

A GEOMETRICAL VIEW OF ULRICH VECTOR BUNDLES

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ABSTRACT. We study geometrical properties of an Ulrich vector bundle \mathcal{E} of rank r on a smooth n -dimensional variety $X \subseteq \mathbb{P}^N$. We characterize ampleness of \mathcal{E} and of $\det \mathcal{E}$ in terms of the restriction to lines contained in X . We prove that all fibers of the map $\Phi_{\mathcal{E}} : X \rightarrow \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ are linear spaces, as well as the projection on X of all fibers of the map $\varphi_{\mathcal{E}} : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}H^0(\mathcal{E})$. Then we get a number of consequences: a characterization of bigness of \mathcal{E} and of $\det \mathcal{E}$ in terms of the maps $\Phi_{\mathcal{E}}$ and $\varphi_{\mathcal{E}}$; when $\det \mathcal{E}$ is big and \mathcal{E} is not big there are infinitely many linear spaces in X through any point of X ; when $\det \mathcal{E}$ is not big, the fibers of $\Phi_{\mathcal{E}}$ and $\varphi_{\mathcal{E}}$ have the same dimension; a classification of Ulrich vector bundles whose determinant has numerical dimension at most $\frac{n}{2}$; a classification of Ulrich vector bundles with $\det \mathcal{E}$ of numerical dimension at most k on a linear \mathbb{P}^k -bundle.

1. INTRODUCTION

Let X be a smooth irreducible variety of dimension $n \geq 1$. Introducing a rank r globally generated vector bundle \mathcal{E} on X gives, as is well known, two ways of investigating the geometry of X . One is via the map

$$\Phi_{\mathcal{E}} : X \rightarrow \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$$

and the other one via the map

$$\varphi_{\mathcal{E}} = \varphi_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)} : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cong \mathbb{P}H^0(\mathcal{E}).$$

The latter map together with the ones associated to multiples $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)$ measure the positivity of \mathcal{E} , for example \mathcal{E} is ample if and only if $\varphi_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)}$ is an embedding for $m \gg 0$ and \mathcal{E} is big if and only if the image of $\varphi_{\mathcal{E}}$ has dimension $n+r-1$. On the other hand, considering the map

$$\lambda_{\mathcal{E}} : \Lambda^r H^0(\mathcal{E}) \rightarrow H^0(\det \mathcal{E})$$

one gets a commutative diagram (see for example [M, §3])

$$(1.1) \quad \begin{array}{ccc} X & \xrightarrow{\Phi_{\mathcal{E}}} & \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E})) \\ \downarrow \varphi_{|\mathrm{Im} \lambda_{\mathcal{E}}|} & & \downarrow P_{\mathcal{E}} \\ \mathbb{P}\mathrm{Im} \lambda_{\mathcal{E}} & \hookrightarrow & \mathbb{P}\Lambda^r H^0(\mathcal{E}) \end{array}$$

where $P_{\mathcal{E}}$ is the Plücker embedding. Thus the positivity of $\det \mathcal{E}$ can be related to the map $\Phi_{\mathcal{E}}$.

There is a very interesting class of vector bundles that has recently attracted a lot of attention, namely that of Ulrich vector bundles. Recall that if $X \subseteq \mathbb{P}^N$, a vector bundle \mathcal{E} on X is an Ulrich vector bundle if $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$.

Ulrich vector bundles are very interesting for several reasons as they play a relevant role both in commutative algebra and in algebraic geometry (see for example [ES, Be, CMRPL] and references therein). The existence problem for Ulrich bundles has implications for a variety of topics, including Boij-Söderberg theory and determinantal representations of Chow forms.

Now an Ulrich vector bundle \mathcal{E} is globally generated, hence one naturally wonders about the positivity properties of \mathcal{E} , of $\det \mathcal{E}$ and the behaviour of the above maps $\Phi_{\mathcal{E}}$ and $\varphi_{\mathcal{E}}$.

It is the purpose of this paper to highlight several nice geometric features of these maps and their relation to the positivity of \mathcal{E} .

The first main result is a characterization of Ulrich vector bundles that are ample, or that have ample determinant, in terms of the above maps and of lines contained in X , as follows.

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Theorem 1.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety and let \mathcal{E} be an Ulrich vector bundle on X . Then the following are equivalent:

- (i) \mathcal{E} is very ample (that is $\varphi_{\mathcal{E}}$ is an embedding).
- (ii) \mathcal{E} is ample.
- (iii) either X does not contain lines or $\mathcal{E}|_L$ is ample on any line $L \subset X$.

Also the following are equivalent:

- (iv) $\Phi_{\mathcal{E}}$ is an embedding.
- (v) $\det \mathcal{E}$ is very ample.
- (vi) $\det \mathcal{E}$ is ample.
- (vii) either X does not contain lines or $\mathcal{E}|_L$ is not trivial on any line $L \subset X$.

As far as we know the best result previously known, is that if $X \subseteq \mathbb{P}^N$ is not covered by lines, then any Ulrich vector bundle on X is big and that if X does not contain any line, then $\det \mathcal{E}$ is ample [Lo, Thm. 1]. Thus Theorem 1 improves and clarifies [Lo, Thm. 1]. Note that one cannot hope to detect bigness simply by restricting to lines (cf. Theorem 1 and Corollary 1 below). For example the spinor bundle on the quadric 3-fold is Ulrich not big, while its restriction to the hyperplane section is Ulrich and big. But they both restrict to lines as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Moreover note that varieties not containing lines are also characterized by the ampleness of a vector bundle, namely of $\Omega_X(2)$ [Br, Prop. 4.2]. A series of examples of application of Theorem 1 are given in Remarks 4.2 and 4.3.

Another very nice property of Ulrich vector bundles is that the fibers of $\Phi_{\mathcal{E}}$ are linear spaces and the fibers of $\varphi_{\mathcal{E}}$ project on X to linear spaces, as we show in the following result.

We set $\nu(\mathcal{E}) := \nu(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ for the numerical dimension.

