ON PARTIALLY AMPLE ULRICH BUNDLES

ANGELO FELICE LOPEZ AND DEBADITYA RAYCHAUDHURY

Abstract. We characterize q-ample Ulrich bundles on a variety $X \subseteq \mathbb{P}^N$ with respect to $(q+1)$-dimensional linear spaces contained in X.

1. Introduction

Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension $n \geq 1$. The study of positivity properties of vector bundles $\mathcal{E}$ on X is a classical one. Starting with Hartshorne's pioneering paper [H], several positivity notions have been introduced, among which, perhaps, the most important one is ampleness. The latter amounts to say that the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample. One possible weakening of this notion, so that some properties are maintained, is q-ampleness, that we now recall (see for example [To] and references therein).

Definition 1.1. Let $q \geq 0$ and let $\mathcal{L}$ be a line bundle on a scheme Y. We say that $\mathcal{L}$ is *q-ample* if for every coherent sheaf $\mathcal{F}$ on Y, there exists an integer $m_0 > 0$ such that $H^i(\mathcal{F}(m\mathcal{L})) = 0$ for $m \geq m_0$ and $i > q$. Let $\mathcal{E}$ be a vector bundle on Y. We say that $\mathcal{E}$ is *q-ample* if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is q-ample.

In this paper we are interested in studying the above notion for a special class of vector bundles, namely for Ulrich bundles, that is bundles $\mathcal{E}$ such that $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$. The importance of Ulrich bundles is well-known (see for example [ES, Be, CMRPL] and references therein). Positivity properties of Ulrich bundles have been studied recently [L, LM, LS, LMS1, LMS2]. In particular, in [LS, Thm. 1], we showed that an Ulrich bundle $\mathcal{E}$ is ample (that is 0-ample) if and only if either X does not contain lines or $\mathcal{E}_{|L}$ is ample on any line $L \subset X$. We prove here a generalization of this result.

Theorem 1.
Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $n \geq 1$. Let $\mathcal{E}$ be an Ulrich vector bundle and let $q \geq 0$ be an integer. Then the following are equivalent:

(i) *$\mathcal{E}$ is q-ample;*
(ii) *either X does not contain a linear space of dimension $q+1$, or $\mathcal{E}_{|M}$ does not have a trivial direct summand for every linear space $M \subseteq X$ of dimension $q+1$;*
(iii) *either X does not contain a linear space of dimension $q+1$, or $h^0(\mathcal{E}^*_{|M}) = 0$ for every linear space $M \subseteq X$ of dimension $q+1$.*

We also have the following consequence.

Corollary 1. *Let $\mathcal{E}$ be an Ulrich vector bundle on $X \subseteq \mathbb{P}^N$. Then:*

(i) *$\mathcal{E}$ is $(n-1)$-ample if and only if $(X, \mathcal{O}_X(1), \mathcal{E}) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$.*
(ii) *If $n \geq 2$, $(X, \mathcal{O}_X(1), \mathcal{E}) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$ and $\rho(X) = 1$, then $\mathcal{E}$ is $(n-2)$-ample.*

In recent years, positivity of vector bundles have been measured by augmented and restricted base loci (see for example [BKKMSU, FM]). In the last section we will ask a question about augmented base loci of Ulrich bundles arising from the above theorem.

2. Notation

Throughout the paper we work over the field $\mathbb{C}$ of complex numbers. A *variety* is by definition an integral separated scheme of finite type over $\mathbb{C}$.

The first author is a member of the GNSAGA group of INdAM and was partially supported by PRIN "Advances in Moduli Theory and Birational Classification".

Mathematics Subject Classification : Primary 14F06. Secondary 14J60.

3. Generalities on vector bundles

In this section we collect some general facts about vector bundles and some notation that will be used later.

Definition 3.1. Let $\mathcal{E}$ be a rank r vector bundle on X. We set $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}(\mathcal{E}))$ with projection map $\pi : \mathbb{P}(\mathcal{E}) \to X$ and tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. If $\mathcal{E}$ is globally generated we define the map determined by $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ as

$$\varphi = \varphi_{\mathcal{E}} = \varphi_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)} : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathcal{E})$$

and we set

$$\Pi_y = \pi(\varphi^{-1}(y)), y \in \varphi(\mathbb{P}(\mathcal{E})) \text{ and } P_x = \varphi(\mathbb{P}(\mathcal{E}_x)), x \in X.$$

We also define the map determined by $\mathcal{E}$ as

$$\Phi = \Phi_{\mathcal{E}} : X \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$$

given, for any $x \in X$, by $\Phi(x) = [P_x] \in \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$.

