Notes for a brief introductory course to Bridgeland theory

Enrico Arbarello

Roma 3, March 2023

[BBDG] A. Beilinson, T. Bernstein, P. Deligne, O. Gabber: *Faisceaux Pervers*.

[H] D. Huybrechts: *Fourier Mukai Transforms in Algebraic Geometry*.

[Br-TC] T. Bridgeland: *Stability conditions on triangulated categories*

[Br-K3] T. Bridgeland: *Stability conditions on K3 surfaces*.

[Ba-Tour] A. Bayer: *A Tour of stability conditions*.

[MS-Lect] E. Macrì, B. Schmidt: *Lectures on Bridgeland Stability*

[Ch-B] F. Charles: *Conditions de stabilité et géométrie birationnelle*.

[BM-local] A. Bayer, E. Macrì : *The space of stability conditions on the local projective plane*

[APPENDIX B)]

[BMS] A. Bayer, E. Macrì, P. Stellari: *Stability conditions on abelian threefolds and some Calabi-Yau threefolds.*

[APPENDIX A)]

[Alp] J. Alper : *some incomplete lecture notes on Bridgeland stability.*

[Ba-Short] A. Bayer : *A short proof of the deformation property of Bridgeland stability conditions.*

SLOPES

- C smooth proj. curve.

$$Z : \mathrm{Coh}(C) \longrightarrow \mathbb{C}$$

$$Z(\mathcal{F}) = -\deg(\mathcal{F}) + i\, \mathrm{rk}(\mathcal{F})$$

$$= -\mathrm{ch}_1(\mathcal{F}) + i\, \mathrm{ch}_0(\mathcal{F})$$

$$\mu(\mathcal{F}) = \frac{d(\mathcal{F})}{r(\mathcal{F})} \quad \text{slope}$$

1) $Z : \mathrm{Coh}(C) \longrightarrow \underset{\Lambda}{\overset{\shortparallel}{H^*_{alg}(X,\mathbb{Z})}} \longrightarrow \mathbb{C}$

$\Lambda = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z})$

2) $\mathcal{F} \neq 0 \quad \mathrm{Im}\, Z \geq 0 \,, \quad \mathrm{Im}\, Z = 0 \Rightarrow \mathrm{Re}\, Z < 0$

- $\dim X \geq 2 \qquad X$ smooth proj. surface

Search for

$Z : \mathrm{coh}(X) \longrightarrow \mathbb{C}$ with 1) and 2) :

$$Z(\ ) = (a_0 + i b_0)\, \mathrm{ch}_0 + (a_1 + i b_1)\, \mathrm{ch}_1 + (a_2 + i b_2)\, \mathrm{ch}_2$$

$\mathcal{F} = \mathcal{O}_C(np), \ p \in C \subset S \qquad \mathrm{Im}\, Z(\mathcal{F}) \geq 0 \quad \Rightarrow b_2 = 0$

$\mathcal{F} = \mathcal{O}_X(n) \qquad n \ll 0 \qquad\qquad \Rightarrow b_1 = 0$

$\underset{n \gg 0}{\mathrm{Re}\, Z(\mathcal{O}_C(np)) < 0} \quad \Rightarrow a_2 = 0$

But then $Z(\mathbb{C}_x) = 0 \Rightarrow$ such a Z doesnot exist.

One idea: go to Gieseker stability.

Another: change abelian category:

Start with $\mathcal{A} = \mathrm{Coh}(X)$, consider $\mathcal{D}^b(\mathcal{A}) = \mathcal{D}^b$

Look for another abelian category $\mathcal{A}' \subset \mathcal{D}^b(\mathcal{A})$

$\boxed{\text{A FEW THINGS ABOUT DER. CAT.}}$

[H]. p 1-52

- $C \in \mathcal{D}^b$, $(C[1])^i = C^{i+1}$.

- distinguished triangles: $A \xrightarrow{f} B \to C \to A[1]$

  iso to $A \xrightarrow{f} B \to C(f) = B \oplus A[1]$

  $\uparrow$ ([H] p.33 Def 2.15)

- 4 Axioms e.g.:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dotted, "\exists g"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

commutative

g not necess. unique

- Corollary: $A \to B \to C \xrightarrow{0} A[1]$

$$\Rightarrow \quad B = A \oplus C$$

- **Example** : $E : E^{-1} \xrightarrow{d} E^0$

$$\begin{tikzcd} \mathrm{Ker}\, d\,[1] \arrow[r, hook, "\iota"] & E \arrow[r] & \mathrm{Coker}\, d \\ H^{-1}(E)[1] \arrow[u, phantom, "\wr"] & & H^0(E) \arrow[u, phantom, "\wr"] \end{tikzcd}$$

dist. triang.

i.e. $\mathrm{Coker}\, d \underset{\text{q.i.}}{\cong} C(\iota)$

<u>Exercise</u>

- **Fact** : $A \to B \to C \to A[1]$

triangle + $A, B, C \in \mathcal{A}$

$\Longleftrightarrow$ $A \hookrightarrow B \twoheadrightarrow C$ exact.

- $E, F \in \mathcal{A} \Longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(E, F[i]) = \mathrm{Ext}^i_{\mathcal{A}}(E, F)$

[H] Prop. 2.56

Corollary : C smooth curve $E \in \mathcal{D}(C)$

$$E \cong \bigoplus H^{-i}(E)[i]$$

Hint: start with the case $E = E^{-1} \to E^0$ then

$H^{-1}(E)[1] \to E \to H^0(E) \xrightarrow{\varphi} H^{-1}[1][1] \ldots$

$\varphi = \mathrm{Hom}(H^0(E), H^{-1}[2])$

$= \mathrm{Ext}^2(\qquad) = 0$

.....

New abelian subcat. $\mathcal{A}' \subseteq \mathcal{D}^b(\mathcal{A})$

t-structure: ([DBBG] p.29-40)

$\left.\begin{matrix}\mathcal{D}^{\leq 0}\\ \mathcal{D}^{\geq 0}\end{matrix}\right\}$ Full subcat. s.t.:

E.g. trivial t-struct.

$\mathcal{D}^{\leq 0} = \{C \mid H^i(C)=0,\ i>0\}$

$\mathcal{D}^{\geq 0} = \{C \mid H^i(C)=0,\ i<0\}$

1) $\mathcal{D}^{\leq 0}[1] \subseteq \mathcal{D}^{\leq 0}$

$\mathcal{D}^{\geq 0}[-1] \subseteq \mathcal{D}^{\geq 0}$

2) $\mathrm{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$

3) $\forall\ C \in \mathcal{D}\ \exists$ $\underset{\mathcal{D}^{\leq 0}}{A} \to C \to \underset{\mathcal{D}^{\geq 1}}{B} \to A[1]$

$\mathcal{A}^\# := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} :=$ heart of t-structure

Def $\left\{\begin{matrix}\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]\\ \mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]\end{matrix}\right.$

Theorem: a) $\exists\ \tau^{\leq n} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}^{\leq n}$,

$\exists\ \tau^{\geq n} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}^{\geq n}$.

b) $\tau^{\leq 0}(C) \longrightarrow C \longrightarrow \tau^{\geq 1}(C)$ triangle:

e.g. trivial t-structure: $\tau^{\leq 0}(C)$: $\cdots \to C^{-1} \to \mathrm{Ker}\, d_0 \to 0 \to 0 \cdots$

$\tau^{\geq 1}(C)$: $0 \to C^0/\mathrm{Ker}\, d_0 \to C^1 \to C^2 \to \cdots$

c) $\mathcal{A}^\#$ is abelian.

Def: $H^n_\# : \mathcal{D} \longrightarrow \mathcal{A}^\#$

$$[n] \circ \tau^{\leq n} \circ \tau^{\geq n} \simeq \tau^{\geq n} \circ \tau^{\leq n} \circ [n]$$

Theorem: $H^n_\#$ is a cohomological functor (long exact seq. etc ...)

Remark. $A \in \mathcal{A}_\# \Leftrightarrow H^0_\#(A) = A$

$H^i_\#(A) = 0, \; i \neq 0$

Def bdd t-struct. $\Leftrightarrow$ $H^n_\#(C) \neq 0$ finit. many n.

Also $H^n_\#(C) = H^0_\#(C[n])$

Given: $A \xrightarrow{f} B \longrightarrow C = C(f) \longrightarrow A[1]$

$$\mathrm{Ker}^\# f = H^{-1}_\#(C) = \tau^{\leq 0}(C)[-1]$$

$$\mathrm{Coker}^\# f = H^0_\#(C) = \tau^{\geq 0}(C)$$

(this criterion is Lemma 3.2 in [Br-TC]) its proof is given as Exercise.

