Notes for a brief introductory course to Bridgeland theory

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[BBDG] A. Beilinson, J. Bernstein, P. Deligne, O. Gabber: *Faisceaux Pervers.*

[H] D. Huybrechts: *Fourier Mukai Transforms in Algebraic Geometry.*

[Br-TC] T. Bridgeland: *Stability conditions on triangulated categories*

[Br-K3] T. Bridgeland: *Stability conditions on K3 surfaces.*

[Ba-Tour] A. Bayer: *A Tour of Stability conditions.*

[MS-lect] E. Macrì, B. Schmidt: *Lectures on Bridgeland Stability*

[Ch-B] F. Charles: *Conditions de stabilité et géométrie birationnelle.*

[BM-local] A. Bayer, E. Macrì : The space of stability conditions on the local projective plane

[APPENDIX B)]

[BMS] A. Bayer, E. Macrì, P. Stellari: stability conditions on abelian threefolds and some Calabi-Yau threefolds.

[APPENDIX A)]

[Alp] J. Alper : some incomplete lecture notes on Bridgeland stability.

[Ba-Short] A. Bayer: A short proof of the deformation property of Bridgeland stability conditions.

[BM-Proj] A. Bayer, E. Macrì: Projectivity and birational geometry of Bridgeland moduli spaces.

[Bot] A. Bottini: Stable sheaves on K3 surfaces via wall crossing.

SLOPES

- C smooth proj. curve.

$$Z : \mathrm{coh}(C) \longrightarrow \mathbb{C}$$

$$Z(\mathcal{F}) = -\deg(\mathcal{F}) + i\, \mathrm{rk}(\mathcal{F})$$

$$= -ch_1(\mathcal{F}) + i\, ch_0(\mathcal{F})$$

$$\mu(\mathcal{F}) = \frac{d(\mathcal{F})}{r(\mathcal{F})} \quad \text{slope}$$

1) $Z : \mathrm{Coh}(C) \longrightarrow H^*_{alg}(X, \mathbb{Z}) \longrightarrow \mathbb{C}$

$$H^*_{alg}(X,\mathbb{Z}) = \Lambda = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z})$$

2) $\mathcal{F} \neq 0$ $\mathrm{Im}\, Z \geq 0$, $\mathrm{Im}\, Z = 0 \Rightarrow \mathrm{Re}\, Z < 0$

- $\dim X \geq 2$ X smooth proj. surface

Search for

$$Z : \mathrm{coh}(X) \longrightarrow \mathbb{C} \quad \text{with 1) and 2):}$$

$$Z(\ ) = (a_0 + i b_0)\, ch_0 + (a_1 + i b_1)\, ch_1 + (a_2 + i b_2)\, ch_2$$

$\mathcal{F} = \mathcal{O}_C(np)$, $p \in C \subset S$ $\quad \mathrm{Im}\, Z(\mathcal{F}) \geq 0 \Rightarrow b_2 = 0$

$\mathcal{F} = \mathcal{O}_X(n)$ $\quad n \ll 0 \quad \Rightarrow b_1 = 0$

$\mathrm{Re}\, Z(\mathcal{O}_C(np)) < 0$ $\;_{n \gg 0}$ $\Rightarrow a_2 = 0$

But then $Z(\mathbb{C}_x) = 0 \Rightarrow$ such a Z does not exist.

One idea: go to Gieseker stability.
Another: change abelian category:
Start with $\mathcal{A} = \mathrm{Coh}(X)$, Consider $\mathcal{D}^b(\mathcal{A}) = \mathcal{D}^b$
Look for another abelian category $\mathcal{A}' \subset \mathcal{D}^b(\mathcal{A})$

$\boxed{\text{A FEW THINGS ABOUT DER. CAT.}}$

[H]. p 1-52

- $C \in \mathcal{D}^b$, $(C[1])^i = C^{i+1}$.

- distinq. triangles: $A \xrightarrow{f} B \to C \to A[1]$

  iso to $A \xrightarrow{f} B \to C(f) = B \oplus A[1]$
  $\uparrow$ ([H] p.33 Def 2.15)

- 4 Axioms e.g.:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dotted, "\exists g"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

commutative

g not necess. unique

- Corollary: $A \to B \to C \xrightarrow{0} A[1]$

$\Rightarrow \quad B = A \oplus C$

- **Example**: $E : E^{-1} \xrightarrow{d} E^0$

$$\begin{array}{ccccc} \operatorname{Ker} d\,[1] & \xrightarrow{c} & E & \longrightarrow & \operatorname{Coker} d \\ \| & & & & \wr\| \\ H^{-1}(E)[1] & & & & H^0(E) \end{array}$$

dist. triang.

i.e. $\operatorname{Coker} d \underset{\text{q.i.}}{\cong} C(c)$

Exercise

- **Fact**: $A \to B \to C \to A[1]$

triangle + $A, B, C \in \mathcal{A}$

$\Longleftrightarrow \quad A \hookrightarrow B \twoheadrightarrow C$ exact.

- $E, F \in \mathcal{A} \Longrightarrow \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(E, F[i]) = \operatorname{Ext}^i_{\mathcal{A}}(E, F)$

[H] Prop. 2.56

Corollary: C smooth curve $\quad E \in \mathcal{D}(C)$

$$E \cong \bigoplus H^{-i}(E)[i]$$

Hint: start with the case $E = E^{-1} \to E^0$ then

$H^{-1}(E)[1] \to E \to H^0(E) \xrightarrow{\varphi} H^{-1}[1][1] \ldots\ldots$

$\varphi = \operatorname{Hom}(H^0(E), H^{-1}[2])$

$= \operatorname{Ext}^2(\qquad) = 0$

.....

New abelian subcat. $\mathcal{A}' \subseteq \mathcal{D}^b(\mathcal{A})$

t-structure: ([DBBG] p.29-40)

$\left.\begin{array}{l}\mathcal{D}^{\leq 0}\\ \mathcal{D}^{\geq 0}\end{array}\right\}$ Full subcat. s.t.:

E.g. trivial t-struct.

$\mathcal{D}^{\leq 0} = \{C \mid H^i(C)=0,\ i>0\}$

$\mathcal{D}^{\geq 0} = \{C \mid H^i(C)=0,\ i<0\}$

1) $\mathcal{D}^{\leq 0}[1] \subseteq \mathcal{D}^{\leq 0}$

$\mathcal{D}^{\geq 0}[-1] \subseteq \mathcal{D}^{\geq 0}$

2) $\mathrm{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$

3) $\forall\ C \in \mathcal{D}\ \ \exists\ \ \underset{\mathcal{D}^{\leq 0}}{A} \to C \to \underset{\mathcal{D}^{\geq 1}}{B} \to A[1]$

$\mathcal{A}^{\#} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} :=$ heart of t-structure

Def $\left\{\begin{array}{l}\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]\\ \mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]\end{array}\right.$

Theorem: a) $\exists\ \tau^{\leq n} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}^{\leq n}$,

$\exists\ \tau^{\geq n} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}^{\geq n}$.

b) $\tau^{\leq 0}(C) \longrightarrow C \longrightarrow \tau^{\geq 1}(C)$ triangle:

[e.g trivial t-structure: $\tau^{\leq 0}(C)$: $\cdots \to C^{-1} \to \mathrm{Ker}\, d_0 \to 0 \to 0 \cdots$

$\tau^{\geq 1}(C)$: $0 \to C^0/\mathrm{Ker}\, d_0 \to C^1 \to C^2 \to \cdots$]

c) $\mathcal{A}^{\#}$ is abelian.

Def: $H^n_\# : \mathcal{D} \longrightarrow \mathcal{A}^\#$

$[n] \circ \tau^{\leq n} \circ \tau^{\geq n} \simeq \tau^{\geq n} \circ \tau^{\leq n} \circ [n]$

Theorem: $H^n_\#$ is a cohomological functor (long exact seq. etc...)

Remark. $A \in \mathcal{A}_\# \iff H^0_\#(A) = A$, $H^i_\#(A) = 0$, $i \neq 0$

Def bdd t-struct. $\iff$ $H^n_\#(C) \neq 0$ finit. many n.

Also $H^n_\#(C) = H^0_\#(C[n])$

Given: $A \xrightarrow{f} B \longrightarrow C = C(f) \longrightarrow A[1]$

$\mathrm{Ker}^\# f = H^{-1}_\#(C) = \tau^{\leq 0}(C)[-1]$

$\mathrm{Coker}^\# f = H^0_\#(C) = \tau^{\geq 0}(C)$

(this criterion is Lemma 3.2 in [Br-TC]) its proof is given as Exercise.)

Taken as a Definition in [MS lect] p.20 Def 5.1

A CRITERION for $\mathcal{A}^\# \subset \mathcal{D}$, full sub-category, to be the HEART of a t-structure in $\mathcal{D}$:

bdd t-struct. determined by its heart

A) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}^\#, \mathcal{A}^\#[k]) = 0 \quad k < 0$

B) $\forall\ E \in \mathcal{D}$, $\exists\ k_1 > k_2 > \dots > k_n$ ⑥

$$(*)\quad \begin{tikzcd}[column sep=small] 0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d, "\cong"'] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[d] \\ & A_1 \arrow[ur, dotted] & A_2 \arrow[ur, dotted] & & & A_n \arrow[ull, dotted] \end{tikzcd}$$

$A_i \in \mathcal{A}^*[k_i]$

"Pf": $\Rightarrow$) Start with bdd t-structure

let $-k_1 < -k_2 < \dots < -k_n$ s.t. $H_*^{-k_i}(E) \neq 0$:

$$\tau^{\leq -k_{i-1}}(E) = E_{i-1} \longrightarrow E_i = \tau^{\leq -k_i}(E) \longrightarrow A_i = H_*^{-k_i}(E)[k_i]$$

(think of trivial t-structure
and assume $(k_1, \dots, k_n) = (1, 2, \dots, n)$)

$\Leftarrow$) Start with $\mathcal{A}_*$ with A) and B).

Define

$$\mathcal{D}^{\leq 0} = \{E \mid A_i = 0 \quad k_i > 0\}$$
$$\mathcal{D}^{\geq 0} = \{E \mid A_i = 0 \quad k_i < 0\}$$

i.e. cut $(*)$ in two.

$\boxed{\text{Torsion pairs}}$ $\rightsquigarrow$ t-structure $\rightsquigarrow$ ${}^{\prime}\mathcal{A}$

([MS lect] § 6)

$\mathcal{F}, \mathcal{T}$ full subcategories of $\mathcal{A}$ $(= \mathrm{Coh}(X))$

1) $\mathrm{Hom}(T, F) = 0 \qquad T \in \mathcal{T},\ F \in \mathcal{F}$

2) $\forall\ E \in \mathcal{A},\ \exists:\ 0 \to \underset{\mathcal{T}}{\underset{\cap}{T(E)}} \to E \to \underset{\mathcal{F}}{\underset{\cap}{F(E)}}$

unique and functorial

$$\boxed{\begin{aligned} {}^{\prime}\mathcal{A} &= \{ E \in \mathcal{D} \mid H^i(E) = 0 \quad i \neq -1, 0,\ H^{-1}(E) \in \mathcal{F},\ H^0(E) \in \mathcal{T} \} \\ &= \left\{ E : E^{-1} \xrightarrow{d} E^0 \;\middle|\; \begin{array}{l} H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E) \\ \text{short exact sequence in } {}^{\prime}\mathcal{A} \end{array} \right\} \\ &= \langle \mathcal{F}[1], \mathcal{T} \rangle \qquad \text{extension closure } \langle\ ,\ \rangle \end{aligned}}$$

${}^{\prime}\mathcal{A}$ <u>abelian</u> (it is the heart of a t-structure)

([MS lect.] § 6)

${}^{\prime}\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ <u>where</u>:

$\mathcal{D}^{\leq 0} = \{ E \in \mathcal{D} \mid H^i(E) = 0,\ i > 0,\ H^0(E) \in \mathcal{T} \}$

$\mathcal{D}^{\geq 0} = \{ E \in \mathcal{D} \mid H^i(E) = 0,\ i < -1,\ H^{-1}(E) \in \mathcal{F} \}$.

