# ON VARIETIES WITH ULRICH TWISTED TANGENT BUNDLES

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ABSTRACT. We study varieties $X \subseteq \mathbb{P}^N$ of dimension $n$ such that $T_X(k)$ is an Ulrich vector bundle for some $k \in \mathbb{Z}$. First we give a sharp bound for $k$ in the case of curves. Then we show that $k \le n+1$ if $2 \le n \le 12$. We classify the pairs $(X, \mathcal{O}_X(1))$ for $k=1$ and we show that, for $n \ge 4$, the case $k=2$ does not occur.

## 1. INTRODUCTION

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. As is well known, the study of vector bundles on $X$ can give important geometrical information about $X$ itself. Regarding this, one of the most interesting family of vector bundles associated to $X$ and its embedding, that received a lot of attention lately, is that of Ulrich vector bundles, that is bundles $\mathcal{E}$ such that $H^i(\mathcal{E}(-p)) = 0$ for all $i \ge 0$ and $1 \le p \le n$. The study of such bundles is closely related with several areas of commutative algebra and algebraic geometry, and often gives interesting consequences on the geometry of $X$ and on the cohomology of sheaves on $X$ (see for example in [ES, Be1, CMRPL] and references therein).

Perhaps the most challenging question in these matters is whether every $X \subseteq \mathbb{P}^N$ carries an Ulrich vector bundle (see for example [ES, page 543]). It comes therefore very natural to ask if usual vector bundles associated to $X$ can be Ulrich. Also, since Ulrich vector bundles are globally generated, it is better to consider twisted versions, by some divisor $D$, of the usual bundles associated to $X$. On the other hand, in order to keep some relation with the embedding and to have a better chance for global generation, we will consider twists by $D = kH$, for some integer $k$. The cases of the (twisted) normal, cotangent, restricted tangent and cotangent bundles have been dealt with in [Lop], with an essentially complete classification.

In this paper we study the more delicate question: for which integers $k$ one has that $T_X(k)$ is an Ulrich vector bundle?

Ulrich vector bundles have special cohomological features, but also numerical ones. This makes the above question rather tricky. It is easy to show that $k \ge 0$ unless $(X, \mathcal{O}_X(1), k) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), -2)$. In the case $k = 0$, a recent result [BMPT, Prop. 4.1, Thm. 4.5] gives a classification: $(X, \mathcal{O}_X(1)) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)), (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ (we will give a new and simple proof in section 8; another proof is given in [C2]). On the other hand, for $k \ge 1$, the question is more subtle as we will see below.

In the case of curves, one sees that $k = 1$ is not possible (see Lemma 4.3(i)), while the cases $k = 2, 3$ can be dealt with on any curve (see Lemma 5.1 and Example 5.3). On the other hand, the following sharp bound holds, showing that for curves $k$ can be as large as wanted.

**Theorem 1.**
*Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible curve of genus $g$. If $T_X(k)$ is an Ulrich line bundle, then*

$$k \le \frac{\sqrt{8g+1}-1}{2} \tag{1.1}$$

*and equality holds if and only if $k$ is even and either $X$ is one of the curves (5.1) lying on a smooth cubic or $X$ is a curve of type $(\frac{k}{2}+1, k+2)$ on a smooth quadric. Also, in both cases, $T_X(k)$ is an Ulrich line bundle, hence the bound is sharp for every even $k \ge 0$. Moreover, if $X$ has general moduli, then $k \le 4$.*

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As far as we know, only curves show this kind of behavior, meaning that $k$ is not bounded in terms of the dimension (a somewhat bad bound can also be given in terms of the degree, see Lemma 4.8). As supporting evidence, we prove the following

**Theorem 2.**
*Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n$ such that $2 \le n \le 12$. If $T_X(k)$ is an Ulrich vector bundle, then $k \le n+1$.*

We should point out that, for $n \ge 2$, we know no examples with $k \ge 2$ and only one example with $k = 1$. As a matter of fact, the case $k = 1$ can be completely characterized, as follows

**Theorem 3.**
*Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. Then $T_X(1)$ is an Ulrich vector bundle if and only if $(X, \mathcal{O}_X(1)) = (S_5, -2K_{S_5})$, where $S_5$ is a Del Pezzo surface of degree 5.*

On the other hand, for $k = 2$, we have

**Theorem 4.**
*Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 4$. Then $T_X(2)$ is not an Ulrich vector bundle.*

We do not know what happens for $k = 2, n = 3$, even though some evidence suggests that it might not be possible. Also, for surfaces, the cases $k = 2, 3$ point out to the possible existence, that needs to be further investigated, of some minimal surfaces of general type, as shown in Lemma 6.1 and Proposition 6.2.

Finally, in any dimension, another interesting case is the one in which $\omega_X$ and $\mathcal{O}_X(1)$ are numerically proportional. This is dealt with in Theorem 4.11, Corollaries 4.12 and 4.13.

## 2. Notation

Throughout the paper we work over the complex numbers. Moreover we henceforth establish the following

**Notation 2.1.**

- $X$ is a smooth irreducible variety of dimension $n \ge 1$.
- $H$ is a very ample divisor on $X$.
- For any sheaf $\mathcal{G}$ on $X$ we set $\mathcal{G}(l) = \mathcal{G}(lH)$.
- $d = H^n$ is the degree of $X$.
- $C$ is a general curve section of $X$ under the embedding given by $H$.
- $S$ is a general surface section of $X$ under the embedding given by $H$, when $n \ge 2$
- $g = g(C) = \frac{1}{2}[K_X H^{n-1} + (n-1)d] + 1$ is the sectional genus of $X$.
- For $1 \le i \le n-1$, let $H_i \in |H|$ be general divisors and set $X_n := X$ and $X_i = H_1 \cap \cdots \cap H_{n-i}$.

## 3. Generalities on Ulrich bundles

We collect some well-known facts, to be used sometimes later.

**Definition 3.1.** Let $\mathcal{E}$ be a vector bundle on $X$. We say that $\mathcal{E}$ is an *Ulrich vector bundle* for $(X, H)$ if $H^i(\mathcal{E}(-p)) = 0$ for all $i \ge 0$ and $1 \le p \le n$.

We have

**Lemma 3.2.** *Let $\mathcal{E}$ be a rank $r$ Ulrich vector bundle for $(X, H)$. Then*

(i) $c_1(\mathcal{E})H^{n-1} = \frac{r}{2}[K_X + (n+1)H]H^{n-1}$.
(ii) *If $n \ge 2$, then* $c_2(\mathcal{E})H^{n-2} = \frac{1}{2}[c_1(\mathcal{E})^2 - c_1(\mathcal{E})K_X]H^{n-2} + \frac{r}{12}[K_X^2 + c_2(X) - \frac{3n^2+5n+2}{2}H^2]H^{n-2}$.
(iii) $\chi(\mathcal{E}(m)) = \frac{rd}{n!}(m+1)\cdots(m+n)$.
(iv) $H^n(\mathcal{E}(m)) = 0$ *if and only if* $m \ge -n$.
(v) $\mathcal{E}^*(K_X + (n+1)H)$ *is also an Ulrich vector bundle for* $(X, H)$.
(vi) $\mathcal{E}$ *is globally generated.*
(vii) $h^0(\mathcal{E}) = rd$.

(viii) *$\mathcal{E}$ is arithmetically Cohen-Macaulay (aCM), that is $H^i(\mathcal{E}(j)) = 0$ for $0 < i < n$ and all $j \in \mathbb{Z}$.*
(ix) *$\mathcal{E}_{|Y}$ is Ulrich on a smooth hyperplane section $Y$ of $X$.*

*Proof.* We have

$$K_{X_i} = (K_X + (n-i)H)_{|X_i}, 1 \le i \le n. \tag{3.1}$$

By [CH, Lemma 2.4(iii)] we have that

$$c_1(\mathcal{E})H^{n-1} = \deg(\mathcal{E}_{|C}) = r(d+g-1)$$

and using (3.1) on $C = X_1$ we have

$$K_X H^{n-1} = 2(g-1) - (n-1)d$$

thus giving (i). To see (ii) observe that the exact sequences, for $1 \le i \le n-1$,

$$0 \to T_{X_i} \to (T_{X_{i+1}})_{|X_i} \to H_{|X_i} \to 0$$

and (3.1) give by induction that

$$c_2(S) = c_2(X_2) = c_2(X)H^{n-2} + (n-2)K_X H^{n-1} + \binom{n-1}{2}d. \tag{3.2}$$

It follows from [C1, Prop. 2.1(2.2)], (3.1), and Noether's formula $12\chi(\mathcal{O}_S) - K_S^2 = c_2(S)$ that

$$\begin{aligned} c_2(\mathcal{E})H^{n-2} &= \tfrac{1}{2}[c_1(\mathcal{E})^2 - c_1(\mathcal{E})(K_X + (n-2)H)]H^{n-2} - r\left(H^n - \tfrac{[K_X+(n-2)]^2H^{n-2}+c_2(S)}{12}\right) = \\ &= \tfrac{1}{2}[c_1(\mathcal{E})^2 - c_1(\mathcal{E})K_X]H^{n-2} - \tfrac{n-2}{2}c_1(\mathcal{E})H^{n-1} - r\left(H^n - \tfrac{[K_X+(n-2)H]^2H^{n-2}+c_2(S)}{12}\right) \end{aligned}$$

Now (ii) follows from the above equation by using (i) and (3.2). Next, (iii) is [CH, Lemma 2.6]. To see (iv) observe that $\mathcal{E}$ is 0-regular, hence it is $q$-regular for every $q \ge 0$ and therefore $H^n(\mathcal{E}(q-n)) = 0$, that is (iv). Also, (v) follows by definition and Serre duality, while (vi) follows by definition, since $\mathcal{E}$ is 0-regular, and [Laz, Thm. 1.8.5]. For (vii), (viii) and (ix) see [ES, Prop. 2.1] (or [Be1, (3.1)]) and [Be1, (3.4)]. □

## 4. $T_X(k)$ Ulrich in any dimension

We start by drawing some consequences on $(X, H, k)$, of cohomological and numerical type, when $T_X(k)$ is an Ulrich vector bundle.

**Lemma 4.1.** *Let $(X, H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)), n \ge 1$. Then $T_X(k)$ is an Ulrich vector bundle if and only if $n = 1$ and $k = -2$.*

*Proof.* The assertion is obvious if $(X, H, k) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), -2)$. Vice versa suppose that $T_X(k)$ is an Ulrich vector bundle. If $(X, H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, it follows by [ES, Prop. 2.1] (or [Be1, Thm. 2.3]) that $T_{\mathbb{P}^n}(k) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus n}$, hence $0 = \det(T_{\mathbb{P}^n}(k)) = \mathcal{O}_{\mathbb{P}^n}(nk+n+1)$, so that $1 = -n(k+1)$, giving $n = 1, k = -2$. □

**Lemma 4.2.** (cohomological conditions)
*Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. If $T_X(k)$ is an Ulrich vector bundle we have:*

(i) *Either $(X, H, k) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), -2)$, or $k \ge 0$.*
(ii) *If $n \ge 2$, then $T_X$ is aCM, that is $H^i(T_X(j)) = 0$ for $1 \le i \le n-1$ and for every $j \in \mathbb{Z}$. In particular $H^i(T_X) = 0$ for $1 \le i \le n-1$.*
(iii) *If $k \ge 1$, then $H^0(T_X) = 0$, hence $X$ has discrete automorphism group.*
(iv) *If $n \ge 2$, then $X$ is infinitesimally rigid, that is $H^1(T_X) = 0$.*
(v) *$H^0(K_X + (n-k-2)H) = 0$ and, if $n \ge 2$, also $H^0(K_X + (n-k-1)H) = 0$.*
(vi) *If $q(X) \ne 0$ then $H^0(K_X + (n-k)H) = 0$.*
(vii) *If $k \le n-1$, then $p_g(X) = 0$.*
(viii) *Let $a(X, H) = \min\{l \in \mathbb{Z} : lH - K_X \ge 0\}$. Then $k \le \frac{a(X,H)(n+2)}{2n} + \frac{n+1}{2}$. Moreover $H^0((\lceil\frac{n(2k-n-1)}{n+2}\rceil - 1)H - K_X) = 0$.*
(ix) *$K_X - kH$ is not big.*

*Proof.* Since $T_X(k)$ is an Ulrich vector bundle, it is globally generated by Lemma 3.2(vi). Now if $k \le -1$ we would have that $0 \ne H^0(T_X(k)) \subseteq H^0(T_X(-1))$. But then the Mori-Sumihiro-Wahl's theorem [MS, Thm. 8], [W, Thm. 1] implies that $(X,H) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)), (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. In the first case we have that $0 = H^i(T_{\mathbb{P}^1}(k-1)) = H^i(\mathcal{O}_{\mathbb{P}^1}(2k)) = 0$ for $i \ge 0$, a contradiction. In the second case apply Lemma 4.1. This proves (i). Now (ii) follows by Lemma 3.2(viii). If $k \ge 1$ we have that $H^0(T_X) \subseteq H^0(T_X(k-1)) = 0$, hence (iii). (iv) is implied by (ii). As for (v), recall that, as is well known, $\Omega^1_X(2)$ is globally generated. Now if $H^0(K_X + (n-k-2)H) \ne 0$ then we get the contradiction

$$0 \ne H^0(\Omega^1_X(2)) \subseteq H^0(\Omega^1_X(K_X + (n-k)H)) = H^n(T_X(k-n))^* = 0.$$

This gives the first part of (v). Similarly, if $q(X) \ne 0$ and $H^0(K_X + (n-k)H) \ne 0$ then we get the contradiction

$$0 \ne H^0(\Omega^1_X) \subseteq H^0(\Omega^1_X(K_X + (n-k)H)) = H^n(T_X(k-n))^* = 0.$$

This gives (vi). Now if $n \ge 2$, consider $Y \in |H|$ smooth. Then $T_X(k)_{|Y}$ is an Ulrich vector bundle on $Y$ by Lemma 3.2(ix), hence $H^{n-1}(T_X(k-n+1)_{|Y}) = 0$. Now the exact sequence

$$0 \to T_Y(k-n+1) \to T_X(k-n+1)_{|Y} \to \mathcal{O}_Y(k-n+2) \to 0$$

implies that $H^{n-1}(\mathcal{O}_Y(k-n+2)) = 0$. Hence, setting $\mathcal{L} = K_X + (n-k-1)H$, we get by Serre's duality that

$$H^0(\mathcal{L}_{|Y}) = H^0(K_Y + (n-k-2)H_{|Y}) = 0.$$

Therefore $H^0(\mathcal{L}(-l)_{|Y}) = 0$ for every $l \ge 0$ and the exact sequences

$$0 \to \mathcal{L}(-l-1) \to \mathcal{L}(-l) \to \mathcal{L}(-l)_{|Y} \to 0$$

show that $h^0(\mathcal{L}(-l-1)) = h^0(\mathcal{L}(-l))$ for every $l \ge 0$. Since this is zero for $l \gg 0$, we get that they are all zero, hence $H^0(K_X + (n-k-1)H) = 0$. This proves the second part of (v). Now, to see (vii), suppose that $k \le n-1$. If $n \ge 2$, we see that (v) gives $H^0(K_X) \subseteq H^0(K_X + (n-k-1)H) = 0$, hence (vii). If $n = 1$ we have that $k \le 0$, hence $X = \mathbb{P}^1$ by (i) and Lemma 4.3(i). Observe that $a(X,H)H - K_X \ge 0$, hence $(a(X,H)H - K_X)H^{n-1} \ge 0$ and using Lemma 4.3(ii), we get

$$a(X,H) \ge \frac{n(2k-n-1)}{n+2}$$

This gives (viii) since, by its own definition, $H^0((a(X,H)-1)H - K_X) = 0$. Finally assume that $K_X - kH$ is big. Then Serre's duality gives $H^0(T_X(k)) = H^n(\Omega^1_X(K_X - kH))^* = 0$ by Bogomolov-Sommese vanishing [Bo, Thm. 4], contradicting Lemma 3.2(vi). □

We recall that a *pseudoeffective (or pseff)* divisor, on a variety $X$, is a divisor $D$ whose class lies in the closure of the effective cone in the space $N^1(X)_{\mathbb{R}}$ of numerical equivalence classes of divisors.