We record some simple but useful facts. The first one is essentially contained in [Tg, Proof of Lemma 2.4, page 426][1].

Lemma 3.2. *Let V be a vector space and let $P \in \mathbb{P}V$ be a point. Let $Y \subset \mathbb{G}(k, \mathbb{P}V)$ be a subvariety such that, for every $y \in Y$, the corresponding k-plane contains P. If $\mathcal{U}$ is the universal subbundle of $\mathbb{G}(k, \mathbb{P}V)$, then $\mathcal{U}_{|Y} \cong \mathcal{O}_Y \oplus \mathcal{G}$, for some rank k vector bundle $\mathcal{G}$ on Y.*

Proof. The assertion being obvious if $\dim V = 1$, we assume that $\dim V \geq 2$. Let $P = \mathbb{P}V_1$, where $V_1 \subseteq V$ is 1-dimensional and choose a splitting $V = V_1 \oplus V'$. We have a closed embedding $j : G(k-1, V') \hookrightarrow G(k, V)$ defined by $j([W]) = [V_1 \oplus W]$, where $V_1 \oplus W \subset V$, or, equivalently, j is the morphism associated to the vector bundle $\mathcal{O}_{G(k-1,V')} \oplus (\mathcal{U}')^*$, where $\mathcal{U}'$ is the universal subbundle of $G(k-1, V')$. Let $G_P = \{[U] \in G(k, V) : P \in \mathbb{P}U\}$. Then j defines an isomorphism $G(k-1, V') \cong G_P$, hence

$$\mathcal{U}_{|G_P} \cong j^*\mathcal{U} \cong \mathcal{O}_{G(k-1,V')} \oplus \mathcal{U}' \cong \mathcal{O}_{G_P} \oplus \mathcal{G}'$$

for some rank k vector bundle $\mathcal{G}'$ on G_P. Since $Y \subseteq G_P$, we get that $\mathcal{U}_{|Y} \cong \mathcal{O}_Y \oplus \mathcal{G}$, where $\mathcal{G} = \mathcal{G}'_{|Y}$. □

Lemma 3.3. *Let $\mathcal{E}$ be a globally generated rank r vector bundle on X.*

(i) *For every $x \in X$ the restriction morphism $\varphi_{|\mathbb{P}(\mathcal{E}_x)} : \mathbb{P}(\mathcal{E}_x) \to P_x$ is an isomorphism onto a linear subspace of dimension $r-1$ in $\mathbb{P}H^0(\mathcal{E})$.*

Now let $y \in \varphi(\mathbb{P}(\mathcal{E}))$. Then:

(ii) *$\pi_{|\varphi^{-1}(y)} : \varphi^{-1}(y) \to X$ is a closed embedding.*
(iii) *$\mathcal{E}_{|\Pi_y} \cong \mathcal{O}_{\Pi_y} \oplus \mathcal{G}$, for some rank $r-1$ vector bundle $\mathcal{G}$ on Π_y.*

Proof. To see (i), observe that we have $\mathbb{P}(\mathcal{E}_x) \cong \mathbb{P}^{r-1}$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)_{|\mathbb{P}(\mathcal{E}_x)} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Let

$$W = \operatorname{Im}\{H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^{r-1}}(1))\}.$$

Being $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ globally generated, we have that so is W, hence $\dim W \geq r$. It follows that $W = H^0(\mathcal{O}_{\mathbb{P}^{r-1}}(1))$ and $\varphi_{|\mathbb{P}(\mathcal{E}_x)} = \varphi_{\mathcal{O}_{\mathbb{P}^{r-1}}(1)}$ is an isomorphism onto its image, which is then a linear subspace of dimension $r-1$ in $\mathbb{P}H^0(\mathcal{E})$. This proves (i) and then (i) implies that π and its differential are injective on the fibers of φ, proving (ii). As for (iii), set $M = \Pi_y$ and consider the globally generated rank r vector bundle $\mathcal{E}_{|M}$ on M. Let

$$U = \operatorname{Im}\{H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \to H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{|M})}(1)\}$$