<u>Example:</u> X smooth proj surface,

$B \in NS(X)$, $\omega \in \mathrm{Amp}(X)$, $E \in \mathrm{Coh}(X)$

$$\mu = \mu_{B,\omega}(E) = \begin{cases} \dfrac{\omega \cdot c_1(E)}{r(E)} - \omega . B, & \text{if } r(E) \neq 0, \\ 0, & \text{if } r(E) = 0. \end{cases}$$

sometimes divide both summands by ω^2

μ-stability as usual;

Harder - Narasimhan:

$$0 = E_0 \subset E_1 \subset \dots \subset E_{n-1} \subset E_n = E$$

$$A_i = E_i / E_{i-1} \quad \mu\text{-stable}$$

$$\mu^+(E) = \mu(A_1) > \dots > \mu(A_n) = \mu^-(E)$$

$$\mathcal{T}^{B,\omega} = \{ E \in \mathrm{Coh}(X) \mid \text{For all } \underset{\text{in } \mathrm{Coh}(X)}{E \twoheadrightarrow F} \quad \mu(F) > 0 \} \supset \text{Torsion}$$

$$\mathcal{F}^{B,\omega} = \{ E \in \mathrm{Coh}(X) \mid \text{For all } \underset{\text{in } \mathrm{Coh}(X)}{F \hookrightarrow E} \quad \mu(F) \leq 0 \} \subset \text{Tor-free}$$

<u>equiv:</u> $\left\{ \dots\dots\dots \quad \text{For all H-N factor } A \text{ of } E \quad \begin{matrix} \mu(A) > 0 \\ \mu(A) \leq 0 \end{matrix} \quad \dots\dots \right\}$

usual rules e.g.: E F semist $\mu(E) > \mu(F) \Rightarrow \mathrm{Hom}(E, F) = 0$

$(\mathcal{T}^{B,\omega}, \mathcal{F}^{B,\omega})$ torsion pair

$$\mathrm{Coh}^{B,\omega} := \{ E \in \mathcal{D} \mid E: E^{-1} \xrightarrow{d} E^0, \quad \underset{\mathcal{F}^{B,\omega}[1]}{\underset{\cap}{H^{-1}(E)[1]}} \to E \to \underset{\mathcal{T}^{B,\omega}}{\underset{\cap}{H^0(E)}} \}.$$

Lemma: Let $E \in \mathcal{T}$, a subobject of E is given by:

1) $B \in \mathcal{T} = \mathcal{T}^{B,\omega}$
2) $f : B \to E$
3) $\ker f \in \mathcal{F}$

Pf.: $f : B \to E$ morphism in $\mathcal{A}' = \mathrm{Coh}^{B,\omega}(X)$

can use the definition of 'Ker f (p.5). Otherwise:

complete f to a triangle $B \xrightarrow{f} E \to C$ in $\mathcal{D}(\mathcal{A})$

$\mathcal{A} = \mathrm{Coh}(X)$

$$H^{-2}(C) \to H^{-1}(B) \to 0 \to H^{-1}(C) \to H^{0}(B) \xrightarrow{H^0(f)} H^0(E) \to H^0(C) \to 0$$

(Fact p.3) $\qquad H^0(E) = E$

f monomorph. $\Leftrightarrow$ $C \in \mathcal{A}'$ $\Rightarrow$ $H^{-2}(C) = 0$ $\Rightarrow$ $H^{-1}(B) = 0$

$\Rightarrow B \in \mathcal{T}$ $\Rightarrow$ $H^0(f) = f$ $\Rightarrow$ $\ker f = H^{-1}(C) \in \mathcal{F}$

Viceversa $\quad B \in \mathcal{T} \Rightarrow H^{-1}(B) = 0 \Rightarrow H^{-2}(C) = 0$

$\Rightarrow H^{-1}(C) = \ker f \in \mathcal{F}$, Also $H^0(E) = E \to H^0(C) \to 0$

$\Rightarrow C \in \mathcal{A}'$ $\qquad \square$

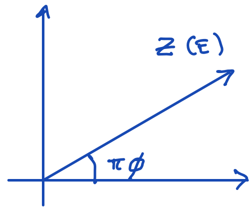
Observe: $\quad B = \beta\omega \quad \beta < 0 \quad \Rightarrow \quad \mathcal{O}_X \in \mathcal{T}^{B,\omega}.$

Next:

DEFINITION: A stability functions on $\mathcal{A}$ abelian cat. (10)

is given by $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ additive

s.t. $\operatorname{Im} Z \geq 0$, $\operatorname{Im} Z = 0 \Rightarrow \operatorname{Re} Z < 0$



Notation:

- $Z(E) = |Z(E)| \, e^{i\pi\phi(E)} \leftarrow$ phase of E $1 \leq \phi < 0$.

$$R = \operatorname{Im} Z \qquad Z = -D + iR$$
$$D = -\operatorname{Re} Z$$

$\nu = \frac{D}{R}$ = slope $\rightsquigarrow$ notion of Z-stability (semistability)

$(R = 0 \quad \nu = +\infty)$

- $E \in \mathcal{A}$ Z-ss $\Leftrightarrow$ no subobj. $F \subset E$ has $\geq$ slope

$\phi = \frac{1}{\pi} \operatorname{arccot}(-\nu)$ [We will later examine H-N and J-H w.r.t. Z (or ν)]

Def Z numeric if Z factors.

$$Z : K_0(\mathcal{A}) \xrightarrow{\lambda} \Lambda \longrightarrow \mathbb{C}$$

$Z(E) = Z(\lambda(E))$ $\qquad$ Λ finite dim lattice

Example: X smooth proj surface

$\Lambda = H^*_{alg}(X, \mathbb{Z})$

- One possibility $\lambda(\ ) = ch(\ )$ [MSlect] sect. 6.2
- or (if X is a K3) $\lambda(\ ) = v(\ ) = ch(\ )\sqrt{td(X)}$

Mukai vector

$\mathcal{A} = \mathrm{Coh}^{B,\omega}$ $\qquad B \in NS(X) \qquad \omega \in \mathrm{Amp}(X)$

$$\boxed{\begin{aligned} ch_0^B &= ch_0 = r \\ ch_1^B &= ch_1 - B\,ch_0 \\ ch_2^B &= ch_2 - B\,ch_1 + \tfrac{B^2}{2}\,ch_0 \end{aligned}}$$

Def: $ch^B = ch \cdot e^{-B}$

$$Z(E) := Z_{B\omega}(E) = -\int \exp(-B - i\omega) \cdot ch(E)$$
$$= -\int \exp(-i\omega) \cdot ch^B(E)$$

$$= \left(-1 + i\omega + \frac{\omega^2}{2},\ ch_0^B(E) + ch_1^B(E) + ch_2^B(E)\right)_2$$

$$= \underbrace{-ch_2^B(E) + \frac{\omega^2}{2}\, ch_0^B(E)}_{(*)} + i\,(\omega \cdot ch_1^B(E))$$
$$i\,(\omega \cdot ch_1^B(E)) = i\,\omega\cdot(c_1(E) - r(E)B) = i\, r(E)\, \mu_{B,\omega}$$

Lemma: Z is a stab function:

Pf.

$$H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E)$$
$$H^{-1}(E)[1] \in \mathcal{F}[1], \qquad H^0(E) \in \mathcal{T}, \qquad \mathrm{Im}(H^0(E)) \geq 0$$

$$Z(H^{-1}(E)[1]) = -Z(H^{-1}(E))$$
$$\mathrm{Im}\, Z(H^{-1}(E)[1]) \geq 0 \qquad \Rightarrow \qquad \mathrm{Im}\, Z \geq 0$$

Suppose: $\mathrm{Im}\, Z(E) = 0 \quad \Rightarrow \mathrm{Im}\, Z(H^0(E)) = \mathrm{Im}\, Z(H^{-1}(E)) = 0$

$\Downarrow$

$r(H^0(E))\,\mu_{B\omega}(H^0(E)) = 0, \quad H^0(E) \in \mathcal{T} \Rightarrow \mu_{B\omega}(H^0(E)) > 0$

$\Rightarrow r(H^0(E)) = 0 \Rightarrow c_1(H^0(E)) = 0 \Rightarrow ch_2(H^0(E)) \geq 0$

$\Rightarrow \mathrm{Re}(H^0(E)) = -ch_2^B(E) \leq 0$.

Want: $\mathrm{Re}\ Z(\underbrace{H^{-1}(E)}_{F}) > 0$

$\mathrm{Im}\ Z(F) = 0 \Rightarrow \mu(F) = 0 \Rightarrow F\ \mu\ ss$

1) $\omega \cdot ch_1^B(F) = 0$, Hodge index theorem $\Rightarrow$
$ch_1^B(F)^2 \leq 0$.

2) Bogomolov [M-S p. 32] F $\mu_{B\omega}$-ss, torsion free,

$$\Rightarrow ch_1^B(F)^2 - 2\,ch_0^B(F) \cdot ch_2^B(F) \geq 0$$

$$\Rightarrow ch_2^B(F) \leq 0 \Rightarrow \mathrm{Re}\ Z(F) > 0 \quad (*)$$
$$\left(\frac{\omega^2}{2} ch_0^B(F) > 0\right)$$

$\square$

<u>Case of K3 surfaces</u> (mostly when $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$)

Here instead of <u>ch</u> use <u>v</u> and Mukai pairing

$$v(E) = \left(ch_0(E),\ ch_1(E),\ ch_2(E) + ch_0(E)\right)$$

$$Z(E) = \langle \exp(B + i\omega), v(E) \rangle$$

$$Z : \Lambda \longrightarrow \mathbb{C}$$

In all relevant results (K3-surf. case)
$\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, so:

$$\boxed{B = \beta \cdot H, \quad \omega = \alpha H} \qquad \beta \in \mathbb{R},\ \alpha \in \mathbb{R}_+$$

$v(E) = v = (r, cH, s)$ $\quad$ $s = ch_2(E) + ch_0(E)$
$r = ch_0(E)$, $cH = ch_1(E)$

$$\boxed{Z_{\beta\alpha}(E) = -s + \beta c H^2 - \frac{r}{2}(\beta^2 - \alpha^2)H^2 + i(c - r\beta)\alpha H^2}$$

Theorem X a K3 surface, $Z_{\beta\alpha}$ is a stability function for $\mathrm{Coh}^{\beta,\alpha}(X)$ if the following condition is satisfied:

(*) $\boxed{\forall\, \delta \in \Lambda, \text{ with } \delta^2 = -2, \ r(\delta) > 0, \text{ and } \mu_{\beta\alpha}(\delta) = 0, \text{ then } \mathrm{Re}\, Z_{\beta\alpha}(\delta) > 0}$

(This always happens if $\alpha^2 H^2 \geq 2$).

Pf. As before up to the point where must show $-\mathrm{Re} Z(H^{-1}(E)) < 0$.
May assume $H^{-1}(E) \neq 0$ since $\mathrm{Im}(H^{-1}(E)) = 0$

$\mu_{\beta\alpha}(H^{-1}(A)) = 0 \Rightarrow H^{-1}(A)$, $\mu_{\beta\alpha}$-ss. torsion free (positive rank). May assume stable $\Rightarrow$ $v(H^{-1}(E))^2 \geq -2$. Set $v(H^{-1}(E)) = v = (r, c_1, s)$

If $v^2 = -2$, set $\delta = H^{-1}(E)$ and (*) $\Rightarrow$ $\mathrm{Re}\, Z(\delta) > 0$. Otherwise $v^2 = c_1^2 - 2rs \geq 0$

Since $c_1 = r\beta H$ get $r^2\beta^2H^2 - 2rs \geq 0$

$$\Rightarrow \quad \operatorname{Re} Z(H^{-1}(E)) = -s + \frac{r}{2}(\alpha^2+\beta^2)H^2 \geq \alpha^2 H^2 \frac{r}{2} > 0$$

$\square$

Recall :
$$\mu_{B,\omega}(E) = \frac{\omega \cdot C_1(E)}{\omega^2 \, r(E)} - \frac{\omega \cdot B}{\omega^2}$$

when $B = \beta H$, $\omega = \alpha H$, $c_1(E) = c(E)\cdot H$

$$\boxed{\mu_{\beta\alpha}(E) = \frac{c(E)}{\alpha\, r(E)} - \frac{\beta}{\alpha}}$$

The expression $\mu_{\beta\alpha}(E) \geq \mu_{\beta\alpha}(F)$

only depends on β. $\Longrightarrow$

we can write $\mathcal{F}^\beta, \mathcal{T}^\beta, \operatorname{Coh}^\beta$

instead of $\mathcal{F}^{\beta,\alpha}, \ldots$

Lemma: For all $x \in X$, $\mathbb{C}_x \in \mathcal{T}^\beta \subset \operatorname{Coh}^\beta(X)$, is minimal (i.e. has no nontrivial subobject.)

Pf. Suppose $0 \to A \to \mathbb{C}_x \to B \to 0$ in $\operatorname{Coh}^\beta(X)$

Then $H^{-1}(A) = 0$ so that $A = H^0(A)$ and we get

$$0 \to H^{-1}(B) \longrightarrow A \longrightarrow \underset{\underset{\mathbb{C}_x}{\parallel}}{H^0(\mathbb{C}_x)} \longrightarrow H^0(B) \to 0$$

exact sequence of sheaves

But then : $\mu_\beta(H^{-1}(B)) = \mu_\beta(H^0(A))$

which is absurd as $H^{-1}(B) \in \mathcal{F}^\beta$, and $A = H^0(A) \in \mathcal{T}^\beta$.

The only way out is : $H^{-1}(B) = 0$,

but then $0 \to A \to \mathbb{C}_x \to H^0(B) \to 0$ is

exact in $Coh(X)$ so that either $A = 0$

or $A = \mathbb{C}_x$. $\square$

The large volume limit. (see [BrK3] § 14)

We limit ourselves to the case X K3

$Pic(X) = \mathbb{Z}\cdot H$, and to $Z_{\beta\alpha} : \Lambda \longrightarrow \mathbb{C}$

$\beta \in \mathbb{R}$ $\alpha \in \mathbb{R}_+$ but valid

for general Pic and for $Z_{B,\omega}$

$B, \omega \in NS(X) \otimes \mathbb{R}$, $\omega \in Amp(X)$.

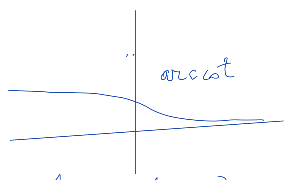
So consider the stab function $Z_{\beta\alpha}$

and the $Z_{\beta,\alpha}$ - semistability : $E \in Coh^\beta(X)$ is Z-s.s. $\Leftrightarrow$

$$\nu = \frac{-\mathrm{Re}\, Z}{\mathrm{Im}\, Z}$$

$A \subset E \Rightarrow \nu_{\beta\alpha}(A) \leq \nu_{\beta\alpha}(E)$

equiv. $\phi_{\beta\alpha}(A) \leq \phi_{\beta\alpha}(E)$.



$\Phi = \frac{1}{\pi} \mathrm{arccot}(-\nu)$

$\sigma(E) = (r, cH, s) \equiv (r, c, s)$

$$\mu_\beta = \frac{c}{r} - \beta$$

if $E \in Coh(X)$ <u>reduced</u> Hilb poly:

$$P_E(m) = \frac{m^2}{2} + \frac{c}{r} m + \frac{s+r}{rH^2} \qquad s+r = \chi(E)$$

Gieseker H-ss:

- $r \neq 0$: $\forall\ A \subset E$ $\quad v(A) \equiv (r', c', s')$

  $\frac{c'}{r'} \leq \frac{c}{r}$ if $\frac{c'}{r'} = \frac{c}{r}$ then $\frac{s'}{r'} \leq \frac{s}{r}$

- $r = 0$: $\frac{s'}{c'} < \frac{s}{c}$

write $\lambda_\beta(E) = \frac{s - \beta c}{r}$

useful formula $(r(E) \neq 0,\ r(A) \neq 0)$

(*)
$$\frac{Z_{\alpha\beta}(E)}{r(E)} - \frac{Z_{\alpha\beta}(A)}{r(A)} = -\left(\lambda_\beta(E) - \lambda_\beta(A)\right) + i\,\alpha\left(\mu_\beta(E) - \mu_\beta(A)\right)$$

Recall:
$$\boxed{Z_{\beta\alpha}(E) = -s + \beta c H^2 - \frac{r}{2}(\beta^2 - \alpha^2) H^2 + i\,(c - r\beta)\,\alpha H^2}$$

Theorem: let $E \in \mathrm{Coh}^\beta(X)$. Assume

- either $r > 0$, and $c - \beta r > 0$ $\quad (\Rightarrow \mu_\beta > 0)$
- or $r = 0$, $c > 0$.