**Lemma 4.3.** (numerical conditions)
*Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. If $T_X(k)$ is an Ulrich vector bundle we have:*

(i) $d = \frac{(n+2)(g-1)}{nk-1}$. *In particular either $(X,H,k) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), -2)$, or $g = k = 0$, or $g \ge 2$.*
(ii) $k = \frac{n+1}{2} + \left(\frac{n+2}{2nd}\right) K_X H^{n-1}$; *equivalently* $K_X H^{n-1} = \frac{n(2k-n-1)}{n+2} d$.
(iii) *If $k < \frac{n+1}{2}$, then $X$ is rationally connected and $H^i(\mathcal{O}_X) = 0$ for every $i \ge 1$.*
(iv) *If $k > \frac{n+1}{2}$, then $-K_X$ is not pseff.*
(v) *$T_X$ is semistable.*
(vi) *If $n \ge 2$, then $K_X^2 H^{n-2} \le \frac{2n}{n-1} c_2(X) H^{n-2}$.*
(vii) *If $n \ge 2$, then*

$$(12kn - 12k^2 + 12k - 3n^2 - 5n - 2)nd + 2(n+12)K_X^2 H^{n-2} + 2(n-12)c_2(X)H^{n-2} = 0.$$

*Proof.* Since $c_1(T_X(k)) = -K_X + nkH$, we get by Lemma 3.2(i) that

$$(-K_X + nkH)H^{n-1} = \frac{n}{2}\left(K_X H^{n-1} + (n+1)d\right)$$

and this gives (ii). Also, using $K_X H^{n-1} = 2(g-1) - (n-1)d$, we get that

$$(nk-1)d = (n+2)(g-1).$$

Now if $nk-1=0$ then $n=k=g=1$, but then $T_X(k)=\mathcal{O}_X(1)$ is not Ulrich. Therefore $nk-1\neq 0$ and $d=\frac{(n+2)(g-1)}{nk-1}$. Hence $g\neq 1$ and if $g=0$ then either $k=0$ or $k\neq 0$ and in the latter case we have that $nk<1$, hence $(X,H,k)=(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1),-2)$ by Lemma 4.2(i). This proves (i). Next, (v) follows since Ulrich vector bundles are semistable by [CH, Thm. 2.9], hence Bogomolov's inequality gives (vi). To see (iii), suppose that $k<\frac{n+1}{2}$. If $n\geq 2$, then (ii) gives that $K_XH^{n-1}<0$, hence $X$ is rationally connected by (v) and [BMQ, Main Thm.] (see also [CP, Thm. 1.1]). Hence, as is well known, $H^i(\mathcal{O}_X)=0$ for every $i\geq 1$. If $n=1$ then $k\leq 0$ and $X=\mathbb{P}^1$ by (i). Thus we get (iii). If $k>\frac{n+1}{2}$, then either $n=1$ and $g\geq 2$ by (i), so that $-K_X$ is not pseff, or $n\geq 2$ and (ii) gives that $K_XH^{n-1}>0$, hence again $-K_X$ is not pseff and we get (iv). To see (vii), observe that

$$c_2(T_X(k))H^{n-2}=c_2(X)H^{n-2}-k(n-1)K_XH^{n-1}+\binom{n}{2}k^2d. \tag{4.1}$$

From Lemma 3.2(ii), we get

$$c_2(T_X(k))H^{n-2}=\left(\frac{n^2k^2}{2}-\frac{n}{24}\left(3n^2+5n+2\right)\right)d-\frac{3nk}{2}K_XH^{n-1}+\left(1+\frac{n}{12}\right)K_X^2H^{n-2}+\frac{n}{12}c_2(X)H^{n-2}. \tag{4.2}$$

Combining (4.1), (4.2) and (ii), we obtain (vii). □

**Definition 4.4.** For $n\geq 1$ we denote by $Q_n$ a smooth quadric in $\mathbb{P}^{n+1}$.

**Lemma 4.5.** *Let $(X,H)=(Q_n,\mathcal{O}_{Q_n}(1)),n\geq 1$. Then $T_X(k)$ is not an Ulrich vector bundle for any integer $k$.*

*Proof.* This follows from the well-known fact that Ulrich vector bundles on quadrics are direct sums of spinor bundles. Alternatively, since $g=0$, it follows by Lemma 4.3(i) that $k=0$ and $2=d=n+2$, a contradiction. □

We will use the nef value of $(X,H)$:

$$\tau(X,H)=\min\{t\in\mathbb{R}:K_X+tH \text{ is nef}\}. \tag{4.3}$$

We observe that in [BS, Def. 1.5.3] the nef value is defined only when $K_X$ is not nef. On the other hand, it makes sense and it will be used, throughout this paper, also when $K_X$ is nef.

A very useful observation is the following.

**Lemma 4.6.** *Let $X\subseteq\mathbb{P}^N$ be a smooth irreducible variety of dimension $n\geq 1$. If $T_X(k)$ is an Ulrich vector bundle, then $\Omega^1_Y(K_{X|Y}+(n+1-k)H_{|Y})$ is globally generated for any smooth subvariety $Y\subseteq X$. Moreover:*

- (i) *If $\pm(K_X+\frac{n(n+1-2k)}{n+2}H)$ is pseff, then $K_X\equiv\frac{n(2k-n-1)}{n+2}H$.*
- (ii) $\tau(X,H)\geq\frac{n(n+1-2k)}{n+2}$.
- (iii) $\tau(X,H)\leq n-\frac{nk}{n+1}$. *In particular, if $K_X$ is not nef, then $k\leq n$.*
- (iv) *If $k\geq n+1$, then $K_X$ is ample.*

*Proof.* Note that $\Omega^1_X(K_X+(n+1-k)H)$ is Ulrich and globally generated by Lemma 3.2(v) and (vi). Since $\Omega^1_X(K_X+(n+1-k)H)$ surjects onto $\Omega^1_Y(K_{X|Y}+(n+1-k)H_{|Y})$, the latter is also globally generated. Moreover so is $\det(\Omega^1_X(K_X+(n+1-k)H)=(n+1)K_X+n(n+1-k)H$, hence we get (iii) and, if $k\geq n+2$, we also deduce that $K_X$ is ample. On the other hand, if $k=n+1$, then $\Omega^1_X(K_X)$ is Ulrich, and we claim that $\det(\Omega^1_X(K_X))=(n+1)K_X$ is ample. In fact, if not, then [LS, Thm. 1] implies that there is a line $L\subset X$ such that $\Omega^1_X(K_X)_{|L}$ is trivial. Hence $(n+1)K_X\cdot L=\deg(\Omega^1_X(K_X)_{|L})=0$. But then we have a surjection $\Omega^1_X(K_X)_{|L}\to\Omega^1_L$, contradicting the fact that $\Omega^1_L$ is not globally generated. This proves (iv). As for (i) and (ii), set $q=\frac{n(n+1-2k)}{n+2}$, so that $(K_X+qH)H^{n-1}=0$ by Lemma 4.3(ii). Now if $\pm(K_X+qH)$ is pseff, then $K_X+qH\equiv 0$ by [FL2, Cor. 3.15] (see also [FL1, Prop. 3.7]), thus proving (i). Also, (i) implies that either $K_X\equiv -qH$ and then $\tau(X,H)=q$, or $K_X+qH$ is not pseff, hence not nef. Therefore, in the latter case, $\tau(X,H)>q$, proving (ii). □

In the case of hypersurfaces we have

**Lemma 4.7.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth irreducible nondegenerate hypersurface. Then $T_X(k)$ is not an Ulrich vector bundle.*

*Proof.* We have $d \ge 2$. If $n = 1$ we have that $K_X = (d-3)H$, $g = \binom{d-1}{2}$ and $k = \frac{3(d-3)}{2} + 1$ by Lemma 4.3(i). Now $0 = H^0(T_X(k-1)) = H^0((-d+2+k)H)$ and therefore $-d+2+k \le -1$, giving the contradiction $d \le 1$. Hence $n \ge 2$ and since $C \subset \mathbb{P}^2$ we have that $g - 1 = \frac{d(d-3)}{2}$ and Lemma 4.3(i) implies that $d = \frac{2(nk-1)}{n+2} + 3$. On the other hand Lemma 4.2(v) gives that

$$0 = H^0(K_X + (n-k-1)H) = H^0((d-k-3)H)$$

and therefore

$$\frac{2(nk-1)}{n+2} - k \le -1$$

that is $k(n-2) + n \le 0$, a contradiction. □

**Lemma 4.8.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. If $T_X(k)$ is an Ulrich vector bundle we have that*

$$k \le \frac{(n+2)(d-4)+4}{4n}.$$

*Proof.* We have $X \subseteq \mathbb{P}H^0(H) = \mathbb{P}^N$. If $N = n$ then $(X, H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and Lemma 4.1 gives that $n = 1$ and $k = -2$. Since $d = 1$ we have that $k = -2 \le -\frac{5}{4} = \frac{(n+2)(d-4)+4}{4n}$. If $N > n$, by Lemma 4.7 we have that $N \ge n+2$ and $C \subset \mathbb{P}^{N-n+1}$ can be projected isomorphically to a non-degenerate smooth irreducible curve in $\mathbb{P}^3$. Then Castelnuovo's bound gives that $g - 1 \le \frac{d(d-4)}{4}$ and Lemma 4.3(ii) implies the required bound on $k$. □

A nice consequence of the above lemmas is the following.

**Proposition 4.9.** *There does not exist any $(X, H, k)$ with $T_X(k)$ an Ulrich vector bundle, when:*

(i) $K_X \equiv 0$.
(ii) $\pm K_X$ *is pseff and* $k = \frac{n+1}{2}$.

*Proof.* Under hypothesis (ii), we get from Lemma 4.6(i) that $K_X \equiv 0$. Thus we will be done if we prove (i). Assume next that $K_X \equiv 0$, so that $k = \frac{n+1}{2}$ by Lemma 4.3(ii) and $n \ge 3$ by Lemma 4.3(i). Since $H - K_X$ is ample, it follows by Kodaira vanishing that $H^i(H) = H^i(K_X + H - K_X) = 0$ for $i > 0$, hence

$$h^0(K_X + H) = \chi(K_X + H) = \chi(H) = h^0(H) \neq 0.$$

On the other hand, Lemma 4.2(v) gives that $h^0(K_X + \frac{n-3}{2}H) = 0$, whence, if $n \ge 5$, we get the contradiction $h^0(K_X + H) = 0$.

It remains to consider the case $n = 3, k = 2$. Note that $p_g(X) = 0$ by Lemma 4.2(vii) and $q(X) = 0$, for otherwise Lemma 4.2(vi) gives that $h^0(K_X + H) = 0$. Therefore $\chi(\mathcal{O}_X) \ge 1$. On the other hand $\chi(\mathcal{O}_X) = \frac{1}{24}c_1(X)c_2(X) = 0$, a contradiction. □

We now prove Theorem 2.

*Proof of Theorem 2.* By the Hodge index theorem we have that $H_{|S}^2 K_S^2 \le (H_{|S} K_S)^2$, that is

$$dK_X^2 H^{n-2} \le (K_X H^{n-1})^2. \tag{4.4}$$

Using Lemma 4.3(vi), (vii) and (4.4) we obtain that

$$0 = (12kn - 12k^2 + 12k - 3n^2 - 5n - 2)nd + 2(n+12)K_X^2 H^{n-2} + 2(n-12)c_2(X)H^{n-2} \le$$

$$\le (12kn - 12k^2 + 12k - 3n^2 - 5n - 2)nd + \frac{3n^2 + 11n + 12}{n} K_X^2 H^{n-2} \le$$

$$\le (12kn - 12k^2 + 12k - 3n^2 - 5n - 2)nd + \frac{3n^2 + 11n + 12}{nd} (K_X H^{n-1})^2$$

which, using Lemma 4.3(ii) becomes

$$4nk^2 - 4n(n+1)k - 3n^2 - 7n - 4 \le 0$$

giving

$$k \le \frac{n^2+n+\sqrt{n^4+5n^3+8n^2+4n}}{2n} < n+2.$$

□

The case $k=0$ is known:

**Theorem 4.10.** *([BMPT, Prop. 4.1, Thm. 4.5])*
*Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. Then $T_X$ is an Ulrich vector bundle if and only if $(X,H) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)), (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.*

We will give a quick alternative proof in section 8.

Next we study the case when $K_X$ and $H$ are proportional.

**Theorem 4.11.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. Suppose that the numerical classes of $H$ and $K_X$ are proportional and that either*

(i) *$1 \le n \le 11$ and either $k \le \frac{n+1}{2}$ or $k \ge n+2$; or*
(ii) *$n = 12$, or*
(iii) *$n \ge 13$ and $k \notin \{n+2, n+3\}$.*

*Then $T_X(k)$ is Ulrich if and only if $(X,H,k)$ is one of the following:*

(1) $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), -2)$,
(2) $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3), 0)$,
(3) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2), 0)$,
(4) $(S_5, -2K_{S_5}, 1)$, *where $S_5$ is a Del Pezzo surface of degree* 5.

*Proof.* In the cases (1)-(4) we have that $T_X(k)$ is Ulrich by Lemma 4.3(i), Theorem 4.10 and Theorem 6.3.

Vice versa, suppose that the numerical classes of $H$ and $K_X$ are proportional and that we are under one of hypotheses (i), (ii) or (iii) and that $T_X(k)$ is Ulrich.

Observe that, since $N^1(X)$ is a torsion free finitely generated abelian group, we can find an ample primitive (that is indivisible) divisor $A$ and some $r, s \in \mathbb{Z}$ such that $s > 0, H \equiv sA$ and $K_X \equiv -rA$. In particular, Lemma 4.3(ii) gives

$$r(n+2) = n(n+1-2k)s. \tag{4.5}$$

If $k \le 0$ we are in cases (1)-(3) by Lemma 4.2(i) and Theorem 4.10. Hence assume that $k \ge 1$. Note that Proposition 4.9 shows that $k \ne \frac{n+1}{2}$.

If (ii) or (iii) holds, since the numerical classes of $H$ and $K_X$ are proportional, Lemma 4.3(vi), (ii) and (vii) imply that

$$4nk^2 - 4n(n+1)k - 3n^2 - 7n - 4 \ge 0$$

so that $k > n+1$. This is a contradiction under hypothesis (ii) by Theorem 2. Under hypothesis (iii), we get that $k \ge n+4$. Then it follows by (4.5) that

$$-r-ks = \frac{(n-2)k-n^2-n}{n+2}s \ge \frac{n-8}{n+2}s > 0$$

hence $K_X - kH = (-r-ks)A$ is ample, contradicting Lemma 4.2(ix). Thus it remains to consider hypothesis (i).

Now assume (i), so that $n \ge 2$ and Theorem 2 implies that it cannot be that $k \ge n+2$. Hence $k < \frac{n+1}{2}$ and Lemma 4.3(ii) implies that $X$ is Fano. Consequently, the numerical and linear equivalence for divisors coincide on $X$ and then $K_X = -rA$ and $H = sA$. We can also assume that $r \le n-1$, for otherwise, as is well known, $(X,H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)), (Q_n, \mathcal{O}_{Q_n}(1))$, contradicting Lemmas 4.1 and 4.5.

Next, set $P_A(t) := \chi(K_X + tA)$, so that $P_A(t) = h^0(K_X + tA)$ whenever $t \ge 1$ is an integer. By Riemann-Roch (see for example [Ho, eq. (1), p. 2], we have

$$P_A(t) = \frac{A^n}{n!}t^n + \frac{A^{n-1}K_X}{2(n-1)!}t^{n-1} + \frac{A^{n-2}(K_X^2 + c_2(X))}{12(n-2)!}t^{n-2} + \dots + (-1)^n\chi(\mathcal{O}_X). \tag{4.6}$$

Note that $n$ is even, for otherwise $n$ and $n+2$ are coprime, and consequently $n$ divides $r$ by (4.5), hence $r \ge n$, a contradiction.