Theorem 2.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$ and degree d . Let \mathcal{E} be a rank r Ulrich vector bundle on X . Then any scheme-theoretic fiber of $\Phi_{\mathcal{E}} : X \rightarrow \mathbb{G}(r-1, rd-1)$ is a linear subspace in \mathbb{P}^N of dimension at least $n - \nu(\det \mathcal{E})$.

Moreover any scheme-theoretic fiber of $\varphi_{\mathcal{E}} : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^{rd-1}$ projects isomorphically onto a linear subspace in \mathbb{P}^N of dimension at least $n + r - 1 - \nu(\mathcal{E})$ contained in X .

It is clear that, unless the image has dimension 1, the fibers of $\Phi_{\mathcal{E}}$ and $\varphi_{\mathcal{E}}$ can actually have different dimensions, see for example Remark 4.5.

The above theorems and the methods used to prove them lead to several consequences in some interesting cases, listed below. See also Remark 4.4 for global generation of adjoint bundles.

Notice that for any globally generated vector bundle \mathcal{E} one has that if $\varphi_{\mathcal{E}}$ (respectively $\Phi_{\mathcal{E}}$) is birational onto its image, then \mathcal{E} (respectively $\det \mathcal{E}$) is big. It is a nice fact that for an Ulrich vector bundle the converse holds.

Corollary 1.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety and let \mathcal{E} be an Ulrich vector bundle on X . Then the following are equivalent:

- (i) $\varphi_{\mathcal{E}}$ is birational onto its image.
- (ii) \mathcal{E} is big.

Similarly, the following are equivalent:

- (iii) $\Phi_{\mathcal{E}}$ is birational onto its image.
- (iv) $\det \mathcal{E}$ is big.

Next, when $\det \mathcal{E}$ is big and \mathcal{E} is not big, we show that there are actually infinitely many linear subspaces of dimension $n + r - 1 - \nu(\mathcal{E})$ contained in X and passing through any point.

We let $\phi(\mathcal{E})$ be the dimension of the general fiber of $\Phi_{\mathcal{E}}$.

Corollary 2.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$ and let \mathcal{E} be a rank r Ulrich vector bundle on X such that $c_1(\mathcal{E})^n > 0$ (or, equivalently, $\phi(\mathcal{E}) = 0$). If \mathcal{E} is not big, then through any point of X there are infinitely many $(n + r - 1 - \nu(\mathcal{E}))$ -dimensional linear subspaces of \mathbb{P}^N contained in X .

On the other hand, we can draw several consequences when $\det \mathcal{E}$ is not big or has lower numerical dimension.

First, we have a result linking the fibers of $\Phi_{\mathcal{E}}$ and $\varphi_{\mathcal{E}}$ when $\det \mathcal{E}$ is not big (that is when $c_1(\mathcal{E})^n = 0$).

Corollary 3.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$ and let \mathcal{E} be a rank r Ulrich vector bundle on X such that $c_1(\mathcal{E})^n = 0$ (or, equivalently, $\phi(\mathcal{E}) \geq 1$). Then

$$\nu(\mathcal{E}) + \phi(\mathcal{E}) = n + r - 1$$

that is the general fibers of $\Phi_{\mathcal{E}}$ and $\varphi_{\mathcal{E}}$ have the same dimension. Moreover $X \subseteq \mathbb{P}^N$ is covered by a family of linear $\mathbb{P}^{\phi(\mathcal{E})}$'s.

We also get a classification of Ulrich vector bundles whose determinant has numerical dimension at most $\frac{n}{2}$.

Corollary 4.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$. Let \mathcal{E} be a rank r vector bundle on X . Then \mathcal{E} is Ulrich with $c_1(\mathcal{E})^{\lfloor \frac{n}{2} \rfloor + 1} = 0$ if and only if $(X, \mathcal{O}_X(1), \mathcal{E})$ is one of the following:

- (i) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$.
- (ii) $(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1), p^*(\mathcal{G}(\det \mathcal{F})))$, where \mathcal{F} is a very ample rank $n - b + 1$ vector bundle on a smooth irreducible projective variety B of dimension b with $1 \leq b \leq \frac{n}{2}$, $p : X \cong \mathbb{P}(\mathcal{F}) \rightarrow B$ is the projection and \mathcal{G} is a rank r vector bundle on B such that $H^q(\mathcal{G} \otimes S^k \mathcal{F}^*) = 0$ for $q \geq 0, 0 \leq k \leq b - 1$.

The above result improves [Lo, Thm. 3], that was proved only in dimension 3 and with several exceptions. Note that the power $\lfloor \frac{n}{2} \rfloor + 1$ in Corollary 4 is sharp, see Remark 4.6.

In the above corollary naturally appears the case when X is a linear \mathbb{P}^k -bundle over a smooth variety. In fact, in this case, we can classify Ulrich vector bundles when the numerical dimension of its determinant does not exceed k .

Corollary 5.

Let B be a smooth irreducible variety of dimension $b \geq 1$ and let \mathcal{F} be a very ample rank $n - b + 1 \geq 1$ vector bundle on B . Let $X = \mathbb{P}(\mathcal{F}) \subset \mathbb{P}^N$ be embedded with the tautological line bundle H and let $p : X \rightarrow B$ be the projection. Let \mathcal{E} be a rank r vector bundle on X .

Then \mathcal{E} is Ulrich with $c_1(\mathcal{E})^{n-b+1} = 0$ if and only if (X, H, \mathcal{E}) is as follows:

- (i) $(\mathbb{P}^b \times \mathbb{P}^{n-b}, \mathcal{O}_{\mathbb{P}^b}(1) \boxtimes \mathcal{O}_{\mathbb{P}^{n-b}}(1), q^*(\mathcal{O}_{\mathbb{P}^{n-b}}(b))^{\oplus r})$, where $q : X \rightarrow \mathbb{P}^{n-b}$ is the second projection.
- (ii) $(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1), p^*(\mathcal{G}(\det \mathcal{F})))$, where \mathcal{G} is a rank r vector bundle on B such that $H^q(\mathcal{G} \otimes S^k \mathcal{F}^*) = 0$ for $q \geq 0$ and $0 \leq k \leq b - 1$. Moreover $b \leq \frac{n}{2}$ and $c_1(\mathcal{G}(\det \mathcal{F}))^b \neq 0$.