[1] See also https://mathoverflow.net/questions/395472/trivial-subbundle-of-universal-bundle-on-the-grassmannian-mathbbgk-n.

so that $\varphi_{|\mathbb{P}(\mathcal{E}_{|M})} = \varphi_U : \mathbb{P}(\mathcal{E}_{|M}) \to \mathbb{P}U$. Set $\Phi_M = \Phi_{\mathcal{E}_{|M}}, \varphi_M = \varphi_{\mathcal{E}_{|M}}$ and, for any $x \in M$, $P_{M,x} = \varphi_M(\mathbb{P}((\mathcal{E}_{|M})_x)$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_{|M}) & \xrightarrow{\varphi_M} & \varphi_M(\mathbb{P}(\mathcal{E}_{|M})) \subset \mathbb{P}H^0(\mathcal{E}_{|M}) \\ & \searrow^{\varphi_U} & \downarrow p \\ & & \varphi_U(\mathbb{P}(\mathcal{E}_{|M})) \subset \mathbb{P}U \end{array}$$

where p is a finite map. For any $x \in M$, there is a $z \in \varphi^{-1}(y)$ such that $x = \pi(z)$. Hence $z \in \mathbb{P}(\mathcal{E}_x) = \mathbb{P}((\mathcal{E}_{|M})_x)$ and therefore $y = \varphi(z) = \varphi_U(z) = p(\varphi_M(z))$, so that $\varphi_M(z) \in p^{-1}(y) \cap P_{M,x}$. Therefore each $(r-1)$-plane $P_{M,x}$ passes through one of the points of $p^{-1}(y)$. On the other hand, the family of these $(r-1)$-planes is just $\Phi_M(M) \subset \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}_{|M}))$, thus it is irreducible. Since $p^{-1}(y)$ is finite and the condition of passing through a point is closed, we deduce that there is a point $y_M \in \mathbb{P}H^0(\mathcal{E}_{|M})$ such that $y_M \in P_{M,x}$ for every $x \in M$. Set $Y = \Phi_M(M) \subset \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}_{|M}))$. It follows by Lemma 3.2 that $\mathcal{U}^*_{|Y} \cong \mathcal{O}_Y \oplus \mathcal{G}$, for some rank $r-1$ vector bundle $\mathcal{G}$ on Y. Since $\mathcal{E}_{|M} = \Phi_M^* \mathcal{U}^*$, this proves (iii). □

4. q-ample vector bundles

We discuss some generalities on q-ample vector bundles.

Definition 4.1. Let $\mathcal{E}$ be a vector bundle on X. We set $q_{\min}(\mathcal{E}) = \min\{q \geq 0 : \mathcal{E} \text{ is } q\text{-ample}\}$.

The definition of $q_{\min}(\mathcal{E})$ implies that $\mathcal{E}$ is q-ample if and only if $q \geq q_{\min}(\mathcal{E})$.

Remark 4.2. We have:

(i) If $\mathcal{E}$ is a globally generated vector bundle on X, then $\mathcal{E}$ is q-ample if and only if $\dim F \leq q$ for every fiber F of $\varphi = \varphi_{\mathcal{E}} : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathcal{E})$.

(ii) If $\mathcal{E}$ is globally generated, then it is n-ample. Also $n + r - 1 - \nu(\mathcal{E}) \leq q_{\min}(\mathcal{E}) \leq n$, where r is the rank of $\mathcal{E}$ and $\nu(\mathcal{E})$ is the numerical dimension of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Proof. (i) is just [S, Prop. 1.7]. The first part of (ii) follows either by [S, Prop. 1.7] or by (i), since $\dim \varphi^{-1}(y) = \dim \Pi_y \leq n$ for every $y \in \varphi(\mathbb{P}(\mathcal{E}))$. Thus $q_{\min}(\mathcal{E}) \leq n$. Since $\mathcal{E}$ is $q_{\min}(\mathcal{E})$-ample, for any fiber F of φ, we have by (i) that $n + r - 1 - \nu(\mathcal{E}) \leq \dim F \leq q_{\min}(\mathcal{E})$. This proves (ii). □

We have the following characterization, which is a special case of [S, Prop. 1.7].