Set $n = 2m$, where $1 \leq m \leq 5$.

If $m = 1$ we have that $n = 2, k = 1$ and we are in case (4) by Theorem 6.3.

We will now exclude the remaining cases for $m$.

Note that we can rewrite (4.5) as

$$(m+1)r = m(2m-2k+1)s. \tag{4.7}$$

Case 1: $m = 2$. In this case $k \leq 2$ and $r \leq 3$.

*(1a):* $k = 1$. We see that $r = 2s$. Thus $(r, s) = (2, 1)$, contradicting Lemma 4.2(v).

*(1b):* $k = 2$. We see that $2s = 3r$. Thus $(r, s) = (2, 3)$ and Lemma 4.2(v) shows that $h^0(A) = 0$. This contradicts [A, Lemma 2] or [K, Thm. 5.1].

Case 2: $m = 3$. In this case $k \leq 3$ and $r \leq 5$.

*(2a):* $k = 1$. We see that $4r = 15s$. Thus, 15 divides $r$ which is clearly impossible.

*(2b):* $k = 2$. We see that $9s = 4r$. Thus, 9 divides $r$ which is a contradiction.

*(2c):* $k = 3$. We see that $3s = 4r$. Thus $(r, s) = (3, 4)$ and Lemma 4.2(v) shows that $h^0(K_X + 8A) = 0$. This is a contradiction by [GL, Thm. 1.2], as $K_X + 8A$ is base-point-free.

Case 3: $m = 4$. In this case $k \leq 4$ and $r \leq 7$.

*(3a):* $k = 1$. We see that $5r = 28s$. Thus, 28 divides $r$ which is clearly impossible.

*(3b):* $k = 2$. We see that $4s = r$. Thus $(r, s) = (4, 1)$ and Lemma 4.2(v) shows $h^0(K_X + 5A) = h^0(H) = 0$, a contradiction.

*(3c):* $k = 3$. We see that $12s = 5r$. Thus, 12 divides $r$ which is absurd.

*(3d):* $k = 4$. We see that $4s = 5r$. Thus, $(r, s) = (4, 5)$.

We have $K_X = -4A$ and $H = 5A$. Hence $H^0(K_X + 15A) = 0$ by Lemma 4.2(v). Then

$$P_A(1) = P_A(2) = P_A(3) = P_A(5) = P_A(10) = P_A(15) = 0, \quad P_A(0) = P_A(4) = 1$$

and

$$P_A(t) = \frac{A^8}{8!}(t-1)(t-2)(t-3)(t-5)(t-10)(t-15)(t^2 + at + b).$$

Therefore

$$1 = P_A(0) = \frac{A^8}{8!}(4500b) \implies \frac{A^8}{8!}b = \frac{1}{4500} \tag{4.8}$$

and calculating the coefficient of $t^7$ in (4.6) we get

$$\frac{A^8}{8!}(a-36) = \frac{A^7 K_X}{2(7!)} \implies a = 20. \tag{4.9}$$

We also know that $P_A(4) = 1$ and that gives us

$$-\frac{A^8}{8!}(396)(16 + 4a + b) = 1.$$

We simplify the above using (4.8) and (4.9) to obtain

$$-38016\frac{A^8}{8!} = 1 + \frac{396}{4500}$$

which is clearly absurd.

Case 4: $m = 5$. In this case $k \leq 5$ and $r \leq 9$.

*(4a):* $k = 1$. We see that $2r = 15s$. Thus, 15 divides $r$ which is impossible.

*(4b):* $k = 2$. We see that $35s = 6r$. Thus, 35 divides $r$ which is also impossible.

*(4c):* $k = 3$. We see that $25s = 6r$. Thus, 25 divides $r$ which is also impossible.

*(4d):* $k = 4$. We see that $5s = 2r$. Thus $(r, s) = (5, 2)$ and Lemma 4.2(v) shows that $h^0(K_X + 10A) = 0$. Then

$$P_A(1) = P_A(2) = P_A(3) = P_A(4) = P_A(6) = P_A(8) = P_A(10) = 0, \quad P_A(0) = P_A(5) = 1$$

so that we obtain

$$P_A(t) = \frac{A^{10}}{10!}(t-1)(t-2)(t-3)(t-4)(t-6)(t-8)(t-10)(t^3 + at^2 + bt + c).$$

Therefore

$$1 = P_A(0) = -\frac{A^{10}}{10!}(11520c) = 1 \implies \frac{A^{10}}{10!}c = -\frac{1}{11520}. \tag{4.10}$$

and calculating the coefficient of $t^9$ in (4.6) we get

$$\frac{A^{10}}{10!}(a-34) = \frac{A^9K_X}{2(9!)} \implies a = 9. \tag{4.11}$$

We also know that $P_A(5) = 1$ and that gives us

$$-\frac{A^{10}}{10!}(360)(125+25a+5b+c) = 1.$$

We simplify the above using (4.10) and (4.11) to obtain

$$\frac{A^{10}}{10!}b = \frac{1}{5(11520)} - \frac{1}{5(360)} - 70\frac{A^{10}}{10!}. \tag{4.12}$$

Finally, calculating the coefficient of $t^8$ in (4.6) we get

$$\frac{A^{10}}{10!}(b-34a+463) = \frac{A^8(K_X^2+c_2(X))}{12(8!)}. \tag{4.13}$$

We simplify (4.13) using (4.11) and (4.12) to obtain

$$67A^{10} + 5A^8c_2(X) + 1302 = 0. \tag{4.14}$$

On the other hand, Lemma 4.3(vii) shows that

$$115A^{10} = A^8c_2(X)$$

and combining with (4.14), we get that $A^{10}$ is negative, which is clearly impossible.

*(4e): $k = 5$.* We see that $5s = 6r$. Thus $(r, s) = (5, 6)$. We have $K_X = -5A$ and $H = 6A$. Again $H^0(K_X + 24A) = 0$ by Lemma 4.2(v). Then

$$P_A(1) = P_A(2) = P_A(3) = P_A(4) = P_A(6) = P_A(12) = P_A(18) = P_A(24) = 0, \quad P_A(0) = P_A(5) = 1.$$

Thus, we obtain

$$P_A(t) = \frac{A^{10}}{10!}(t-1)(t-2)(t-3)(t-4)(t-6)(t-12)(t-18)(t-24)(t^2+at+b).$$

Then

$$1 = P_A(0) = \frac{A^{10}}{10!}(746496b) \implies \frac{A^{10}}{10!}b = \frac{1}{746496}. \tag{4.15}$$

calculating the coefficient of $t^9$ in (4.6) we get

$$\frac{A^{10}}{10!}(a-70) = \frac{A^9K_X}{2(9!)} \implies a = 45. \tag{4.16}$$

We also know that $P_A(5) = 1$ and that gives us

$$\frac{A^{10}}{10!}(41496)(25+5a+b) = 1.$$

We simplify the above using (4.15) and (4.16) to obtain

$$A^{10} = \left(\frac{705000}{746496}\right)(10!)$$

which is clearly absurd since $A^{10}$ is an integer. □

**Corollary 4.12.** *Suppose that $K_X = eH, e \in \mathbb{Z}$ (hence, in particular, if $\mathrm{Pic}(X) = \mathbb{Z}H$). Then $T_X(k)$ is Ulrich if and only if $(X, H, k) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), -2)$.*

*Proof.* This follows by [Lop, Prop. 4.1(i)]. We give another proof. We have that $e = \frac{n(2k-n-1)}{n+2}$ by Lemma 4.3(ii). If $k \le \frac{n+1}{2}$ it follows by Theorem 4.11 that $(X, H, k) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), -2)$. Now assume that $k > \frac{n+1}{2}$, so that $e \ge 1$. If $n \ge 2$, Lemma 4.2(v) gives that $k \ge n + e$, hence $k(n-2) + n \le 0$, a contradiction. Then $n = 1$ and $e = \frac{2(k-1)}{3}$. But $0 = H^0(T_X(k-1)) = H^0((k-1-e)H)$, hence $e \ge k$, so that $k \le -2$, contradicting $k > 1$. □

**Corollary 4.13.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 2$ with $T_X(k)$ is Ulrich. Suppose that there is an ample line bundle $A$ on $X$ such that $K_X = rA$, $H = sA$ for some $r, s \in \mathbb{Z}$ (hence, in particular, if $\operatorname{Pic}(X) \cong \mathbb{Z}$). Let $m(H, A) := \min\{m \ge 0 : H^0(mH + qA) \ne 0$ for all $q \ge 1\}$. Then:*

(i) *$m(H, A) > \frac{(n-2)k-2}{n+2}$ and, if $n \ge 3$, then $k < \frac{(n+2)m(H,A)+2}{n-2}$.*
(ii) *If $A$ is effective, then $n = 2$.*
(iii) *If $m(H, A) \le n - 3$, then $n \le 11$.*

*Proof.* Observe first that, if $n \ge 3$, then $(n-2)k - 2 \ge k - 2 \ge 0$: In fact if $k \le 1$ we have a contradiction by Theorem 4.11. Now Lemma 4.3(ii) implies that

$$r(n+2) = n(2k-n-1)s \tag{4.17}$$

and Lemma 4.2(v) gives

$$0 = H^0(K_X + (n-k-1)H) = H^0\left(\frac{(nk-2k-2)s}{n+2}A\right). \tag{4.18}$$

To see (i), notice that it is obvious for $n = 2$, for $m(H, A) \ge 0$ by definition. If $n \ge 3$ we see by (4.18) that $\frac{(nk-2k-2)s}{n+2} \in \mathbb{Z}$ and we can write $\frac{(nk-2k-2)s}{n+2} = as + b$ for some $a, b \in \mathbb{Z}$ with $a \ge 0, 0 \le b < s$. Since $H^0(aH + bA) = 0$ by (4.18), we get that

$$\frac{(n-2)k-2}{n+2} - 1 < a \le m(H, A) - 1$$

giving (i). Now suppose that $A$ is effective. If $n \ge 3$ we know that $(n-2)k - 2 \ge 0$, contradicting (4.18). This proves (ii). To see (iii), notice that if $n \ge 12$, then $k \ge n+2$ by Theorem 4.11(ii) and (iii). Hence $\frac{(n-2)k-2}{n+2} > n - 3$ and (4.18) gives that

$$0 = H^0\left(\frac{(nk-2k-2)s}{n+2}A\right) = H^0((n-3)H + qA)$$

for some $q \ge 1$, contradicting the hypothesis $m(H, A) \le n - 3$. □

## 5. Curves

Throughout this section we will have that $X \subseteq \mathbb{P}^N$ is a smooth irreducible curve.

It follows by Lemma 4.3(i) and Theorem 4.10 that if $n = 1$ and $T_X(k)$ is an Ulrich line bundle, then $(X, H, k) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), -2), (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3), 0)$ or $k \ge 2$ and $g \ge 2$.

We will give below examples with $k = 2, 3$, essentially on any curve. Then we will give a sharp bound on $k$ depending on the genus.

The case $k = 2$ can be characterized.

**Lemma 5.1.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible curve. Then $T_X(2)$ is Ulrich if and only if there exists $M \in \operatorname{Pic}(X)$ such that $H^i(M) = 0$ for $i \ge 0$ and $H = K_X + M$ is very ample.*

*Proof.* If $T_X(2)$ is Ulrich, set $M = H - K_X$. Then $H^i(M) = H^i(T_X(1)) = 0$ for $i \ge 0$ and $K_X + M = H$ is very ample. Vice versa, let $M \in \operatorname{Pic}(X)$ be such that $H^i(M) = 0$ for $i \ge 0$ and $H = K_X + M$ is very ample. Then $H^i(T_X(1)) = H^i(M) = 0$ for $i \ge 0$, so that $T_X(2)$ is Ulrich. □

*Example* 5.2. The case $k = 2$ occurs on a curve $X$ if and only if $g \ge 3$.

*Proof.* If $T_X(2)$ is Ulrich, we have by Lemma 4.3(i) that $g \ge 3$ unless $g = 2, d = 3$. But the latter case does not occur, as there is no smooth curve of degree 3 and genus 2 in $\mathbb{P}^N$.

Suppose that $g \ge 3$ and let $M \in \operatorname{Pic}(X)$ be such that $H^i(M) = 0$ for $i \ge 0$. We claim that $H := K_X + M$ is very ample. In fact $\deg M = g - 1$ by Riemann-Roch, hence $\deg H = 3g - 3$. If $g \ge 4$, then $\deg H \geq 2g+1$, hence $H$ is very ample. If $g=3$ we have that $\deg H = 2g$ and, as is well known, $H$ is very ample unless $H = K_X + P + Q$ for two points $P, Q \in X$. But then $M = P + Q$ is effective, a contradiction. Thus the claim is proved and, since $H = K_X + M$, we have $H^i(T_X(1)) = H^i(M) = 0$ for $i \geq 0$, so that $T_X(2)$ is Ulrich. □

*Example* 5.3. The case $k=3$ occurs on any curve $X$ with (necessarily) odd genus $g \geq 9$. This was suggested to us by E. Sernesi, whom we thank.

*Proof.* Let $d = \frac{3(g-1)}{2}$. We claim that a general $H \in \mathrm{Pic}^d(X)$ is very ample. In fact, first observe that $H^1(H) = 0$, for otherwise $K_X - H \geq 0$. But $K_X - H$ is a general line bundle of degree $\frac{g-1}{2} \leq g-1$, hence $h^0(K_X - H) = 0$. Now, if $H$ were not very ample, there will be two points $p, q \in X$ such that $h^0(H-p-q) \geq h^0(H) - 1$. But this can be rewritten, by Riemann-Roch, as $h^1(H-p-q) \geq 1$, that is $K_X - H + p + q \geq 0$. Hence there are some points $p_1, \dots, p_{\frac{g+3}{2}} \in X$ such that

$$K_X - H + p + q \sim p_1 + \ldots + p_{\frac{g+3}{2}}$$

that is

$$H \sim K_X - p_1 - \ldots - p_{\frac{g+3}{2}} + p + q.$$

This means that $H$ is in the image of the morphism $h : X^{\frac{g+7}{2}} \to \mathrm{Pic}^d(X)$ sending $(p_1, \dots, p_{\frac{g+3}{2}}, p, q)$ to $K_X - p_1 - \ldots - p_{\frac{g+3}{2}} + p + q$. But $\dim \mathrm{Im} h \leq \frac{g+7}{2} < g$, contradicting that $H$ is general. This proves that there is a non-empty open subset $W$ of $\mathrm{Pic}^d(X)$ such that any $H \in W$ is very ample.

Consider the surjective morphism $\psi : \mathrm{Pic}^d(X) \to \mathrm{Pic}^{3g-3}(X)$ given by $\psi(L) = 2L$ and the isomorphism $\varphi : \mathrm{Pic}^{3g-3}(X) \to \mathrm{Pic}^{g-1}(X)$ given by $\varphi(L) = L - K_X$. Let $U$ be the non-empty open subset of $\mathrm{Pic}^{g-1}(X)$ such that $H^i(M) = 0$ for $i \geq 0$ for any $M \in U$. Now let $H \in \psi^{-1}(\varphi^{-1}(U)) \cap V \cap W$. Then $H$ is very ample and $2H = K_X + M$. In the embedding given by $H$ we have that $H^i(T_X(2)) = H^i(-K_X + 2H) = H^i(M) = 0$ for $i \geq 0$, hence $T_X(3)$ is Ulrich. □

*Remark* 5.4. If $k \geq 1$ and $g-1$ is a prime number, then $k \in \{2, 4\}$.

*Proof.* By Lemma 4.3(i) we get that $(k-1)d = 3(g-1)$ and $g \geq 2$, hence $d \geq 4$. If 3 does not divide $k-1$ we get that 3 divides $d$ and $(k-1)\frac{d}{3} = g-1$, so that $k=2$. If 3 divides $k-1$ we get that $\frac{k-1}{3}d = g-1$, so that $k=4$. □

*Example* 5.5. Every odd $k \geq 3$ occurs.