Then $\exists\ \alpha_0 \in \mathbb{R}_+$,

$$\left(E \text{ is } Z_{\beta\alpha}\text{-ss},\ \forall\ \alpha \geq \alpha_0\right) \iff \left(E \text{ is Gieseker } H\text{-ss sheaf}\right)$$

Pf.

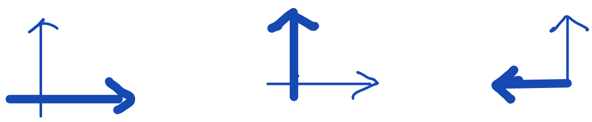
$$\frac{\mathrm{Re}\, Z_{\beta\alpha}}{\mathrm{Im}\, Z_{\beta\alpha}} = \frac{-s + \left(\beta c - \frac{r}{2}\left(\beta^2 - \alpha^2\right)\right) H^2}{(c - \beta r)\,\alpha . H^2} \begin{cases} \to +\infty \\ \to 0 \\ \to 1 \end{cases}$$

$$\alpha \longrightarrow +\infty \quad \left\{ \begin{array}{ll} \mathrm{supp}(E) = X, & r > 0 \\ \dim \mathrm{Supp}(E) = 1, & r = 0,\ c \neq 0 \\ \dim \mathrm{Supp}(E) = 0, & r = c = 0 \end{array} \right\} \quad \phi(\alpha) \to \begin{cases} 0 \\ \frac{1}{2} \\ 1 \end{cases}$$

$$\Downarrow$$

$\mathrm{Im}\, Z = 0$, $\mathrm{Re}\, Z < 0$ $\quad \Rightarrow \phi = 1$

$$\phi = \frac{1}{\pi} \operatorname{arccot} \frac{\mathrm{Re}\, Z}{\mathrm{Im}\, Z}$$



$\boxed{\text{Case } r(E) > 0}$

$$H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E)$$

$H^{-1}(E)$ torsion free $\Rightarrow$ $\lim\limits_{\alpha \to +\infty} \phi\left(H^{-1}(E)\right) = 0$

$\Rightarrow \lim\limits_{\alpha \to +\infty} \phi\left(H^{-1}(E)[1]\right) = 1$. Also $r(E) > 0 \Rightarrow$ $\lim\limits_{\alpha \to +\infty} \phi(E) = 0$, so $H^{-1}(E)[1]$ destabilizes E for $\alpha \gg 0$, against the hypothesis. Absurd.

$\Rightarrow H^{-1}(E)=0 \Rightarrow E=H^0(E)$ is a sheaf (18)

Suppose not Gieseker stable, then $\exists A\subset E$

$\mu_\beta(A)>\mu_\beta(E)$ or $\mu(A)_\beta=\mu_\beta(E)$
and $\lambda_\beta(A)>\lambda_\beta(E)$

suppose $\mu_\beta(A)>\mu_\beta(E)$, i.e. $\boxed{\mu_\beta(A)=\varrho\,\mu_\beta(E) \quad \varrho>1}$

$$\frac{Z(A)}{r(A)} = L_A+\alpha^2 M + i\mu_\beta(A)\alpha H^2,$$
$$\frac{Z(E)}{r(E)} = L_E+\alpha^2 M + i\mu_\beta(E)\alpha H^2,$$

$\phi(A)<\phi(E)$ for $\alpha\gg 0 \iff$

$$\frac{L_A+\alpha^2 M}{\varrho\,\mu_\beta(E)\,\alpha H^2} \overset{(*)}{>} \frac{L_E+\alpha^2 M}{\mu_\beta(E)\,\alpha H^2} \Rightarrow \varrho<1$$

case $\mu_\beta(A)=\mu_\beta(E)$ use (*) p. 16

$\square$

$\boxed{\text{Case } r(E)=0}$ easier.

$(*)$ $\phi=\frac{1}{\pi}\operatorname{arccot}\left(\frac{\operatorname{Re} Z}{\operatorname{Im} Z}\right)$ $\quad \phi\downarrow$

PRESTABILITY CONDITIONS

Definition: $\mathcal{A}$ abelian, $Z: K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ stab function,

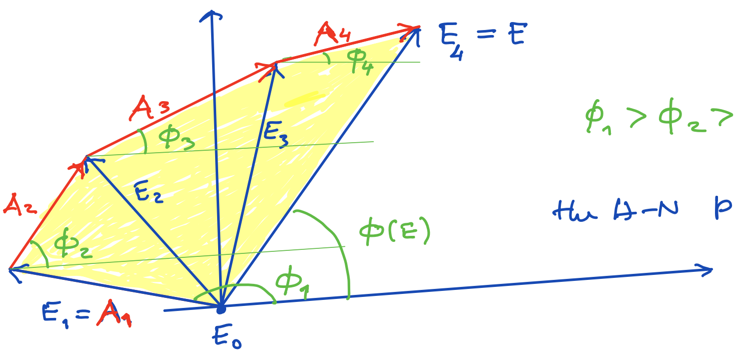
the pair $\sigma = (Z, \mathcal{A})$ is a prestability condition

if every object $E \in \mathcal{A}$ has a Harder–Narasimhan

filtration: $\quad 0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$

with: $E_i \in \mathcal{A}$, $\quad A_i = E_i / E_{i-1}$, Z-semistable

and $\quad \underset{\phi^+(E)}{\underset{\shortparallel}{\phi(A_1)}} > \cdots > \underset{\phi^-(E)}{\underset{\shortparallel}{\phi(A_n)}}$



$\phi_1 > \phi_2 > \phi_3 > \phi_4$

the H–N polygon

Harder Narasimhan

I)

Theorem: [Br TC] p.323

$Z : K(\mathcal{A}) \to \mathbb{C}$ stab function s.t.

a) $\nexists$ infinite seq $\subset E_{k+1} \subset E_k \subset \cdots \subset E_n = E$

$\cdots > \phi(E_{k+1}) > \phi(E_k) > \cdots$

b) $\nexists$ infinite seq. $E_1 \twoheadrightarrow E_2 \twoheadrightarrow E_3 \twoheadrightarrow \cdots$

$\cdots > \phi(E_k) > \phi(E_{k+1}) > \cdots$

Then HN holds for Z.

"pf" Given $E \in \mathcal{A}$, find a maximal destabilizing quotient $E \twoheadrightarrow B$

i.e. $\forall$ $E \twoheadrightarrow B'$ $\phi(B') \geq \phi(B)$

with $=$ iff $E \to B \to B'$.

Corollary: [Br K3] (Proposition 7.1) $(Coh^\beta, Z_{\alpha\beta})$ $\boxed{\alpha, \beta \in \mathbb{Q}}$ is a prestability condition i.e. H.N. holds.

Pf. Must show a) and b).

suppose a) does not hold. $\exists$ short exact

$$0 \to E_{i+1} \to E_i \to F_i \to 0, \quad \text{Since } \operatorname{Im} Z \geq 0,$$

$\operatorname{Im} Z(E_{i+1}) < \operatorname{Im} Z(E_i)$

Image Z discrete $\Rightarrow$ $\operatorname{Im} Z(E_i) = \text{const}$ $i \gg 0$

$\Rightarrow \operatorname{Im} Z(F_i) = 0 \Rightarrow \operatorname{Re} Z(F_i) < 0 \Rightarrow$

$\operatorname{Re} Z(E_{i+1}) > \operatorname{Re} Z(E_i)$, but $\phi(E_{i+1}) > \phi(E_i)$.

b) Similar (harder).

II)

Theorem: [MS lect] (Prop 4.10)

$Z : K(\mathcal{A}) \longrightarrow \mathbb{C}$ stab funct.

- $\mathcal{A}$ noetherian category,
- Image $(\operatorname{Im} Z) \subset \mathbb{C}$ discrete,

then Z satisfies H N.

"Pf" graph proof following Shatz, as revived by Bayer ([Ba-Towr], Theorem 2.1.6.).

Corollary $(\mathrm{Coh}^\beta, Z_{\alpha\beta})$ $(\alpha, \beta) \in \mathbb{Q}$ is a prestability condition

Proof. Must show Coh^β is a Noetherian category. [MS lect] Lemma 6.17

Slicings

Definition: A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ (e.g. $\mathcal{D} = \mathcal{D}(X)$) is the datum of a full subcat. $\mathcal{P}(\phi) \subset \mathcal{D}$, $\forall \phi \in \mathbb{R}$, s.t.

1) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1] \quad \forall \phi \in \mathbb{R}$

2) $\mathrm{Hom}(E_1, E_2) = 0 \qquad E_i \in \mathcal{P}(\phi_i)$, $\phi_1 > \phi_2$.

3) $\forall \; E \in \mathcal{D}$ $\exists$ diagr. of dist. triang

$$(*) \quad \begin{tikzcd} 0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\ & A_1 \arrow[ul, dashed, "{[1]}"] & & A_2 \arrow[ul, dashed, "{[1]}"] & & & & A_n \arrow[ul, dashed, "{[1]}"] & \end{tikzcd}$$

$A_i \in \mathcal{P}(\phi_i)$, $\phi_1 > \cdots > \phi_n$ (Harder Narasimhan)

$\phi_1 = \phi^+(E)$, $\phi_n = \phi^-(E)$

Definition: $\mathcal{P}(\phi)$ = semi-stable objects of phase ϕ

Remark: H-N is unique up to iso.

Suppose

$$\begin{tikzcd} E'_0 \arrow[r] & E'_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E'_m = E \arrow[dl] \\ A_1 & & A_m & \end{tikzcd}$$

$E'_0 \cong 0$

Suppose $\phi(E_1') \geq \phi(E_1)$

k minimal s.t. $\exists\, E_1' \to E_k$

Then: $E_1' \to E_k \to A_k$

$$\phi(A_k) \geq \phi(E_1') \geq \phi(E_1) \geq \phi(A_k)$$

$\Rightarrow$ all $=$. But $\phi(E_1) = \phi(E_k/E_{k-1}) = \phi(A_k)$
$$\phi(E_1) = \phi(A_1)$$

$\Rightarrow A_k = E_1$ i.e. $k=1$ so that $E_1' \xrightarrow{\cong} E_1$. $\square$

Thus: it makes sense to set:

$$\phi_1 = \phi^+(E), \quad \phi_n = \phi^-(E)$$

$$\phi_i = \phi_i(E)$$

<u>Notice:</u>
If everything takes place in an abelian category, so that Triangles are exact sequence get $E_1' = E_1$ and therefore uniqueness.

Definition: $I \subset \mathbb{R}$ interval

$$\mathcal{P}(I) = \{\text{extension-closure of the } \mathcal{P}(\phi),\ \phi \in I.\}$$

$$\Rightarrow \boxed{E \in \mathcal{P}(\phi^-(E), \phi^+(E))}$$

Remark: $\mathcal{P}((a,b)) = \{E \in \mathcal{D} \mid a < \phi^-(E) \le \phi^+(E) < b\} \cup \{0\}$

$= \{E \in \mathcal{D} \mid \text{each } \phi_i(E) \in (a,b)\} \cup \{0\}$

Proposition: $\mathcal{P}((\phi, \phi+1])$ heart of a t-structure ($\Rightarrow$ Abelian)

$$\mathcal{D}^{\leq 0} := \mathcal{P}(>\varphi), \quad \mathcal{D}^{\geq 0} := \mathcal{P}(\leq \varphi+1)$$

$$\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \mathcal{P}((\varphi, \varphi+1])$$

([Ch-B] Proposition 1.9)

Remark: $\mathcal{P}(\phi)$ is abelian

Pf. $A \xrightarrow{f} B$ in $\mathcal{P}(\phi] \in \mathcal{P}((\phi-1, \phi])$ abelian

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & I = \operatorname{im} f \arrow[ur] \arrow[d] & \\ & A_n(I) \in \mathcal{P}(\phi^-(I)) & \end{tikzcd}$$

$\phi(A) = \phi \leq \phi^-(I)$

Also $\phi^+(I) \leq \phi(B) = \phi$

$\Rightarrow \phi^+(I) = \phi^-(I) = \phi$, $I \in \mathcal{P}(\phi)$

Ker and Coker similar.

Theorem: The following are equivalent

a) the datum $\sigma = (\mathcal{P}, Z)$ where $\mathcal{P}$ is a slicing of $\mathcal{D}$, and $Z : K_0(\mathcal{D}) \longrightarrow \mathbb{C}$ is a homomorphism s.t. $\forall\ \phi \in \mathbb{R}$ and $E \in \mathcal{P}(\phi)$, $Z(E) \in \mathbb{R}_+ e^{i\pi\phi}$

b) a prestability condition $(\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a t-structure on $\mathcal{D}$

Pf Idea:

$$(\mathcal{P}, Z) \rightsquigarrow (\mathcal{A}, Z) \qquad \mathcal{A} = \mathcal{P}((0,1])$$

$$(\mathcal{A}, Z) \rightsquigarrow (\mathcal{P}, Z) \qquad \mathcal{P} = \{\mathcal{P}(\phi)\},$$

where $\mathcal{P}(\phi) =$ s.s. obj of phase ϕ in $\mathcal{A}$ if $\varphi \in (0,1]$

otherwise $\mathcal{P}(\phi) = \mathcal{P}(\phi)[n]$, if $\phi \in (-n+1, n]$

also $Z(E[n]) = (-1)^n Z(E) = e^{\pi i n} Z(E)$

But one has to deal with H-N in both directions.

a) $\Rightarrow$ b) $\mathcal{A} = \mathcal{P}((0,1])$ is the heart of a t-structure. Restrict Z to $\mathcal{A}$ and get a stability function. Must check H-N for $\mathcal{A}$.