Proposition 4.3. *Let X be a smooth variety of dimension $n \geq 1$. Let $\mathcal{E}$ be a globally generated vector bundle on X and let $q \geq 0$ be an integer. Then the following are equivalent:*

(i) *$\mathcal{E}$ is q-ample;*

(ii) *$\mathcal{E}_{|Z}$ does not have a trivial direct summand for every subvariety $Z \subseteq X$ of dimension $q+1$;*

(iii) *$h^0(\mathcal{E}^*_{|Z}) = 0$ for every subvariety $Z \subseteq X$ of dimension $q+1$.*

Proof. The equivalence (ii)-(iii) follows by [O, Lemma 3.9]. As for the equivalence (i)-(ii), assume first that $\mathcal{E}_{|Z}$ does not have a trivial direct summand for every subvariety $Z \subseteq X$ of dimension $q+1$. If $\mathcal{E}$ is not q-ample, there exists by Remark 4.2(i) an $y \in \varphi(\mathbb{P}(\mathcal{E}))$ such that $\dim \varphi^{-1}(y) \geq q+1$. Set $M = \Pi_y$. By Lemma 3.3(ii) we have that $M \cong \varphi^{-1}(y)$, hence $\dim M \geq q+1$. Also, Lemma 3.3(iii) implies that $\mathcal{E}_{|M} \cong \mathcal{O}_M \oplus \mathcal{G}$, for some vector bundle $\mathcal{G}$ on M. But then, for any subvariety $Z \subseteq M$ with $\dim Z = q+1$, we have that $\mathcal{E}_{|Z} \cong \mathcal{O}_Z \oplus \mathcal{G}_{|Z}$, contradicting the hypothesis. Vice versa, assume that $\mathcal{E}$ is q-ample and let $Z \subseteq X$ be a subvariety of dimension $q+1$. If $\mathcal{E}_{|Z} \cong \mathcal{O}_Z \oplus \mathcal{G}$, for some vector bundle $\mathcal{G}$ on Z, then $Z \cong \mathbb{P}(\mathcal{O}_Z) \subseteq \mathbb{P}(\mathcal{E}_{|Z}) \subseteq \mathbb{P}(\mathcal{E})$ and

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)_{|\mathbb{P}(\mathcal{O}_Z)} \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}_{|Z})}(1)_{|\mathbb{P}(\mathcal{O}_Z)} \cong \mathcal{O}_{\mathbb{P}(\mathcal{O}_Z)}(1) \cong \mathcal{O}_Z$$

hence $\varphi(\mathbb{P}(\mathcal{O}_Z))$ is a point. Therefore φ has a fiber of dimension at least $q+1$, contradicting Remark 4.2(i). □

5. Proofs of the main results

In the case of Ulrich vector bundles, we can do better than Proposition 4.3.

Proof of Theorem 1. Recall that $\mathcal{E}$ is globally generated since it is 0-regular. The equivalence (ii)-(iii) follows by [O, Lemma 3.9]. As for the equivalence (i)-(ii), assume first that $\mathcal{E}$ is q-ample. Then either X does not contain a linear space of dimension $q+1$ or it follows by Proposition 4.3 that $\mathcal{E}_{|M}$ does not have a trivial direct summand for every linear space $M \subseteq X$ of dimension $q+1$. To see the converse, let $y \in \varphi(\mathbb{P}(\mathcal{E}))$ and let $\Pi_y = \pi(\varphi^{-1}(y))$, so that $\Pi_y \cong \varphi^{-1}(y)$ by Lemma 3.3(ii). By [LS, Thm. 2] we have that Π_y is a linear space contained in X. Now, if X does not contain a linear space of dimension $q+1$, then $\dim \varphi^{-1}(y) = \dim \Pi_y \le q$ for every $y \in \varphi(\mathbb{P}(\mathcal{E}))$. Hence $\mathcal{E}$ is q-ample by Remark 4.2(i). On the other hand, assume that $\mathcal{E}_{|M}$ does not have a trivial direct summand for every linear space $M \subseteq X$ of dimension $q+1$. If $\mathcal{E}$ is not q-ample, there exists by Remark 4.2(i) an $y \in \varphi(\mathbb{P}(\mathcal{E}))$ such that $\dim \varphi^{-1}(y) \ge q+1$. Hence $\dim \Pi_y \ge q+1$, and picking a linear subspace $M \subseteq \Pi_y$ with $\dim M = q+1$, we get a contradiction by Lemma 3.3(iii). □

We also have.