*Proof.* Let $E$ be an elliptic curve, let $D$ be a divisor of degree 3 on $E$ and let $S = E \times \mathbb{P}^1$ with two projections $\pi_1 : S \to E, \pi_2 : S \to \mathbb{P}^1$. Set $C_0 = \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Then $H = C_0 + \pi_1^* D$ is very ample on $S$. Let $M \in \mathrm{Pic}^0(E)$ be not 2-torsion and let $B = \frac{k-1}{2}D + M$. Again $H_1 := (k+2)C_0 + \pi_1^* B$ is very ample on $S$. Set

$$\mathcal{L} = -K_S + (k-1)H = (k+1)C_0 + \pi_1^*(2B - 2M)$$

so that

$$\mathcal{L} - H_1 = -C_0 + \pi_1^*(B - 2M)$$

while

$$\mathcal{L} - 2H_1 = -(k+3)C_0 + \pi_1^*(-2M)$$

and it is easily seen by the Künneth formula, that $H^i(\mathcal{L} - pH_1) = 0$ for $i \geq 0, 1 \leq p \leq 2$. Hence, if $X \in |H_1|$ is a smooth irreducible curve, the exact sequence

$$0 \to \mathcal{L} - 2H_1 \to \mathcal{L} - H_1 \to T_X(k-1) \to 0$$

shows that $H^i(T_X(k-1)) = 0$ for $i \geq 0$, that is $T_X(k)$ is an Ulrich line bundle. □

We now give a bound for $k$.

We first analyze a special case. We use the notation $(a; b_1, b_2, b_3, b_4, b_5, b_6) \in \mathbb{Z}^7$ for the divisor $a\varepsilon^* L - \sum_{i=1}^6 b_i E_i$ on a smooth cubic $W \subset \mathbb{P}^3$, where $\varepsilon : W \to \mathbb{P}^2$ is the blow up in six points, no three collinear and not on a conic, with exceptional divisors $E_i$ and $L$ is a line in $\mathbb{P}^2$.

**Lemma 5.6.** *Let $X \subset \mathbb{P}^3$ be a smooth irreducible curve of genus 3 and degree 6 lying on a smooth cubic $W$. Then $T_X(2)$ is Ulrich if and only if $X$ is linearly equivalent to one of the following divisors on $W$:*

$$(4;1,1,1,1,1,1),(5;2,2,2,1,1,1),(6;3,2,2,2,2,1),(7;3,3,3,2,2,2),(8;3,3,3,3,3,3). \tag{5.1}$$

*Proof.* Let $D = -K_W$. We have that $D \cdot X = 6$ and $X^2 = 10$. Thus Riemann-Roch gives that $\chi(2D - X) = 0$. Further, $D(2D - X) = 0$, whence $H^0(2D - X) = 0$, for otherwise $X \sim 2D$ and then $X^2 = 12$, a contradiction. Also, $D(-3D + X) = -3$, whence $H^0(-3D + X) = 0$, which, be Serre's duality, is $H^2(2D - X) = 0$. Thus, also $H^1(2D - X) = 0$. Now the exact sequence

$$0 \to 2D - 2X \to 2D - X \to T_X(1) \to 0$$

gives that

$$h^1(T_X(1)) = h^2(2D - 2X) = h^0(3K_W + 2X)$$

by Serre's duality. Since $\deg T_X(1) = 2$, we deduce by Riemann-Roch, that $T_X(2)$ is Ulrich if and only if $H^1(T_X(1)) = 0$, hence

$$T_X(2) \text{ is Ulrich if and only if } H^0(3K_W + 2X) = 0. \tag{5.2}$$

Let $(a; b_1, \cdots, b_6)$ with $b_1 \geq b_2 \geq \cdots \geq b_6 \geq 0$ be the class of $X$. It follows from the assumption on degree and genus that (A.7) holds, whence $X$ is as in (5.1) by Lemma A.2. Using (5.2), it remains to show that $H^0(3K_W + 2X) = 0$ in all of these cases.

In case $(4;1,1,1,1,1,1)$, we have that $3K_W + 2X = (-1;1,1,1,1,1,1)$ is clearly not effective.

In case $(5;2,2,2,1,1,1)$, we have that $3K_W + 2X = (1;-1,-1,-1,1,1,1)$. If it were effective, then so would be $(1;0,0,0,1,1,1)$, a contradiction since no three blown-up points are collinear.

In case $(6;3,2,2,2,2,1)$, we have that $3K_W + 2X = (3;3,1,1,1,1,-1)$. Assume it is effective. Intersecting with $(1;1,1,0,0,0,0)$ we see that $(2;2,0,1,1,1,-1)$ must be effective. Intersecting the latter with $(1;1,0,1,0,0,0)$ we conclude that $(1;1,0,0,1,1,-1)$ must be effective, hence also $(1;1,0,0,1,1,0)$, a contradiction since no three blown-up points are collinear.

In case $(7;3,3,3,2,2,2)$, we have that $3K_W + 2X = (5;3,3,3,1,1,1)$. Assume it is effective. Intersecting with $(1;1,1,0,0,0,0)$ we see that $(4;2,2,3,1,1,1)$ must be effective. Intersecting the latter with $(1;1,0,1,0,0,0)$ we conclude that $(3;1,2,2,1,1,1)$ must be effective. Finally, intersecting with $(1;0,1,1,0,0,0)$ we conclude that $(2;1,1,1,1,1,1)$ must be effective, a contradiction since the blown-up points do not lie on a conic.

In case $(8;3,3,3,3,3,3)$, we have that $3K_W+2X = (7;3,3,3,3,3,3)$. Observe that $D(3K_W+2X) = 3$. Thus, if $3K_W + 2X$ were effective, it would contain a divisor $\Gamma$ that is either irreducible, or is a union of three lines, or is a union of a line and a conic. Now $p_a(\Gamma) = -3$, hence $\Gamma$ is not irreducible. On the other hand, since the first coefficient of a line in $W$ is at most 2 and of a conic at most 3, we see that $\Gamma$, whose first coefficient is 7, is not a union of three lines nor of a line and a conic. This contradiction shows that $3K_W + 2X$ is not effective. □

Now we give the sharp bound.

*Proof of Theorem 1.* First assume that $g = 0$. Then Lemma 4.3(i) gives that $k \leq 0$, that is the required bound, and if equality holds, then Theorem 4.10 shows that $X$ is a curve of type $(1,2)$ on a smooth quadric.

Thus, by Lemma 4.3(i), we can now assume that $g \geq 2$.

Observe that $h^0(H) \geq 4$. In fact, the only possibility remaining is that $h^0(H) = 3$. But then $K_X = (d-3)H$, $g = \binom{d-1}{2}$ and $k = \frac{3(d-3)}{2} + 1$ by Lemma 4.3(i). Now $0 = H^0(T_X(k-1)) = H^0((-d+2+k)H)$ and therefore $-d + 2 + k \leq -1$, giving the contradiction $d \leq 1$.

Now, if $X$ has general moduli, since it has a $g^3_d$, the Brill-Noether theorem implies that $\rho(g,3,d) \geq 0$, that is $d \geq \frac{3g+12}{4}$. By Lemma 4.3(i) we get that $\frac{3(g-1)}{k-1} \geq \frac{3g+12}{4}$, that gives $k \leq 4$. This proves the last assertion of the theorem.

Turning to the first assertion, let $X \subseteq \mathbb{P}^N$ be a smooth irreducible curve of genus $g \geq 2$ such that $T_X(k)$ is an Ulrich line bundle and assume that

$$k \geq \frac{\sqrt{8g+1} - 1}{2}.$$

Using Lemma 4.3(i), the above inequality can be rephrased as

$$g \geq \frac{2}{9}d^2 - d + 1. \tag{5.3}$$

Consider a general linear projection $X'$ of $X$ to $\mathbb{P}^3$. Note that $X' \cong X$, hence $T_{X'}(k)$ is Ulrich. We first observe that $X'$ cannot be a complete intersection (hence, in particular, $X'$ is nondegenerate), for otherwise $T_{X'}(k) = lH$ for some $l \in \mathbb{Z}$. Now $T_{X'}(k)$, being Ulrich, is globally generated by Lemma 3.2(vi), hence $l \geq 0$. Also $0 = H^0(T_{X'}(k-1)) = H^0((l-1)H)$ and therefore $l = 0$. Hence Lemma 3.2(vii) gives that $d = h^0(T_{X'}(k)) = 1$, a contradiction.

Using Lemma 4.3(i) and Castelnuovo's bound, we get that either $(d, g, k) = (6, 3, 2)$ or $d \geq 7$.

Suppose that $d \geq 7$.

We aim to show that $X'$ must lie on a smooth quadric.

To this end, observe that (5.3) and Lemma 4.3(i) imply that

$$g > \begin{cases} \frac{1}{6}d(d-3)+1 & \text{if } d \equiv 0 \pmod 3 \\ \frac{1}{6}d(d-3)+\frac{1}{3} & \text{if } d \equiv 1, 2 \pmod 3 \end{cases} \tag{5.4}$$

unless $d = 9$ and $g = 10$. But in the latter case it is easy to show that if $X'$ does not lie on a quadric, then it is a complete intersection of two cubics, a contradiction. Therefore (5.4) and [Ha2, Thm. 3.2] give that $X'$ lies on a quadric $Q$. Moreover $Q$ is smooth, for otherwise it must be a cone, $d = 2b+1$ is odd and $g = b^2 - b$ by [Ha1, Ex. V.2.9]. But then Lemma 4.3(i) gives that $4(k-1) = 6b - 9 - \frac{3}{2b+1}$, and therefore $b = 1$, a contradiction.

Thus $X'$ is a curve of type $(a, b)$ on $Q$, with $2 \leq a \leq b$. In particular $X'$ is linearly normal, hence $X = X'$. In the exact sequence

$$0 \to \mathcal{O}_Q(k+1-2a, k+1-2b) \to \mathcal{O}_Q(k+1-a, k+1-b) \to T_X(k-1) \to 0$$

since $H^0(T_X(k-1)) = 0$, we get that

$$H^0(\mathcal{O}_Q(k+1-2a, k+1-2b)) = H^0(\mathcal{O}_Q(k+1-a, k+1-b))$$

hence $k+1-b \leq -1$, for otherwise $k+1-a \geq k+1-b \geq 0$, but then $X$ is a base-component of $|\mathcal{O}_Q(k+1-a, k+1-b)|$, contradicting the fact that this linear system is base-point-free. Therefore $b \geq k+2$. Moreover Lemma 4.3(i) can be rewritten now as

$$(a+b)(k-1) = 3((a-1)(b-1)-1)$$

that is

$$a = \frac{b(k+2)}{3b-k-2}$$

and it is readily seen that $b \geq k+2$ is equivalent to $a \leq \frac{k}{2}+1$. Therefore $b \geq 2a$. But the maximum genus of a curve of type $(a, b)$ with $b \geq 2a$ and degree $d$ is attained when $b = \frac{2}{3}d$. Therefore

$$g \leq (\frac{1}{3}d-1)(\frac{2}{3}d-1) = \frac{2}{9}d^2 - d + 1.$$

This shows that the inequality in (5.3) cannot be strict, and therefore $g \leq \frac{2}{9}d^2 - d + 1$, which is equivalent to (1.1). Moreover, if equality holds in (1.1), then it holds in (5.3) and therefore $X$ is a curve of type $(a, b)$ with $b = \frac{2}{3}d$, hence $b = 2a$ and $2a = b \geq k+2 \geq 2a$, so that $k$ is even, $a = \frac{k}{2}+1$ and $b = k+2$.

Next consider the only remaining case, $(d, g, k) = (6, 3, 2)$.

Again $X'$ is linearly normal, hence $X = X'$. Also we have equality in (5.3) and if $X$ lies on a quadric, then it must be of type $(2, 4)$ and we are done in this case. Suppose therefore that $X$ does not lie on a quadric. Then it is easily seen that $\mathcal{J}_{X/\mathbb{P}^3}(3)$ is 0-regular, hence globally generated, and we get that $X$ is contained in a smooth cubic. Therefore $X$ is one of the curves (5.1) by Lemma 5.6 and $T_X(2)$ is Ulrich.

Finally, to show that the bound (1.1) is sharp for every even $k \geq 0$, let $X$ be a curve of type $(\frac{k}{2}+1, k+2)$ on a smooth quadric $Q \subset \mathbb{P}^3$, so that $k = \frac{\sqrt{8g+1}-1}{2}$. It remains to show that $T_X(k)$ is Ulrich. Set $k = 2c$. We have

$$T_X(k-1) = -K_X + (k-1)H = \mathcal{O}_Q(c, -1)_{|X}$$

and the exact sequence

$$0 \to \mathcal{O}_Q(-1,-2c-3) \to \mathcal{O}_Q(c,-1) \to \mathcal{O}_Q(c,-1)_{|X} \to 0$$

shows that $H^i(\mathcal{O}_Q(c,-1)_{|X}) = 0$ for $i \geq 0$, since $H^i(\mathcal{O}_Q(c,-1)) = H^i(\mathcal{O}_Q(-1,-2c-3)) = 0$ for $i \geq 0$. Hence $T_X(k)$ is Ulrich. □

## 6. Surfaces

Throughout this section we will have that $X \subseteq \mathbb{P}^N$ is a smooth irreducible surface.

We start by a characterization.

**Lemma 6.1.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible surface. Then $T_X(k)$ is an Ulrich vector bundle if and only if*

(i) $d = \frac{4(g-1)}{2k-1}$

(ii) $HK_X = \frac{(2k-3)d}{2}$.

(iii) $K_X^2 = 5\chi(\mathcal{O}_X) + \frac{(k-1)(k-2)d}{2}$.

(iv) $H^0(T_X(k-1)) = 0$.

(v) $H^2(T_X(k-2)) = 0$.

*Proof.* Note that (i) and (ii) are equivalent, since $HK_X = 2(g-1) - d$. Now (ii) and (iii) are the conditions (2.2) in [C1, Prop. 2.1]. Hence the lemma follows by loc. cit. □

Now we show the possible cases.

**Proposition 6.2.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible surface. If $T_X(k)$ is an Ulrich vector bundle, the following hold:*

(i) $0 \leq k \leq 3$.

*Moreover, either*

(ii) $k = 0$ *and* $(X, H) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, *or*

(iii) $k = 1$ *and $X$ is a Del Pezzo surface of degree* 5, *or*

(iv) $k = 2, q = 0$ *and $X$ is a minimal surface of general type, or*

(v) $k = 3$, *$X$ is a minimal surface of general type with* $2K_X \equiv 3H, K_X^2 = \frac{9d}{4}, \chi(\mathcal{O}_X) = \frac{d}{4}$. *Moreover $X$ is a ball quotient.*

*Proof.* We have that $k \geq 0$ by Lemma 4.2(i). Now $H^1(T_X) = 0$ by Lemma 4.2(iv), that is $X$ is infinitesimally rigid and [BC, Thm. 1.3] implies that either $X$ is a minimal surface of general type or $X$ is a Del Pezzo surface of degree $j \geq 5$. In the latter case we have that $HK_X < 0$ hence either $k = 0$ and we get (ii) by Theorem 4.10, or $k = 1$ and $K_X^2 = 5$ by Lemma 6.1(ii),(iii). This gives (iii). On the other hand, if $X$ is a minimal surface of general type then $HK_X > 0$, hence $k \geq 2$ by Lemma 6.1(ii). Next, the Hodge index theorem $H^2K_X^2 \leq (HK_X)^2$ can be rewritten, using Lemma 6.1(ii),(iii) as

$$\chi(\mathcal{O}_X) \leq \frac{(2k^2-6k+5)d}{20}.$$

Similarly, the Bogomolov-Miyaoka-Yau inequality $K_X^2 \leq 9\chi(\mathcal{O}_X)$ can be rewritten as

$$\chi(\mathcal{O}_X) \geq \frac{(k^2-3k+2)d}{8}.$$

Combining we get that

$$\frac{(k^2-3k+2)d}{8} \leq \frac{(2k^2-6k+5)d}{20}$$

and this gives that $k \leq 3$ and moreover that, if $k = 3$, then equality holds in both inequalities. Hence, when $k = 3$ we have, as is well known, that $X$ is a ball quotient and that $H^2K_X \equiv (HK_X)H$, that is $2K_X \equiv 3H$. Then $K_X^2 = \frac{9d}{4}$ and $\chi(\mathcal{O}_X) = \frac{d}{4}$. Thus (i) and (v) are proved. Alternatively (i) follows by Theorem 2. To see (iv) observe that since $k = 2$ we have by the above that $X$ is a minimal surface of general type. Now if $p_g = 0$ then $q = 0$ by [Be2, Lemma VI.1 and Prop. X.1]. If $p_g \neq 0$ we have an inclusion $H^0(\Omega_X^1) \subseteq H^0(\Omega_X^1(K_X))$ hence $q = h^0(\Omega_X^1) \leq h^0(\Omega_X^1(K_X)) = h^2(T_X) = 0$ since $T_X(2)$ is Ulrich. This proves (iv). □

We now characterize the case $k = 1$ for surfaces.