Since, given $E \in \mathcal{A} = \mathcal{P}((0,1])$, we have:

$$0 \leq \phi^-(E) \leq \phi^+(E) \leq 1$$

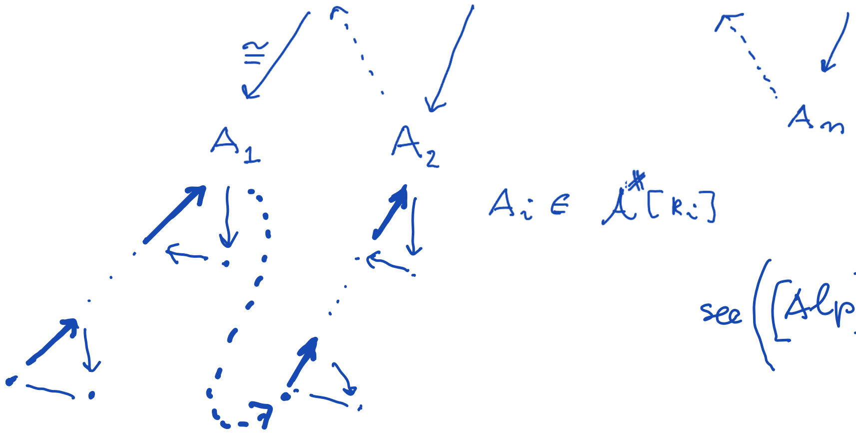
the Harder-Narasimhan in the definition of $\mathcal{P}$ actually takes place in $\mathcal{A}$.

b) $\Rightarrow$ a) take H-N in the CRITERION of p. 5 then in each $\mathcal{A}[k_i]$.

Use the H-N for Z in $\mathcal{A}$.

Can put these together via octahedron

$$0 = E_0 \to E_1 \to E_2 \to \cdots \to E_{n-1} \to E_n = E$$



$A_i \in \mathcal{A}^{\#}[k_i]$

see ([Alp], Prop. 7.3)

GROUP ACTION

$\mathrm{Stab}(X)_{pre} = \{\text{prestab. cond. on } X\}$.

$\widetilde{GL_2^+}(\mathbb{R}) :=$ univ. cover of $GL_2^+(\mathbb{R})$

acts on $\mathrm{Stab}(X)_{pre}$

$\tilde{g} \in \widetilde{GL_2^+}(\mathbb{R})$ $\quad \tilde{g} = (f, g)$ $\quad g \in GL_2^+(\mathbb{R})$

$f : \mathbb{R} \to \mathbb{R}$, increasing $\quad f(\phi+1) = f(\phi)+1$

such that:

$g|_{S^1} : S^1 \to \mathbb{R}^2 \setminus \{0\} / \mathbb{R}^+ = S^1$, $\quad f : \underset{\underset{S^1}{\parallel}}{\mathbb{R}/\mathbb{Z}} \to \underset{\underset{S^1}{\parallel}}{\mathbb{R}/\mathbb{Z}}$

coincide

Then $\tilde{g} \cdot (\mathcal{P}, Z) = (\mathcal{P}', Z')$, where

$Z' = g^{-1} \circ Z$, $\quad \mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

[M-S lect] Example 5.17. $\quad C$ smooth curve

$\mathrm{Stab}(X)_{pre} = \widetilde{GL_2^+}(\mathbb{R}) \cdot [\text{standard stab cond}]$

Bridgeland stability conditions

Def: A Bridgeland stability condition on $\mathcal{D}$ is the datum of a prestability condition $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A}$ is a heart in $\mathcal{D}$, satisfying the support property, that is:

i.e. $\exists\ c > 0$, s.t. $\forall\ E$ semistable,

$$|Z(v(E))| \geq c\, \|v(E)\|$$

$\| \ \|$ some norm on Λ.

Remark: $Z : K_0(\mathcal{A}) \xrightarrow{v} \Lambda \longrightarrow \mathbb{C}$

$\Rightarrow$ Finitely many $Z(E) = Z(v(E))$ E semistable in a compact $K \subset \mathbb{C}$

$$V_K = \{ v(E) : Z(v(E)) \in K \} \subset \{ v(E) : |Z(v(E))| < A \}$$

$\xRightarrow{\text{via supp. prop.}} \|v(E)\| < A' \Rightarrow V_K$ finite

The supp prop. says that there is no sequence of s.s. $\{E_n\}$ such that $\lim_{n\to\infty} Z(E_n) = z_0$ otherwise in a ball around z_0 of radius K we would have

$$K > |Z(E_n)| \geq C\|v(E_n)\|$$

which is absurd being Λ discrete

As a consequence get Jordan-Hölder in the abelian category $\mathcal{P}(\phi)$.

Pf. Suffices to prove that $\mathcal{P}(\phi)$ is Artinian (it is also Noetherian)

Suppose: $\dots \subset A_i \subset \dots \subset A_1 = A \in \mathcal{P}(\phi)$

$0 \to A_{i+1} \to A_i \to A_i/A_{i+1} \to 0 \in \mathcal{P}(\phi)$

$Z(A_i/A_{i+1}) = r e^{i\pi\varphi} = (r_i - r_{i+1}) e^{i\pi\varphi} \Rightarrow r_i \downarrow \Rightarrow$

$Z(A_i/A_{i+1}) \to 0$, against support property.

Example: C smooth curve. Look at

$Coh^{\sqrt{2}}(C)$, $Z(E) = i(\deg E + \sqrt{2}\, \mathrm{rk}\, E)$. It is a pre-stability condition without supp. property.
(Every $E \in Coh^{\sqrt{2}}(C)$ is s.s. of Z-slope)

Kontsevich - Soibelman

The following two conditions are equivalent:

1) $\exists$ $\|\ \|$ on $\Lambda_{\mathbb{R}}$ s.t.

$\forall$ E Z-s.s. $|Z(v(E))| \geq C \|v(E)\|$

2) $\exists$ quadratic form Q on $\Lambda_{\mathbb{R}}$ s.t. $Q|_{\ker Z} < 0$ and

$\forall$ E <u>Z-semistable</u> $Q(E) \geq 0$

<u>Pf</u>.

1) $\Rightarrow$ 2)

Set $Q(v) = |Z(v)|^2 - C\|v\|^2$

2) $\Rightarrow$ 1) [Ch-B] Proposition 1.13.

Remark.

$$\underbrace{\Lambda_{\mathbb{R}}}_{\dim = \rho+2} = \underbrace{\hat{\Lambda}_{\mathbb{R}}}_{\dim 2} \oplus \underbrace{\operatorname{Ker} Z}_{\dim \rho}$$

$\hat{\Lambda}_{\mathbb{R}} \cong \mathbb{C}$

We are assuming $Z : \hat{\Lambda}_{\mathbb{R}} \xrightarrow{\cong} \mathbb{C}$

$p : \Lambda_{\mathbb{R}} \to \operatorname{Ker} Z$ can write:

$$Q(v) = Q'_{\mathbb{C}}(v) - \|p(v)\|^2_{\operatorname{Ker} Z}$$

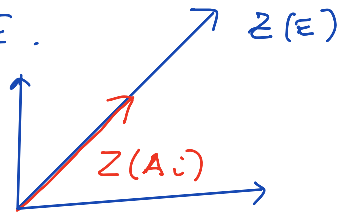
$\| \; \|_{\operatorname{Ker} Z}$ induced by $-Q$ on $\operatorname{Ker} Z$

Q' a quadratic form on $\mathbb{R}^2 = \mathbb{C}$.

Remark. Enough to check 2).

$\forall$ stable E.

$A_1, \dots, A_n$ J-H factors of E.



Suppose $Q(v(A_i)) \geq 0$,

$$Q'(Z(E)) = \left(\sum_i \sqrt{Q'(Z(A_i))} \right)^2$$

$$\sum_i \sqrt{Q'(Z(A_i))} \underset{\text{by hypothesis}}{\geq} \sum_i \|p(v(A_i))\|_{\operatorname{Ker} Z} \underset{\text{triangle in } \operatorname{Ker} Z}{\geq} \|p(v(E))\|_{\operatorname{Ker} Z}$$

Example $(\mathrm{Coh}^{\beta}, Z_{\alpha\beta})$ is a stability condition, when $\alpha, \beta \in \mathbb{Q}$.

Pf. Must verify support property

$v \in \mathrm{Ker}\, Z_{\alpha\beta}$ iff 0 $\Rightarrow \begin{cases} c - r\beta = 0 \\ -s + c\beta H^2 - \frac{r}{2}(\beta^2 - \alpha^2) H^2 \end{cases}$

$$\Rightarrow v = \left(r, r\beta H, \frac{r}{2}(\beta^2+\alpha^2)H^2\right)$$

$$\langle v, v\rangle = r^2\beta^2H^2 - r^2(\beta^2+\alpha^2)H^2 = -r^2\alpha^2H^2 < 0$$

Write $\Lambda_{\mathbb{R}} = \Lambda'_{\mathbb{R}} \oplus \mathrm{Ker}\, Z \leftarrow$ dim S

$p : \Lambda_{\mathbb{R}} \longrightarrow \mathrm{Ker}\, Z$, $\mathrm{Sign}(\langle\, ,\, \rangle) = (2, S)$

Choose $\|\ \|$ on $\Lambda'_{\mathbb{R}}$ s.t.

$$\langle v, v\rangle = \|Z(v)\|^2_{\mathbb{C}} - \|p(v)\|^2_{\mathrm{Ker}\, Z}$$

$\leftarrow$ norm induced by $\langle\, ,\, \rangle$ on $\mathrm{Ker}\, Z$

would like $\langle v(E), v(E)\rangle \geq 0$

$\forall$ E stable. But $\langle v(E), v(E)\rangle \geq -2$

must take care of $\delta \in \Delta = \{v \mid v^2 = -2\}$

observe that $Z(\delta) \neq 0$, by $(*)$ p. 13

$$\Rightarrow \quad \|Z(\delta)\|_{\mathbb{C}}^2 + 2 = \|p(\delta)\|_{\operatorname{Ker} Z}^2$$

$\forall$ K only finite no of δ with

$\|Z(\delta)\| < K \Rightarrow \exists\ C > 0$, s.t. $\forall\ \delta \in \Delta$

$\|Z(\delta)\| \geq C$. set

$$\boxed{Q_Z(v) = \langle v, v\rangle + \frac{2}{C^2}\|Z(v)\|_{\mathbb{C}}^2} =$$

$$= \frac{C^2+2}{C^2}\|Z(v)\|_{\mathbb{C}}^2 - \|p(v)\|_{\operatorname{Ker} Z}^2$$

Q_Z <u>gives support property</u>

Topology on Stab(X) (34)

Notation : $\mathcal{D} = \mathcal{D}(X)$ $\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}(X)$

$$\mathrm{Slice}(\mathcal{D}) = \left\{ \text{slicings } \mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}} \right\}$$

$$d(\mathcal{P}, \mathcal{Q}) = \sup_{0 \neq E \in \mathcal{D}} \left\{ \left|\phi^+_{\mathcal{P}}(E) - \phi^+_{\mathcal{Q}}(E)\right|, \left|\phi^-_{\mathcal{P}}(E) - \phi^-_{\mathcal{Q}}(E)\right| \right\}$$

$$= \inf_{\varepsilon \geq 0} \left\{ \forall \in \mathbb{R},\ \mathcal{Q}(\phi) \subset \mathcal{P}([\phi-\varepsilon, \phi+\varepsilon]) \right\}$$

$$= \sup_{\phi \in \mathbb{R}} = \left\{ \left|\phi^+_{\mathcal{Q}}(E) - \phi\right|, \left|\phi - \phi^-_{\mathcal{Q}}(E)\right| \,\middle|\, E \in \mathcal{P}(\phi) \right\}$$

$$\mathrm{Stab}(\mathcal{D}) \xrightarrow{(p,q)} \mathrm{Slice}(\mathcal{D}) \times \mathrm{Hom}(\Lambda, \mathbb{C})$$

$$(\mathcal{P}, Z) \longmapsto (\mathcal{P}, Z)$$

Definition: Top. on $\mathrm{Stab}(\mathcal{D})$ is the coarsest top making (p,q) continuous.

Lemma $\widetilde{GL}_2(\mathbb{R})$ acts continuously on $\mathrm{Stab}(X)$

Theorem (Bridgeland) ("deformation th.")

For each connected component Σ of $\mathrm{Stab}(X)$, there is a linear subspace $V(\Sigma) \subset \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C})$ such that

$$q : \Sigma \longrightarrow V(\Sigma) \subseteq \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C})$$
$$(P, Z) \longmapsto Z$$

is a local homeomorphism. Σ is called full if $V(\Sigma) = \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C})$.

$\Rightarrow$ A full Σ has the structure of a complex manifold of $\mathbb{C}$-dimension equal to $\mathrm{rk}(\Lambda)$.

Lemma $(P, Z), (Q, Z) \in \mathrm{Stab}(X)$

$d(P, Q) < 1 \Rightarrow P = Q$.