Finally, in the case of $\mathbb{P}^b \times \mathbb{P}^{n-b}$, a more precise result can be proved, see Corollary 4.9.

2. NOTATION AND STANDARD FACTS ABOUT ULRICH VECTOR BUNDLES

We collect in this section some useful definitions, notation and facts.

Given a nef line bundle \mathcal{L} on a smooth variety X we denote by

$$\nu(\mathcal{L}) = \max\{k \geq 0 : c_1(\mathcal{L})^k \neq 0\}$$

the numerical dimension of \mathcal{L} . Recall that when \mathcal{L} is globally generated $\nu(\mathcal{L})$ is the codimension of the general fiber of the morphism induced by \mathcal{L} .

Definition 2.1. Let X be a smooth irreducible variety and let \mathcal{E} be a vector bundle on X . We say that \mathcal{E} is *nef (big, ample, very ample)* if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef (big, ample, very ample). If \mathcal{E} is nef, we define the numerical dimension of \mathcal{E} by $\nu(\mathcal{E}) := \nu(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. We will denote by $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$ the projection map.

Definition 2.2. Let X be a smooth irreducible variety of dimension $n \geq 1$ and let \mathcal{E} be a globally generated rank r vector bundle on X . We set $\phi(\mathcal{E})$ to be the dimension of the general fiber of $\Phi_{\mathcal{E}} : X \rightarrow \mathbb{G}(r - 1, \mathbb{P}H^0(\mathcal{E}))$. For every $x \in X$ we set $P_x = \varphi_{\mathcal{E}}(\mathbb{P}(\mathcal{E}_x))$.

Note that diagram (1.1) implies that

$$(2.1) \quad \phi(\mathcal{E}) = n - \nu(\det \mathcal{E}).$$

Moreover, note that the point $\Phi(x) \in \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ corresponds to the linear subspace $P_x \subseteq \mathbb{P}H^0(\mathcal{E})$. We will write this as

$$\Phi(x) = [P_x]$$

in order to distinguish the subspace $P_x \subseteq \mathbb{P}H^0(\mathcal{E})$ with the point $[P_x] \in \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$.

A simple fact that will be useful is the following.

Remark 2.3. For every $x \in X$ the restriction morphism

$$\varphi|_{\mathbb{P}(\mathcal{E}_x)} : \mathbb{P}(\mathcal{E}_x) \rightarrow P_x$$

is an isomorphism onto a linear subspace of dimension $r-1$ in \mathbb{P}^{rd-1} . In particular $\pi|_F : F \rightarrow X$ is a closed embedding on any scheme theoretic fiber F of φ .

We now turn to a few generalities on Ulrich vector bundles.

Definition 2.4. Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$ and let \mathcal{E} be a vector bundle on X . We say that \mathcal{E} is an *Ulrich vector bundle* if $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$.

Observe that if $X \subseteq \mathbb{P}^N$ is a smooth irreducible variety of degree d and \mathcal{E} is a rank r Ulrich vector bundle on X , then \mathcal{E} is 0-regular in the sense of Castelnuovo-Mumford. Hence \mathcal{E} is globally generated (see for instance [La, Thm. 1.8.5]). Moreover \mathcal{E} is ACM and $h^0(\mathcal{E}) = rd$ (see for example [ES, Prop. 2.1] or [Be, (3.1)]). These facts will be often used without further reference.

3. SEPARATION LEMMAS

We start with some preliminary results, the goal being understanding the separation properties of the linear systems $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ and $|\mathrm{Im} \lambda_{\mathcal{E}}|$ in the case \mathcal{E} is an Ulrich vector bundle.

Lemma 3.1. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$. Let \mathcal{E} be a rank r Ulrich vector bundle on X . Let $s \in \{1, \dots, n\}$ and let H_1, \dots, H_s be hyperplane sections of X . Let $X_s = H_1 \cap \dots \cap H_s$ and suppose that X_s is of dimension $n-s$.*

Then $H^i(\mathcal{E}|_{X_s}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n-s$, that is $\mathcal{E}|_{X_s}$ is an Ulrich sheaf for $(X_s, \mathcal{O}_{X_s}(1))$. Moreover

$$H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_{X_s})$$

is an isomorphism.

Proof. For $0 \leq j \leq s$ set $X_0 = X$ and $X_j = H_1 \cap \dots \cap H_j$. We prove that, for any $0 \leq j \leq s$, the following hold:

$$(3.1) \quad H^i(\mathcal{E}|_{X_j}(-p)) = 0 \text{ for all } i \geq 0 \text{ and } 1 \leq p \leq n-j,$$

and that

$$(3.2) \quad H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_{X_j}) \text{ is an isomorphism.}$$

If $j=0$ note that (3.1) is just the fact that \mathcal{E} is Ulrich and (3.2) is obvious. If $j \geq 1$ assume that (3.1) and (3.2) hold for $j-1$ and consider the exact sequence

$$0 \rightarrow \mathcal{E}|_{X_{j-1}}(-p-1) \rightarrow \mathcal{E}|_{X_{j-1}}(-p) \rightarrow \mathcal{E}|_{X_j}(-p) \rightarrow 0.$$

Then clearly (3.1) holds for j and this also gives that $H^0(\mathcal{E}|_{X_{j-1}}) \rightarrow H^0(\mathcal{E}|_{X_j})$ is an isomorphism. Therefore so is $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_{X_j})$. This proves (3.2) for j and the lemma. \square

We now apply the previous lemma to study the injectivity of $\varphi_{\mathcal{E}}$ and $\Phi_{\mathcal{E}}$ and of their differentials.

First we study the case when two points, possibly infinitely near, do not belong to a line contained in X .