Taken as a Definition in [MS lect] p.20 Def 5.1

A CRITERION for $\mathcal{A}^\# \subset \mathcal{D}$, full sub-category, to be the HEART of a t-structure in $\mathcal{D}$:

bdd t-struct. determined by its heart

A) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}^\#, \mathcal{A}^\#[k]) = 0 \quad k < 0$

B) $\forall \; E \in \mathcal{D}, \; \exists \; k_1 > k_2 > \dots > k_n$

$$\begin{tikzcd}
(*) & 0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d, "\cong"'] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[d] \\
 & & A_1 & A_2 \arrow[ul, dashed] & & & A_n \arrow[ull, dashed]
\end{tikzcd}$$

$A_i \in \mathcal{A}^{*}[k_i]$

"Pf": $\Rightarrow$) Start with bdd t-structure

let $-k_1 < -k_2 < \dots < -k_n$ s.t. $H_*^{-k_i}(E) \neq 0$:

$$\tau^{\leq -k_{i-1}}(E) = E_{i-1} \longrightarrow E_i = \tau^{\leq -k_i}(E) \longrightarrow A_i = H_*^{-k_i}(E)[k_i]$$

(think of trivial t-structure and assume $(k_1, \dots, k_n) = (1, 2, \dots, n)$)

$\Leftarrow$) Start with $\mathcal{A}_*$ with A) and B).

Define

$$\mathcal{D}^{\leq 0} = \{ E \mid A_i = 0 \quad k_i > 0 \}$$
$$\mathcal{D}^{\geq 0} = \{ E \mid A_i = 0 \quad k_i < 0 \}$$

i.e. cut $(*)$ in two.

$\boxed{\text{Torsion pairs}} \rightsquigarrow$ t-structure $\rightsquigarrow$ $'\mathcal{A}$

([MS lect] § 6)

$\mathcal{F}, \mathcal{T}$ full subcategories of $\mathcal{A}$ $(= \mathrm{Coh}(X))$

1) $\mathrm{Hom}(T, F) = 0 \quad T \in \mathcal{T}, \quad F \in \mathcal{F}$

2) $\forall \; E \in \mathcal{A}, \; \exists : 0 \to \underset{\mathcal{T}}{\underset{\rotatebox{-90}{$\in$}}{T(E)}} \to E \to \underset{\mathcal{F}}{\underset{\rotatebox{-90}{$\in$}}{F(E)}}$

unique and functorial

$'\mathcal{A} = \{ E \in \mathcal{D} \mid H^i(E) = 0 \quad i \neq -1, 0, \; H^{-1}(E) \in \mathcal{F}, \; H^0(E) \in \mathcal{T} \}$

$= \left\{ E : \; E^{-1} \xrightarrow{d} E^0 \;\middle|\; \begin{array}{l} H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E) \\ \text{short exact sequence in } '\mathcal{A} \end{array} \right\}$

$= \langle \mathcal{F}[1], \mathcal{T} \rangle$ extension closure $\langle \, , \, \rangle$

$'\mathcal{A}$ abelian (it is the heart of a t-structure)

([MS lect.] § 6)

$'\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ where :

$\mathcal{D}^{\leq 0} = \{ E \in \mathcal{D} \mid H^i(E) = 0, \; i > 0, \; H^0(E) \in \mathcal{T} \}$

$\mathcal{D}^{\geq 0} = \{ E \in \mathcal{D} \mid H^i(E) = 0, \; i < -1, \; H^{-1}(E) \in \mathcal{F} \}$.

Example: X smooth proj surface,

$B \in NS(X)$, $\omega \in Amp(X)$, $E \in Coh(X)$

$$\mu = \mu_{B,\omega}(E) = \begin{cases} \dfrac{\omega\cdot c_1(E)}{r(E)} - \omega.B, & \text{if } r(E) \neq 0, \\ 0, & \text{if } r(E) = 0. \end{cases}$$

Sometimes divide both summands by ω^2

μ-stability as usual;

Harder-Narasimhan:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_m = E$$

$A_i = E_i / E_{i-1}$ μ-stable

$$\mu^+(E) = \mu(A_1) > \cdots > \mu(A_n) = \mu^-(E)$$

$\mathcal{T}^{B,\omega} = \{ E \in Coh(X) \mid \text{For all } \underset{\text{in } Coh(X)}{E \twoheadrightarrow F} \quad \mu(F) > 0 \}$ $\supset$ Torsion

$\mathcal{F}^{B,\omega} = \{ E \in Coh(X) \mid \text{For all } \underset{\text{in } Coh(X)}{F \hookrightarrow E}, \quad \mu(F) \leq 0 \}$ $\subset$ Tor-free

equiv: $\left\{ \ldots\ldots\ldots \text{ For all H-N factor } A \text{ of } E \quad \begin{matrix} \mu(A) > 0 \\ \mu(A) \leq 0 \end{matrix} \quad \ldots\ldots \right\}$

usual rules e.g.: E F semist $\mu(E) > \mu(F) \Rightarrow \operatorname{Hom}(E, F) = 0$

$(\mathcal{T}^{B,\omega}, \mathcal{F}^{B,\omega})$ torsion pair

$$Coh^{B,\omega} := \{ E \in \mathcal{D} \mid E : E^{-1} \xrightarrow{d} E^0, \ \underset{\substack{\cap \\ \mathcal{F}^{B,\omega}[1]}}{H^{-1}(E)[1]} \to E \to \underset{\substack{\cap \\ \mathcal{T}^{B,\omega}}}{H^0(E)} \}.$$

Lemma: Let $E \in \mathcal{T}$, a subobject of E is given by:
1) $B \in \mathcal{T} = \mathcal{T}^{B,\omega}$
2) $f : B \to E$
3) $\ker f \in \mathcal{F}$

Pf.: $f : B \to E$ morphism in $'\mathcal{A} = \mathrm{Coh}^{B,\omega}(X)$ can use the definition of $'\ker f$ (p.⑤). Otherwise: complete f to a triangle $B \xrightarrow{f} E \to C$ in $D(\mathcal{A})$

$\mathcal{A} = \mathrm{Coh}(X)$

$$\begin{tikzcd}
H^{-2}(C) \arrow[r] & H^{-1}(B) \arrow[r] & 0 \arrow[r] & H^{-1}(C) \arrow[r] & H^0(B) \arrow[r, "H^0(f)"'] & H^0(E) \arrow[r] \arrow[d, equal] & H^0(C) \arrow[r] & 0 \\
 & & & & & E & &
\end{tikzcd}$$

(Fact p.③)

f monomorph. $\Leftrightarrow$ $C \in {}'\mathcal{A}$ $\Rightarrow$ $H^{-2}(C) = 0$ $\Rightarrow$ $H^{-1}(B) = 0$

$\Rightarrow B \in \mathcal{T} \Rightarrow H^0(f) = f \Rightarrow \ker f = H^{-1}(C) \in \mathcal{F}$

Viceversa $B \in \mathcal{T} \Rightarrow H^{-1}(B) = 0 \Rightarrow H^{-2}(C) = 0$

$\Rightarrow H^{-1}(C) = \ker f \in \mathcal{F}$, Also $H^0(E) = E \to H^0(C) \to 0$

$\Rightarrow C \in {}'\mathcal{A}$ $\square$

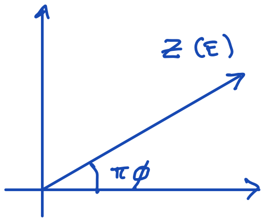
Observe: $B = \beta\omega$ $\quad \beta < 0 \Rightarrow \mathcal{O}_x \in \mathcal{T}^{B,\omega}$.

Next :

DEFINITION: A stability functions on $\mathcal{A}$ abelian cat.

is given by $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ additive

s.t. $\Im Z \geq 0$, $\Im Z = 0 \Rightarrow \Re Z < 0$



Notation:

- $Z(E) = |Z(E)| \, e^{i\pi\phi(E)} \leftarrow$ phase of E $1 \leqq \phi < 0$.