Pf $E \in P(\phi)$. We want $E \in Q(\phi)$

$d(\mathcal{P},\mathcal{Q})<1 \Rightarrow \phi-1 \leq \phi^-_{\mathcal{Q}}(E) \leq \phi^+_{\mathcal{Q}}(E) < \phi+1$

$\mathcal{P}(\phi) \subset \mathcal{Q}(\phi-1, \phi+1)$ ⊛

For the record: given $F \in \mathcal{Q}(\psi)$

(*) $\psi-1 \leq \phi^-_{\mathcal{P}}(F) \leq \phi^+_{\mathcal{P}}(F) < \psi+1$

$\mathcal{Q}(\psi) \subset \mathcal{P}(\psi-1, \psi+1)$ ⊛

Claim: $\phi-1 \leq \phi^-_{\mathcal{Q}}(E) \leq \phi \leq \phi^+_{\mathcal{Q}}(E) \leq \phi+1$

Pf. Suppose $\phi^-_{\mathcal{Q}}(E) > \phi$ already remarked p. 24

$r e^{\pi i \phi} = Z(E) = \sum Z(E_j/E_{j+1}) = \sum r_j e^{i\pi \phi_j}$, $r_j>0$

$\phi_j > \phi$ absurd. ($\phi \leq \phi^+_{\mathcal{Q}}$ analogous) similar

want: $\phi = \phi^+_{\mathcal{Q}}(E)$ (and $\phi = \phi^-_{\mathcal{Q}}(E)$)

Suppose $\phi < \phi^+_{\mathcal{Q}}(E) = \psi \Rightarrow \exists$

$F \in \mathcal{Q}(\psi)$,
$$\begin{array}{ccccc} F & \xrightarrow[*\,0]{HN} & E & \longrightarrow & A \\ \cap & & | & & \cap \\ \mathcal{Q}(\phi,\phi+1) & & \phi & & \mathcal{Q}(\phi-1,\phi] \end{array}$$ ⊛

use (*). As before $\psi \leq \phi^+_{\mathcal{P}}(F) = \gamma$

$$\begin{tikzcd} \mathcal{P}(\gamma) \ni G \arrow[r, "HN"', "\neq 0"] & F \arrow[r] & B \end{tikzcd}$$
$$G \in \mathcal{P}(\psi, \psi+1) \qquad F \ \text{—}\ \psi \qquad B \in \mathcal{P}(\psi-1, \psi]$$

But: $G \longrightarrow F \longrightarrow E$ $\qquad \psi > \phi \implies$

$G \in \mathcal{P}(\psi, \psi+1)$, $E \in \mathcal{P}(\phi)$

$$\begin{tikzcd} & G_{\psi} \arrow[d] \arrow[dl, dashed] & \\ A[-1] \arrow[r] & F \arrow[r] & E \end{tikzcd}$$

$(\phi-2, \phi-1]$

Absurd since $G \to F$ is not zero. $\square$

Lemma Assume that Q has sign $= (2, \operatorname{rk}\Lambda - 2)$. Consider $\sigma = (\mathcal{Z}, Z)$ Bridg. stab. cond. with support property given by Q. Then $\exists$ $g \in GL_2^+(\mathbb{R})$ s.t. $\forall$ $v \in \Lambda$
$$Q(v) = |g\, Z(v)|^2 + Q(p(v))$$
where $p : \Lambda_{\mathbb{R}} \longrightarrow \operatorname{Ker} Z$, is the orthogonal projection.

Pf $\qquad \Lambda_{\mathbb{R}} = \Lambda'_{\mathbb{R}} \oplus \operatorname{Ker} Z$; $\quad Z : \Lambda'_{\mathbb{R}} \xrightarrow{\text{iso}} \mathbb{C}$

i.e. $\Lambda'_{\mathbb{R}} = (\operatorname{Ker} Z)^{\perp}$

Write $Q(v) = \| Z(v) \|^2 + Q(p(v))$ and let g send $\| \ \|_Q$ to $| \ |_{\mathbb{C}}$.

Corollary. Up to the action of $GL_2(\mathbb{R})$ can assume $g = 1$

Bridgeland deformation theorem

$$\mathcal{Z} : \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C})$$
$$\sigma = (\mathcal{A}, Z) \longmapsto Z$$

is a local homeomorphism.

<u>Plan:</u> given Z' close to Z search for a σ' close to σ lifting Z', i.e. $\sigma' = (\mathcal{A}', Z')$.

<u>Hope for the best:</u> $\sigma' = (\mathcal{A}, Z')$ with same $\mathcal{A}$ and some Q giving the support property: not quite, but almost.

set: $\quad U = \{ Z' \in \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \mid Q|_{\ker Z'} < 0 \}$

$U \subset \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C})$ is an open neighbourhood of Z.

may assume U connected.

[<u>Recall</u> $Q|_{\ker Z} < 0$, $Q(E) \geq 0$, $\forall$ E σ-semistable]

let V be the connected component of $q^{-1}(U)$ containing σ.

<u>Theorem</u> $\quad q|_V : V \longrightarrow U$ <u>is a top. covering.</u>

We will only prove that q is a local homeomorphism around σ, and that $\forall\ \tau \in \mathcal{V}$, near σ, Q gives the support property for τ.

Bayer's idea: Look at the following linear section of $\mathcal{V}$.

$$W = \left\{ Z' \in \mathcal{V} \;\middle|\; Z'|_{(\operatorname{Ker} Z)^\perp} = Z|_{(\operatorname{Ker} Z)^\perp} \right\}$$

Remark $Z' \in W \iff Z' = Z + u \circ p$ where $u : \operatorname{Ker} Z \to \mathbb{C}$, with $\|u\| < 1$, $\| \ \|$ being the operator norm given by:

$$\|u\| = \sup_{w \in \operatorname{Ker} Z} \frac{|u(w)|}{-Q(w)} \qquad \left[\begin{array}{l} |\cdot| \text{ usual on } \mathbb{C} \\ -Q(\cdot), \text{ on } \operatorname{Ker} Z \end{array}\right]$$

Pf. Being in W entails two conditions:

(1) $Z' - Z|_{\operatorname{Ker} Z^\perp} = 0 \quad \left[\iff \begin{array}{l} Z' - Z = u \circ p \\ \Lambda \xrightarrow{p} \operatorname{Ker} Z \xrightarrow{u} \mathbb{C} \end{array} \right],$

(2) $Z' \in \mathcal{V} \quad \left[\iff Q|_{\operatorname{Ker} Z'} < 0 \right].$

Assume that $Z' = Z + u \circ p$ and $\|u\| < 1$. Then (1) is satisfied and for $w \in \operatorname{Ker} Z'$. We have

$Q(w) = |Z(w)|^2 + Q(p(w)) = |u(p(w))| + Q(p(w)) < 0$

$\Rightarrow Z' \in U$ so that $Z' \in W$. Viceversa, assume $Z' \in W$

Then $Z' = Z + u \circ p$. Suppose $\|u\| \geq 1$. Can assume $\exists\ w \in \operatorname{Ker} Z'$

with $Q(w) = -1$, $u(w) \geq 1$. Let $v \in (\operatorname{Ker} Z)^\perp$ s.t.

$Z(v) = -u(w) \in \mathbb{C}$, ($Z$ is surj.). Then

$$Z'(v+w) = Z(v+w) + u \circ p\,(v+w) =$$
$$= -u(w) + u(w) = 0$$

But $Q(v+w) = Q(v) + Q(w) = Q(v) - 1 = |Z(v)|^2 - 1$
$= |u(w)|^2 - 1 \geq 0$

against (2). $\square$

<u>Next step</u>: Since Z' is close to Z can choose $g \in GL_2^+(\mathbb{R})$, close to 1, such that gZ' and Z coincide on $(\operatorname{Ker} Z)^\perp$.

Substitute Z' with gZ' and assume

$Z' = Z + u \circ p + i v \circ p$ with u and v in $\operatorname{Hom}(\operatorname{Ker} Z, \mathbb{R})$, and close to 0: the old "u" is now $u + iv$. Set $Z_1 = Z + u \circ p$. Then Z_1 is close to Z.

<u>Lemma</u>: <u>Given</u> $\sigma = (A, Z)$ <u>and</u> $Z_1 = Z + u \circ p$ <u>with</u> $u \in \operatorname{Hom}(\Lambda_{\mathbb{R}}, \mathbb{R})$, <u>close to 0</u>, <u>then one can lift Z_1 to a stab. condition σ_1 close to σ</u>.

Go back to $Z_1 = Z + u\circ p$. By the lemma, we can replace Z with Z_1, and σ with σ_1, and assume that:

$$Z' = Z + i v \circ p'$$

with p' close to p. Now, up to replacing Z' with gZ', with g close to 1, we can assume $p' = p$. We must lift Z'.

We have

$$-iZ' = -iZ + v\circ p$$

Let $\tilde{\gamma} \in \widetilde{GL}_2(\mathbb{R})$ be a lifting of (mult by i) $\in GL_2^+(\mathbb{R})$.

Temporarily replace σ and Z with $-\tilde{\gamma}\sigma$ and iZ. Then the lemma gives a lifting of iZ to a stability condition τ, close to $-\tilde{\gamma}\sigma$. We can then take as a lifting of Z, the stab. condition $\sigma' = -\tilde{\gamma}^{-1}\tau$, which is close to σ.

The final step is to prove the lemma.

Sketch of proof. We start with

$\sigma = (\mathcal{A}, Z)$, and $Z' = Z + u \circ p$, u close to 0

and real.

Claim: $\sigma' = (\mathcal{A}, Z')$ is a stab condition, close to σ, with support property given by Q (the quadratic form Q gives the support property for σ). Must prove:

a) Z' stab funct.

b) H-N for Z'.

c) Q gives support property for $(\mathcal{A}, Z')$.

Z has supp prop w.r.t. Q

the key, in the three cases, is the support property for σ, given by Q.

Now one can read section 5 in [Ba-Short].

To give an idea we prove the simplest of the three: item a)

Pf. u real $\Rightarrow$ $\operatorname{Im} Z' = \operatorname{Im} Z \geq 0$

Suppose $\mathrm{Im}\, Z'(E) = \mathrm{Im}\, Z(E) = 0$, $E \in \mathcal{A}$.

Then $Z(E) \in \mathbb{R}_{<0}$, and E is σ-semistable.

Thus, if $v = v(E)$ we have:

$$Q(v) = |Z(v)|^2 + Q(p(v)) \geqslant 0$$

i.e.
$$\underset{\displaystyle -Z(v)}{\overset{\shortparallel}{|Z(v)|}} \geqslant \sqrt{-Q(p(v))}$$

$$Z'(E) = Z(E) + u \circ p(v) < Z(E) + \|u\| \cdot \sqrt{-Q(p(v))}$$

$$\left(\sup_{w \in \operatorname{Ker} Z} \left(\frac{|u(w)|^2}{-Q(w)} \right) = \|u\|^2 \right) \qquad \leq Z(E) - Z(E) = 0 \qquad \square$$

The proof of b) consists in showing that for $E \in \mathcal{A}$ and $c \in \mathbb{R}$ there are only a finite number of $v \in \Lambda$ with $v = v(F)$, $F \subset E$, and

$$\mathrm{Re}\, (Z + u \circ p)(F) < c.$$

(Not too bad). Point c) is more subtle.

Stability conditions on K3 surfaces

X a K3, $\langle\, , \rangle$ Mukai pairing

on $\Lambda = N(X) = H^0(X) \oplus NS(X) \oplus H^4(X)$

$\langle\, , \rangle : \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \xrightarrow{\cong} \Lambda_{\mathbb{C}}$

$$\begin{tikzcd}
\mathrm{Stab}(X) \arrow[r, "q"] \arrow[dr, bend right, "\pi"'] & \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \arrow[d, "\langle\, , \rangle"] \\
& \Lambda_{\mathbb{C}} = N(X) \otimes \mathbb{C}
\end{tikzcd} \qquad (*)$$

$\sigma = (\mathcal{A}, Z) \longmapsto Z$

$\pi(\sigma) : Z(\;) = \langle \pi(\sigma), \; \rangle$

$$\mathcal{P}(X) = \left\{ \Omega = \xi + i\eta \;\middle|\; \left(\langle\, , \rangle \big|_{\mathbb{R}\xi \oplus \mathbb{R}\eta} \right) > 0 \right\}$$

$GL_2^+(\mathbb{R})$ acts freely on $N(X) \otimes \mathbb{R}^2$.

A section of this action:

$$\mathcal{Q}(X) = \left\{ \Omega \in \mathcal{P}(X) \;\middle|\; \langle \Omega, \Omega \rangle = 0, \langle \Omega, \bar{\Omega} \rangle > 0, r(\;) = 1 \right\}$$

r: projection on $H^0(X) \subset \Lambda_{\mathbb{R}}$

$$NS(X) \otimes \mathbb{C} \longleftrightarrow \mathcal{Q}(X)$$

$$(\omega, \beta) \longmapsto \exp(B + i\omega) =$$

$$= \left(1, B + i\omega, \tfrac{1}{2}(B^2 - \omega^2) + i\, B \cdot \omega \right)$$

$\omega^2 > 0$

(Later: $B = \beta H$, $\omega = \alpha H$)

$\mathcal{P}^+(X)$ = conn. comp of $\mathcal{P}$ containing $\mathcal{Z}(X)$ (45)

$$\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta} \delta^\perp \subset \mathcal{N}(X) \otimes \mathbb{C}$$

$$\Delta = \{ \delta \in \mathcal{N}(X) \otimes \mathbb{C} \mid \delta^2 = -2 \}$$

<u>Theorem</u> (Bridgeland) <u>there exists a conn. comp</u> $\mathrm{Stab}^\dagger(X)$ of $\mathrm{Stab}(X)$ <u>and a covering map</u>

$$\mathrm{Stab}^\dagger(X) \xrightarrow{\pi} \mathcal{P}_0^+(X) \subset \Lambda_{\mathbb{C}}$$

<u>covering group</u>:

$$\Gamma = \{ \gamma \in \mathrm{Aut}(\mathcal{D}(X)) \mid \gamma \text{ preserves } \mathrm{Stab}^\dagger \}$$

Moreover let:

$$V_{NS}(X) = \left\{ \left(\mathrm{Coh}^{B,\omega}(X), Z_{B,\omega}\right) \;\middle|\; B, \omega \in NS(X), \omega^2 > 0 \right\}$$

then: $\widetilde{GL}_2(\mathbb{R}) \cdot V_{NS} = U(X)$

where $U(X) = \left\{ \sigma \in \mathrm{Stab}(X) \;\middle|\; \begin{array}{l} \mathbb{C}_x \ \sigma\text{-stab of} \\ \text{phase} = 1, \forall x \in X, \\ \text{and } \pi(\sigma) \in \mathcal{P}_0^+(X) \end{array} \right\}$

and $U(X) = \mathrm{Stab}^\dagger(X) \setminus \left\{ \begin{array}{l} \text{real codimension} \\ 1 \text{ submanifolds} \end{array} \right\}$

One idea: E stable of phase, $0 < \phi < 1$,

$\mathbb{C}_x$ stable of phase 1 $\Rightarrow$ $\mathrm{Hom}^i(E, \mathbb{C}_x) = 0$

$i < 0$. Serre duality $\Rightarrow$ $\mathrm{Hom}^i(E, \mathbb{C}_x) = \mathrm{Hom}^{2-i}(\mathbb{C}_x, E)^\vee$
$= 0, \quad i \geq 2$

Bridgeland-Macìocia
$\Longrightarrow$ $E : E^{-1} \to E^0$.
(Homological Alg.) $\quad$ ([Br-K3] Proposition 10.3).