Lemma 3.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$ and degree d . Let \mathcal{E} be a rank r Ulrich vector bundle on X . Let $Z \subset X$ be a 0-dimensional subscheme of length 2 and let $L \subset \mathbb{P}^N$ be the line generated by Z and suppose that $L \not\subset X$. Then*

$$H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_Z)$$

is surjective. Now assume that $Z = \{x, x'\}$ with $x \neq x'$. Then

$$P_x \cap P_{x'} = \emptyset$$

hence, in particular

$$\Phi_{\mathcal{E}}(x) \neq \Phi_{\mathcal{E}}(x').$$

Proof. Let H_1, \dots, H_n be general hyperplane sections containing Z . Then $X_n = H_1 \cap \dots \cap H_n$ is a 0-dimensional subscheme of X of length d containing Z , as $L \not\subset X$. By Lemma 3.1 we have that $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_{X_n})$ is surjective, hence so is $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_Z)$. Therefore $h^0(\mathcal{I}_{Z/X} \otimes \mathcal{E}) = h^0(\mathcal{E}) - 2r = r(d - 2)$. If $Z = \{x, x'\}$ with $x \neq x'$, considering the isomorphism

$$H^0(\mathcal{I}_{\mathbb{P}(\mathcal{E}_x) \cup \mathbb{P}(\mathcal{E}_{x'})/\mathbb{P}(\mathcal{E})} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cong H^0(\mathcal{I}_{\{x, x'\}/X} \otimes \mathcal{E})$$

we deduce that $\dim\langle P_x, P_{x'} \rangle = 2r - 1$, hence that $P_x \cap P_{x'} = \emptyset$. \square

In the case two points, possibly infinitely near, belong to a line contained in X , we will use the following.

Lemma 3.3. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$. Let \mathcal{E} be a rank r Ulrich vector bundle on X . Let $L \subseteq X$ be a line. Then*

$$H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_L)$$

is surjective. Moreover either $(\Phi_{\mathcal{E}})|_L$ is an isomorphism onto its image or it is constant and, in that case, $\mathcal{E}|_L$ is trivial.

Proof. The lemma being obvious if $n = 1$, we suppose that $n \geq 2$ and let H_1, \dots, H_n be general hyperplane sections (of X) containing L . By Lemma 3.1 we have that $H^1(\mathcal{E}|_{X_{n-1}}(-1)) = 0$ and the map

$$H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_{X_{n-1}})$$

is an isomorphism. Now $X_{n-1} = C \cup L$, where C is a curve not containing L or $C = \emptyset$, and in the latter case we are done. Let $D = C \cap L$. From the exact sequence

$$0 \rightarrow \mathcal{E}|_L(-D)(-1) \rightarrow \mathcal{E}|_{X_{n-1}}(-1) \rightarrow \mathcal{E}|_C(-1) \rightarrow 0$$

we find that $H^1(\mathcal{E}|_C(-1)) = 0$. Moreover there is an effective divisor D' on C such that $H_n \cap C = D + D'$ and the exact sequence

$$0 \rightarrow \mathcal{E}|_C(-1) \rightarrow \mathcal{E}|_C(-D) \rightarrow \mathcal{E}|_{D'}(-D) \rightarrow 0$$

implies that $H^1(\mathcal{E}|_C(-D)) = 0$. But then the exact sequence

$$0 \rightarrow \mathcal{E}|_C(-D) \rightarrow \mathcal{E}|_{X_{n-1}} \rightarrow \mathcal{E}|_L \rightarrow 0$$

gives that the map

$$H^0(\mathcal{E}|_{X_{n-1}}) \rightarrow H^0(\mathcal{E}|_L)$$

is surjective and we deduce that the map

$$H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_L)$$

is also surjective.

Consider next the inclusion $\mathbb{P}(\mathcal{E}|_L) \subseteq \mathbb{P}(\mathcal{E})$ so that $(\varphi_{\mathcal{E}})|_{\mathbb{P}(\mathcal{E}|_L)} = \varphi_{\mathcal{E}|_L}$. Let

$$\mathcal{E}|_L \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$

with $a_i \geq 0$ for every $1 \leq i \leq r$ (since $\mathcal{E}|_L$ is globally generated). Then the map

$$\lambda_{\mathcal{E}|_L} : \Lambda^r H^0(\mathcal{E}|_L) \rightarrow H^0(\det(\mathcal{E}|_L))$$

is surjective. We have diagram (1.1) applied to $\mathcal{E}|_L$:

$$\begin{array}{ccc} L & \xrightarrow{\Phi_{\mathcal{E}|_L}} & \mathbb{G}(r-1, h^0(\mathcal{E}|_L) - 1) \\ \downarrow \varphi_{\det(\mathcal{E}|_L)} & & \downarrow P_{\mathcal{E}|_L} \\ \mathbb{P}H^0(\det(\mathcal{E}|_L)) & \hookrightarrow & \mathbb{P}\Lambda^r H^0(\mathcal{E}|_L) \end{array}$$

Now if there is an $i \in \{1, \dots, r\}$ with $a_i > 0$, then $\varphi_{\det(\mathcal{E}|_L)}$ is an isomorphism on its image, and therefore so is $\Phi_{\mathcal{E}|_L}$. Therefore also $(\Phi_{\mathcal{E}})|_L$ is an isomorphism onto its image.

On the other hand, if $a_i = 0$ for every $1 \leq i \leq r$, it follows that $\varphi_{\det(\mathcal{E}|_L)}$ maps L to a point, and then so do $\Phi_{\mathcal{E}|_L}$ and $(\Phi_{\mathcal{E}})|_L$. \square

Lemma 3.4. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 1$. Let \mathcal{E} be a rank r Ulrich vector bundle on X . Let F be a scheme-theoretic fiber of $\Phi_{\mathcal{E}}$. Let $Z \subset F$ be a 0-dimensional subscheme of length 2 and let $L \subset \mathbb{P}^N$ be the line generated by Z . Then $L \subseteq F$. In particular F is a linear space in \mathbb{P}^N .*

Proof. By Lemma 3.2 we have that if $L \not\subseteq X$ then $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_Z)$ is surjective, hence

$$h^0(\mathcal{I}_{Z/X} \otimes \mathcal{E}) = h^0(\mathcal{E}) - 2r \leq h^0(\mathcal{E}) - r - 1$$

and Z is separated by $\Phi_{\mathcal{E}}$ (see for instance [A, Prop. 2.4]), a contradiction. Therefore $L \subseteq X$ and Z is not separated by $(\Phi_{\mathcal{E}})|_L$. Thus $(\Phi_{\mathcal{E}})|_L$ is constant by Lemma 3.3 and then $L \subseteq F$. Since this happens for every two distinct points in F_{red} (if there are) we find that F_{red} is a linear space in \mathbb{P}^N . Moreover, by the same reason, F must be smooth, hence F is a linear space in \mathbb{P}^N . \square

4. PROOF OF THE MAIN RESULTS

We first prove Theorem 1.