**Theorem 6.3.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible surface. Then $T_X(1)$ is an Ulrich vector bundle if and only if $X$ is a Del Pezzo surface of degree 5 and $H = -2K_X$. Moreover in the latter case $T_X(1)$ is very ample.*

*Proof.* If $T_X(1)$ is an Ulrich vector bundle, then Proposition 6.2 implies that $X$ is a Del Pezzo surface of degree 5 and $H^2 + 2HK_X = 0$. Let $\varepsilon : X \to \mathbb{P}^2$ be the blow-up map, with exceptional divisors $E_i$ over the points $P_i \in \mathbb{P}^2, 1 \le i \le 4$ and let $L$ be a line in $\mathbb{P}^2$. Then we can write

$$H \sim a\varepsilon^* L - \sum_{i=1}^{4} b_i E_i$$

and, as $H$ is very ample, we have, without loss of generality,

$$b_1 \ge b_2 \ge b_3 \ge b_4 \ge 1, a \ge b_1 + b_2 + 1$$

and $H^2 + 2HK_X = 0$ is

$$a^2 - 6a + 4 = \sum_{i=1}^{4} (b_i - 1)^2.$$

Setting $c_i = b_i - 1$, we get by Lemma A.1 the following possibilities:

$$(a; b_1, b_2, b_3, b_4) \in \{(6; 3, 1, 1, 1), (6; 2, 2, 2, 2), (7; 4, 2, 2, 1), (9; 4, 4, 4, 3)\}.$$

In the case $(6; 2, 2, 2, 2)$ we have that $H = -2K_X$. We now exclude the other cases.

Let $H = 6\varepsilon^* L - 3E_1 - E_2 - E_3 - E_4$. We will prove that $h^2(T_X(-1)) = h^0(\Omega^1_X(H + K_X)) \ne 0$, so that $T_X(1)$ cannot be an Ulrich vector bundle. To this end observe that, since $\varepsilon^*\Omega^1_{\mathbb{P}^2} \subset \Omega^1_X$, we will be done in this case if we prove that

$$H^0(\varepsilon^*\Omega^1_{\mathbb{P}^2}(H + K_X)) \ne 0.$$

Now $H + K_X = 3\varepsilon^* L - 2E_1$, hence

$$H^0(\varepsilon^*\Omega^1_{\mathbb{P}^2}(H + K_X)) \cong H^0(\mathcal{I}_Z \otimes \Omega^1_{\mathbb{P}^2}(3))$$

where $Z \subset \mathbb{P}^2$ is the 0-dimensional subscheme of length 2 supported on $P_1$. Finally

$$h^0(\mathcal{I}_Z \otimes \Omega^1_{\mathbb{P}^2}(3)) \ge h^0(\Omega^1_{\mathbb{P}^2}(3)) - 6 = 2 > 0$$

and we are done in this case.

Consider now the exact sequences, for any $1 \le i \le 4$,

$$0 \to \mathcal{O}_{E_i}(-E_i) \to \Omega^1_{X|E_i} \to \Omega^1_{E_i} \to 0$$

that is

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1) \to \Omega^1_{X|E_i} \to \mathcal{O}_{\mathbb{P}^1}(-2) \to 0$$

from which we get, for any $1 \le i \le 4$, that

$$h^1(\Omega^1_{X|E_i}) = 1 \tag{6.1}$$

and

$$H^1(\Omega^1_{X|E_i} \otimes \mathcal{O}_{\mathbb{P}^1}(2)) = 0. \tag{6.2}$$

In the two remaining cases we will prove that $h^1(T_X(-1)) = h^1(\Omega^1_X(H + K_X)) \ne 0$.

Let $H = 7\varepsilon^* L - 4E_1 - 2E_2 - 2E_3 - E_4$.

Note that $H + K_X - E_4 + E_1 = 4\varepsilon^* L - 2E_1 - E_2 - E_3 - E_4$ is very ample by [DR, Cor. 4.6], hence

$$H^2(\Omega^1_X(H + K_X - E_4 + E_1)) = 0$$

by Bott vanishing [T, Thm. 2.1]. Then the exact sequence

$$0 \to \Omega^1_X(H + K_X - E_4) \to \Omega^1_X(H + K_X - E_4 + E_1) \to \Omega^1_{X|E_1}(H + K_X - E_4 + E_1) \to 0$$

and (6.2) imply that $H^2(\Omega^1_X(H + K_X - E_4)) = 0$. Now the exact sequence

$$0 \to \Omega^1_X(H + K_X - E_4) \to \Omega^1_X(H + K_X) \to \Omega^1_{X|E_4}(H + K_X) \to 0$$

and (6.1) imply that

$$h^1(\Omega^1_X(H+K_X)) \ge h^1(\Omega^1_{X|E_4}(H+K_X)) = h^1(\Omega^1_{X|E_4}) = 1.$$

Let $H = 9\varepsilon^* L - 4E_1 - 4E_2 - 4E_3 - 3E_4$.

Let $C \in |\varepsilon^* L - E_2 - E_3|$ be the strict transform of a line through $P_2$ and $P_3$. Note that $H + K_X - C + E_1 = 5\varepsilon^* L - 2E_1 - 2E_2 - 2E_3 - 2E_4$ is very ample by [DR, Cor. 4.6], hence

$$H^2(\Omega^1_X(H+K_X-C+E_1)) = 0$$

by Bott vanishing [T, Thm. 2.1]. Then the exact sequence

$$0 \to \Omega^1_X(H+K_X-C) \to \Omega^1_X(H+K_X-C+E_1) \to \Omega^1_{X|E_1}(H+K_X-C+E_1) \to 0$$

and (6.2) imply that $H^2(\Omega^1_X(H+K_X-C)) = 0$. Now the exact sequence

$$0 \to \mathcal{O}_C(-C) \to \Omega^1_{X|C} \to \Omega^1_C \to 0$$

that is

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1) \to \Omega^1_{X|C} \to \mathcal{O}_{\mathbb{P}^1}(-2) \to 0$$

gives that $h^1(\Omega^1_{X|C}) = 1$. Finally, from the exact sequence

$$0 \to \Omega^1_X(H+K_X-C) \to \Omega^1_X(H+K_X) \to \Omega^1_{X|C}(H+K_X) \to 0$$

using that $(H+K_X)C = 0$, we get that

$$h^1(\Omega^1_X(H+K_X)) \ge h^1(\Omega^1_{X|C}(H+K_X)) = h^1(\Omega^1_{X|C}) = 1.$$

This completes the proof under the assumption that $T_X(1)$ is an Ulrich vector bundle.

Suppose now that $X$ is a Del Pezzo surface of degree 5 and $H = -2K_X$. Setting $k = 1$ in Lemma 6.1, we have that $d = 4(g-1)$ and, in order to verify that $T_X(1)$ is an Ulrich vector bundle, we need to check that $H^0(T_X) = H^2(T_X(-1)) = 0$. The first vanishing is well known. As for the second, we first observe that for $i < 2$ we have

$$h^i(T_X(-1)) = h^{2-i}(\Omega^1_X(H+K_X)) = h^{2-i}(\Omega^1_X(-K_X)) = 0$$

by Bott vanishing [T, Thm. 2.1]. Therefore $h^2(T_X(-1)) = \chi(T_X(-1)) = d - 4(g-1) = 0$. Finally, as $X$ does not contain lines in the embedding given by $H = -2K_X$, we have that $T_X(1)$ is very ample by [LS, Thm. 1] □

## 7. Properties of complete intersections

We collect some properties inherited by the complete intersections $X_i$ of $X$ (as in Notation 2.1), when $T_X(k)$ is an Ulrich vector bundle.

**Lemma 7.1.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. Then $q(X) = q(X_i)$ for $2 \le i \le n$.*

*Proof.* By Kodaira vanishing we have that $H^1(\mathcal{O}_{X_{i+1}}(-1)) = H^2(\mathcal{O}_{X_{i+1}}(-1)) = 0$ as long as $2 \le i \le n-1$. Then the exact sequences

$$0 \to \mathcal{O}_{X_{i+1}}(-1) \to \mathcal{O}_{X_{i+1}} \to \mathcal{O}_{X_i} \to 0$$

imply that $h^1(\mathcal{O}_{X_{i+1}}) = h^1(\mathcal{O}_{X_i})$ for every $2 \le i \le n-1$, hence $q(X) = q(X_i)$ for $2 \le i \le n$. □

**Lemma 7.2.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. Suppose that $k \le n-2$ and that $T_X(k)$ is Ulrich. Then $H^i(\mathcal{O}_{X_i}) = 0$ for all $i$ such that $\max\{1, k+1\} \le i \le n-1$.*

*Proof.* Assume that $\max\{1, k+1\} \le i \le n-1$. Since $T_{X|X_i}(k)$ is Ulrich, it follows by Lemma 3.2(iv) that $H^i(T_{X|X_i}(k+m)) = 0$ for all $m \ge -i$, hence $H^i(T_{X|X_i}(-1)) = 0$. Now the exact sequence

$$0 \to T_{X_i}(-1) \to T_{X|X_i}(-1) \to \mathcal{O}_{X_i}^{\oplus(n-i)} \to 0$$

implies that $H^i(\mathcal{O}_{X_i}) = 0$. □

**Lemma 7.3.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. Suppose that $k \le n-1$ and that $T_X(k)$ is Ulrich. Assume that $H^i(\mathcal{O}_X)=0$ for all $i \ge 1$. Then $H^i(\mathcal{O}_{X_j})=0$ for all $i \ge 1$ and for all $j$ such that* $\max\{1,k+1\} \le j \le n$.

*Proof.* Assume that $i \ge 1$ and $\max\{1,k+1\} \le j \le n$. We prove the lemma by induction on $n-j \ge 0$.

If $n-j=0$ then $X_j = X_n = X$ and $H^i(\mathcal{O}_X)=0$ for all $i \ge 1$ just by our assumption.

Next suppose that $n-j \ge 1$, so that $\max\{1,k+1\} \le j \le n-1$, hence, in particular $k \le n-2$. Consider the exact sequence

$$0 \to \mathcal{O}_{X_{j+1}}(-1) \to \mathcal{O}_{X_{j+1}} \to \mathcal{O}_{X_j} \to 0.$$

If $j=i$, we have that $H^j(\mathcal{O}_{X_j})=0$ by Lemma 7.2. Also, we have by induction that $H^i(\mathcal{O}_{X_{j+1}})=0$. Now $H^{i+1}(\mathcal{O}_{X_{j+1}}(-1))=0$ by Kodaira vanishing if $i+1<j+1$ and by dimension reasons if $i+1>j+1$. Thus $H^i(\mathcal{O}_{X_j})=0$ if $i \ne j$ and we are done. □

We now collect some properties of the $X_i$'s that hold when $T_X(1)$ is Ulrich.

**Lemma 7.4.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 2$. Suppose that $T_X(1)$ is an Ulrich vector bundle. Then:*

(i) $H^1(\mathcal{O}_{X_i})=0$ *for* $2 \le i \le n$.
(ii) $H^2(\mathcal{O}_{X_i})=0$ *for* $1 \le i \le n$.
(iii) $H^1(\mathcal{O}_{X_i}(1))=0$ *for* $1 \le i \le n$.
(iv) $H^2(\mathcal{O}_{X_i}(1))=0$ *for* $1 \le i \le n$.
(v) $h^0(\mathcal{O}_{X_i}(1))=d-g+i$ *for* $1 \le i \le n$.
(vi) $d \ge n+3$.

*Proof.* We have $H^i(\mathcal{O}_X)=0$ for $i \ge 1$ by Lemma 4.3(iii). Now (i) follows by Lemma 7.1 and (ii) follows by Lemma 7.3. To see (iii) observe that, if $i=1$ we have that $X_1=C$ and $T_X(1)_{|C}$ is an Ulrich vector bundle on $C$ by Lemma 3.2(ix), hence $H^1(T_{X|C})=0$. Then the exact sequence

$$0 \to T_C \to T_{X|C} \to \mathcal{O}_C(1)^{\oplus(n-1)} \to 0$$

shows that $H^1(\mathcal{O}_C(1))=0$. If $i \ge 2$, since $H^1(\mathcal{O}_{X_i})=0$ by (i), the exact sequences

$$0 \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_i}(1) \to \mathcal{O}_{X_{i-1}}(1) \to 0 \tag{7.1}$$

imply by induction that $H^1(\mathcal{O}_{X_i}(1))=0$ and we get (iii). Now (iv) is obvious for $i=1$, while, for $i \ge 2$, the exact sequences (7.1) and (ii) show by induction that $H^2(\mathcal{O}_{X_i}(1))=0$. This proves (iv). Note that (v) follows for $i=1$ by Riemann-Roch and (iii). For $i \ge 2$, the exact sequences (7.1) and (i) show by induction that $h^0(\mathcal{O}_{X_i}(1))=1+h^0(\mathcal{O}_{X_{i-1}}(1))=d-g+i$, that is (v). Finally, to see (vi), observe that $g-1=\frac{n-1}{n+2}d$ by Lemma 4.3(i), hence $g \ge 2$ and (v) gives that $\frac{3d}{n+2}=h^0(\mathcal{O}_C(1)) \ge 3$, so that $d \ge n+2$. Moreover, if equality holds, we get that $g=n$ and $h^0(\mathcal{O}_X(1))=n+2$ by (v), hence $X \subset \mathbb{P}H^0(H)=\mathbb{P}^{n+1}$ is a hypersurface of degree $n+2$, so that $K_X=0$, contradicting Lemma 4.3(ii). Hence (vi) is proved. □

## 8. $T_X(k)$ Ulrich and special varieties in adjunction theory

In this section we exclude some special varieties frequently arising in adjunction theory, under the hypothesis that $T_X(k)$ is an Ulrich vector bundle. The cases $(X,H)=(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)),(Q_n,\mathcal{O}_{Q_n}(1))$ have been already treated in Lemmas 4.1 and 4.5.

We start by recalling the following (see [BS, I]).

**Definition 8.1.** Let $E$ be an effective divisor on $(X,H)$. The divisor $E$ is called *exceptional*

(i) *of type 1* if $(E,H_{|E}) \cong (\mathbb{P}^{n-1},\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and $N_{E/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$,
(ii) *of type 2* if $(E,H_{|E}) \cong (\mathbb{P}^{n-1},\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and $N_{E/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-2)$,
(iii) *of type 3* if $(E,H_{|E}) \cong (Q_{n-1},\mathcal{O}_{Q_{n-1}}(1))$ and $N_{E/X} \cong \mathcal{O}_{Q_{n-1}}(-1)$,
(iv) *of type 4* if $(E,H_{|E})$ is a linear $\mathbb{P}^{n-2}$-bundle over a smooth curve $B$ and $(N_{E/X})_{|F} \cong \mathcal{O}_{\mathbb{P}^{n-2}}(-1)$, where $F$ is a fiber of the structure morphism $E \to B$.

Often these exceptional divisors will not be present under the condition that $T_X(k)$ is Ulrich. To see this we first prove

**Lemma 8.2.** *Let $W$ be a variety of dimension $s \ge 1$ and let $\mathcal{O}_W(1)$ be a very ample line bundle. Then $\Omega^1_W(1)$ is not globally generated if:*

(i) $(W, \mathcal{O}_W(1)) \cong (\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1))$.