$R = \Im Z \qquad Z = -D + iR$

$D = -\Re Z$

$\nu = \frac{D}{R} = \text{slope} \rightsquigarrow$ Notion of Z-stability (semistability)

$(R = 0 \qquad \nu = +\infty)$

- $E \in \mathcal{A}$ Z-ss $\Leftrightarrow$ no subobj. $F \subset E$ has $\geqq$ slope

$\phi = \frac{1}{\pi} \operatorname{arccot}(-\nu)$

[We will later examine H-N and J-H w.r.t. Z (or ν)]

Def Z numeric if Z factors.

$$Z : K_0(\mathcal{A}) \xrightarrow{\lambda} \Lambda \longrightarrow \mathbb{C}$$

finite dim lattice

$$Z(E) = Z(\lambda(E))$$

Example: X smooth proj surface

$\Lambda = H^*_{alg}(X, \mathbb{Z})$

- One possibility $\lambda(\,) = ch(\,)$ [MSlect] sect. 6.2
- or (if X is a K3) $\lambda(\,) = v(\,) = ch(\,)\sqrt{td(X)}$

Mukai vector

$\mathcal{A} = \mathrm{Coh}^{B,\omega}$ $\qquad B \in NS(X) \qquad \omega \in \mathrm{Amp}(X)$

$$\begin{aligned} ch_0^B &= ch_0 = r \\ ch_1^B &= ch_1 - B\cdot ch_0 \\ ch_2^B &= ch_2 - B\cdot ch_1 + \frac{B^2}{2} ch_0 \end{aligned}$$

Def : $ch^B = ch \cdot e^{-B}$

$$Z(E) := Z_{B\omega}(E) = -\int \exp(-B - i\omega)\cdot ch(E)$$

$$= -\int \exp(-i\omega)\cdot ch^B(E)$$

$$= \left(-1 + i\omega + \frac{\omega^2}{2},\ ch_0^B(E) + ch_1^B(E) + ch_2^B(E)\right)_2$$

$$= \underbrace{-ch_2^B(E) + \frac{\omega^2}{2} ch_0^B(E)}_{(*)} + i(\omega\cdot ch_1^B(E))$$

$$\begin{aligned} &\quad \Vert \\ &i\omega\cdot(c_1(E) - r(E)B) \\ &\quad \Vert \\ &i\, r(E)\, \mu_{B,\omega} \end{aligned}$$

Lemma : Z is a stab. function:

Pf.

$$\begin{tikzcd} H^{-1}(E)[1] \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & E \arrow[r] & H^0(E) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ \mathcal{F}[1] & & \mathcal{T} \end{tikzcd}$$

$Im(H^0(E)) \geq 0$

$Z(H^{-1}(E)[1]) = -Z(H^{-1}(E))$

$Im\, Z(H^{-1}(E)[1]) \geq 0$

$\Rightarrow \quad Im\, Z \geq 0$

Suppose: $Im\, Z(E) = 0 \qquad \Rightarrow Im\, Z(H^0(E)) = Im\, Z(H^{-1}(E)) = 0$

$$\Downarrow$$

$r(H^0(E))\, \mu_{B\omega}(H^0(E)) = 0, \quad H^0(E) \in \mathcal{T} \Rightarrow \mu_{B\omega}(H^0(E)) > 0$

$\Rightarrow r(H^0(E)) = 0 \Rightarrow c_1(H^0(E)) = 0 \Rightarrow ch_2(H^0(E)) \geq 0$

$\Rightarrow Re(H^0(E)) = -ch_2^B(E) \leq 0$.

Want: $\mathrm{Re}\ Z(\underbrace{H^{-1}(E)}_{F}) > 0$

$\mathrm{Im}\ Z(F) = 0 \Rightarrow \mu(F) = 0 \Rightarrow F\ \mu\ ss$

1) $\omega . ch_1^B(F) = 0$, Hodge index theorem $\Rightarrow$
$ch_1^B(F)^2 \leq 0$.

2) Bogomolov [M-S p.32] $F\ \mu_{B\omega}$-ss, torsion free,

$$\Rightarrow ch_1^B(F)^2 - 2\, ch_0^B(F) . ch_2^B(F) \geq 0$$

$$\Rightarrow ch_2^B(F) \leq 0 \Rightarrow \mathrm{Re}\ Z(F) > 0 \quad (*)$$

$$\left(\frac{\omega^2}{2} ch_0^B(F) > 0 \right)$$

$\square$

Case of K3 surfaces (mostly when $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$)

Here instead of $\underline{ch}$ use $\underline{v}$ and Mukai pairing

$$v(E) = (ch_0(E), ch_1(E), ch_2(E) + ch_0(E))$$

$$Z(E) = \langle \exp(B + i\omega), v(E) \rangle$$

$$Z : \Lambda \longrightarrow \mathbb{C}$$

In all relevant results (K3-surf. case)
$\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, so:

$\boxed{B = \beta \cdot H, \quad \omega = \alpha H}$ $\beta \in \mathbb{R},\ \alpha \in \mathbb{R}_+$

$v(E) = v = (r, cH, s)$

$s = ch_2(E) + ch_0(E)$
$r = ch_0(E)$, $cH = ch_1(E)$

$$\boxed{Z_{\beta\alpha}(E) = -s + \beta c H^2 - \frac{r}{2}(\beta^2 - \alpha^2)H^2 + i(c - r\beta)\alpha H^2}$$

Theorem X a K3 surface, $Z_{\beta\alpha}$ is a stability function for $\mathrm{Coh}^{\beta,\alpha}(X)$ if the following condition is satisfied:

(*) $\boxed{\forall\, \delta \in \Lambda, \text{ with } \delta^2 = -2,\ r(\delta) > 0, \text{ and } \mu_{\beta\alpha}(\delta) = 0, \text{ then } \mathrm{Re}\, Z_{\beta\alpha}(\delta) > 0}$

(This always happens if $\alpha^2 H^2 \geq 2$).

Pf. As before up to the point where must show – $\mathrm{Re}Z(H^{-1}(E)) < 0$.

May assume $H^{-1}(E) \neq 0$ since $\mathrm{Im}(H^{-1}(E)) = 0$

$\mu_{\beta\alpha}(H^{-1}(A)) = 0 \Rightarrow H^{-1}(A)$, $\mu_{\beta\alpha}$-ss. torsion free (positive rank). May assume stable $\Rightarrow$ $v(H^{-1}(E))^2 \geq -2$. Set $v(H^{-1}(E)) = v = (r, c_1, s)$

If $v^2 = -2$, set $\delta = H^{-1}(E)$ and (*) $\Rightarrow$ $\mathrm{Re}\, Z(\delta) > 0$. Otherwise $v^2 = c_1^2 - 2rs \geq 0$

Since $c_1 = r\beta H$ get $r^2\beta^2H^2 - 2rs \geq 0$

$$\Rightarrow \quad \mathrm{Re}\, Z(H^{-1}(E)) = -s + \frac{r}{2}(\alpha^2+\beta^2)H^2 \geq \alpha^2 H^2 \frac{r}{2} > 0$$

□

Recall : $$\mu_{B,\omega}(E) = \frac{\omega . c_1(E)}{\omega^2\, r(E)} - \frac{\omega . B}{\omega^2}$$

when $B = \beta H$, $\omega = \alpha H$, $c_1(E) = c(E)\cdot H$

$$\boxed{\mu_{\beta\alpha}(E) = \frac{c(E)}{\alpha\, r(E)} - \frac{\beta}{\alpha}}$$

The expression $\mu_{\beta\alpha}(E) \geq \mu_{\beta\alpha}(F)$

only depends on β. $\Longrightarrow$

we can write $\mathcal{F}^\beta, \mathcal{T}^\beta, \mathrm{Coh}^\beta$

instead of $\mathcal{F}^{\beta\alpha}, \ldots$

Lemma : For all $x \in X$, $\mathbb{C}_x \in \mathcal{T}^\beta \subset \mathrm{Coh}^\beta(X)$, is minimal (i.e. has no nontrivial subobject.)

Pf. Suppose $0 \to A \to \mathbb{C}_x \to B \to 0$ in $\mathrm{Coh}^\beta(X)$

Then $H^{-1}(A) = 0$ so that $A = H^0(A)$ and we get

$$0 \to H^{-1}(B) \longrightarrow A \longrightarrow \underset{\underset{\mathbb{C}_x}{\parallel}}{H^0(\mathbb{C}_x)} \longrightarrow H^0(B) \to 0$$

exact sequence of sheaves

But then : $\mu_\beta(H^{-1}(B)) = \mu_\beta(H^0(A))$

which is absurd as $H^{-1}(B) \in \mathcal{F}^\beta$, and $A = H^0(A) \in \mathcal{T}^\beta$.

The only way out is : $H^{-1}(B) = 0$,

but then $0 \to A \to \mathbb{C}_x \to H^0(B) \to 0$ is exact in $Coh(X)$ so that either $A = 0$ or $A = \mathbb{C}_x$. $\square$

The large volume limit. (see [BrK3] § 14)

We limit ourselves to the case X K3

$Pic(X) = \mathbb{Z} \cdot H$, and to $Z_{\beta\alpha} : \Lambda \longrightarrow \mathbb{C}$

$\beta \in \mathbb{R}$ $\alpha \in \mathbb{R}_+$ but valid

for general Pic and for $Z_{B,\omega}$

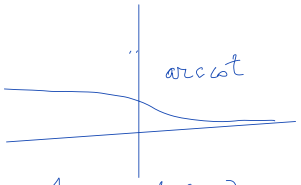
$B, \omega \in NS(X) \otimes \mathbb{R}$, $\omega \in Amp(X)$.

So consider the stab function $Z_{\beta\alpha}$

and the $Z_{\beta,\alpha}$-semistability : $E \in Coh^\beta(X)$ is Z-s.s. $\Leftrightarrow$

$A \subset E \Rightarrow \nu_{\beta\alpha}(A) \leq \nu_{\beta\alpha}(E)$ $\qquad \nu = \dfrac{-\mathrm{Re}\, Z}{\mathrm{Im}\, Z}$

equiv. $\phi_{\beta\alpha}(A) \leq \phi_{\beta\alpha}(E)$.