Proof of Theorem 1. Obviously (i) implies (ii). If (ii) holds and $L \subset X$ is a line, then

$$(4.1) \quad \mathcal{O}_{\mathbb{P}(\mathcal{E}|_L)}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\mathbb{P}(\mathcal{E}|_L)}$$

is ample, hence $\mathcal{E}|_L$ is ample. Thus we get (iii). Now assume (iii) and let $Z \subset \mathbb{P}(\mathcal{E})$ be a 0-dimensional subscheme of length 2. We will prove that

$$(4.2) \quad r_Z : H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_Z) \text{ is surjective.}$$

If $Z \subset \mathbb{P}(\mathcal{E}_x)$ for some $x \in X$ then we have the following commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) & \xrightarrow{r_Z} & H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_Z) \\ \downarrow f & & \downarrow \cong \\ H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)) & \xrightarrow{r_{Z,x}} & H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)|_Z) \end{array}$$

where f is surjective (see Remark 2.3) and $r_{Z,x}$ is surjective because $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1) \cong \mathcal{O}_{\mathbb{P}^{r-1}}(1)$ is very ample. Therefore r_Z is surjective in this case.

Hence we can assume that Z is not contained in any fiber of π and let $\pi(Z) \subset X$ be the corresponding 0-dimensional subscheme of length 2. Let L be the line in \mathbb{P}^N generated by $\pi(Z)$. If $L \not\subseteq X$, which holds in particular if X does not contain lines, then we know by Lemma 3.2 that $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_{\pi(Z)})$ is surjective, hence so is r_Z . If $L \subset X$ it follows by Lemma 3.3 that $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_L)$ is surjective. On the other hand $\mathcal{E}|_L$ is ample, hence very ample and therefore $H^0(\mathcal{E}|_L) \rightarrow H^0(\mathcal{E}|_{\pi(Z)})$ is surjective. But then again $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_{\pi(Z)})$ is surjective, hence so is r_Z . Thus (i) is proved.

Next, if (iv) holds, then diagram (1.1) gives that $|\mathrm{Im} \lambda_{\mathcal{E}}|$ is very ample, hence also (v) holds. Obviously (v) implies (vi). If (vi) holds and $L \subset X$ is a line, then $\det(\mathcal{E}|_L) = (\det \mathcal{E})|_L$ is ample, hence $\mathcal{E}|_L$ is not trivial. Thus we get (vii). Finally assume (vii). Let $Z \subset X$ be a 0-dimensional subscheme of length 2. We will prove that

$$(4.3) \quad h_Z : \mathrm{Im} \lambda_{\mathcal{E}} \rightarrow H^0((\det \mathcal{E})|_Z) \cong \mathbb{C}^2 \text{ is surjective.}$$

This gives that $|\text{Im}\lambda_{\mathcal{E}}|$ is very ample and, together with diagram (1.1), implies (iv). Let L be the line in \mathbb{P}^N generated by Z . If $L \not\subset X$, which holds in particular if X does not contain lines, then we know by Lemma 3.2 that $g_Z : H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_Z) \cong \mathbb{C}^{2r}$ is surjective, where r is the rank of \mathcal{E} . Consider Z as two possibly infinitely near points x, x' . The surjectivity of g_Z implies that we can find, for every $1 \leq i \leq r$, sections $\sigma_i \in H^0(\mathcal{E})$ such that $\sigma_i(x) = 0$ and $\{\sigma_1(x'), \dots, \sigma_r(x')\}$ are linearly independent (and similarly swapping x and x'). Now this gives that

$$\sigma_{1|Z}(x) \wedge \dots \wedge \sigma_{r|Z}(x) = 0, \sigma_{1|Z}(x') \wedge \dots \wedge \sigma_{r|Z}(x') \neq 0$$

and similarly swapping x and x' . Thus h_Z is surjective.

If $L \subset X$ it follows by Lemma 3.3 that $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_L)$ is surjective and that, being $\mathcal{E}|_L$ globally generated, that $\lambda_{\mathcal{E}|_L}$ is surjective. This gives the following commutative diagram

$$\begin{array}{ccc} \Lambda^r H^0(\mathcal{E}) & \xrightarrow{\lambda_{\mathcal{E}}} & \text{Im}\lambda_{\mathcal{E}} \hookrightarrow H^0(\det \mathcal{E}) \\ \downarrow & & \downarrow h_L \\ \Lambda^r H^0(\mathcal{E}|_L) & \xrightarrow{\lambda_{\mathcal{E}|_L}} & H^0(\det(\mathcal{E}|_L)) \end{array}$$

hence h_L is surjective. Now $\det(\mathcal{E}|_L)$ is very ample hence

$$H^0(\det(\mathcal{E}|_L)) \rightarrow H^0(\det(\mathcal{E}|_L)|_Z) = H^0((\det \mathcal{E})|_Z)$$

is surjective and composing with h_L we get that h_Z is surjective. Thus (4.3) is proved and (iv) holds. Alternatively, the fact that (iv) holds assuming (vii) can be proved as follows. By a standard separation criterion (see for instance [A, Prop. 2.4]), to get that $\Phi_{\mathcal{E}}$ is an embedding one just needs to prove that

$$(4.4) \quad h^0(\mathcal{I}_{Z/X} \otimes \mathcal{E}) \leq h^0(\mathcal{E}) - r - 1.$$