Remark: Suppose, for simplicity, $B = \beta H$, $\omega = \alpha H$ then an easy computation shows that if $Z_{\beta,\alpha}$, $Z_{\beta'\alpha'}$ and $g \in GL_2^+(\mathbb{R})$ are such that

$g Z_{\beta,\alpha} = Z_{\beta',\alpha'}$ then $g = 1$.

Main property of $\mathcal{S}$ and $\mathcal{S}_0$:

<u>support property</u>: Fix $\| \ \|$ on $\Lambda_{\mathbb{C}}$, $\langle , \rangle$ Mukai pairing. <u>Then</u>:

$$(*) \left[\begin{array}{ll} \Omega \in \mathcal{S}(X) \Rightarrow & \exists\, r>0, \text{ s.t. } \\ & |\langle u,v\rangle| \leq r \|u\| \, |\langle \Omega, v\rangle| \\ & \forall\, u \in \Lambda_{\mathbb{C}},\ \forall\, v,\ v^2 \geq 0 \\ \Omega \in \mathcal{S}_0 & \underline{\text{same}} \text{ with } v^2 \geq -2 \end{array} \right]$$

(<u>think of</u> $\langle \Omega, v\rangle = Z(v)$ under $\langle , \rangle : \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \leftrightarrow \Lambda_{\mathbb{C}}$)

<u>CLAIM</u>:

$$(**) \left[\Omega \in \mathcal{S}(X) \Rightarrow \begin{array}{l} \exists r>0 \text{ s.t. } \|v\| \leq r |\langle \Omega, v\rangle| \\ \quad (= r|Z(v)|) \\ \forall\, v \text{ s.t. } v \gtrsim 0,\ v^2 \geq -2 \end{array} \right] \Rightarrow (*)$$

This is the usual support property where the condition on v^2 replaces the semistability condition on E.

<u>Pf</u> (CLAIM) $\Lambda_{\mathbb{R}} = \mathcal{N}(X) \otimes \mathbb{R}$ $\operatorname{sg}(\langle , \rangle) = (2, \rho-2)$

Let $\Omega \in \mathcal{S}(X)$, $\Omega = \xi + i\eta$ choose basis of $\Lambda_{\mathbb{R}}$

$$\begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_n \\ + & + & - & & - \\ \xi, & \eta & & & \end{array}$$

set $\|v\| = v_1^2 + \cdots + v_n^2$

$v_i = e_i$- component of v.

C-S $\Rightarrow$ $|\langle u, v\rangle| = |u_1 v_1 + u_2 v_2 - u_3 v_3 - \dots| \leq \|u\| \|v\|$ $\square$

Pf of $(**)$ $\Omega = \xi + i\eta = a_1 e_1 + i a_2 e_2$

$|\langle \Omega, v\rangle|^2 = |\langle \xi, v\rangle + i\langle \eta, v\rangle|^2 = a_1^2 v_1^2 + a_2^2 v_2^2 \geqslant a_2^2 (v_1^2 + v_2^2)$
say

Suppose $v \in N(X)_{\mathbb{R}}$ $v^2 \geq 0$ $\Rightarrow$ $v_1^2 + v_2^2 - \sum_{i \geq 3} v_i^2 \geqslant 0$

$\Rightarrow$ $2(v_1^2 + v_2^2) \geqslant \|v\|^2$ $\Rightarrow$ $|\langle \Omega, v\rangle| \geqslant \frac{a_2^2}{2} \|v\|^2$

When $v \in \Delta$ $\|v\|^2 = 2(v_1^2 + v_2^2 + 1)$

Must check $C|\langle \Omega, v\rangle|^2 \geqslant 1$ for some $c > 0$. But if $|\langle \Omega, v\rangle| \leq 1$

then $\|v\|^2 < K$ so finitely many

and $\Omega \in \mathcal{P}_0$ $\Rightarrow$ $|\langle \Omega, \delta\rangle| \neq 0$. $\square$

CONSEQUENCE $\#\{v \in N(X) \mid v^2 \geq -2,\ |\langle \Omega, v\rangle| \leq m\} < \infty$

$|\langle \Omega, v\rangle| = |Z(v)|$

Definition: $\mathrm{Stab}^*(X)$ good c.c. if $\exists\, \sigma \in \mathrm{Stab}^*(X)$ s.t. $\pi(\sigma) \in \mathcal{P}_0(X)$ (e.g. $\sigma = \sigma_{B,\omega}$)

Remark: By the definition of the topology on $\mathrm{Stab}(X)$, given $A \in \mathcal{D}$

then $\Phi^+_\sigma(A)$ and $\Phi^-_\sigma(A)$ are continuous on $\mathrm{Stab}(X)$. Since A is s.s. $\Longleftrightarrow$ $\Phi^+_\sigma(A) = \Phi^-_\sigma(A)$ we get

Corollary: $\{\sigma \in \mathrm{Stab}(X) \mid A \text{ is } \sigma\text{-semistable}\}$ is closed in $\mathrm{Stab}(X)$

this is in contrast with Gieseker semistability which is not closed.

Example: (Saccà) Gieseker semistability is not closed

X K3, $NS(X) = e\mathbb{Z} \oplus f\mathbb{Z}$

$e^2 = f^2 = 0$, $ef = 2$, I_Z ideal sheaf $\ell(Z) = n$

$v(I_Z) = (1, 0, 1-n)$, $n \gg 0$. $\exists$ non split ext

$$0 \to \mathcal{O}_X \to E \to L \otimes I_Z \to 0 \qquad L = \mathcal{O}(f-e)$$

Chambers in $\mathrm{Amp}(X)$: for which $H \in \mathrm{Amp}(X)$ is E H-(Gieseker)-semistable? Measure for H-ss: coeff. of reduced Hilb poly:

$\mu_H(F) = \dfrac{H \cdot c_1(F)}{\mathrm{rk}\, F}$ in case of parity $\dfrac{\chi(F)}{\mathrm{rk}\, F}$

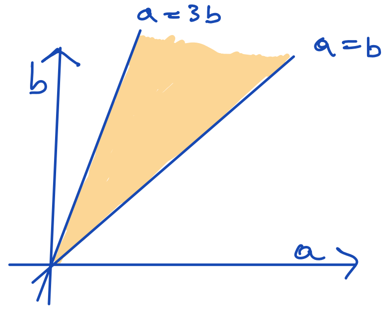
$c_1(E) = c_1(L) = f - e$. Write $H = ae + bf$

$\mu_H(E) = \frac{(ae+bf)(f-e)}{2} = \frac{a-b}{2}$, Set $H_0 = e+f$

$aH_0 = a(e+f)$ $\quad \mu_{H_0}(E) = 0 = \mu_{H_0}(\mathcal{O}_X)$

but $\frac{\chi(\mathcal{O}_X)}{1} = 2 > \frac{\chi(E)}{2} = 2 - n$ $\quad n \gg 0$.

So E is H_0-unstable.
i.e. unstable along the wall $a=b$, but H-stable for H in the shaded region



Pf. Suppose G destabilizes

$$\begin{array}{c} G \\ \downarrow \\ 0 \to \mathcal{O}_X \to E \to J_Z \otimes L \to 0 \end{array}$$

G pure rk 1 subsheaf (saturated), $c_1(G) = \Gamma$

$a > b \Rightarrow \mu_H(E) = \frac{a-b}{2} > 0$. If $G \xrightarrow{0} J_Z \otimes L$ then $G \to \mathcal{O}_X \Rightarrow \mu_H(G) \leq 0$. So can assume $G \to E \to J_Z \otimes L$ not zero $\Rightarrow L \otimes \Gamma^{-1} \geqslant 0$

$G \cong \mathcal{O}_X(\Gamma) \otimes J_W$. Also $L \neq \mathcal{O}(\Gamma)$ since extension is non split. Only to worry about Γ s.t.

$$\frac{L\cdot H}{2} \leq \Gamma \cdot H < L \cdot H \quad (*)$$

$L \otimes \Gamma^{-1}$ effect. $L \otimes \Gamma^{-2}$ not effect. $\Rightarrow \Gamma = f - ce$, $c \geqslant 2$. For H in the range above no Γ satisfies $(*)$.

BACK TO WALLS AND CHAMBERS

Fix $v \in \Lambda$ and an arbitrary set

$$S \subseteq \{ E \in \mathcal{D} \mid v(A) = v \}$$

We will decompose $\mathrm{Stab}(X)$ in walls and chambers according to whether objects $E \in S$ are σ-stable, strictly semistable or unstable. Let's start with the strictly semistable. Walls are defined in terms of those. Write

$$\sigma = (\mathcal{A}, Z) \qquad Z = Z_\sigma$$

suppose $E \in S$ is strictly σ-semistable then

$\exists \quad F \subset E \quad$ s.t. $\quad Z(F) = c\, Z(E)$, $c \in \mathbb{R}_+$

$$\Longleftrightarrow \qquad \mathrm{Im} \frac{Z(F)}{Z(E)} = 0$$

$$\Longleftrightarrow \qquad \mathrm{Re}\, Z(F)\, \mathrm{Im}\, Z(E) - \mathrm{Re}\, Z(E)\, \mathrm{Im}\, Z(F) = 0$$

If $\gamma = v(F)$ then this equation can be written as

$$(*) \quad \mathrm{Re}(Z(\gamma))\, \mathrm{Im}(Z(v)) - \mathrm{Re}(Z(v))\, \mathrm{Im}(Z(\gamma)) = 0 .$$

For fixed v and γ this is a real quadratic equation satisfied by

Z's in $\mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C})$

which defines a real hypersurface

$$Q_\gamma \subseteq \mathrm{Hom}(\Lambda_{\mathbb{R}}, \mathbb{C}) \cong \Lambda_{\mathbb{C}}$$

and, under the projection,

$$\pi : \mathrm{Stab}(X)^* \longrightarrow \Lambda_{\mathbb{C}}$$

a real codimension 1, not necessarily closed, submanifold

$$(**) \quad W_\gamma^{S,v} = \pi^{-1}(Q_\gamma) \subset \mathrm{Stab}(X)^*$$

this is a <u>wall</u>. In conclusion

For all $\sigma = (\mathcal{A}, Z) \in W_\gamma$, there exists $E \in S$ <u>strictly σ-semistable</u> by virtue of an F with $v(F) = \gamma$.

(Actually, we could have started with $v = v(E)$ and γ, and equation $(**)$ then defines, via $(*)$, a <u>virtual wall</u>, which becomes <u>effective</u> only when there is an $F \subset E$ with $v(F) = \gamma$.)

Formally walls are defined by

$$W_\gamma^{S,v} = \left\{ \sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(X) \;\middle|\; \begin{array}{l} \exists\, \phi \in \mathbb{R},\ \exists\, F \hookrightarrow E \text{ in } \mathcal{P}(\phi), \\ E \in S,\ v(F) = \gamma,\ \gamma \nparallel v \end{array} \right\}$$

and the set of walls is denoted by $\mathcal{W}^{S,v}$ or simply by $\mathcal{W}$. Thus

$$\bigcup_{W \in \mathcal{W}} = \{ \sigma \in \mathrm{Stab}(X) \mid \exists\ E \in S \text{ which is strictly } \sigma\text{-semistable} \}$$

Theorem : (Bridgeland) Fix v and S as above.

1) the collection of walls in $\mathcal{W}$ is locally finite

2) Let C be a connected component of $\mathrm{Stab}(X) \setminus \bigcup_{W \in \mathcal{W}} W$

Let $\sigma, \tau \in C$. Then $E \in S$ is σ-stable (resp. σ-semistable) $\iff$ it is τ-stable (resp. τ-semistable).

3) $\sigma \in C$, $E \in S$, E σ-semistable, v primitive $\implies$ E σ-stable

4) If W is a wall there exists $E \in S$ semistable on W and unstable in one of the two chambers adjacents to W

5) the action of $\widetilde{GL_2^+}(\mathbb{R})$ preserves walls and chambers (obvious).

Lemma: Given $\varepsilon > 0$, can cover $\mathrm{Stab}(X)$ with open sets U s.t.

1) $\forall$ pair $\sigma = (P, Z)$, $\tau = (Q, W)$ in U: $Q((\varepsilon-\eta, \varepsilon+\eta)) \supset P((\varepsilon, 1-\varepsilon))$

$0 < \varepsilon - \eta < 1 - \varepsilon + \eta < 1$

2) if $f : A \hookrightarrow B$ in $P(\varepsilon, 1-\varepsilon) \subset A_\sigma$ with $\mathrm{coker}\, f \in P(\varepsilon, 1-\varepsilon)$ then $f : A \hookrightarrow B$ in $A_\tau = Q(0, 1]$.

Pf 1) Suffices $U = \{\sigma \mid d(\sigma, \rho) < \frac{\eta}{2}\}$ some $\rho \in \mathrm{Stab}(X)$, $\varepsilon + \eta = b$,

$|\phi^\pm_\sigma(E) - \phi^\pm_\tau(E)| < \eta$ $\quad \sigma, \tau \in U$.