$\phi = \frac{1}{\pi} \mathrm{arccot}(-\nu)$

$\sigma(E) = (r, cH, s) \equiv (r, c, s)$

$$\mu_\beta = \frac{c}{r} - \beta$$

if $E \in Coh(X)$ reduced Hilb poly :

$$P_E(m) = \frac{m^2}{2} + \frac{c}{r} m + \frac{s+r}{rH^2} \qquad s+r = \chi(E)$$

Gieseker H-ss :

- $r \neq 0$: $\forall \; A \subset E$ $\qquad v(A) = (r', c', s')$

  $\frac{c'}{r'} \leq \frac{c}{r}$ $\qquad$ if $\frac{c'}{r'} = \frac{c}{r}$ then $\frac{s'}{r'} \leq \frac{s}{r}$

- $r = 0$: $\frac{s'}{c'} < \frac{s}{c}$

write $\lambda_\beta(E) = \frac{s - \beta c}{r}$

useful formula $(r(E) \neq 0, \; r(A) \neq 0)$

(*)

$$\frac{Z_{\alpha\beta}(E)}{r(E)} - \frac{Z_{\alpha\beta}(A)}{r(A)} = -\left(\lambda_\beta(E) - \lambda_\beta(A)\right) + i\alpha\left(\mu_\beta(E) - \mu_\beta(A)\right)$$

Recall :

$$\boxed{Z_{\beta\alpha}(E) = -s + \beta c H^2 - \frac{r}{2}(\beta^2 - \alpha^2)H^2 + i(c - r\beta)\alpha H^2}$$

Theorem: let $E \in Coh^{\beta}(X)$ Assume

- either $r>0$, and $c-\beta r>0$ $(\Rightarrow \mu_\beta > 0)$
- or $r=0$, $c>0$.

Then $\exists\ \alpha_0 \in \mathbb{R}_+$,

$\left(E \text{ is } Z_{\beta\alpha}\text{-ss},\ \forall\ \alpha \geq \alpha_0\right) \Leftrightarrow \left(E \text{ is Gieseker } H\text{-ss sheaf}\right)$

Pf.

$$\frac{\operatorname{Re} Z_{\beta\alpha}}{\operatorname{Im} Z_{\beta\alpha}} = \frac{-s + \left(\beta c - \frac{r}{2}\left(\beta^2-\alpha^2\right)\right)H^2}{(c-\beta r)\,\alpha . H^2} \begin{cases} \to +\infty \\ \to 0 \\ \to 1 \end{cases}$$

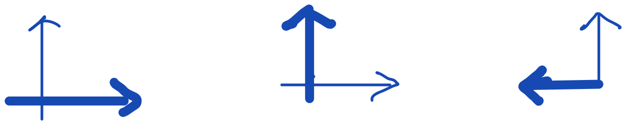
$$\alpha \longrightarrow +\infty \quad \left\{ \begin{array}{l} \operatorname{Supp}(E) = X,\ r>0 \\ \dim \operatorname{Supp}(E) = 1,\ r=0,\ c \neq 0 \\ \dim \operatorname{Supp}(E) = 0,\ r=c=0 \end{array} \right\} \quad \phi(\alpha) \to \begin{cases} 0 \\ \frac{1}{2} \\ 1 \end{cases}$$

$$\Downarrow$$

$\operatorname{Im} Z = 0$

$\operatorname{Re} Z < 0$ $\Rightarrow \phi = 1$

$$\phi = \frac{1}{\pi} \operatorname{arccot} \frac{\operatorname{Re} Z}{\operatorname{Im} Z}$$



$\boxed{\text{Case } r(E)>0}$

$$H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E)$$

$H^{-1}(E)$ torsion free $\Rightarrow \lim\limits_{\alpha \to +\infty} \phi(H^{-1}(E)) = 0$

$\Rightarrow \lim\limits_{\alpha \to +\infty} \phi(H^{-1}(E)[1]) = 1$. Also $r(E)>0 \Rightarrow \lim\limits_{\alpha \to +\infty} \phi(E) = 0$, so $H^{-1}(E)[1]$ destabilizes E for $\alpha \gg 0$, against the hypothesis. Absurd.

$\Rightarrow H^{-1}(E)=0 \quad \Rightarrow \quad E = H^0(E)$ is a sheaf

Suppose not Gieseker stable, then $\exists\ A \subset E$

$\mu_\beta(A) > \mu_\beta(E)$ or $\mu(A)_\beta = \mu_\beta(E)$
and $\lambda_\beta(A) > \lambda_\beta(E)$

suppose $\mu_\beta(A) > \mu_\beta(E)$, i.e. $\boxed{\mu_\beta(A) = \rho\,\mu_\beta(E) \quad \rho > 1}$

$$\frac{Z(A)}{r(A)} = L_A + \alpha^2 M + i\,\mu_\beta(A)\,\alpha H^2,$$

$$\frac{Z(E)}{r(E)} = L_E + \alpha^2 M + i\,\mu_\beta(E)\,\alpha H^2,$$

$\phi(A) < \phi(E)$ for $\alpha >> 0 \iff$

$$\frac{L_A + \alpha^2 M}{\rho\,\mu_\beta(E)\,\alpha H^2} \overset{(*)}{>} \frac{L_E + \alpha^2 M}{\mu_\beta(E)\,\alpha H^2} \quad \Rightarrow \rho < 1$$

case $\mu_\beta(A) = \mu_\beta(E)$ use $(*)$ p. 16

$\square$

$\boxed{\text{Case } r(E)=0}$ easier.

(*) $\phi = \frac{1}{\pi}\operatorname{arccot}\left(\frac{\Re Z}{\Im Z}\right) \qquad \phi \downarrow$

PRESTABILITY CONDITIONS

Definition: $\mathcal{A}$ abelian, $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ stab function,

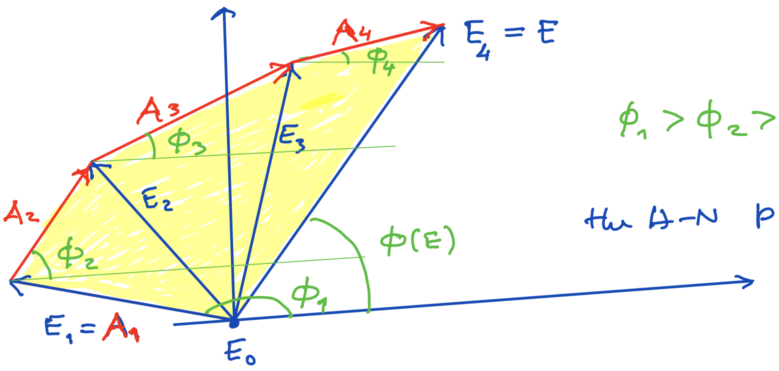
the pair $\sigma = (Z, \mathcal{A})$ is a prestability condition

if every object $E \in \mathcal{A}$ has a Harder–Narasimhan

filtration: $0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$

with: $E_i \in \mathcal{A}$, $A_i = E_i / E_{i-1}$, Z-semistable

and
$$\underset{\phi^+(E)}{\underset{\shortparallel}{\phi(A_1)}} > \cdots > \underset{\phi^-(E)}{\underset{\shortparallel}{\phi(A_n)}}$$



$\phi_1 > \phi_2 > \phi_3 > \phi_4$

the H-N polygon

Harder Narasimhan

I)

Theorem : [Br TC] p.323

$Z: K(\mathcal{A}) \to \mathbb{C}$ stab function s.t.

a) $\nexists$ infinite seq $\subset E_{k+1} \subset E_k \subset \cdots \subset E_n = E$

$\dots > \phi(E_{k+1}) > \phi(E_k) > \cdots$

b) $\nexists$ infinite seq. $E_1 \twoheadrightarrow E_2 \twoheadrightarrow E_3 \twoheadrightarrow \cdots$

$\cdots > \phi(E_k) > \phi(E_{k+1}) > \cdots$

Then HN holds for Z.

"Pf" Given $E \in \mathcal{A}$, find a maximal destabilizing quotient $E \twoheadrightarrow B$

i.e. $\forall$ $E \twoheadrightarrow B'$ $\phi(B') \geq \phi(B)$

with $=$ iff $E \to B \to B'$.

Corollary: [Br K3] (Proposition 7.1) $(\mathrm{Coh}^\beta, Z_{\alpha\beta})$ $\boxed{\alpha, \beta \in \mathbb{Q}}$

is a prestability condition i.e. H.N. holds.

