{P_\gamma}(0,2)|$ and of $|\mathcal{O}_{P_\ell}(2,0)|$ consists of 6 and 2 disjoint lines, respectively. Hence a general $S \in |\mathcal{I}_{\Gamma_\ell/T}(1,2)|$ contains a configuration of 8 disjoint lines, say $N_1, \ldots, N_8$, such that

(25) $\Gamma_\ell + N_1 + N_2 = S \cdot P_\ell \in |\mathcal{O}_S(1,0)|$ and $\Gamma_\ell + N_3 + \cdots + N_8 = S \cdot P_\gamma \in |\mathcal{O}_S(-1,2)|$

(using (19) and Lemma 5.2). By (25), we also get

$$2\Gamma_\ell + N_1 + \cdots + N_8 \in |\mathcal{O}_S(0,2)|,$$

whence $N_1 + \cdots + N_8$ is divisible by 2 in $\operatorname{Pic} S$. One easily checks that

$$\mathcal{O}_S(1,2)(-\Gamma_\ell) \sim \mathcal{O}_S(0,2) + N_1 + N_2 \sim \mathcal{O}_S(2,0) + N_3 + \cdots + N_8$$

is a genus 11 polarization having zero intersection with all $N_1, \ldots, N_8$. The fact that S is of non-standard type is an immediate consequence of (25). $\qquad\square$

By the considerations at the beginning of the section, any smooth genus 11 Nikulin surface of nonstandard type is an element of $|\mathcal{O}_T(1,2)|$ and defines a smooth Γ_ℓ mapping $2 : 1$ to a line ℓ on $\mathbb{P}^2$ under p. It moreover comes equipped with 6 horizontal rational curves $N_3 \cup \cdots \cup N_8$, and thus determines 6 points on γ.

Lemma 5.6. *Fix a general line $\ell \subset \mathbb{P}^2$ and six general points $p_3, \ldots, p_8$ on γ. Let $N_i = P_\gamma \cap p_T^{-1}(p_i)$, $i = 3, \ldots, 8$. Then $\dim |\mathcal{I}_{\Gamma_\ell + N_3 + \cdots + N_8/T}(1,2)| = 6$.*

Proof. The statement follows from the ideal sequence

(26) $\quad 0 \longrightarrow \mathcal{I}_{P_\gamma/T}(1,2) \longrightarrow \mathcal{I}_{\Gamma_\ell + N_3 + \cdots + N_8/T}(1,2) \longrightarrow \mathcal{I}_{\Gamma_\ell + N_3 + \cdots + N_8/P_\gamma}(1,2) \longrightarrow 0,$

along with (22) and the fact that $\mathcal{I}_{\Gamma_\ell + N_3 + \cdots + N_8/P_\gamma}(1,2) \simeq \mathcal{O}_{P_\gamma}$ by Lemma 5.3. $\qquad\square$

We consider the $\mathbb{P}^6$-bundle $\mathcal{P}$ over $(\mathbb{P}^2)^\vee \times \operatorname{Sym}^6(\gamma)$, whose fiber over the point $(\ell, p_3 + \cdots + p_8)$ is the linear system $|\mathcal{I}_{\Gamma_\ell + N_3 + \cdots + N_8/T}(1,2)|$ with $N_i = P_\gamma \cap p_T^{-1}(p_i)$. Our construction provides a dominant rational moduli map

$$f : \mathcal{P} \dashrightarrow \mathcal{F}_{11}^{\mathbf{N},ns},$$

and the fibers are orbits of the group of automorphisms of T that fix the exceptional divisor P_γ, namely, of the group of automorphisms of $\gamma \subset \mathbb{P}^3$. In particular $\mathcal{F}_{11}^{\mathbf{N},ns}$ is birational to $\mathcal{P}/\operatorname{Aut}(\gamma)$.

Theorem 5.7. *The moduli space $\mathcal{F}_{11}^{\mathbf{N},ns}$ is rational.*

Proof. Since there are no non-trivial automorphisms of $\mathbb{P}^1$ mapping a set of 6 general points to itself, $\mathcal{P}/\operatorname{Aut}(\gamma)$ is birational to a $\mathbb{P}^6$-bundle over $(\mathbb{P}^2)^\vee \times (\operatorname{Sym}^6(\gamma))/\operatorname{Aut}(\gamma)$. It is then enough to recall that $\operatorname{Sym}^6(\mathbb{P}^1))/\operatorname{Aut}(\mathbb{P}^1)$ is birational to the moduli space $\mathcal{M}_2$ of genus 2 curves, which is known to be rational, cf. [Ig]. $\qquad\square$

5.2. **The fibre of the Prym-Nikulin map χ_{11}^{ns}.** The genus 11 case of Theorem 1.1 is a consequence of the following:

Lemma 5.8. *Let (S, M, H) be a general member of $\mathcal{F}_{11}^{\mathbf{N},ns}$. For any $C \in |H|$, the linear system $|\mathcal{I}_{C/T}(1, 2)|$ is a pencil of nonisomorphic non-standard Nikulin surfaces of genus 11.*

Proof. The ideal sequence of $C \subset S \subset T$ twisted by $\mathcal{O}_T(1, 2)$ becomes

$$(27) \qquad 0 \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{I}_{C/T}(1, 2) \longrightarrow \mathcal{O}_S(\Gamma) \longrightarrow 0,$$

by Proposition 5.5. As a consequence, the 1-dimensional linear system $|\mathcal{I}_{C/T}(1, 2)|$ contains $C \cup \Gamma$ as its base locus and thus parametrizes Nikulin surfaces again by Proposition 5.5. Let $S', S'' \in |\mathcal{I}_{C/T}(1, 2)|$ be two distinct points parametrizing isomorphic Nikulin surfaces. Then there exists $\alpha \in \text{Aut}(T)$ such that $\alpha(S') = S''$, $\alpha(\Gamma) = \Gamma$ and $\alpha(C) = C$. In particular, such an α would induce a non-trivial automorphism of C. Note that the image of $m_{11}^{\mathbf{N},ns}$ has dimension at least $22 - 1 = 21$, which is an upper bound for the dimension of any component of the locus in $\mathcal{M}_{11}$ of curves with a non-trivial automorphism, cf. [Co]. However, this bound is reached only by the hyperelliptic locus and $[C]$ does not lie in it as its Clifford index is 4 by [KLV, Prop. 2.3]. $\square$

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