Hence $E \in P(\varepsilon, 1-\varepsilon) \Rightarrow E \in Q(0, 1)$.

2) $f : A \hookrightarrow B$ in $A_\tau \iff C(f) \in A_\tau$

Pf. 1) Walls are locally finite.

Take U as above. Enough to prove that if U is small then

$\{\gamma \in \Lambda \mid W_\gamma \cap U \neq \phi\}$ is a finite set. (54)

The function $\begin{array}{l} U \longrightarrow \mathbb{C} \\ \tau \longmapsto |Z_\tau(v)| \end{array}$ is

continuous. So for U small can

assume $|Z_\tau(v)| \leq M$. $W_\gamma \cap U \neq \phi \Longrightarrow$

$\exists$ $F \hookrightarrow E$ in $\mathcal{A}(\phi)$ with $v(F) = \gamma$

$\Rightarrow$ $|Z(F)| < |Z(E)| \leq M$ but

support $\Rightarrow$ $|Z(F)| \geq C \|v(F)\| = C\|\gamma\|$

$\Rightarrow$ the above set is finite

2) Let C be a chamber. It is enough to prove that, given $E \in S$,

$\Sigma' = \{\tau \in C \mid E \ \tau\text{-semistable}\}$ is open.

(we already know it is closed)

(and we leave the assertion about stability as an exercise.)

Let then $\sigma = (\beta, Z_\sigma) \in \Sigma$. It suffices to show the $\exists$ U open $\ni \sigma$ s.t.

$U \subset \Sigma$. Take $U \subset C$ as above.

Assume $\phi_\sigma(E) = \frac{1}{2}$, $E \in P(\frac{1}{2})$,

and $\forall\ \tau = (Q, Z_\tau) \in U$, $E \in Q(\frac{1}{3}, \frac{2}{3})$

Assume for some τ, E not τ-semistable

$\frac{1}{3} < \phi_\tau^-(E) \leq \phi_\tau(E) \leq \phi_\tau^+(E) < \frac{2}{3}$ let F τ-ss s.t.

$\phi_\tau(F) = \phi_\tau^+(E)$ $\quad F \xhookrightarrow{f} E$ in $Q(\frac{1}{3}, \frac{2}{3})$

with $\operatorname{Coker} f \in Q(\frac{1}{3}, \frac{2}{3})$ (Look at HN polygon)

$\Rightarrow$ $\quad F \xhookrightarrow{f} E$ monomorphism in $Q_{\tau'}(0,1)$ $\forall\ \tau' \in U$

$$\operatorname{Im}\left(Z_\tau(F) / Z_\tau(E)\right) > 0$$

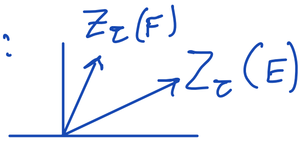
$\operatorname{Im}\left(Z_{\tau'}(F) / Z_{\tau'}(E)\right)$ continuous for $\tau' \in U$

never zero since $U \subset C$ $\Rightarrow$

$$\operatorname{Im}\left(\phi_{\tau'}(F) / \phi_{\tau'}(E)\right) > 0, \quad \forall\ \tau' \in U$$

against the hypothesis that E is σ-semistable. $\square$

4) For a σ on a wall, and $F \xhookrightarrow{f} E$ an injection in $\mathcal{A}(\phi)$, with $v(F)=\gamma$, take τ very close to σ and on the side where $\mathrm{Im}\left(Z_\tau(F)/Z_\tau(E)\right)>0$ then $F \xhookrightarrow{f} E$ is still an injection (in $\mathcal{A}_\tau$) and $\phi_\tau(F)>\phi_\tau(E)$:



3) Suppose that v is primitive, that σ is in a chamber $\mathcal{C}$ and that $E\in S$ is σ-semistable. Suppose E is not σ-stable. Let $\{A_i\}$ be the set of J-H factors of E, then not all the $v(A_i)$'s can be parallel to v, otherwise v would not be primitive. So, there is one such that $v(A_i)=\gamma\neq v$. On the other hand,

$$\mathrm{Im}\left(\frac{Z_\sigma(A_i)}{Z_\sigma(E)}\right)=\mathrm{Im}\left(\frac{Z_\sigma(\gamma)}{Z_\sigma(v)}\right)=0,$$ against the hypothesis $E\in\mathcal{C}$. $\square$

Moduli spaces

X a K3 surface, $v \in \Lambda = H^*_{alg}(X, \mathbb{Z})$ a Mukai vector. Consider the wall and chamber decomposition with respect to $S = \{v\}$. A stability condition $\sigma \in \mathrm{Stab}^+(X)$ is called <u>v-generic</u> if it belongs to a chamber of this decomposition.

Set $\sigma = (\mathcal{A}, Z)$. Two objects E, E' in $\mathcal{A}$ are said to be <u>S-equivalent</u> (<u>Seshadri-equivalent</u>) iff $\oplus A_i \cong \oplus A_i'$ where $\{A_i\}$ (resp. $\{A_i'\}$) is the set of JH factors of E (resp. E')

Given $\sigma = (\mathcal{P}, Z)$, $v \in \Lambda$

$\phi = \frac{1}{\pi} \operatorname{arccot}(-v)$ i.e.

$-v = \operatorname{cotan} \pi\phi$, notice that

$-v = \operatorname{cotan} \pi(\phi+1)$

so that v does not completely determine ϕ.

<u>Def</u>

$$M_{\sigma,\phi}(v) = \left\{ E \in \mathcal{P}(\phi),\ v(E) = v \right\} \Big/ S\text{-equiv.}$$

Also recall that

$$\bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) = \bigcup_{\substack{0 < \phi \leq 1 \\ n \in \mathbb{Z}}} \left\{ \mathcal{P}(\phi) \subset \mathcal{P}(0,1][n] \right\}$$

Theorem (Mukai, O'Grady, Yoshioka, Toda, Huybrechts)

X, v, σ as above.

1) v primitive, σ v-generic

$M_{\sigma,\phi}(v)$ has the structure of a smooth projective hyperKähler variety

It is non-empty $\iff$ $v^2 \geq -2$

and $\dim M_{\sigma\phi}(v) = v^2+2$.

2) v arbitrary, σ v-generic

then $M_{\sigma\phi}(v)$ is a, possibly singular, projective, symplectic variety of dimension v^2+2.

3) (Alper, Halpern-Leistner, Heinloth

Bayer, Lahoz, Macrì, Nuer, Perry, Stellari [BLM+], Lieblich, Abramovich, Polishchuk)

v, σ arbitrary $M_{\sigma\phi}(v)$ is a complete algebraic space of $\dim = v^2+2$

Walls and Chambers in the (α,β)-plane.

$$\boxed{X \text{ a } K3,\ \operatorname{Pic} X = \mathbb{Z}\cdot H}$$

Let us go back to the stability conditions $\sigma_{\beta,\alpha} = (\mathrm{Coh}^{\beta}, Z_{\beta\alpha})$, $(\beta,\alpha) \in \mathbb{R}\times\mathbb{R}_+$, where, for $v(E) = (r, cH, s)$:

$$Z_{\beta\alpha}(v(E)) = -s + c\beta H^2 - \frac{r}{2}(\alpha^2 - \beta^2)H^2 + i(c-\beta r)\alpha H^2,$$

$$\mu_{\beta}(v(E)) = \frac{(c-\beta r)\alpha H^2}{r}$$

$(\mathrm{Coh}^{\beta}, Z_{\beta\alpha}) \in \mathrm{Stab}(X) \iff$

(*) $\boxed{\forall\, \delta \in \Lambda, \text{ with } \delta^2 = -2,\ r(\delta) > 0, \text{ and } \mu_{\beta\alpha}(\delta) = 0, \text{ then } \operatorname{Re} Z_{\beta\alpha}(\delta) > 0}$

(This always happens if $\alpha^2 H^2 \geq 2$).

δ in $(*)$, $\delta = (r, c, s)$, $\delta^2 = c^2 - rs = -2$

$r(\delta) > 0$

$c = \beta r$

$$\delta_{\beta, r} = \left(r, \; r\beta H, \; \frac{\beta^2 r^2 H^2 + 2}{2r}\right)$$

$$\delta_\beta = \delta_{\beta,1} = \left(1, \; \beta H, \; \frac{\beta^2 H^2 + 2}{2}\right)$$

$$\mathrm{Ker}\, Z_{\beta\alpha} = \left(r, \; \beta r H^2, \; \frac{r}{2}(\alpha^2 + \beta^2) H^2\right)$$

(By the Remark on p.46 $\mathrm{Ker}\, Z_{\beta\alpha}$ determines $Z_{\beta,\alpha}$ as an element of $V_{NS}(X)$)

$$Z_{\beta,\alpha} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 = \mathbb{C}$$

$\mathbb{R}^3$ accomodates both $\mathrm{Ker}\, Z_{\beta\alpha}$ and Mukai vectors.

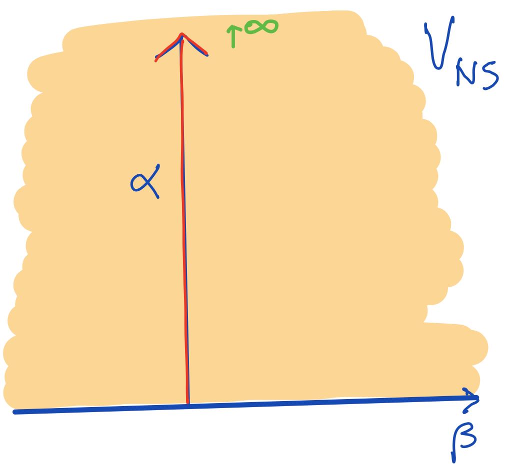
(62)

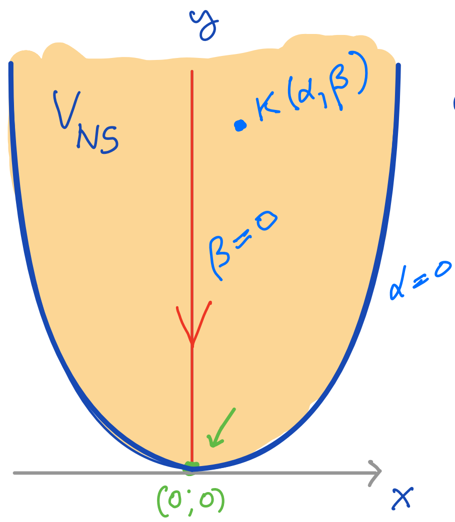
$$\eta : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}^3$$
$$(\beta, \alpha) \longmapsto \left(r, \beta r H^2, \frac{r}{2}(\alpha^2+\beta^2)H^2\right)$$

$$q : \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}^2$$
$$(c, r, s) \longmapsto \left(\frac{c}{s}, \frac{r}{s}\right)$$

$$K(\beta,\alpha) = q\,\eta\,(\beta,\alpha) = \left(\frac{2\beta}{H^2(\alpha^2+\beta^2)}, \frac{2}{H^2(\alpha^2+\beta^2)}\right)$$

$$\mathbb{R} \times \mathbb{R}_+ \xrightarrow{K} \mathbb{R}^2$$
$$(\beta, \alpha) \longmapsto (x, y)$$





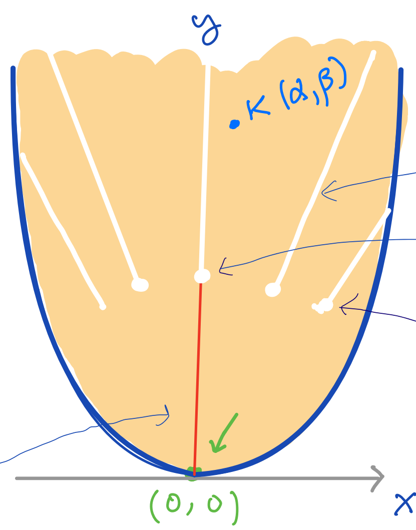
$$y = x^2\frac{H^3}{2}$$

"Bad" spherical objects:

$$\delta_{\beta,r} = \left(r,\ r\beta H,\ \frac{\beta^2 r^2 H^2 + 2}{2r}\right)$$

(they cover the white rays)

$$\alpha_0 = \sqrt{\frac{2}{H^2}}$$



$y = x^2 \frac{H^2}{2}$

$y = \beta x$

$(0,1) = q(\delta_{0,1}) = K(0, \alpha_0)$

$q(\delta_{\beta,1}) = K(\beta, \alpha_0)$

They are "bad" since no point $K(\beta,\alpha)$ can belong to one of the white rays, otherwise $\delta_{\beta r}$ would belong to $\operatorname{Ker} Z_{\beta\alpha}$, which is exactly what is excluded by condition (*) on p. 59. In other words, the white rays do not belong to $V_{NS}(x)$

$$q(\delta_{\beta r}) = \left(\frac{2r^2\beta}{r^2\beta^2H^2+2},\ \frac{2r^2}{r^2\beta^2H^2+2}\right)$$

$$q(\delta_{\beta,1}) = \left(\frac{2\beta}{\beta^2H^2+2},\ \frac{2}{\beta^2H^2+2}\right)$$

$$K(0,\alpha),\ \alpha > \alpha_0,\ Z_{\beta\alpha}(\delta_{\beta r}) > 0 \Rightarrow \sigma_{0,\alpha} \in V_{NS}(x)$$

$\alpha \gg 0$

$\alpha \gg 0 \iff K(\beta,\alpha)$ near $(0,0) \iff \left\{ \begin{array}{l} E \text{ with } v(E)=v \\ \text{Bridg. ss} = \\ \text{Gieseker s.s.} \end{array} \right\}$ 64

$\lim_{\alpha \to +\infty} K(\beta,\alpha) = (0,0)$

Walls W_γ^v :

$\exists\ F \subset E \subset Coh^\beta(X) \qquad v(F)=\gamma \not\parallel v = v(E)$