This clearly holds when the line L generated by Z is not contained in X , since Lemma 3.2 gives that $h^0(\mathcal{I}_{Z/X} \otimes \mathcal{E}) = h^0(\mathcal{E}) - 2r$. On the other hand, if $L \subset X$, we know by Lemma 3.3 that $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_L)$ is surjective. Therefore $h^0(\mathcal{I}_{Z/X} \otimes \mathcal{E}) = h^0(\mathcal{E}) - \text{rk } \psi$, where $\psi : H^0(\mathcal{E}|_L) \rightarrow H^0(\mathcal{E}|_Z)$. Now $\mathcal{E}|_L$ is globally generated and non-trivial, hence we can write $\mathcal{E}|_L \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$ with $a_1 \geq \dots \geq a_r \geq 0$ and $a_1 \geq 1$. Since $\text{Ker } \psi = H^0(\mathcal{E}|_L(-Z))$, letting k be the number of $i \in \{1, \dots, r\}$ such that $a_i \geq 2$, we get

$$\text{rk } \psi = \sum_{i=1}^r (a_i + 1) - \sum_{i=1}^k (a_i - 1) = \sum_{i=k+1}^r a_i + r + k \geq r + 1$$

hence (4.4) holds. \square

Remark 4.1. With the same method of proof of Theorem 1 one can show that if \mathcal{E} is a rank r ample Ulrich vector bundle on $X \subset \mathbb{P}^N$ then, the Seshadri constants, on any $x \in X$, satisfy:

- (i) $\varepsilon(\mathcal{E}, x) \geq 1$.
- (ii) $\varepsilon(\det \mathcal{E}, x) \geq r$.

The same can be proven using Theorem 1, [BSS] and [FM].

In the next remarks we give a non-exhausting list of examples of application of Theorem 1 and Corollary 1.

Remark 4.2. The following are examples of varieties that do not contain lines but are known to carry an Ulrich vector bundle, which, by Theorem 1, is then very ample.

- (i) (X, H) a polarized abelian surface or a general polarized K3 surface of genus $g \geq 3$ (Ulrich vector bundles exist by [Be, Prop. 5.1] and [F, Thm. 1]).
- (ii) X a general degree d hypersurface in \mathbb{P}^{n+1} with $d \geq 2n$ (Ulrich vector bundles exist by [HUB]).
- (iii) X any variety that carries an Ulrich vector bundle for a very ample L and $H = mL, m \geq 2$ (Ulrich vector bundles exist by [Be, Cor. 3.2]). Among these examples we have all the Veronese varieties.

Remark 4.3. The following are examples of varieties that contain lines and of an Ulrich vector bundle \mathcal{E} , which is ample (respectively not ample but not trivial) on any line, hence Theorem 1 gives that \mathcal{E} is very ample (resp. \mathcal{E} is not ample but $\det \mathcal{E}$ is very ample).

- (i) Let Y, Z be smooth varieties of dimension n and m , let L, M be two very ample line bundles on Y, Z and let \mathcal{F} be an Ulrich vector bundle for (Y, L) and \mathcal{G} an Ulrich vector bundle for (Z, M) . Assume that Y is covered by lines, while Z does not contain lines. Let $X = Y \times Z$ and $H = L \boxtimes M$. Then $\mathcal{E} = \mathcal{F}(m) \boxtimes \mathcal{G}$ is an Ulrich vector bundle for (X, H) by [Be, (3.5)]. Moreover X is covered by lines and $\mathcal{E}|_R \cong \mathcal{F}|_R(m)^{\oplus \text{rk} \mathcal{G}}$ is ample on any line $R \subset X$. Therefore \mathcal{E} is very ample by Theorem 1.
- (ii) Let (X, H) be as in (i) but let now $\mathcal{E} = \mathcal{F} \boxtimes \mathcal{G}(n)$, which is an Ulrich vector bundle by [Be, (3.5)]. Moreover assume that $\mathcal{F}|_R$ is not ample but not trivial on any line $R \subset Y$ (for example \mathcal{F} a spinor bundle on a quadric Y). Then the same holds for \mathcal{E} , so that \mathcal{E} is not ample but $\det \mathcal{E}$ is very ample by Theorem 1.
- (iii) Let C be a smooth curve and let \mathcal{F} be a very ample vector bundle on C . Let $X = \mathbb{P}(\mathcal{F})$ and $H = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$. Assume that either C is not rational or $C \cong \mathbb{P}^1$ and $\mathcal{F} \cong \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 2$ for every $1 \leq i \leq s$. It follows by [H, Thm. A] that X carries a rank r Ulrich vector bundle \mathcal{E} such that $\mathcal{E}|_R \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$ is ample on any line $R \subset X$. Therefore \mathcal{E} is very ample by Theorem 1. Examples of these varieties include rational normal scrolls or more generally scrolls in \mathbb{P}^N .
- (iv) Let $Q \subset \mathbb{P}^{n+1}$ be a smooth quadric. For $n \geq 7, n \neq 10$ any Ulrich vector bundle \mathcal{E} on Q is big by [LMS1, Thm. 1] but not ample and $\det \mathcal{E}$ is very ample.
- (v) Let X be either a smooth cubic or quartic fourfold in \mathbb{P}^5 , or a smooth complete intersection of type $(2, 2), (2, 3)$ in \mathbb{P}^6 or of type $(2, 2, 2)$ in \mathbb{P}^7 . Then X is covered by lines and [LMS2, Cor. 1] implies that any Ulrich vector bundle \mathcal{E} on X is big with $\det \mathcal{E}$ very ample (Ulrich vector bundles exist by [HUB]).
- (vi) Let $d \in \{3, \dots, 8\}$ and let $X \subset \mathbb{P}^{d+1}$ be a Fano threefold of index 2. Then X is covered by lines unless $d = 8$ and X carries a big Ulrich vector bundle by [Be, Prop. 6.1] and Corollary 2. Furthermore, if $d \in \{3, 4, 5\}$ then every Ulrich vector bundle on X is big by Corollary 2 as $\det \mathcal{E}$ is very ample for any $(X, \mathcal{O}_X(1)) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ such that $\rho(X) = 1$.
- (vii) On the other hand, some examples of non-big Ulrich vector bundles can be found in [LM, LMS1, LMS2].

Next we prove Theorem 2.