Pf. Must show a) and b).

suppose a) does not hold. ∃ short exact

$0 \to E_{i+1} \to E_i \to F_i \to 0$, Since $\operatorname{Im} Z \geq 0$,

$\operatorname{Im} Z(E_{i+1}) < \operatorname{Im} Z(E_i)$

Image Z discrete ⇒ $\operatorname{Im} Z(E_i) = \text{const}$ $i >> 0$

⇒ $\operatorname{Im} Z(F_i) = 0$ ⇒ $\operatorname{Re} Z(F_i) < 0$ ⇒

$\operatorname{Re} Z(E_{i+1}) > \operatorname{Re} Z(E_i)$, but $\phi(E_{i+1}) > \phi(E_i)$.

b) Similar (harder).

II)

Theorem: [MS lect] (Prop 4.10)

$Z : K(\mathcal{A}) \longrightarrow \mathbb{C}$ stab funct.

- $\mathcal{A}$ noetherian category,
- Image $(\operatorname{Im} Z) \subset \mathbb{C}$ discrete,

then Z satisfies HN.

"Pf" graph proof following Shatz, as revived by Bayer ([Ba-Tow], Theorem 2.1.6.).

Corollary $(\operatorname{Coh}^\beta, Z_{\alpha\beta})$ $(\alpha, \beta) \in \mathbb{Q}$ is a prestability condition

Proof. Must show Coh^β is a Noetherian category. [MS lect] Lemma 6.17

Slicings

Definition: A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ (e.g. $\mathcal{D}=\mathcal{D}(X)$) is the datum of a full subcat. $\mathcal{P}(\phi)\subset\mathcal{D}$, $\forall\,\phi\in\mathbb{R}$, s.t.

1) $\mathcal{P}(\phi+1)=\mathcal{P}(\phi)[1]$ $\quad\forall\,\phi\in\mathbb{R}$

2) $\mathrm{Hom}(E_1,E_2)=0$ $\quad E_i\in\mathcal{P}(\phi_i)$, $\phi_1>\phi_2$.

3) $\forall\; E\in\mathcal{D}$ $\;\exists$ diagr. of dist. triang

(*)

$$\begin{tikzcd}
0=E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n=E \arrow[dl] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & A_2 \arrow[ul, dashed, "{[1]}"] & & & & A_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

$A_i\in\mathcal{P}(\phi_i)$, $\quad\phi_1>\cdots>\phi_n$ $\qquad$ (Harder Narasimhan)

$\phi_1=\phi^+(E)$, $\quad\phi_n=\phi^-(E)$

Definition: $\mathcal{P}(\phi)$ = semi-stable objects of phase ϕ

Remark: H-N is unique up to iso.

Suppose

$$\begin{tikzcd}
E'_0 \arrow[r] \arrow[d, equal] & E'_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E'_m=E \arrow[d] \\
0 & A_1 & & A_m
\end{tikzcd}$$

㉓

Suppose $\phi(E_1') \geq \phi(E_1)$

k minimal s.t. $\exists\, E_1' \to E_k$

Then: $E_1' \to E_k \to A_k$

$$\phi(A_k) \geq \phi(E_1') \geq \phi(E_1) \geq \phi(A_k)$$

$\Rightarrow$ all $=$. But $\underset{\overset{\shortparallel}{\phi(A_1)}}{\phi(E_1)} = \phi(E_k/E_{k-1}] = \phi(A_k)$

$\Rightarrow A_k = E_1$ i.e. $k=1$ so that $E_1' \xrightarrow{\cong} E_1$. $\square$

<u>Thus: it makes sense to set:</u>

$$\phi_1 = \phi^+(E), \quad \phi_n = \phi^-(E)$$

$$\phi_i = \phi_i(E)$$

<u>Notice:</u> If everything takes place in an abelian category, so that Triangles are exact sequence, get $E_1' = E_1$ and therefore uniqueness.

Definition: $I \subset \mathbb{R}$ interval

$$\mathcal{P}(I) = \left\{ \text{extension-closure of the } \mathcal{P}(\phi),\ \phi \in I. \right\}$$

$$\Rightarrow \boxed{E \in \mathcal{P}(\phi^-(E), \phi^+(E))}$$

Remark: $\mathcal{P}((a,b)) = \{ E \in \mathcal{D} \mid a < \phi^-(E) \leq \phi^+(E) < b \} \cup \{0\}$

$= \{ E \in \mathcal{D} \mid \text{each } \phi_i(E) \in (a,b) \} \cup \{0\}$

Proposition: $\mathcal{P}((\phi, \phi+1])$ heart of a t-structure

($\Rightarrow$ Abelian)

$$\mathcal{D}^{\leq 0} := \mathcal{P}(>\varphi), \quad \mathcal{D}^{\geqq 0} := \mathcal{P}(\leq \varphi+1)$$

$$\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \mathcal{P}((\varphi, \varphi+1])$$

([Ch-B] Proposition 1.9)

Remark: $\mathcal{P}(\phi)$ is abelian

Pf. $A \xrightarrow{f} B$ in $\mathcal{P}(\phi] \in \mathcal{P}((\phi-1, \phi])$ abelian

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ I = \operatorname{Im} f \arrow[ur] \arrow[d] & \\ A_n(I) \in \mathcal{P}(\phi^-(I)) & \end{tikzcd}$$

$$\phi(A) = \phi \leq \phi^-(I)$$

Also $\phi^+(I) \leq \phi(B) = \phi$

$$\Rightarrow \phi^+(I) = \phi^-(I) = \phi, \quad I \in \mathcal{P}(\phi)$$

Ker and CoKer similar.

Theorem: The following are equivalent

a) the datum $\sigma = (\mathcal{P}, Z)$ where $\mathcal{P}$ is a slicing of $\mathcal{D}$, and $Z : K_0(\mathcal{D}) \longrightarrow \mathbb{C}$ is a homomorphism s.t. $\forall\ \phi \in \mathbb{R}$ and $E \in \mathcal{P}(\phi)$, $Z(E) \in \mathbb{R}_+ e^{i\pi\phi}$

b) a prestability condition $(\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a t-structure on $\mathcal{D}$

Pf Idea:

$$(\mathcal{P}, Z) \rightsquigarrow (\mathcal{A}, Z) \qquad \mathcal{A} = \mathcal{P}((0,1])$$

$$(\mathcal{A}, Z) \rightsquigarrow (\mathcal{P}, Z) \qquad \mathcal{P} = \{\mathcal{P}(\phi)\},$$

where $\mathcal{P}(\phi) =$ s.s. obj of phase ϕ in $\mathcal{A}$ if $\varphi \in (0,1]$

otherwise $\mathcal{P}(\phi) = \mathcal{P}(\phi)[n]$, if $\phi \in (-n+1, n]$

also $Z(E[n]) = (-1)^n Z(E) = e^{\pi i n} Z(E)$

But one has to deal with H-N in both directions.

a) ⇒ b) $\mathcal{A} = \mathcal{P}((0,1])$ is the heart of a t-structure. Restrict Z to $\mathcal{A}$ and get a stability function. Must check H-N for $\mathcal{A}$.

Since, given $E \in \mathcal{A} = \mathcal{P}((0,1])$, we have:

$$0 \leq \phi^-(E) \leqq \phi^+(E) \leq 1$$

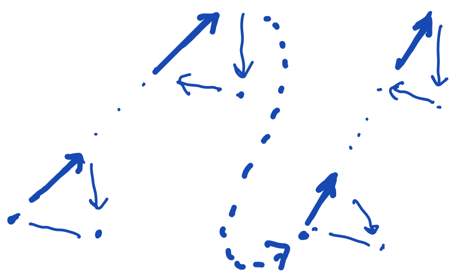
the Harder-Narasimhan in the definition of $\mathcal{P}$ actually takes place in $\mathcal{A}$.

b) ⇒ a) take H-N in the CRITERION of p. 5 then in each $\mathcal{A}[k_i]$.

Use the H-N for Z in $\mathcal{A}$.

Can put these together via octahedron

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[d, "\cong"'] \arrow[r] & E_2 \arrow[d] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[d] \\
& A_1 & A_2 \arrow[ul, dashed] & & & A_n \arrow[ul, dashed]
\end{tikzcd}$$



$A_i \in \mathcal{A}^{\#}[k_i]$

see ([Alp], Prop. 7.3)

GROUP ACTION

$\mathrm{Stab}(X)_{pre} = \{\text{prestab. cond. on } X\}$.