$v(F) = (r', c', s')\ , \quad v = (r, c, s)$

$$\operatorname{Re} Z(v) \operatorname{Im} Z(\gamma) - \operatorname{Re} Z(\gamma) \operatorname{Im} Z(v) = 0$$

$$\Updownarrow$$

$$2(cs' - c's) - 2\beta\,(rs' - r's) + (rc' - r'c) H^2 (\beta^2 + \alpha^2) = 0$$

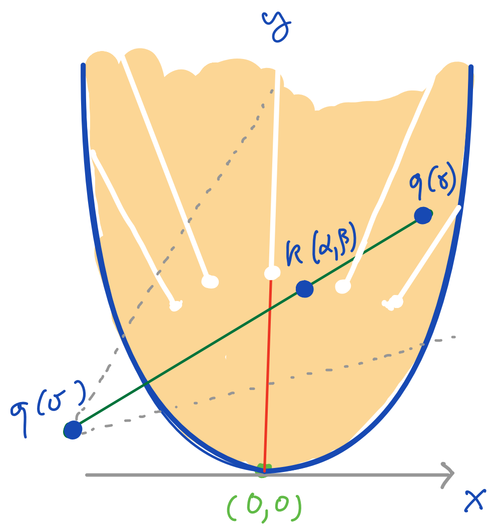
circle in the (β,α)-plane

$$\Updownarrow$$

$$\det \begin{pmatrix} 2 & 2\beta & H^2(\beta^2+\alpha^2) \\ r & c & s \\ r' & c' & s' \end{pmatrix} = 0$$

$\iff$ $v(E)$, $v(F)$, $\operatorname{Ker} Z_{\beta\alpha}$ are coplanar in $\mathbb{R}^3$

$\Longleftrightarrow$ $q(v), q(\gamma), \kappa(\beta, \alpha)$ are collinear in $\mathbb{R}^2$ (x, y)



Walls W_γ^v all go through $q(v)$, in fact W_γ^v is given by $\overline{q(v), q(\gamma)} \cap V_{NS}$

An Example : (HILBERT-CHOW)
([BM-proj] Sect. 10, [Bot 21] Sect. 5)
(We will study this in details later) Let $v = (1, 0, 1-n)$, $(v^2 + 2 = n)$ so that $v = v(I_Y)$, Y a 0-dim subscheme of length n.

For $\alpha \gg 0$, $\beta < 0$, we have that $\sigma = \sigma_{\beta\alpha} = (\mathrm{Coh}^\beta, Z_{\beta\alpha})$ lies in V_{NS}.

Since $c(Y) - \beta r(Y) = -\beta > 0$, we have that $I_Y \in \mathcal{T}^\beta$.

Moreover, by the Theorem on p. 17, since I_Y is H-Gieseker stable it is also $\sigma_{\alpha\beta}$-stable. As $\dim M_\sigma(v) = n$, we get:

$$M_\sigma(v) \cong M_H(v) \cong \mathrm{Hilb}^{[n]}(X).$$
$$J_Y \longmapsto J_Y \longmapsto Y.$$

Going to the (Hilbert-Chow) wall $\beta = 0$, we have that $J_Y[1] \in \mathcal{F}^0[1] \subset \mathrm{Coh}^{0,\alpha}(X)$ and we have a triangle in $\mathrm{coh}^0(X)$

$$\mathcal{O}_Y \to J_Y[1] \to \mathcal{O}_X[1]$$

which is therefore an exact sequence in $\mathrm{Coh}^{0,\alpha}(X)$. We have: $\mathrm{Im}\, Z_{0\alpha}(\mathcal{O}_Y) =$

$$= \mathrm{Im}\, Z_{0\alpha}(J_Y[1]) = \mathrm{Im}\, Z_{0\alpha}(\mathcal{O}_X[1]) = 0$$

So that they are all semistable and they all belong to $\mathcal{P}_{\sigma_{0,\alpha}}(1)$. Also $J_Y[1]$ is strictly $\sigma_{0,\alpha}$-semist. while $\mathcal{O}_X[1]$ is $\sigma_{0\alpha}$-stable. (67)

On the other hand if $Y = n_1 p_1 + \dots + n_k p_k$ as a cycle then $\mathcal{O}_Y$ is S-equivalent to

$$\mathcal{O}_{p_1}^{\oplus n_1} \oplus \dots \oplus \mathcal{O}_{p_k}^{n_k}$$

[Example $Y = 3p$: $\mathcal{O}_p \hookrightarrow \mathcal{O}_{2p} \hookrightarrow \mathcal{O}_{3p}$ with J-H factors $\cong \mathcal{O}_p$]

Hence $J_Y[1] \underset{S_{\sigma_0}}{\sim} J_{Y'}[1] \iff cl(Y) = cl(Y')$.

On the other hand $J_Y[1] \in Coh^0(X) \iff J_Y \in Coh^0(X)[-1]$ and the Hilbert–Chow morphism is given by:

$$\begin{array}{ccc} Hilb^n(X) = M_{\sigma_{\beta\alpha,1}}(v) & \longrightarrow & M_{\sigma_{0\alpha,0}}(v) = Chow^n(X) \\ Y \longleftrightarrow J_Y & \longmapsto & [J_Y] \longleftrightarrow cl(Y) \end{array}$$

$v = (1, 0, 1-n)$

The Brill Noether wall

C curve of genus g.

$V^r_d(C) = \{ L \in \mathrm{Pic}^d(C) \mid h^0(L) = r+1 \}$

$W^r_d(C) = \{ L \in \mathrm{Pic}^d(C) \mid h^0(L) \geq r+1 \} = \overline{V^r_d(C)}$

$\rho = g - (r+1)(g-d+r) = g - h^0(L)h^1(L)$

$\mu_0 : H^0(L) \otimes H^0(KL^{-1}) \longrightarrow H^0(K)$

μ_0 injective $\Rightarrow \rho > 0$

TH. (Griffiths-Harris): C general curve of genus g:

1) $\rho < 0 \iff V^r_d(C) = \phi$.

2) $\rho \geq 0 \Rightarrow \dim V^r_d(C) = \rho$.

TH. (Gieseker): C general curve of genus g:

3) $\mathrm{Ker}\, \mu_0 = 0$

TH. (Lazarsfeld): X K3, $C \in |H|$ smooth.
$[C]$ not decomposable then 1), 2) hold.
If $C \in |H|$ general then 3) holds

<u>pf</u> (Lazarsfeld) $0 \to E_L^\vee \to H^0(L) \otimes \mathcal{O}_X \to \iota_* L \to 0$

$\iota : C \hookrightarrow X$

- $\chi(\mathrm{End}(E_L)) = 2 - 2\rho \implies \rho \geq 0$
  $\mathrm{End}(E_L) \cong \mathbb{C}$
- C general (Pareschi) $\Rightarrow \mathrm{End}(E_L) = \mathbb{C} \oplus \mathrm{Ker}\, \mu_0$

(A. Bayer) : X a K3 (for simplicity)
$\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, $C \in |H|$ smooth, $\iota : C \hookrightarrow X$
$L \in \mathrm{Pic}^d(C)$, $\iota_* L$ torsion sheaf on X
$v(\iota_* L) = v = (0, H, d - g + 1)$.

<u>Detour</u> : $M_H(v) = \left\{ \begin{array}{l} E \text{ supported on } C \in |H| \\ v(E) = v \end{array} \right\}$

Gieseker moduli space

π

that is $E = \iota_* L$, C not necess. smooth. If C is smooth

$\pi^{-1}(C) = \mathrm{Pic}^d(C)$

$|H|$

$M_H(v)$ smooth compact hyperKähler of dim. $v^2 + 2 = 2g$

Define: $\mathcal{V}_d^r \subset M_H(v)$ by:

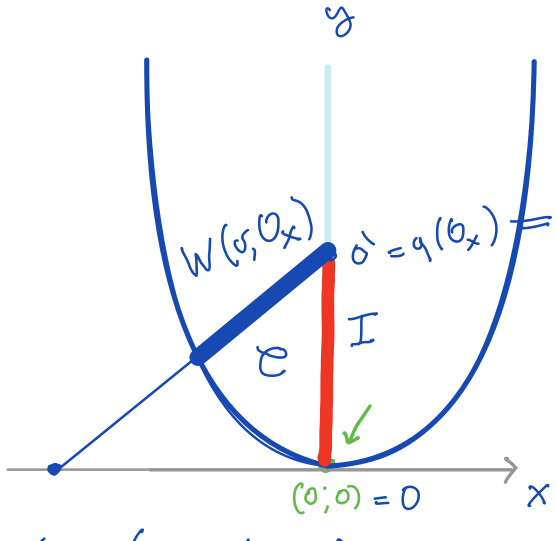
$|H|_{\text{smooth}}$ $\quad \mathcal{V}_d^r \cap \pi^{-1}(C) = V_d^r(C)$

For 1), 2) suffices:

1) $\rho < 0 \iff \mathcal{V}_d^r = \emptyset$

2) $\rho \geq 0 \iff \dim \mathcal{V}_d^r = \rho + g$.

Look at $V_{NS}(X)$



$y = x^2 \frac{H^3}{2}$ $\qquad q(\mathcal{O}_X) = q(v(\mathcal{O}_X))$

$\mathbb{R}(0, \alpha_0)$

$\alpha_0 = \sqrt{\frac{2}{H^2}}$

$q(v) = \left(0, \frac{1}{d-g+1}\right)$

We will see that $W(v, \mathcal{O}_X)$ is a wall.

Notice: $\boxed{\sigma(\alpha, \beta) \in W(v, \mathcal{O}_X) \Rightarrow -1 < \beta < 0}$

We will limit ourselves to the proof of 2), just to connect BN theory to the theory of walls and chambers.

Claim A)

No wall through $q(v)$ cutting $I = \{\beta = 0\}$ (see Figure above)

Pf Suppose $i_* L$ strictly $\sigma_{0\alpha}$-destabilized. Let $A = A_1$ be the first J-H factor of $i_* L$. So that $A \hookrightarrow i_* L$ in $Coh^0(X)$. But $\operatorname{Im} Z_{0\alpha}(i_* L) = (c - \beta r)\alpha H^2 = \alpha H^2 > 0 \Rightarrow i_* L \in \mathcal{T}^0(X)$. By the standard Lemma, $A \in \mathcal{T}^0(X)$. Now $\operatorname{Im} Z(A) = c(A)\,\alpha H^2$, so that $c(A) > 0$.
If $\{A_i\}_{i=1}^n$ are the J-H-factors. Then $\sum \operatorname{Im} Z(A_i) = \operatorname{Im} Z(A)$. So we have $\operatorname{Im} Z(A_i) \geq 0$ (A_i are st.) $\operatorname{Im} Z(A_1) > 0$. So that $c(A_1) = 1$ $c(A_i) = 0$ $i \neq 1$ $\Rightarrow$ $\nu_{0\alpha}(A_i) = \infty$
But $\nu_{0\alpha}(A_i) = \nu_{0\alpha}(A) < \infty$. Hence $n = 1$. $\square$

Claim A) $\Longrightarrow$ $W(v, O_x)$ bounds the "Gieseker Chamber" in the sense that (see figure above) including $\mathcal{C}$

In the region $\mathcal{C}$ $\beta = 0$, we have $M_{\sigma_{\beta\alpha}}(v) = M_H(v)$

Now we look at what happens when we hit the wall $W(v, \mathcal{O}_X)$ coming from inside of $\mathcal{C}$.

Claim B) $W(v, \mathcal{O}_X)$ is a "Brill Noether" wall

that is: let $C \in |H|_{smooth}$ $L \in \mathrm{Pic}^d(C)$ $v(\iota_* L) = v$. Then:

$\iota_* L$ is destabilized by $\sigma_{\beta\alpha} \in W(v, \mathcal{O}_X)$ via a short exact sequence in $\mathrm{Coh}^\beta(X)$

$$0 \to \mathcal{O}_X^{r+1} \to \iota_* L \to B \to 0$$

with B stable along $W(v, \mathcal{O}_X)$

$$\Updownarrow$$

$\iota_* L \in V_d^r(C)$ i.e. $h^0(L) = r+1$

(Euristically: $B = E_L^\vee[1]$).

Pf $\Downarrow$) Apply $\mathrm{Hom}(\mathcal{O}_X, \;)$ to the above sequence. Get $h^0(L) = r+1 + h^0(B) = r+1$, since $\mathrm{Hom}(\mathcal{O}_X, B) = 0$, as B is stable along $W(v, \mathcal{O}_X)$, $\phi_\sigma(B) = \phi_\sigma(\mathcal{O}_X)$ $\sigma \in W(v, \mathcal{O}_X)$ and $v(B) \neq v(\mathcal{O}_X)$

II) Consider the evaluation map:

$$(*) \quad H^0(L)\otimes \mathcal{O}_X \xrightarrow{f} i_* L$$

Claim C) $\mathcal{O}_X$ is stable along $W(v, \mathcal{O}_X)$

Claim D) f is an injection in $\mathrm{Coh}^{\beta}(X)$ $\forall \beta$, s.t. $\sigma_{\beta\alpha} \in W(v, \mathcal{O}_X)$.

Claim C) + **Claim D** $\Longrightarrow$

$$0 \to H^0(L)\otimes \mathcal{O}_X \longrightarrow i_* L \longrightarrow B \to 0$$

is exact in $\mathrm{Coh}^{\beta}(X)$ $\forall \beta$ s.t. $\sigma_{\beta\alpha} \in W(v, \mathcal{O}_X)$.

Claim E) B is stable along $W(v, \mathcal{O}_X)$

Corollary (BN) $\rho(L) \geq 0$

Pf $v(B) = v(i_* L) - (r+1)(1,0,1)$
$= (-(r+1), H, d-g-r)$

$v^2(B) = 2\rho(L) - 2$, $[\rho(L) = g-(r+1)(d-g+r)]$, B stable $\Rightarrow$

$\dim M_{\sigma_{\beta\alpha}}(v(B)) = v^2(B)+2 = 2\rho(L) \geq 0$. $\square$