Proof of Corollary 1. First we prove (i). If $\mathcal{E}$ is not $(n-1)$-ample, then it follows by Theorem 1 that $X = \mathbb{P}^n$ and $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{G}$, for some vector bundle $\mathcal{G}$ on X. But then $\mathcal{O}_X$ is Ulrich and therefore $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$ by [ACLR, Lemma 4.2](vi) and [ES, Prop. 2.1] (or [Be, Thm. 2.3]). On the other hand, if $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$, then $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}^{r-1} \times \mathbb{P}^n$ and $\varphi = \pi_1 : \mathbb{P}^{r-1} \times \mathbb{P}^n \to \mathbb{P}^{r-1}$ has n-dimensional fibers, hence $\mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ is not $(n-1)$-ample by Remark 4.2(i). This proves (i). As for (ii), using Theorem 1, we just need to prove that $X \subset \mathbb{P}^N$ does not contain linear spaces of dimension $n-1$ unless $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$. To this end, let A be the ample generator of $N^1(X)$ and let $H \in |\mathcal{O}_X(1)|$, so that $H \equiv hA$. If X contains a linear space M of dimension $n-1$, then $M \equiv aA$ for some integer $a \ge 1$, and therefore

$$1 = MH^{n-1} = ah^{n-1}A^n$$

hence $a = h = A^n = 1$ and then $H^n = 1$, so that $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$ by [ES, Prop. 2.1] (or [Be, Thm. 2.3]). This proves (ii). □

6. Augmented base loci of Ulrich bundles

Given a vector bundle $\mathcal{E}$, it follows by [BKKMSU, Thm. 1.1] that $\mathbf{B}_+(\mathcal{E}) \neq \emptyset$ if and only if $\mathcal{E}$ is not ample if and only if $\mathcal{E}$ is not 0-ample. More generally, given $q \ge 0$, we have by Proposition 4.3 that $\mathcal{E}$ is not q-ample if and only if there exists a subvariety $Z \subseteq X$ of dimension $q+1$ such that $\mathcal{E}_{|Z}$ has a trivial direct summand. For any such subvariety, we have that $Z \cong \mathbb{P}(\mathcal{O}_Z) \subseteq \mathbb{P}(\mathcal{E})$ and, since $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)_{|\mathbb{P}(\mathcal{O}_Z)} = \mathcal{O}_{\mathbb{P}(\mathcal{O}_Z)}(1) \cong \mathcal{O}_Z$, it follows that $\mathbb{P}(\mathcal{O}_Z) \subseteq \mathbf{B}_+(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. If $\pi : \mathbb{P}(\mathcal{E}) \to X$ is the natural map, then [BKKMSU, Prop. 3.2] implies that

$$Z = \pi(\mathbb{P}(\mathcal{O}_Z)) \subseteq \pi(\mathbf{B}_+(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))) = \mathbf{B}_+(\mathcal{E}).$$

It is well-known, using for example [BBP, Prop. 2.3], that one cannot expect, in general, that $\mathbf{B}_+(\mathcal{E})$ is the union of all such Z's, already in the case of line bundles.

Now assume that $\mathcal{E}$ is Ulrich and not ample. It follows by [LS, Thm. 1] that there is a line $L \subseteq X$ such that $\mathcal{E}_{|L}$ is not ample. It was recently proved by Buttinelli [Bu, Thm. 2] that

$$\mathbf{B}_+(\mathcal{E}) = \bigcup_L L$$

where L runs among all lines contained in X such that $\mathcal{E}_{|L}$ is not ample. Equivalently L runs among all lines contained in X such that $\mathcal{E}_{|L}$ has a trivial direct summand. This is the case $q = 0$ of a more general question. In fact, when $\mathcal{E}$ is not q-ample, we have by Theorem 1 that there is a linear space $M \subseteq X$ of dimension $q+1$ such that $\mathcal{E}_{|M}$ has a trivial direct summand. As above, this implies that $M \subseteq \mathbf{B}_+(\mathcal{E})$. Question: is $\mathbf{B}_+(\mathcal{E})$ the union of all such M's?