(ii) *$(W, \mathcal{O}_W(1))$ is a (possibly singular) quadric hypersurface in $\mathbb{P}^{s+1}$ and $s \ge 2$.*

(iii) *$(W, \mathcal{O}_W(1))$ is a smooth Del Pezzo variety, $s \ge 2$ and $(W, \mathcal{O}_W(1)) \notin \{(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)), (Q_2, \mathcal{O}_{Q_2}(2)), (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\}$.*

*Proof.* (i) follows from $\det(\Omega^1_{\mathbb{P}^s}(1)) = \mathcal{O}_{\mathbb{P}^s}(-1)$. To see (ii), observe that the restricted Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^{s+1}|W}(1) \to H^0(\mathcal{O}_W(1)) \otimes \mathcal{O}_W \to \mathcal{O}_W(1) \to 0$$

implies that $H^0(\Omega^1_{\mathbb{P}^{s+1}|W}(1)) = 0$. Now the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{s+1}}(-3) \to \mathcal{O}_{\mathbb{P}^{s+1}}(-1) \to \mathcal{O}_W(-1) \to 0$$

implies that $H^1(\mathcal{O}_W(-1)) = 0$ and the dual normal bundle sequence

$$0 \to \mathcal{O}_W(-1) \to \Omega^1_{\mathbb{P}^{s+1}|W}(1) \to \Omega^1_W(1) \to 0$$

gives that $H^0(\Omega^1_W(1)) = 0$, hence $\Omega^1_W(1)$ is not globally generated. Next, to see (iii), observe that, from the classification of Del Pezzo varieties [IP, Thm. 3.3.1] it follows, for the surface section $W_2$, that $(W_2, \mathcal{O}_{W_2}(1)) \notin \{(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)), (Q_2, \mathcal{O}_{Q_2}(2))\}$. Hence, as is well known, $W_2$, and hence $W$, contains a line $L$. But now the surjection $\Omega^1_W(1) \to \Omega^1_L(1) = \mathcal{O}_{\mathbb{P}^1}(-1)$ gives that $\Omega^1_W(1)$ is not globally generated. □

Now

**Lemma 8.3.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. Assume that $T_X(k)$ is an Ulrich vector bundle. We have:*

(i) *If $k \ge 1$ and $n \ge 2$, then $(X, H)$ does not contain any exceptional divisor of type 1.*

(ii) *If $k \ge 2$ and $n \ge 2$, then $(X, H)$ does not contain any exceptional divisor of type 2.*

(iii) *If $k \ge 2$ and $n \ge 3$, then $(X, H)$ does not contain any exceptional divisors of types 3 or 4.*

*Proof.* Let $E$ be an exceptional divisor. It follows from Lemma 4.6 that $\Omega^1_E(K_{X|E} + (n+1-k)H_{|E})$ is globally generated. Now

$$\Omega^1_E(K_{X|E} + (n+1-k)H_{|E}) \cong \begin{cases} \Omega^1_{\mathbb{P}^{n-1}}(2-k) & \text{if } E \text{ is of type 1;} \\ \Omega^1_{\mathbb{P}^{n-1}}(3-k) & \text{if } E \text{ is of type 2;} \\ \Omega^1_{Q_{n-1}}(3-k) & \text{if } E \text{ is of type 3.} \end{cases}$$

Further, when $E$ is of type 4, let $F$ be a fiber of the structure morphism of $E$. Again it follows from Lemma 4.6 that $\Omega^1_F(K_{X|F} + (n+1-k)H_{|F}) \cong \Omega^1_{\mathbb{P}^{n-2}}(3-k)$ is globally generated. Consequently, we draw the conclusions from Lemma 8.2. □

We now recall

**Definition 8.4.** We say that $(X, H)$ is a *linear $\mathbb{P}^k$-bundle* over a smooth variety $B$ if $(X, H) \cong (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$, where $\mathcal{F}$ is a very ample vector bundle on $B$ of rank $k+1$.

We say that $(X, H)$ as above is a *scroll (respectively a quadric fibration; respectively a Del Pezzo fibration) over a normal variety $Y$ of dimension $m$* if there exists a surjective morphism with connected fibers $\phi : X \to Y$ such that $K_X + (n-m+1)H = \phi^*\mathcal{L}$ (respectively $K_X + (n-m)H = \phi^*\mathcal{L}$; respectively $K_X + (n-m-1)H = \phi^*\mathcal{L}$), with $\mathcal{L}$ ample on $Y$.

We now use the fibration to exclude several varieties as above, when $T_X(k)$ is an Ulrich vector bundle.

**Lemma 8.5.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 1$. Assume that $T_X(k)$ is an Ulrich vector bundle. Let $f : X \to B$ be a fibration onto a normal variety $B$ of dimension $m \ge 1$, with general fiber $F$. Then:*

(i) *If $m \le \min\{n-1, k+1\}$, then $(F, H_{|F}) \ne (\mathbb{P}^{n-m}, \mathcal{O}_{\mathbb{P}^{n-m}}(1))$.*

(ii) *If $m \le \min\{n-2, k\}$, then $(F, H_{|F}) \ne (Q_{n-m}, \mathcal{O}_{Q_{n-m}}(1))$.*

(iii) *if $m \le \min\{n-2, k-1\}$, then $(F, H_{|F})$ is not a Del Pezzo variety, unless $(F, H_{|F}) \in \{(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)), (Q_2, \mathcal{O}_{Q_2}(2)), (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\}$.*

*Proof.* We have that

$$\Omega^1_F(K_F+(n+1-k)H_{|F}) \cong \begin{cases} \Omega^1_{\mathbb{P}^{n-m}}(m-k) & \text{if } (F,H_{|F})=(\mathbb{P}^{n-m},\mathcal{O}_{\mathbb{P}^{n-m}}(1)); \\ \Omega^1_{Q_{n-m}}(m-k+1) & \text{if } (F,H_{|F})=(Q_{n-m},\mathcal{O}_{Q_{n-m}}(1)); \\ \Omega^1_F(m-k+2) & \text{if } (F,H_{|F}) \text{ is a Del Pezzo variety.} \end{cases}$$

Now $\Omega^1_F(K_F+(n+1-k)H_{|F})$ is globally generated by Lemma 4.6. Hence, Lemma 8.2 gives that, in each of the three cases, the inequality in $m,n,k$ is not satisfied. □

We get a very useful consequence.

**Lemma 8.6.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 2$. If $T_X(k)$ is an Ulrich vector bundle, then $K_X+(n-1)H$ is nef and $H^0(K_X+(n-1)H) \ne 0$, unless $(X,H,k)=(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(2),0)$ (the latter case actually occurs, see Theorem 4.10).*

*Proof.* Recall that $H^0(K_X+(n-1)H) \ne 0$ if and only if $K_X+(n-1)H$ is nef by [BS, Cor. 7.2.8]. Now if $K_X+(n-1)H$ is not nef, it follows by [BS, Prop.'s 7.2.2, 7.2.3 and 7.2.4] that $(X,H)$ is either $(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)),(Q_n,\mathcal{O}_{Q_n}(1))$, a linear $\mathbb{P}^{n-1}$-bundle over a smooth curve or $(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(2))$. The first three cases are excluded by Lemmas 4.1, 4.5 and 8.5(i), while in the fourth case we have $g=0$, hence $k=0$ by Lemma 4.3(i). □

We can now prove Theorem 4.10.

*Proof of Theorem 4.10.* The assert is clear if either $(X,H)=(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(3))$ or $(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(2))$. Vice versa assume that $T_X$ is Ulrich for $H$. If $n=1$, since $T_X=-K_X$ is globally generated by Lemma 3.2(vi), we have that $X$ is either $\mathbb{P}^1$ or an elliptic curve. Now the latter is excluded by Lemma 4.3(i), while in the former case $T_X=\mathcal{O}_{\mathbb{P}^1}(2)$ Ulrich implies that $H=\mathcal{O}_{\mathbb{P}^1}(3)$. Now assume that $n \ge 2$. Then Lemma 8.6 gives that either $(X,H)=(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(2))$, or $K_X+(n-1)H$ is nef, leading, by Lemma 4.3(ii), to the contradiction

$$0 \le (K_X+(n-1)H)H^{n-1} = -\frac{2d}{n+2}.$$

□

The following result will also be useful.

**Lemma 8.7.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 2$. Suppose that $k \ge 1$ and that $T_X(k)$ is an Ulrich vector bundle. Assume that $X \cong \mathbb{P}(\mathcal{F})$ is a projective bundle over a normal projective variety $B$ of dimension $1 \le m \le n-1$. Then $B$ is smooth and $\mathcal{F}$ is simple. In particular, if $m=1$, then $q(X) \ne 0$.*

*Proof.* Let $\pi : X \cong \mathbb{P}(\mathcal{F}) \to B$ be the structure morphism and let $\xi$ be the tautological bundle of $\mathbb{P}(\mathcal{F})$. By twisting $\mathcal{F}$ with a sufficiently ample line bundle we can assume that $\xi$ is ample. Then [BS, Prop. 3.2.1] implies that $B$ is smooth. Since $H^0(T_X)=0$, the cohomology of the exact sequence

$$0 \to T_{X/B} \to T_X \to \pi^*T_B \to 0$$

gives that $H^0(T_{X/B})=0$. Now the cohomology of the exact sequence

$$0 \to \mathcal{O}_X \to \pi^*\mathcal{F}^* \otimes \xi \to T_{X/B} \to 0$$

implies that

$$h^0(\mathcal{F} \otimes \mathcal{F}^*) = h^0(\pi^*\mathcal{F}^* \otimes \xi) = h^0(\mathcal{O}_X) = 1.$$

Now if $m=1$ and $q(X)=0$ we have that $B \cong \mathbb{P}^1$, hence $\mathcal{F}$ cannot be simple since $\operatorname{rk}\mathcal{F}=n \ge 2$. □

Next we prove three results for $k=2$.

For the first one, in order to apply the results of T. Fujita in [F1], we give the following definition, that coincides with the one in [F1] when $B$ is smooth.

**Definition 8.8.** Let $f : X \to B$ be a fibration over a curve, $L$ an ample line bundle on $X$ such that on the general fiber $F$ we have that $K_F=-(n-2)L_{|F}$. We say that $f$ is *minimal* if there is a line bundle $\mathcal{L}$ on $B$ such that $K_X+(n-2)L=f^*\mathcal{L}$.

Then we have

**Lemma 8.9.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 4$. Suppose that $k \ge 2$ and that $k = 2$ if $n = 4$. Moreover assume that $T_X(k)$ is an Ulrich vector bundle. Then:*

(i) *$(X, H)$ is not a Del Pezzo fibration over a smooth curve.*
(ii) *If $n = 4$ and $K_X + 2H$ is ample, then $(X, K_X + 2H)$ is not a minimal $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$-fibration over a smooth curve.*

*Proof.* For the sake of contradiction, let $L$ be $H$ in case (i) and $K_X + 2H$ in case (ii). Assume that we have a fibration $f : X \to B$ over a smooth curve $B$ such that $(X, L)$ is a Del Pezzo fibration in case (i) (see Definition 8.4) and $(X, L)$ is a minimal $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$-fibration in case (ii). Note that $f$ is minimal also in case (i) by Definition 8.4.

Let $F$ be a general fiber of $f$. In case (i) we have that $F$ is a smooth variety of dimension $n - 1$ and $K_F = K_{X|F} = -(n-2)L_{|F}$, hence $F$ is a Del Pezzo variety. Since $L = H$, Lemma 8.5 implies that $(F, H_{|F}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$, hence $n = 4$. Thus $(F, L_{|F})$ is the same in both cases.

We now claim that every fiber of $f$ is irreducible. Indeed, if not, let $F_0$ be a reducible fiber. Since $F_0$ is connected, it must be singular, hence we can apply [F1, Table (2.20)]. It follows that we are in case (2.17) of [F1, Table (2.20)], the degree of $F_0$ is 8 and, if $D$ is an irreducible component of $F_0$, then $(D, L_{|D})$ is a scroll over $\mathbb{P}^1$ and $K_{X|D} = -2L_{|D}$. Denoting a fiber of the structure morphism $D \to \mathbb{P}^1$ by $F' \cong \mathbb{P}^2$, we obtain $L_{|F'} = \mathcal{O}_{\mathbb{P}^2}(1)$ and $K_{X|F'} = -2L_{|F'} = \mathcal{O}_{\mathbb{P}^2}(-2)$. Set $H_{|F'} = \mathcal{O}_{\mathbb{P}^2}(a)$. In case (ii) we have that

$$\mathcal{O}_{\mathbb{P}^2}(1) = L_{|F'} = (K_X + 2H)_{|F'} = \mathcal{O}_{\mathbb{P}^2}(-2 + 2a)$$

a contradiction. In case (i) we have that $L = H$ and $a = 1$. But now Lemma 4.6 gives that $\Omega^1_{\mathbb{P}^2}(K_{X|F'} + 3H_{|F'}) \cong \Omega^1_{\mathbb{P}^2}(1)$ is globally generated, contradicting Lemma 8.2(i). Thus every fiber of $f$ is irreducible.

Now [F1, (4.8)] implies that every fiber of $f$ is $\mathbb{P}^3$. Since $B$ is a smooth curve, it follows, as is well known, that $X$ is a projective bundle over $B$. On the other hand, since $n = 4$, we have that $k = 2$ and $q(X) = 0$ by Lemma 4.3(iii), contradicting Lemma 8.7. □

**Lemma 8.10.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension 4. Suppose that $K_X + 2H$ is ample and that $T_X(2)$ is an Ulrich vector bundle. Then $(X, K_X + 2H)$ is not a quadric fibration over a smooth curve.*

*Proof.* Suppose that $(X, K_X + 2H)$ is a quadric fibration $\pi : X \to B$ over a smooth curve. Since $\chi(\mathcal{O}_X) = 1$ by Lemma 4.3(iii), it follows from [Lan, Sect. 1.1, eq. (8)] that $B \cong \mathbb{P}^1$. Moreover, [Lan, Sect. 0.1] gives that if $\pi_*(K_X + 2H) \cong \bigoplus_{i=0}^{4} \mathcal{O}_{\mathbb{P}^1}(a_i)$ and $e = \sum_{i=0}^4 a_i$, then there is $b \in \mathbb{Z}$ such that

$$(K_X + 2H)^4 = 2e - b$$

by [Lan, Sect. 1.1, eq. (3)] and

$$K_X^i (K_X + 2H)^{4-i} = (-3)^i 2e + (-3)^{i-1}(-4i + 2ie + (3 - 2i)b) \text{ for } 1 \le i \le 4$$

by [Lan, Sect. 1.1, eq. (4)]. Solving these five equations we obtain

$$K_X H^3 = 4e - 28b - 104 \text{ and } d = H^4 = 16b + 64$$

and therefore Lemma 4.3(iii) gives

$$13d = 48(2 + e). \tag{8.1}$$

Since $T_X(2)$ is Ulrich we have that $H^4(T_X(-2)) = 0$ and the exact sequence

$$0 \to T_{X/\mathbb{P}^1}(-2) \to T_X(-2) \to (\pi^* T_{\mathbb{P}^1})(-2) \to 0$$

implies that $H^4((\pi^* T_{\mathbb{P}^1})(-2)) = 0$. Hence, by Serre duality

$$0 = h^4((\pi^* T_{\mathbb{P}^1})(-2)) = h^0((\pi^* \mathcal{O}_{\mathbb{P}^1}(-2))(K_X + 2H)) = h^0(\pi_*(K_X + 2H) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = \sum_{i=0}^{4} h^0(\mathcal{O}_{\mathbb{P}^1}(a_i - 2))$$

and therefore $a_i \le 1$ for $0 \le i \le 4$. But then $e \le 5$ and (8.1) gives that $1 \le e + 2 \le 7$ is divisible by 13, a contradiction. □

**Lemma 8.11.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension 4. Suppose that $K_X + 2H$ is ample and that $T_X(2)$ is an Ulrich vector bundle. Then $(X, K_X + 2H)$ is not a linear $\mathbb{P}^2$-bundle over a smooth surface.*

*Proof.* Assume by contradiction that we have a $\mathbb{P}^2$-bundle structure $\pi : X \cong \mathbb{P}(\mathcal{F}) \to B$ onto a smooth surface $B$, with $K_X + 2H = \xi$, the tautological bundle, where $\mathcal{F}$ is a rank 3 vector bundle on $B$. Then $H = a\xi - \pi^*M$ for some $a \in \mathbb{Z}$ and $M \in \operatorname{Pic}(B)$, so that

$$\xi = K_X + 2H = (2a-3)\xi + \pi^*(K_B + c_1(\mathcal{F}) - 2M)$$

giving $a = 2$ and $2M = K_B + c_1(\mathcal{F})$, thus

$$H \equiv 2\xi - \frac{1}{2}\pi^*(K_B + c_1(\mathcal{F})). \tag{8.2}$$

We will also use Grothendieck's relation $\sum_{j=0}^{3}(-1)^j \xi^{3-j}\pi^* c_j(\mathcal{F}) = 0$, that is

$$\xi^3 = \xi^2\pi^*c_1(\mathcal{F}) - \xi\pi^*c_2(\mathcal{F}). \tag{8.3}$$

Since $\xi^2 f = 1$ for every fiber $f$ of $\pi$, we get from (8.3) that

$$\xi^3\pi^*c_1(\mathcal{F}) = c_1(\mathcal{F})^2, \xi^3\pi^*K_B = K_Bc_1(F) \text{ and } \xi^4 = c_1(\mathcal{F})^2 - c_2(\mathcal{F}). \tag{8.4}$$

We first collect some invariants of $X$ and $B$.