$\widetilde{GL_2^+}(\mathbb{R}) :=$ univ. cover of $GL_2^+(\mathbb{R})$

acts on $\mathrm{Stab}(X)_{pre}$

$\tilde{g} \in \widetilde{GL_2^+}(\mathbb{R})$ $\quad \tilde{g} = (f, g)$ $\quad g \in GL_2^+(\mathbb{R})$

$f : \mathbb{R} \to \mathbb{R}$, increasing $\quad f(\phi+1) = f(\phi)+1$

such that:

$$g|_{S^1} : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\} / \mathbb{R}^+ = S^1, \quad f : \overset{S^1}{\overset{\|}{\mathbb{R}/\mathbb{Z}}} \longrightarrow \overset{S^1}{\overset{\|}{\mathbb{R}/\mathbb{Z}}}$$

coincide

Then $\tilde{g} \cdot (\mathcal{P}, Z) = (\mathcal{P}', Z')$, where

$Z' = g^{-1} \circ Z$, $\quad \mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

[M-S lect] Example 5.77. $\quad C$ smooth curve

$\mathrm{Stab}(X)_{pre} = \widetilde{GL_2^+}(\mathbb{R}) \cdot [\text{Standard stab cond}]$

Bridgeland stability conditions

Def : A Bridgeland stability condition on $\mathcal{D}$ is the datum of a prestability condition $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A}$ is a heart in $\mathcal{D}$, satisfying the support property, that is :

i.e. $\exists \; c > 0$, s.t. $\forall \; E$ semistable,

$$\left| Z(v(E)) \right| \geq c \, \| v(E) \|$$

$\| \; \|$ some norm on Λ.

Remark : $Z : K_0(\mathcal{A}) \xrightarrow{v} \Lambda \longrightarrow \mathbb{C}$

$\Rightarrow$ Finitely many $Z(E) = Z(v(E))$ E semistable

in a compact $K \subset \mathbb{C}$

$$V_K = \{ v(E) : Z(v(E)) \in K \} \subset \{ v(E) : |Z(v(E))| < A \}$$

$\xRightarrow{\text{via supp. prop.}} \| v(E) \| < A' \Rightarrow V_K$ finite

The supp prop. says that there is no sequence of <u>s.s.</u> $\{E_n\}$ such that $\lim_{n\to\infty} Z(E_n) = z_0$ otherwise in a ball around z_0 of radius K we would have

$$K > |Z(E_n)| \geq C\|v(E_n)\|$$

which is absurd being Λ discrete

As a consequence get <u>Jordan – Hölder</u> <u>in the abelian category $\mathcal{P}(\phi)$.</u>

<u>Pf.</u> Suffices to prove that $\mathcal{P}(\phi)$ is Artinian (it is also Noetherian)

Suppose: $\cdots \subset A_i \subset \cdots \subset A_1 = A \in \mathcal{P}(\phi)$

$0 \to A_{i+1} \to A_i \to A_i/A_{i+1} \to 0 \in \mathcal{P}(\phi)$

$Z(A_i/A_{i+1}) = r e^{i\pi\varphi} = (r_i - r_{i+1}) e^{i\pi\varphi} \Rightarrow r_i \downarrow \Rightarrow$ property.

$Z(A_i/A_{i+1}) \to 0$, against support

<u>Example</u>: C smooth curve. Look at

$\mathrm{Coh}^{\sqrt{2}}(C)$, $Z(E) = i(\deg E + \sqrt{2}\,\mathrm{rk}\, E)$. It is a stability condition without supp. property.
(Every $E \in \mathrm{Coh}^{\sqrt{2}}(C)$ is s.s. of Z-slope)

Kontsevich - Soibelman

the following two conditions are equivalent:

1) $\exists \; \| \; \|$ on $\Lambda_{\mathbb{R}}$ s.t.

$\forall$ E Z-s.s. $|Z(v(E))| \geq C \|v(E)\|$

2) $\exists$ quadratic form Q on $\Lambda_{\mathbb{R}}$ s.t. $Q|_{\ker Z} < 0$ and

$\forall$ E Z-semistable $Q(E) \geq 0$

Pf.

1) $\Rightarrow$ 2)

Set $Q(v) = |Z(v)|^2 - C\|v\|^2$

2) $\Rightarrow$ 1) [Ch-B] Proposition 1.13.

Remark. $$\underset{\dim = \rho+2}{\Lambda_{\mathbb{R}}} = \underset{\dim 2}{\hat{\Lambda}_{\mathbb{R}}} \oplus \underset{\dim \rho}{\operatorname{Ker} Z}$$

$\hat{\Lambda}_{\mathbb{R}} \cong \mathbb{C}$

We are assuming $Z : \hat{\Lambda}_{\mathbb{R}} \xrightarrow{\cong} \mathbb{C}$

$p : \Lambda_{\mathbb{R}} \longrightarrow \operatorname{Ker} Z$ can write:

$$Q(v) = Q'_{\mathbb{C}}(v) - \|p(v)\|^2_{\operatorname{Ker} Z}$$

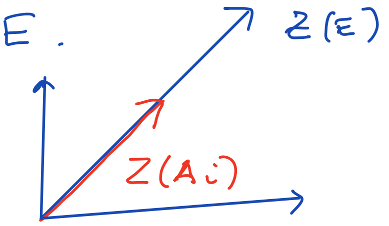
$\| \, \|_{\operatorname{Ker} Z}$ induced by $-Q$ on $\operatorname{Ker} Z$

Q' a quadratic form on $\mathbb{R}^2 = \mathbb{C}$.

Remark. Enough to check 2).

$\forall$ stable E.

$A_1, \ldots, A_n$ J-H factors of E.



Suppose $Q(v(A_i)) \geq 0$,

$$Q'(Z(E)) = \left(\sum_i \sqrt{Q'(Z(A_i))} \right)^2$$

$$\sum_i \sqrt{Q'(Z(A_i))} \underset{\text{by hypothesis}}{\geq} \sum_i \|p(v(A_i))\|_{\operatorname{Ker} Z} \underset{\text{triangle in Ker } Z}{\geq} \|p(v(E))\|_{\operatorname{Ker} Z}$$

Example $(\mathrm{Coh}^{\beta}, Z_{\alpha\beta})$ is a stability condition, when $\alpha, \beta \in \mathbb{Q}$.

Pf. Must verify support property

$$v \in \operatorname{Ker} Z_{\alpha\beta} \quad \Longrightarrow \quad \begin{cases} c - r\beta = 0 \\ -s + c\beta H^2 - \frac{r}{2}(\beta^2 - \alpha^2) H^2 \end{cases}$$

$\neq$

0

$$\Longrightarrow \quad v = \left(r, r\beta H, \frac{r}{2}(\beta^2 + \alpha^2) H^2\right)$$

$$\langle v, v \rangle = r^2\beta^2H^2 - r^2(\beta^2+\alpha^2)H^2 = -r^2\alpha^2H^2 < 0$$

Write $\Lambda_{\mathbb{R}} = \Lambda'_{\mathbb{R}} \oplus \operatorname{Ker} Z \leftarrow \dim \rho$

$p : \Lambda_{\mathbb{R}} \longrightarrow \operatorname{Ker} Z$, $\operatorname{Sign}(\langle\,,\,\rangle) = (2, \rho)$

Choose $\|\ \|$ on $\Lambda'_{\mathbb{R}}$ s.t.

$$\langle v, v \rangle = \|Z(v)\|^2_{\mathbb{C}} - \|p(v)\|^2_{\operatorname{Ker} Z} \leftarrow \text{norm induced by } \langle , \rangle \text{ on } \operatorname{Ker} Z$$

would like $\langle v(E), v(E) \rangle \geq 0$

$\forall$ E stable. But $\langle v(E), v(E) \rangle \geq -2$

Must take care of $\delta \in \Delta = \{ v \mid v^2 = -2 \}$

observe that $Z(\delta) \neq 0$, by $(*)$ p. 13

$$\Rightarrow \quad \| Z(\delta) \|_{\mathbb{C}}^2 + 2 = \| p(\delta) \|_{\ker Z}^2$$

$\forall K$ only finite no of δ with

$\| Z(\delta) \| < K \Rightarrow \exists\, C > 0$, s.t. $\forall \delta \in \Delta$

$\| Z(\delta) \| \geq C$. set

$$\boxed{Q_Z(v) = \langle v, v \rangle + \frac{2}{C^2} \| Z(v) \|_{\mathbb{C}}^2} =$$

$$= \frac{C^2 + 2}{C^2} \| Z(v) \|_{\mathbb{C}}^2 - \| p(v) \|_{\ker Z}^2$$

Q_Z gives support property

Topology on Stab(X)

Notation : $\mathcal{D} = \mathcal{D}(X)$ $\quad$ $\operatorname{Stab}(\mathcal{D}) = \operatorname{Stab}(X)$

$$\operatorname{Slice}(\mathcal{D}) = \left\{ \text{slicings } \mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}} \right\}$$

$$d(\mathcal{P}, \mathcal{Q}) = \sup_{E\in\mathcal{D}} \left\{ \left|\phi^{+}_{\mathcal{P}}(E) - \phi^{+}_{\mathcal{Q}}(E)\right| , \left|\phi^{-}_{\mathcal{P}}(E) - \phi^{-}_{\mathcal{Q}}(E)\right| \right\}$$

$$= \inf_{\varepsilon \geqq 0} \left\{ \forall \in \mathbb{R},\ \mathcal{Q}(\phi) \subset \mathcal{P}([\phi-\varepsilon, \phi+\varepsilon]) \right\}$$

$$= \sup_{\phi\in\mathbb{R}} = \left\{ \left|\phi^{+}_{\mathcal{Q}}(E) - \phi\right|, \left|\phi - \phi^{-}_{\mathcal{Q}}(E)\right| \;\middle|\; E \in \mathcal{P}(\phi) \right\}$$

$$\begin{tikzcd}
\operatorname{Stab}(\mathcal{D}) \arrow[r, "{(p,q)}"] & \operatorname{Slice}(\mathcal{D}) \times \operatorname{Hom}(\Lambda, \mathbb{C}) \\
(\mathcal{P}, Z) \arrow[r, mapsto] & (\mathcal{P}, Z)
\end{tikzcd}$$

Definition: Top. on $\operatorname{Stab}(\mathcal{D})$ is the coarsest top making (p, q) continuous.