References

[ACLR] V. Antonelli, G. Casnati, A. F. Lopez, D. Raychaudhury. *On varieties with Ulrich twisted conormal bundles.* Preprint 2023, arXiv:2306.00113. 4

[Be] A. Beauville. *An introduction to Ulrich bundles.* Eur. J. Math. **4** (2018), no. 1, 26-36. 1, 4

[BBP] S. Boucksom, A. Broustet, G. Pacienza. *Uniruledness of stable base loci of adjoint linear systems via Mori theory.* Math. Z. **275** (2013), no. 1-2, 499-507. 4

[BKKMSU] T. Bauer, S. J. Kovàcs, A. Küronya, E. C. Mistretta, T. Szemberg, S. Urbinati. *On positivity and base loci of vector bundles.* Eur. J. Math. **1**(2015), no. 2, 229-249. 1, 4

[Bu] V. Buttinelli . *Positivity of Ulrich bundles in the ample and free case.* Preprint 2024, arXiv:2403.03139. 4

[CMRPL] L. Costa, R. M. Miró-Roig, J. Pons-Llopis. *Ulrich bundles.* De Gruyter Studies in Mathematics, **77**, De Gruyter 2021. 1

[ES] D. Eisenbud, F.-O. Schreyer. *Resultants and Chow forms via exterior syzygies.* J. Amer. Math. Soc. **16** (2003), no. 3, 537-579. 1, 4

[FM] M. Fulger, T. Murayama. *Seshadri constants for vector bundles.* J. Pure Appl. Algebra **225** (2021), no. 4, 106559, 35 pp. 1

[H] R. Hartshorne. *Ample vector bundles.* Inst. Hautes Études Sci. Publ. Math. No. **29** (1966), 63-94. 1

[L] A. F. Lopez. *On the positivity of the first Chern class of an Ulrich vector bundle.* Commun. Contemp. Math. **24** (2022), no. 9, Paper No. 2150071, 22 pp. 1

[LM] A. F. Lopez, R. Muñoz. *On the classification of non-big Ulrich vector bundles on surfaces and threefolds.* Internat. J. Math. **32** (2021), no. 14, Paper No. 2150111, 18 pp. 1

[LMS1] A. F. Lopez, R. Muñoz, J. C. Sierra. *Non-big Ulrich bundles: the classification on quadrics and the case of small numerical dimension.* Preprint 2022, arXiv:2201.06019. To appear on Manuscripta Math.. 1

[LMS2] A. F. Lopez, R. Muñoz, J. C. Sierra. *On the classification of non-big Ulrich vector bundles on fourfolds.* Preprint 2022, arXiv:2205.10143. To appear on Ann. Sc. Norm. Super. Pisa Cl. Sci.. 1

[LS] A. F. Lopez, J. C. Sierra. *A geometrical view of Ulrich vector bundles.* Int. Math. Res. Not. IMRN(2023), no. 11, 9754-9776. 1, 4

[O] G. Ottaviani. *Varietà proiettive di codimensione piccola.* Quaderni INdAM. Aracne, 1995. 3, 4

[S] A. J. Sommese. *Submanifolds of Abelian varieties.* Math. Ann. **233** (1978), no. 3, 229-256. 3

[Tg] H. Tango. *On (n-1)-dimensional projective spaces contained in the Grassmann variety Gr(n,1).* J. Math. Kyoto Univ. **14** (1974), 415-460. 2

[To] B. Totaro. *Line bundles with partially vanishing cohomology.* J. Eur. Math. Soc. (JEMS) **15** (2013), no. 3, 731-754. 1

[Tu] L. W. Tu. *The connectedness of degeneracy loci.* Banach Center Publ., **26**, Part 2, PWN—Polish Scientific Publishers, Warsaw, 1990, 235-248.

Angelo Felice Lopez, Dipartimento di Matematica e Fisica, Università di Roma Tre, Largo San Leonardo Murialdo 1, 00146, Roma, Italy. e-mail `angelo.lopez@mat.uniroma3.it`

Debaditya Raychaudhury, Department of Mathematics, University of Arizona, 617 N Santa Rita Ave., Tucson, AZ 85721, USA. email: `draychaudhury@math.arizona.edu`