Proof of Theorem 2. It follows by Lemma 3.4 and (2.1) that any scheme-theoretic fiber of $\Phi_{\mathcal{E}}$ is a linear space of dimension at least $\phi(\mathcal{E}) = n - \nu(\det \mathcal{E})$. Now let $F = \varphi_{\mathcal{E}}^{-1}(\varphi_{\mathcal{E}}(z))$ be any scheme-theoretic fiber of $\varphi_{\mathcal{E}}$, so that $\dim F \geq n + r - 1 - \nu(\mathcal{E})$. We want to show that the scheme-theoretic image $\pi(F) \subseteq X \subseteq \mathbb{P}^N$ is a linear space. To this end it is enough to show, as done in Lemma 3.4, that for any 0-dimensional subscheme $Z \subset \pi(F)$ of length 2, then the line $L := \langle Z \rangle$ is contained in $\pi(F)$. By the second part of Remark 2.3 there is a 0-dimensional subscheme $Z' \subset F$ of length 2 such that $Z = \pi(Z')$. By hypothesis we have that $H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{Z'})$ is not surjective, that is $H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_Z)$ is not surjective. Hence Lemma 3.2 gives that $L \subset X$. Now Lemma 3.3 implies that also $H^0(\mathcal{E}|_L) \rightarrow H^0(\mathcal{E}|_Z)$ is not surjective. Thus $\mathcal{E}|_L$ must have a maximal trivial direct summand, say $\mathcal{O}_{\mathbb{P}^1}^{\oplus k}$ for some integer $1 \leq k \leq r$ and $\mathbb{P}(\mathcal{E}|_L)$ gets mapped by $\varphi_{\mathcal{E}}$ onto a rational normal scroll that is a cone with vertex $V = \mathbb{P}^{k-1} \subset \mathbb{P}^{r-d-1}$ when $k \leq r - 1$ or onto $V = \mathbb{P}^{r-1}$ when $k = r$. It follows that $\varphi_{\mathcal{E}}(F) \subseteq V$. For any $x'' \in L$ we have that $V \subseteq \varphi_{\mathcal{E}}(\pi^{-1}(x''))$, hence there exists $z'' \in \pi^{-1}(x'')$ such that $\varphi_{\mathcal{E}}(z'') = \varphi_{\mathcal{E}}(z)$. Therefore $z'' \in F$ and $x'' = \pi(z'') \in \pi(F)$. \square

Remark 4.4. Using Theorem 2 and [BSS, Rmk. 1] we get that if \mathcal{E} is an Ulrich vector bundle on $X \subseteq \mathbb{P}^N$ and H is a hyperplane divisor, then $K_X + tH$ is globally generated provided that

$$t \geq \max\{n - \nu(\det \mathcal{E}) + 1, \nu(\det \mathcal{E}) + 1\}.$$

Remark 4.5. Let $X \subset \mathbb{P}^7$ be the Del Pezzo surface of degree 7, so that X is the blow-up of the plane in two points. Let E_1, E_2 be the exceptional divisors and let \tilde{H} be the inverse image of a line in the plane. Let $\mathcal{E} = \mathcal{O}_X(H + E_1 - E_2) = \mathcal{O}_X(3\tilde{H} - 2E_2)$. Then \mathcal{E} is an Ulrich line bundle on X by [Be, Prop. 4.1(i)]. Now $\Phi_{\mathcal{E}}$ is birational onto its image and behaves differently on the three lines contained in X : E_1 is contracted, E_2 maps to a conic and $\tilde{H} - E_1 - E_2$ maps to a line.

Now we prove the corollaries.

Proof of Corollary 1. Obviously (i) implies (ii) and (iii) implies (iv) by (1.1). Vice versa assume that \mathcal{E} (respectively $\det \mathcal{E}$) is big. Then $\nu(\mathcal{E}) = n + r - 1$ (respectively $\nu(\det \mathcal{E}) = n$). Hence the general fiber of $\varphi_{\mathcal{E}}$ (respectively of $\Phi_{\mathcal{E}}$) is 0-dimensional, so it is a point by Theorem 2. Therefore $\varphi_{\mathcal{E}}$ (respectively $\Phi_{\mathcal{E}}$) is birational onto its image. \square

Proof of Corollary 2. Set $\nu = \nu(\mathcal{E})$. Since $\dim \operatorname{Im} \varphi = \nu$ we have that $\dim \varphi^{-1}(y) \geq n + r - \nu - 1$ for any $y \in \operatorname{Im} \varphi$.

Let $x \in X$ be a general point and let

$$Z = \{z \in X : P_z \cap P_x \neq \emptyset\}.$$

We claim that $\dim Z \geq n + r - \nu$.

To see this, consider the incidence correspondence

$$\mathcal{I} = \{(y, z) \in P_x \times X : y \in P_z\} \subset P_x \times X$$

together with its projections

$$\begin{array}{ccc} & \mathcal{I} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ P_x & & X \end{array} .$$

Let $y \in P_x$. Then $(y, x) \in \mathcal{I}$, hence π_1 is surjective. Next observe that $\dim \pi_1^{-1}(y) \geq n + r - \nu - 1$.

In fact we know that $\dim \varphi^{-1}(y) \geq n + r - \nu - 1$. Since π is injective on $\varphi^{-1}(y)$ by Remark 2.3, we get that $\dim \pi(\varphi^{-1}(y)) \geq n + r - \nu - 1$. Also

$$\pi(\varphi^{-1}(y)) \subseteq \{z \in X : y \in P_z\} \cong \pi_1^{-1}(y) :$$

if $z \in \pi(\varphi^{-1}(y))$ then there is $u \in \varphi^{-1}(y)$ such that $z = \pi(u)$. But $u \in \mathbb{P}(\mathcal{E}_z)$, hence $y = \varphi(u) \in P_z$.