Lemma $\widetilde{GL}_2(\mathbb{R})$ acts continuously on $\operatorname{Stab}(X)$

Theorem (Bridgeland) ("deformation th.")

For each connected component Σ of $\mathrm{Stab}(X)$, there is a linear subspace $V(\Sigma) \subset \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C})$ such that

$$\begin{aligned} q : \Sigma &\longrightarrow V(\Sigma) \subseteq \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \\ (\mathcal{P}, Z) &\longmapsto Z \end{aligned}$$

is a local homeomorphism. Σ is called full if $V(\Sigma) = \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C})$.

$\Rightarrow$ A full Σ has the structure of a complex manifold of $\mathbb{C}$-dimension equal to $\mathrm{rk}(\Lambda)$.

Lemma $(\mathcal{P}, Z), (\mathcal{Q}, Z) \in \mathrm{Stab}(X)$

$$d(\mathcal{P}, \mathcal{Q}) < 1 \quad \Rightarrow \quad \mathcal{P} = \mathcal{Q}.$$

Pf $E \in \mathcal{P}(\phi)$. We want $E \in \mathcal{Q}(\phi)$

$$d(\mathcal{P},\mathcal{Q})<1 \Rightarrow \phi-1 \le \phi^-_{\mathcal{Q}}(E) \le \phi^+_{\mathcal{Q}}(E) < \phi+1$$

$$\mathcal{P}(\phi) \subset \mathcal{Q}(\phi-1,\phi+1) \quad ⊛$$

For the record: given $F \in \mathcal{Q}(\psi)$

$$(*) \qquad \psi-1 \le \phi^-_{\mathcal{P}}(F) \le \phi^+_{\mathcal{P}}(E) < \psi+1$$

$$\mathcal{Q}(\psi) \subset \mathcal{P}(\psi-1,\psi+1) \quad ⊛$$

Claim: $\phi-1 \le \phi^-_{\mathcal{Q}}(E) \le \phi \le \phi^+_{\mathcal{Q}}(E) \le \phi+1$

Pf Suppose $\phi^-_{\mathcal{Q}}(E) > \phi$ already remarked p. 24

$$r e^{\pi i \phi} = Z(E) = \sum Z(E_j/E_{j+1}) = \sum r_j e^{i\pi\phi_j} \qquad r_j>0$$

$\phi_j > \phi$ absurd. ($\phi \le \phi^+_{\mathcal{Q}}$ analogous) similar

want: $\phi = \phi^+_{\mathcal{Q}}(E)$ (and $\phi = \phi^-_{\mathcal{Q}}(E)$)

Suppose $\phi < \phi^+_{\mathcal{Q}}(E) = \psi \Rightarrow \exists$

$F \in \mathcal{Q}(\psi)$,

$$\begin{tikzcd} F \arrow[r, "HN", "\neq 0"'] \arrow[d, phantom, sloped, "\in"] & E \arrow[r] \arrow[d, dash] & A \arrow[d, phantom, sloped, "\in"] \\ \mathcal{Q}(\phi,\phi+1) & \phi & \mathcal{Q}(\phi-1,\phi] \end{tikzcd} \quad ⊛$$

use $(*)$. As before $\psi \le \phi^+_{\mathcal{P}}(F) = \gamma$

$$\boxed{\begin{tikzcd} \mathcal{P}(\gamma) \ni G \arrow[r, "HN", "\neq 0"'] & F \arrow[r] & B \\ \cap & \mid & \\ \mathcal{P}(\psi,\psi+1) & \psi & \mathcal{P}(\psi-1,\psi] \end{tikzcd}}$$

But: $$\begin{tikzcd} G \arrow[r] & F \arrow[r] & E \\ \cap & & \cap \\ \mathcal{P}(\psi,\psi_{+1}) & & \mathcal{P}(\phi) \end{tikzcd}$$ $\psi > \phi \implies$

$$\begin{tikzcd} & G \;\psi \arrow[d] \arrow[dl, dashed] & \\ A[-1] \arrow[r] & F \arrow[r] & E \\ (\phi-2,\phi-1) & & \end{tikzcd}$$

Absurd since $G \to F$ is not zero. $\square$

Lemma Assume that Q has sign $= (2, \mathrm{rk}\,\Lambda - 2)$. Consider $\sigma = (\mathcal{P}, Z)$ Bridg. stab. cond. with support property given by Q. Then $\exists$ $g \in GL_2^+(\mathbb{R})$ s.t. $\forall$ $v \in \Lambda$

$$Q(v) = |g\,Z(v)|^2 + Q(p(v))$$ where

$p : \Lambda_{\mathbb{R}} \to \mathrm{Ker}\,Z$, is the orthogonal projection

Pf $$\Lambda_{\mathbb{R}} = \underset{\substack{\| \\ (\mathrm{Ker}\,Z)^{\perp}}}{\Lambda'_{\mathbb{R}}} \oplus \mathrm{Ker}\,Z \; ; \quad Z : \underset{\substack{\| \\ (\mathrm{Ker}\,Z)^{\perp}}}{\Lambda'_{\mathbb{R}}} \xrightarrow{iso} \mathbb{C}$$ i.e.

write $Q(v) = \|Z(v)\|^2 + Q(p(v))$ and let g send $\| \ \|_{\mathbb{C}}$ to $| \ |_{\mathbb{C}}$.

Corollary. Up to the action of $GL_2(\mathbb{R})$ can assume $g = 1$

Bridgeland deformation theorem

$$\begin{tikzcd} Z : \mathrm{Stab}(\mathcal{D}) \arrow[r] & \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \\ \sigma = (\mathcal{A}, Z) \arrow[r, mapsto] & Z \end{tikzcd}$$

is a local homeomorphism.

Plan: given Z' close to Z search for a σ' close to σ lifting Z', i.e. $\sigma' = (\mathcal{A}', Z')$.

Hope for the best: $\sigma' = (\mathcal{A}, Z')$ with some $\mathcal{A}$ and some Q giving the support property : not quite, but almost.

set : $$U = \left\{ Z' \in \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \;\middle|\; Q\big|_{\mathrm{Ker}\, Z'} < 0 \right\}$$

$U \subset \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C})$ is an open neighbourhood of Z.

may assume U connected.

[Recall $Q|_{\mathrm{Ker}\, Z} < 0$, $Q(E) \geq 0$, $\forall$ E σ-semistable]

let V be the connected component of $q^{-1}(U)$ containing σ.

Theorem $q|_V : V \longrightarrow U$ is a top. covering.

We will only prove that q is a local homeomorphism around σ, and that $\forall\ \tau \in V$, near σ, Q gives the support property for τ.

Bayer's idea: Look at the following linear section of U.

$$W = \left\{ Z' \in U \;\middle|\; Z'\big|_{(\operatorname{Ker} Z)^{\perp}} = Z\big|_{(\operatorname{Ker} Z)^{\perp}} \right\}$$

Remark $Z' \in W \iff Z' = Z + u \circ p$ where $u : \operatorname{Ker} Z \to \mathbb{C}$, with $\|u\| < 1$, $\|\ \|$ being the operator norm given by:

$$\|u\| = \sup_{w \in \operatorname{Ker} Z} \frac{|u(w)|}{-Q(w)} \qquad \left[\begin{array}{l} |\cdot| \text{ usual on } \mathbb{C} \\ -Q(\cdot), \text{ on } \operatorname{Ker} Z \end{array}\right]$$

Pf. Being in W entails two conditions:

$$① \quad Z' - Z\big|_{\operatorname{Ker} Z^{\perp}} = 0 \iff \left[\begin{array}{l} Z' - Z = u \circ p \\ \Lambda \xrightarrow{p} \operatorname{Ker} Z \xrightarrow{u} \mathbb{C} \end{array}\right],$$

$$② \quad Z' \in U \iff \left[Q\big|_{\operatorname{Ker} Z'} < 0 \right].$$

Assume that $Z' = Z + u \circ p$ and $\|u\| < 1$. Then ① is satisfied and for $w \in \operatorname{Ker} Z'$. We have