**Claim 8.12.** *We have:*

(i) $K_XH^3 = -\frac{2}{3}d$.
(ii) $\chi(\mathcal{O}_X) = 1$.
(iii) $\chi(\mathcal{O}_X(H)) = 2 + \chi(\mathcal{O}_S) - \frac{d}{6}$.
(iv) $h^0(K_X + 2H) = \chi(\mathcal{O}_S) - 1$.
(v) $\chi(\mathcal{O}_B) = 1$.

*Proof.* (i) is obtained by Lemma 4.3(ii). Now Lemma 4.3(iii) gives that $H^i(\mathcal{O}_X) = 0$ for $i \geq 1$, hence $H^i(\mathcal{O}_B) = 0$ for $i \geq 1$, giving (ii) and (v). Next, to see (iii), consider the exact sequences

$$0 \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_i}(H) \to \mathcal{O}_{X_{i-1}}(H) \to 0$$

for $i = 4, 3$. They give $\chi(\mathcal{O}_X(H)) = 1 + \chi(\mathcal{O}_{X_3}(H)) = 2 + \chi(\mathcal{O}_S(H))$ and (iii) follows by Riemann-Roch since $H^2_{|S} = d$ and $H_{|S}K_S = (K_X + 2H)H^3 = \frac{4}{3}d$ by (i). To see (iv), observe that, since $R^j\pi_*(-\xi) = 0$ for every $j \geq 0$, we have that $H^i(K_X + H) = H^i(-\xi + \pi^*(K_B + c_1(\mathcal{F}) - M)) = 0$ for every $i$. Hence the exact sequence

$$0 \to K_X + H \to K_X + 2H \to K_{X_3} + H_{|X_3} \to 0$$

implies that

$$h^0(K_X + 2H) = h^0(K_{X_3} + H_{|X_3}). \tag{8.5}$$

Now we have $q(S) = 0$ by Lemma 7.1 and $H^i(K_{X_3}) = 0$ for $i = 0, 1$ by Serre duality and Lemma 7.3. Hence the exact sequence

$$0 \to K_{X_3} \to K_{X_3} + H_{|X_3} \to K_S \to 0$$

shows that

$$\chi(\mathcal{O}_S) - 1 = p_g(S) = h^0(K_S) = h^0(K_{X_3} + H_{|X_3})$$

and we get (iv) by (8.5). □

We continue the proof of the lemma.

Next, we collect some relations among the invariants related to $\xi, K_B$ and the Chern classes of $\mathcal{F}$.

**Claim 8.13.** *The following identities hold:*

(i) $d - 6c_1(\mathcal{F})^2 + 16c_2(\mathcal{F}) - 6K_B^2 + 4K_Bc_1(\mathcal{F}) = 0$.
(ii) $K_Bc_1(\mathcal{F}) - 8 + 2\chi(\mathcal{O}_S) - c_1(\mathcal{F})^2 + 2c_2(\mathcal{F}) = 0$.
(iii) $K_B^2 - 1 + c_1(\mathcal{F})^2 - 3c_2(\mathcal{F}) = 0$.
(iv) $3K_B^2 + c_1(\mathcal{F})^2 - 2c_2(\mathcal{F}) - 30 = 0$.
(v) $4 - K_Bc_1(\mathcal{F}) + \frac{9}{4}K_B^2 + \frac{7}{4}c_1(\mathcal{F})^2 - 5c_2(\mathcal{F}) - \chi(\mathcal{O}_S) + \frac{1}{6}d = 0$.

*Proof.* We have by (8.2) and (8.4) that

$$d = H^4 = (2\xi - \frac{1}{2}\pi^*(K_B + c_1(\mathcal{F})))^4 = 16(c_1(\mathcal{F})^2 - c_2(\mathcal{F})) + 6K_B^2 - 4K_Bc_1(\mathcal{F}) - 10c_1(\mathcal{F})^2$$

that is (i). To see (ii) observe that, since $\pi_*\xi = \mathcal{F}$ and $R^j\pi_*\xi = 0$ for $j > 0$ we have $H^i(\mathcal{F}) = H^i(\xi) = H^i(K_X + 2H) = 0$ for $i > 0$ by Kodaira vanishing. Also, by Claim 8.12(iv), (v) and Riemann-Roch we get

$$\chi(\mathcal{O}_S) - 1 = h^0(K_X + 2H) = h^0(\xi) = h^0(\mathcal{F}) = \chi(\mathcal{F}) = 3 - \frac{1}{2}K_Bc_1(\mathcal{F}) + \frac{1}{2}c_1(\mathcal{F})^2 - c_2(\mathcal{F})$$

that is (ii). Next, consider the exact sequences

$$0 \to T_{X/B} \to T_X \to \pi^*T_B \to 0 \tag{8.6}$$

and

$$0 \to \mathcal{O}_X \to \pi^*\mathcal{F}^*(\xi) \to T_{X/B} \to 0.$$

Since $T_X(2H)$ is Ulrich, we have that $\chi(T_X) = 0$, hence, using Claim 8.12(ii) we get

$$\chi(T_B) = \chi(\pi^*T_B) = -\chi(T_{X/B}) = -\chi(\pi^*\mathcal{F}^*(\xi)) + 1 = -\chi(\mathcal{F} \otimes \mathcal{F}^*) + 1. \tag{8.7}$$

On the other hand, by Riemann-Roch and Claim 8.12(v), $\chi(T_B) = 2K_B^2 - 10$ and $\chi(\mathcal{F} \otimes \mathcal{F}^*) = 9 + 2c_1(\mathcal{F})^2 - 6c_2(\mathcal{F})$. Replacing in (8.7) gives (iii).

Finally, to see (iv), we first compute $c_1(S^2(\mathcal{F})) = 4c_1(\mathcal{F})$ and $c_2(S^2(\mathcal{F})) = 5c_1(\mathcal{F})^2 + 5c_2(\mathcal{F})$, so that

$$c_1(S^2(\mathcal{F})(-M)) = -3K_B + c_1(\mathcal{F}), c_2(S^2(\mathcal{F})(-M)) = \frac{15}{4}K_B^2 - \frac{5}{2}K_Bc_1(\mathcal{F}) - \frac{5}{4}c_1(\mathcal{F})^2 + 5c_2(\mathcal{F}).$$

Now Riemann-Roch gives

$$\chi(S^2(\mathcal{F})(-M)) = 6 - K_Bc_1(\mathcal{F}) + \frac{9}{4}K_B^2 + \frac{7}{4}c_1(\mathcal{F})^2 - 5c_2(\mathcal{F}). \tag{8.8}$$

On the other hand, $\chi(S^2(\mathcal{F})(-M)) = \chi(2\xi - \pi^*M) = \chi(\mathcal{O}_X(H)) = 2 + \chi(\mathcal{O}_S) - \frac{d}{6}$ by Claim 8.12(iii). Using (8.8) we get (v). □

We now conclude the proof of the lemma.

Solving the five equations in Claim 8.13 we get

$$K_B^2 = -\frac{7}{48}d + 7 \text{ and } K_Bc_1(F) = -\frac{5}{48}d + 9. \tag{8.9}$$

In particular $d \geq 48$. On the other hand, using Claim 8.12(i), we get

$$\mu(T_X) = \frac{-K_XH^3}{4} = \frac{1}{6}d$$

and, using (8.2)

$$\mu(\pi^*T_B) = \frac{c_1(\pi^*T_B)H^3}{2} = \frac{-\pi^*K_B\left[2\xi - \frac{1}{2}\pi^*(K_B + c_1(\mathcal{F}))\right]^3}{2} = -\frac{8\xi^3\pi^*K_B - 6\xi^2\pi^*(K_B^2 + K_Bc_1(\mathcal{F}))}{2}.$$

Now (8.4) and (8.9) give

$$\mu(\pi^*T_B) = -K_Bc_1(F) + 3K_B^2 = -\frac{1}{3}d + 12.$$

Since $T_X$ is semistable by Lemma 4.3(v), we deduce by (8.6) that

$$\frac{1}{6}d \leq -\frac{1}{3}d + 12$$

that is $d \leq 24$, a contradiction. □

## 9. $T_X(1)$ Ulrich in any dimension

We study the case $k=1$ in any dimension. We start analyzing the properties of the curve section $C$ and of the surface section $S$.

**Lemma 9.1.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 3$. If $T_X(1)$ is an Ulrich vector bundle, then $d \ge 9$ except, possibly, when $d=8, g=5, n=4$ and $h^0(\mathcal{O}_C(1))=4$.*

*Proof.* By Lemma 4.3(i) we know that $g \ge 2$ and that

$$(n-1)d = (n+2)(g-1). \tag{9.1}$$

By Lemma 7.4(v) we have $d-g+1 = h^0(\mathcal{O}_C(1)) \ge 3$, hence $g \le d-2$. Also, if equality holds, then $h^0(\mathcal{O}_C(1)) = 3$, so that $d-2=g=\binom{d-1}{2}$, thus $d=3$ and $g=1$, a contradiction. Therefore $2 \le g \le d-3$, hence $d \ge 5$. But if $d \le 8$ the only possibility given by (9.1) is $d=8, g=5, n=4$ and $h^0(\mathcal{O}_C(1))=4$. □

**Lemma 9.2.** *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 3$. If $T_X(1)$ is an Ulrich vector bundle we have:*

(i) $K_S H_{|S} = \frac{n-4}{n+2}d$.
(ii) $q(S) = p_g(S) = 0$.
(iii) $K_S^2 = -\frac{3(n-2)}{2(n+2)}d - \frac{n-12}{2}$.
(iv) *$S$ is rational.*

*Proof.* (i) follows by Lemma 4.3(ii), while (ii) follows by Lemma 4.3(iii), Lemma 7.1 and Lemma 7.4(ii). Note now that the equation in Lemma 4.3(vii) can be rewritten as

$$3(n-2)d + 2(n+2)K_S^2 + (n+2)(n-12) = 0$$

giving (iii). Finally assume that $S$ is not ruled, so that $\kappa(S) \ge 0$. Then Lemma 8.6 gives that $K_S + H_{|S} = (K_X + (n-1)H)_{|S}$ is nef, hence $K_S(K_S + H_{|S}) \ge 0$, that is $K_S^2 \ge -K_S H_{|S} = -\frac{n-4}{n+2}d$ by (i). Then (iii) gives

$$-\frac{3(n-2)}{2(n+2)}d - \frac{n-12}{2} = K_S^2 \ge -\frac{n-4}{n+2}d$$

so that

$$(n+2)(d+n-12) \le 0.$$

Since $n \ge 3$ it follows that $d \le 9$, and using Lemma 9.1 we deduce that either $d=9, n=3$ or $d=8, g=5, n=4$ and $h^0(\mathcal{O}_C(1))=4$. In the first case we get a contraction by Lemma 4.3(i), while in the second case $d+n-12=0$, hence $K_S^2=0$. As $C \subset \mathbb{P}^3$ we deduce that $S \subset \mathbb{P}^4$. But this contradicts the well-known formula for the invariants of a surface in $\mathbb{P}^4$. Therefore $S$ is ruled, hence rational by (ii) and (iv) is proved. □

We are now ready to prove Theorem 3.

*Proof of Theorem 3.* If $n=1$ we know by Lemma 4.3(i) that $T_X(1)$ is not an Ulrich vector bundle. If $n=2$ this is Theorem 6.3. Suppose next that $n \ge 3$. Note that $H^0(T_X)=0$ by Lemma 4.2(iii), hence $X$ is neither $\mathbb{P}^n$ nor $Q_n$. Also $q(X)=0$ by Lemma 4.3(iii). We have that $(X,H)$ is not:

(1) A projective bundle over a smooth curve by Lemma 8.7.
(2) A Del Pezzo manifold by Lemma 4.3(i), since otherwise $g=1$.
(3) A hyperquadric fibration over a smooth curve (in the sense of [I]), by Lemma 8.5(ii).
(4) A linear $\mathbb{P}^{n-2}$-bundle over a smooth surface, by Lemma 8.5(i).

Also observe that $X$ does not contain any exceptional divisor of type 1 by Lemma 8.3(i). Hence $(X,H)$ is isomorphic to its reduction $(X',H')$ (see [I, (0.11)]). It follows by [I, Thm. (1.7)] that $K_X+(n-2)H$ is nef. Hence $S$ is minimal and rational by Lemma 9.2(iv), a contradiction since a minimal rational surface does not have nef canonical bundle. Thus the case $n \ge 3$ does not occur and the theorem is proved. □

## 10. $T_X(2)$ Ulrich in any dimension

We prove Theorem 4.

*Proof Theorem 4.* It follows by Lemma 4.2(iii) that $H^0(T_X) = 0$, hence $X$ is neither $\mathbb{P}^n$ nor $Q_n$. Note that $H^i(\mathcal{O}_X) = 0$ for $i \ge 1$ by Lemma 4.3(iii) and $K_X$ is not nef, since Lemma 4.3(ii) gives that $K_X H^{n-1} = \frac{n(3-n)}{n+2} d < 0$.

We divide the proof according to the value of $\tau(X,H)$ (see (4.3)). We will also use the notions of first and second reduction of $(X,H)$, as defined in [BS, Defs. 7.3.3 and 7.5.7].

Case A: $\tau(X,H) \ge n-1$.

This case does not occur since Lemma 4.6(iii) implies that $\tau(X,H) \le n - \frac{2n}{n+1} < n-1$.

Case B: $n-2 \le \tau(X,H) < n-1$.

Then $K_X + (n-1)H$ is ample, hence the first reduction exists and is isomorphic to $(X,H)$. Therefore [BS, Thm. 7.3.4] implies that $\tau(X,H) = n-2$ and then [BS, Thm. 7.5.3] gives that $(X,H)$ is one of the following:

(B.1) a Mukai variety,
(B.2) a Del Pezzo fibration over a smooth curve,
(B.3) a quadric fibration over a normal surface,
(B.4) a scroll over a normal threefold,
(B.5) $(X,H)$ contains an exceptional divisor of type 2, 3, or 4.

Now, the case (B.1) is ruled out by Corollary 4.12. Case (B.2) is excluded for $n=4$ by Lemma 8.9(i) and for $n \ge 5$ by Lemma 8.5(iii). Also the cases (B.3) and (B.4) are ruled out by Lemma 8.5(ii) and (i). Finally the case (B.5) is excluded by Lemma 8.3(ii) and (iii).

Thus also Case B does not occur.

Case C: $\tau(X,H) < n-2$.

Then the first and second reductions exist and are both isomorphic to $(X,H)$, since $K_X + (n-2)H$ is ample.