$Q(w) = |Z(w)|^2 + Q(p(w)) = |u(p(w))| + Q(p(w)) < 0$

$\Rightarrow$ $Z' \in U$ so that $Z' \in W$. Viceversa, assume $Z' \in W$. Then $Z' = Z + u \circ p$. Suppose $\|u\| \geq 1$. Can assume $\exists$ $w \in \operatorname{Ker} Z'$ with $Q(w) = -1$, $u(w) \geq 1$. Let $v \in (\operatorname{Ker} Z)^{\perp}$ s.t. $Z(v) = -u(w) \in \mathbb{C}$, ($Z$ is surj.). Then

$$Z'(v+w) = Z(v+w) + u \circ p\,(v+w) =$$
$$= -u(w) + u(w) = 0$$

But $Q(v+w) = Q(v) + Q(w) = Q(v) - 1 = |Z(v)|^2 - 1$
$= |u(w)|^2 - 1 \geq 0$

against ②. □

Next step: Since Z' is close to Z can choose $g \in GL_2^+(\mathbb{R})$, close to 1, such that gZ' and Z coincide on $(\operatorname{Ker} Z)^{\perp}$. Substitute Z' with gZ' and assume $Z' = Z + u \circ p + i\, v \circ p$ with u and v in $\operatorname{Hom}(\operatorname{Ker} Z, \mathbb{R})$, and close to 0: the old "u" is now $u + iv$. Set $Z_1 = Z + u \circ p$. Then Z_1 is close to Z.

Lemma: Given $\sigma = (\mathcal{A}, Z)$ and $Z_1 = Z + u \circ p$ with $u \in \operatorname{Hom}(\Lambda_{\mathbb{R}}, \mathbb{R})$, close to 0, then one can lift Z_1 to a stab. condition σ_1 close to σ.

Go back to $Z_1 = Z + u \circ p$. By the lemma, we can replace Z with Z_1, and σ with σ_1, and assume that:

$$Z' = Z + i v \circ p'$$

with p' close to p. Now, up to replacing Z' with gZ', with g close to 1, we can assume $p' = p$. We must lift Z'.

We have

$$-i Z' = -i Z + v \circ p$$

Let $\tilde{\gamma} \in \widetilde{GL}_2(\mathbb{R})$ be a lifting of (mult by i) $\in GL_2^+(\mathbb{R})$.

Temporarily replace σ and Z with $-\tilde{\gamma}\sigma$ and iZ. Then the Lemma gives a lifting of iZ to a stability condition τ, close to $-\tilde{\gamma}\sigma$. We can then take as a lifting of Z, the stab. condition $\sigma' = -\tilde{\gamma}^{-1}\tau$, which is close to σ.

The final step is to prove the Lemma.

Sketch of proof. We start with $\sigma = (\mathcal{A}, Z)$, and $Z' = Z + u \circ p$, u close to 0 and real.

Claim: $\sigma' = (\mathcal{A}, Z')$ is a stab condition, close to σ, with support property given by Q (the quadratic form Q gives the support property for σ). Must prove:

a) Z' stab funct.

b) H-N for Z'.

c) Q gives support property for $(\mathcal{A}, Z')$.

Z has supp prop w.r.t. Q

the key, in the three cases, is the support property for σ, given by Q.

Now one can read section 5 in [Ba-Short].

To give an idea we prove the simplest of the three: item a)

Pf. u real $\Rightarrow$ $\operatorname{Im} Z' = \operatorname{Im} Z \geq 0$

Suppose $\operatorname{Im} Z'(E) = \operatorname{Im} Z(E) = 0$, $E \in \mathcal{A}$.

Then $Z(E) \in \mathbb{R}_{<0}$, and E is σ-semistable.

Thus, if $v = v(E)$ we have:

$$Q(v) = |Z(v)|^2 + Q(p(v)) \geq 0$$

i.e. $$\underset{\underset{-Z(v)}{\|}}{|Z(v)|} \geq \sqrt{-Q(p(v))}$$

$$Z'(E) = Z(E) + u \circ p(v) < Z(E) + \|u\| \cdot \sqrt{-Q(p(v))}$$

$$\left(\sup_{w \in \operatorname{Ker} Z} \left(\frac{|u(w)|^2}{-Q(w)}\right) = \|u\|^2\right) \qquad \leq Z(E) - Z(E) = 0 \qquad \square$$

The proof of b) consists in showing that for $E \in \mathcal{A}$ and $c \in \mathbb{R}$ there are only a finite number of $v \in \Lambda$ with $v = v(F)$, $F \subset E$, and

$$\operatorname{Re}\,(Z + u \circ p)(F) < c\,,$$

(Not too bad). Point c) is more subtle.

Stability conditions on K3 surfaces

X a K3 , $\langle , \rangle$ Mukai pairing

on $\Lambda = \mathcal{N}(X) = H^0(X) \oplus NS(X) \oplus H^4(X)$

$\langle , \rangle : \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \xrightarrow{\cong} \Lambda_{\mathbb{C}}$

$$\begin{tikzcd}
\mathrm{Stab}(X) \arrow[rr, "q"] \arrow[ddrr, bend right, "\pi"'] & & \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \arrow[dd, "{\langle , \rangle}", "(*)"'] \\
\sigma = (\mathcal{A}, Z) \arrow[r, mapsto] & Z & \\
 & & \Lambda_{\mathbb{C}} = \mathcal{N}(X) \otimes \mathbb{C}
\end{tikzcd}$$

$\pi(\sigma) : Z(\ ) = \langle \pi(\sigma), \ \rangle$

$\mathcal{P}(X) = \{ \Gamma = u + iv \mid \langle u, v \rangle > 0 \}$

$GL_2^+(\mathbb{R})$ acts freely on $\mathcal{N}(X) \otimes \mathbb{R}^2$

section of this action:

$\mathcal{Q}(X) = \{ \Gamma \in \mathcal{P}(X) \mid \langle \Gamma, \Gamma \rangle = 0, \langle \Gamma, \bar{\Gamma} \rangle > 0, r(\Gamma) = 1 \}$

$$\begin{tikzcd}
NS(X) \otimes \mathbb{C} \arrow[r, leftrightarrow] & \mathcal{Q}(X) \\
(\omega, \beta) \arrow[r, mapsto] & \exp(B + i\omega) =
\end{tikzcd}$$

$= (1, B + i\omega, \frac{1}{2}(B^2 - \omega^2) + i\, B.\omega)$

$\omega^2 > 0$

(Later: $B = \beta H$, $\omega = \alpha H$)

$\mathcal{P}^+(X)$ = conn. comp of $\mathcal{P}$ containing $Z(X)$

$$\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta} \delta^\perp \subset \mathcal{N}(X) \otimes \mathbb{C}$$

$$\Delta = \{ \delta \in \mathcal{N}(X) \otimes \mathbb{C} \mid \delta^2 = -2 \}$$

Theorem (Bridgeland) there exists a conn. comp $\mathrm{Stab}^\dagger(X)$ of $\mathrm{Stab}(X)$ and a covering map

$$\mathrm{Stab}^\dagger(X) \xrightarrow{\pi} \mathcal{P}_0^+(X) \subset \Lambda_{\mathbb{C}}$$

Covering group:

$$\Gamma = \{ \gamma \in \mathrm{Aut}(\mathcal{D}(X)) \mid \gamma \text{ preserves } \mathrm{Stab}^\dagger \}$$

Moreover let :

$$V_{NS}(X) = \{ (\mathrm{Coh}^{B,\omega}(X), Z_{B,\omega}) \mid B, \omega \in NS(X), \omega^2 > 0 \}$$

then : $\widetilde{GL}_2(\mathbb{R}) \cdot V_{NS} = U(X)$

where $U(X) = \left\{ \sigma \in \mathrm{Stab}(X) \;\middle|\; \begin{array}{l} \mathbb{C}_x \ \sigma\text{-stab of} \\ \text{phase} = 1, \forall x \in X, \\ \text{and } \pi(\sigma) \in \mathcal{P}_0^+(X) \end{array} \right\}$

and $U(X) = \mathrm{Stab}^+(X) \setminus \left\{ \begin{array}{l} \text{real codimension} \\ 1 \text{ submanifolds} \end{array} \right\}$

One idea : E stable of phase, $0 < \phi < 1$,

$\mathbb{C}_x$ stable of phase 1 $\Rightarrow$ $\mathrm{Hom}^i(E, \mathbb{C}_x) = 0$

$i < 0$. Serre duality $\Rightarrow$ $\mathrm{Hom}^i(E, \mathbb{C}_x) = \mathrm{Hom}^{2-i}(\mathbb{C}_x, E)^\vee$

$= 0, \quad i \geq 2$

Bridgeland-Maciocia

$\Longrightarrow$ $E : E^{-1} \to E^0$.

(Homological Alg.)

[Br-K3] Proposition 10.3.