We first claim that $K_{X_3}$ is not nef. In fact, assume that $K_{X_3}$ is nef. On the one hand, $\chi(\mathcal{O}_{X_3}) = 1$ by Lemma 7.3. On the other hand, $3c_2(X_3) - c_1(X_3)^2$ is pseff by [M, Thm. 1.1], hence $3c_2(X_3)K_{X_3} \ge K_{X_3}^3 \ge 0$. But then Riemann-Roch gives $\chi(\mathcal{O}_{X_3}) = -\frac{1}{24} c_2(X_3) K_{X_3} \le 0$, a contradiction.

Hence $K_{X_3}$ is not nef and [BS, Prop. 7.9.1] gives the following cases:

(C.1) $n=5$ and $(X, K_X + 3H)$ is a linear $\mathbb{P}^4$-bundle over a smooth curve.
(C.2) $n=4$ and $(X, K_X + 2H)$ is a Del Pezzo.
(C.3) $n=4$ and $(X, K_X + 2H)$ is a quadric fibration over a smooth curve.
(C.4) $n=4$ and $(X, K_X + 2H)$ is a scroll over a normal surface.
(C.5) $n=4$ and $(X,H)$ contains an exceptional divisor of type 2.
(C.6) $n=4$ and $(X, K_X + 2H)$ is a $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$-fibration over a curve.

In case (C.1) we have a contradiction by Lemma 8.7. In case (C.2) we have $4K_X + 6H = 0$, hence $4K_X H^3 + 6H^4 = 0$ and Lemma 4.3(ii) gives the contradiction $d = 0$. Cases (C.3) and (C.5) do not occur by Lemmas 8.10 and 8.3(ii). In Case (C.6) we observe that the fibration is obtained in [F2, (4.6.1)] by contracting an extremal ray, hence it is minimal (see Definition 8.8) and the image is a normal, hence smooth, curve. Thus this case is excluded by Lemma 8.9(ii).

Hence we are left with case (C.4). We have a surjective morphism $\pi : X \to B$ and denoting by $F$ a general fiber, we have $(F, (K_X + 2H)_{|F}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Now all fibers of $\pi$ are 2-dimensional by [BS, Thm. 14.1.1], hence we get by [BS, Prop. 3.2.1] that $B$ is a smooth surface and $(X, K_X + 2H)$ is a linear $\mathbb{P}^2$-bundle over $B$. But this case is excluded by Lemma 8.11.

This concludes the proof of the theorem. □

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## Appendix A. Some numerical lemmas

**Lemma A.1.** *Let $(a; c_1, c_2, c_3, c_4) \in \mathbb{Z}^5$ be such that $c_1 \ge c_2 \ge c_3 \ge c_4 \ge 0, a \ge c_1 + c_2 + 3$ and*

$$a^2 - 6a + 4 = c_1^2 + c_2^2 + c_3^2 + c_4^2. \tag{A.1}$$

*Then $(a; c_1, c_2, c_3, c_4) \in \{(6; 2, 0, 0, 0), (6; 1, 1, 1, 1), (7; 3, 1, 1, 0), (9; 3, 3, 3, 2)\}$.*

*Proof.* We have

$$a - 3 \ge c_1 + c_2. \tag{A.2}$$

Now, (A.1) and (A.2) imply that $(a-3)^2-5=c_1^2+c_2^2+c_3^2+c_4^2 \geq (c_1+c_2)^2-5$, that is

$$c_3^2+c_4^2 \geq 2c_1c_2-5. \tag{A.3}$$

But $2c_3^2 \geq c_3^2+c_4^2$ and $2c_1c_2 \geq 2c_2^2$. Consequently, we get

$$5 \geq 2(c_2-c_3)(c_2+c_3). \tag{A.4}$$

Thus, one of the following should happen:

($\alpha$) $c_2=c_3$.
($\beta$) $c_2=c_3+1$.

First assume that case ($\beta$) holds.

Then (A.4) yields $2c_3 \leq 1$ which gives $c_3=0, c_2=2$ and hence $c_4=o$. The (A.1) gives

$$(a+c_1-3)(a-c_1-3)=6.$$

Thus we have one of the following possibilities

$$a+c_1=4, a-c_1=9, \text{ or } a+c_1=5, a-c_1=6, \text{ or } a+c_1=6, a-c_1=5, \text{ or } \ a+c_1=9, a-c_1=4$$

but none of them have integer solutions.

Assume now that case ($\alpha$) holds.

Set $c_2=c_3=c$. From (A.3), we obtain $c^2+c_4^2 \geq 2c_1c-5$. Since $c \geq c_4$, we get

$$5 \geq 2c(c_1-c). \tag{A.5}$$

The above implies one of the following happens:

($\alpha$1) $c_2=c_3=c_4=0$.
($\alpha$2) $c_1=c_2=c_3$.
($\alpha$3) $c_2=c_3=1$. This case has two sub-cases, namely $c_1=2,3$.
($\alpha$4) $c_2=c_3=1$, $c_1=3$.

Suppose we are in case ($\alpha$1).

Then (A.1) gives $(a-3)^2-5=c_1^2$, so that $(a+c_1-3)(a-c_1-3)=5$. In this case, either $a+c_1=4, a-c_1=8$, giving the contradiction $c_1=-2$, or $a+c_1=8, a-c_1=4$, giving $a=6, c_1=2$ and the solution $(6;2,0,0,0)$.

Suppose we are in case ($\alpha$2).

Using (A.3) we conclude

$$5 \geq (c-c_4)(c+c_4). \tag{A.6}$$

As before, we obtain the following cases:

($\alpha$21) $c=c_1=c_2=c_3=c_4$.
($\alpha$22) $c=c_4+1$.
($\alpha$23) $c=c_4+2$.

We first deal with ($\alpha$21). In this case, from (A.1), we obtain

$$(a+2c-3)(a-2c-3)=5.$$

Thus, we have either $a+2c=4, a-2c=8$, giving the contradiction $c=-1$, or $a+2c=8, a-2c=4$, giving the solution $(6;1,1,1,1)$.

We now deal with ($\alpha$22). From (A.6) we obtain $c+c_4-2 \leq 5$, hence $c_4 \leq 2$. Thus $(c,c_4) \in \{(3,2),(2,1),(1,0)\}$ and using (A.1) we see that it has no integer solutions except in the first case, giving the solution $(9;3,3,3,2)$.

We now deal with ($\alpha$23). As before, in this case we have $c_4 \leq 0$. This implies $c=2, c_4=0$. But then (A.1) does not have any integer solution.

This concludes case ($\alpha$2).

Suppose we are in case ($\alpha$3).

We know that $(c_1,c_2,c_3,c_4) \in \{(1,1,1,1),(2,1,1,0),(3,1,1,1),(3,1,1,0)\}$. Using (A.1) we see that we have no integer solutions except in the last case, giving $(7;3,1,1,0)$.

Suppose we are in case ($\alpha$4).

Then $(c_1,c_2,c_3,c_4) \in \{(3,2,2,2),(3;2,2,1),(3;2,2,0)\}$ and (A.1) has no integer solutions.

This concludes case ($\alpha$) and the proof. □

**Lemma A.2.** *Let* $\mathbf{z} = (a; b_1, b_2, b_3, b_4, b_5, b_6) \in \mathbb{Z}^7$ *with* $b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 \geq b_6$ *satisfying the following*

$$a^2 - \sum_{i=1}^{6} b_i^2 = 10, \quad 3a - \sum_{i=1}^{6} b_i = 6. \tag{A.7}$$

*Then* $\mathbf{z} \in \{(4; 1, 1, 1, 1, 1, 1), (5; 2, 2, 2, 1, 1, 1), (6; 3, 2, 2, 2, 2, 1), (7; 3, 3, 3, 2, 2, 2), (8; 3, 3, 3, 3, 3, 3)\}$.

*Proof.* We first use the Cauchy-Scwartz's inequality $(\sum_{i=1}^{6} b_i)^2 \leq 6(\sum_{i=1}^{6} b_i^2)$ to obtain $(a^2 - 12a + 32) \leq 0$ whence $4 \leq a \leq 8$. We further observe that

$$\sum_{i=1}^{6} (b_i^2 - b_i) = a^2 - 3a - 4. \tag{A.8}$$

Also, $(b_i^2 - b_i) \geq 0$ for all $i \geq 1$, and $b_1 > 0$ as $3a - 6 > 0$ for $a \geq 4$.

<u>Case 1: $a = 4$.</u> We have $\sum_{i=1}^{6} (b_i^2 - b_i) = 0$ whence $|b_i| \leq 1$ for all $i$. Since $\sum_{i=1}^{6} b_i = 6$, we have $b_i = 1$ for all $i$.

<u>Case 2: $a = 5$.</u> We have $\sum_{i=1}^{6} (b_i^2 - b_i) = 6$ whence $|b_i| \leq 3$. Also, $\sum_{i=1}^{6} b_i^2 = 15$ and $\sum_{i=1}^{6} b_i = 9$.
Subcase 2.1) $b_1 = 3$. In this case $\sum_{i=2}^{6} (b_i^2 - b_i) = 0$ whence $|b_i| \leq 1$ for all $i \geq 2$. Consequently $\sum_{i=1}^{6} b_i \leq 8$ which is a contradiction.
Subcase 2.2) $b_1 \leq 2$. In this case we must have $b_1 = b_2 = b_3 = 2$. Consequently $\sum_{i=4}^{6} (b_i^2 - b_i) = 0$ whence $|b_i| \leq 1$ for all $i \geq 4$ whence the only solution is $\mathbf{z} = (5; 2, 2, 2, 1, 1, 1)$.

<u>Case 3: $a = 6$.</u> We have $\sum_{i=1}^{6} (b_i^2 - b_i) = 14$ whence $|b_i| \leq 4$. Also $\sum_{i=1}^{6} b_i^2 = 26$ and $\sum_{i=1}^{6} b_i = 12$.
Subcase 3.1) $b_1 = 4$. Then $\sum_{i=2}^{6} (b_i^2 - b_i) = 2$ whence $|b_i| \leq 2$ for $i \geq 2$. Consequently $b_2 = b_3 = b_4 = 2$. But then $\sum_{i=1}^{6} b_i^2 \geq 28$ which is a contradiction.
Subcase 3.2) $b_1 = 3$.
3.2.1) $b_2 = 3$. In this case $\sum_{i=3}^{6} (b_i^2 - b_i) = 2$ whence $|b_i| \leq 2$ for $i \geq 3$. Consequently, $b_3 = b_4 = 2$. Thus $b_5 + b_6 = 2$ and $b_5^2 + b_6^2$ which is a contradiction.
3.2.2) $b_2 \leq 2$. In this case we have the only solution $\mathbf{z} = (6; 3, 2, 2, 2, 2, 1)$.
Subcase 3.3) $b_1 \leq 2$. In this case $b_i = 2$ for all $i$ whence $\sum_{i=1}^{6} b_i^2 = 24$ which is a contradiction.

<u>Case 4: $a = 7$.</u> We have $\sum_{i=1}^{6} (b_i^2 - b_i) = 24$ whence $|b_i| \leq 5$. Also, $\sum_{i=1}^{6} b_i^2 = 39$ and $\sum_{i=1}^{6} b_i = 15$.
Subcase 4.1) $b_1 = 5$. Then $\sum_{i=2}^{6} (b_i^2 - b_i) = 4$ whence $|b_i| \leq 2$ for all $i \geq 2$. Consequently $b_i = 2$ for all $i$, thus $\sum_{i=1}^{6} b_i^2 = 45$ which is a contradiction.
Subcase 4.2) $b_1 = 4$.
4.2.1) $b_2 = 4$. Then $\sum_{i=3}^{6} (b_i^2 - b_i) = 0$ whence $|b_i| \leq 1$ for all $i \geq 3$. Consequently $\sum_{i=1}^{6} b_i \leq 12$ which is a contradiction.
4.2.2) $b_2 = 3$.
4.2.2.1) $b_3 = 3$. Then $\sum_{i=4}^{6} (b_i^2 - b_i) = 0$ whence $|b_i| \leq 1$ for $i \geq 4$. Consequently $\sum_{i=1}^{6} b_i \leq 13$ which is a contradiction.
4.2.2.2) $b_3 \leq 2$. Then $b_i = 2$ for all $i \geq 3$ whence $\sum_{i=1}^{6} b_i^2 = 41$ which is a contradiction.
4.2.3) $b_2 \leq 2$. Then $\sum_{i=1}^{6} b_i \leq 14$ which is a contradiction.
Subcase 4.3) $b_1 = 3$. Then $b_2 = b_3 = 3$.
4.3.1) $b_4 = 3$. Then $\sum_{i=5}^{6} (b_i^2 - b_i) = 0$ whence $|b_i| \leq 1$ for $i \geq 5$. Consequently $\sum_{i=1}^{6} b_i \leq 14$ which is a contradiction.
4.3.2) $b_4 \leq 2$. Then $b_4 = b_5 = b_6 = 2$. We get only one solution $\mathbf{z} = (7; 3, 3, 3, 2, 2, 2)$.
Subcase 4.4) $b_1 \leq 2$. Then $\sum_{i=1}^{6} b_i \leq 12$ which is a contradiction.

<u>Case 5: $a = 8$.</u> We have $\sum_{i=1}^{6} (b_i^2 - b_i) = 36$ whence $|b_i| \leq 6$. Also, $\sum_{i=1}^{6} b_i^2 = 54$ and $\sum_{i=1}^{6} b_i = 18$.
Subcase 5.1) $b_1 = 6$. Then $\sum_{i=2}^{6} (b_i^2 - b_i) = 6$ whence $|b_i| \leq 3$ for $i \geq 2$.
5.1.1) $b_2 = 3$. In this case $\sum_{i=3}^{6} (b_i^2 - b_i) = 0$ whence $|b_i| \leq 1$ for $i \geq 3$. Consequently $\sum_{i=1}^{6} b_i \leq 13$ which is a contradiction.
5.1.2) $b_2 \leq 2$. In this case $\sum_{i=1}^{6} b_i \leq 16$ which is a contradiction.
Subcase 5.2) $b_1 = 5$. Then $\sum_{i=2}^{6} (b_i^2 - b_i) = 16$ whence $|b_i| \leq 4$.

5.2.1) $b_2 = 4$. Then $\sum_{i=3}^6 (b_i^2 - b_i) = 4$ whence $|b_i| \le 2$ for $i \ge 3$. Thus $\sum_{i=1}^6 b_i \le 17$ which is a contradiction.

5.2.2) $b_2 = 3$ which implies $b_3 = 3$. Then $\sum_{i=4}^6 (b_i^2 - b_i) = 4$ whence $|b_i| \le 2$ for $i \ge 4$. Consequently $\sum_{i=1}^6 b_i \le 17$ which is a contradiction.

5.2.3) $b_2 \le 2$. Then $\sum_{i=1}^6 b_i \le 15$ which is a contradiction.

Subcase 5.3) $b_1 = 4$.

5.3.1) $b_2 = 4$.

5.3.1.1) $b_3 = 4$. Then $\sum_{i=4}^6 (b_i^2 - b_i) = 0$ whence $|b_i| \le 1$ for $i \ge 4$. Consequently $\sum_{i=1}^6 b_i \le 15$ which is a contradiction.

5.3.1.2) $b_3 = 3$ which implies $b_4 = 3$. Thus $\sum_{i=5}^6 (b_i^2 - b_i) = 0$ whence $|b_i| \le 1$ for $i \ge 5$. Consequently $\sum_{i=1}^6 b_i \le 16$ which is a contradiction.

5.3.1.3) $b_3 \le 2$. Then $\sum_{i=1}^6 b_i \le 16$ which is a contradiction.

5.3.2) $b_2 = 3$. Then $b_3 = b_4 = b_5 = 3$ and $b_6 = 2$. Consequently $\sum_{i=1}^6 b_i^2 = 56$ which is a contradiction.

5.3.3) $b_2 \le 2$. Then $\sum_{i=1}^6 b_i \le 14$ which is a contradiction.

Subcase 5.4) $b_1 \le 3$. In this case we have the only solution $\mathbf{z} = (8; 3, 3, 3, 3, 3, 3)$. $\square$

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