Therefore $\dim \pi_1^{-1}(y) \geq n + r - \nu - 1$ and then $\dim \mathcal{I} \geq n + 2r - \nu - 2$. Pick an irreducible component $\mathcal{I}_1 \subseteq \mathcal{I}$ such that $\dim \mathcal{I}_1 \geq n + 2r - \nu - 2$. Let $z \in \pi_2(\mathcal{I}_1)$. Then $\pi_2^{-1}(z) \cong P_x \cap P_z$ hence $\dim \pi_2^{-1}(z) \leq r - 1$, therefore $\pi_2(\mathcal{I}_1)$ is irreducible and $\dim \pi_2(\mathcal{I}_1) \geq n + r - \nu - 1 \geq 1$ because \mathcal{E} is not big. Now let $z \in \pi_2(\mathcal{I}_1)$ be general. Then $z \neq x$. If $\dim \pi_2^{-1}(z) = r - 1$ then $P_x = P_z$ hence $\Phi(x) = \Phi(z)$. But $x \in X$ is a general point and then Theorem 2 implies that $x = z$, a contradiction. Hence $\dim \pi_2^{-1}(z) \leq r - 2$ and this gives that $\dim \pi_2(\mathcal{I}_1) \geq n + r - \nu$. On the other hand $\pi_2(\mathcal{I}_1) \subseteq Z$, so that $\dim Z \geq n + r - \nu$.

Given any $z \in Z, z \neq x$ it follows by Theorem 2 that $\pi(\varphi^{-1}(y)) \subset X$ is a linear subspace of dimension at least $n + r - \nu - 1$ in \mathbb{P}^N for any $y \in P_z \cap P_x$. Moreover

$$Z = \bigcup_{z \in Z \setminus \{x\}} \bigcup_{y \in P_z \cap P_x} \pi(\varphi^{-1}(y)).$$

As $\dim Z \geq n + r - \nu$ we find that through a general point of X there are infinitely many $(n + r - 1 - \nu)$ -dimensional linear subspaces contained in X . Since this is a closed condition, the same holds for any $x \in X$. \square

Proof of Corollary 3. As above we set $\Phi = \Phi_{\mathcal{E}}, \varphi = \varphi_{\mathcal{E}}, \phi = \phi(\mathcal{E})$ and $\nu = \nu(\mathcal{E})$. Note that $c_1(\mathcal{E})^n = 0$ is equivalent to $\phi \geq 1$ by (2.1). Let F be a general fiber of Φ , so that we know by Theorem 2 that $F = \mathbb{P}^{\phi} \subseteq \mathbb{P}^N$ and that X is covered by a family of linear \mathbb{P}^{ϕ} 's. Let $\Phi(F) = [P] \in \mathbb{G}(r - 1, rd - 1)$. For a general $y \in \operatorname{Im} \varphi$ set

$$W_y = \{x \in X : y \in P_x\}.$$

Note that $\pi|_{\varphi^{-1}(y)} : \varphi^{-1}(y) \rightarrow W_y$ is an isomorphism by Remark 2.3, hence

$$\dim W_y = \dim \varphi^{-1}(y) = n + r - 1 - \nu.$$

We will now prove that $F = W_y$ for any $y \in P$. This gives the corollary.

First $F \subseteq W_y$. In fact if $x \in F$ then $[P] = \Phi(F) = \Phi(x) = [P_x]$ so that $y \in P = P_x$, hence $x \in W_y$.

Now assume that there exists $x \in W_y \setminus F$. For any $x' \in F$ we have that $\langle x, x' \rangle \subset X$. In fact if $\langle x, x' \rangle$ is not contained in X , then Lemma 3.2 implies that $P_x \cap P_{x'} = \emptyset$. On the other hand $[P] = \Phi(F) = \Phi(x') = [P_{x'}]$, hence $y \in P = P_{x'}$. But $x \in W_y$, hence $y \in P_x$, a contradiction.

Therefore $\mathbb{P}^{\phi+1} = \langle x, F \rangle \subset X$. Now the restriction of Φ to $\mathbb{P}^{\phi+1} = \langle x, F \rangle$ is a morphism that contracts a hyperplane $F = \mathbb{P}^{\phi}$ to a point, hence it is constant. But this gives the contradiction $x \in F$.

Therefore $F = W_y$ and the corollary is proved. \square

Proof of Corollary 4. If $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (i) or (ii) it follows by [Lo, Lemma 4.1] that \mathcal{E} is Ulrich and obviously $c_1(\mathcal{E})^{\lfloor \frac{n}{2} \rfloor + 1} = 0$.

Assume now that \mathcal{E} is Ulrich with $c_1(\mathcal{E})^{\lfloor \frac{n}{2} \rfloor + 1} = 0$.

Since $\lfloor \frac{n}{2} \rfloor + 1 \leq n$ we have that $(\det \mathcal{E})^n = 0$. If $\rho(X) = 1$, then $\det \mathcal{E} \cong \mathcal{O}_X$ and this gives that $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$. Hence $rd = h^0(\mathcal{E}) = r$ and we are in case (i).

Therefore we may assume that $\rho(X) \geq 2$.

Since $\nu(\det \mathcal{E}) \leq \frac{n}{2}$, we deduce from (2.1) that $\phi(\mathcal{E}) \geq \frac{n}{2}$. By Theorem 2 we know that X is covered by a family of $\mathbb{P}^{\phi(\mathcal{E})}$'s, hence [S2, Main Thm.] implies that there is a smooth irreducible projective variety B of dimension $b \geq 1$, a very ample rank $n - b + 1$ vector bundle \mathcal{F} on B such that $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$. Let $p : \mathbb{P}(\mathcal{F}) \rightarrow B$ be the projection. Then [S2, Main Thm.] gives that there is a nonempty open subset U of X such that

$$\mathbb{P}^{\phi(\mathcal{E})} = \Phi^{-1}(\Phi(u)) \subseteq p^{-1}(p(u)) = \mathbb{P}^{n-b} \text{ for every } u \in U.$$

In particular we have that $\phi(\mathcal{E}) \leq n - b$, hence $b \leq n - \phi(\mathcal{E}) \leq \frac{n}{2}$.

We actually claim that all fibers of Φ and p coincide.

Let $x \in X$. Then $p^{-1}(p(x)) = \mathbb{P}^{n-b}$ and let us show that $\Phi(p^{-1}(p(x)))$ is a point. In fact, if not, then

$$\frac{n}{2} \leq \phi(\mathcal{E}) \leq n - b = \dim \Phi(p^{-1}(p(x))) \leq \dim \Phi(X) = n - \phi(\mathcal{E}) \leq \frac{n}{2}$$

hence $\Phi(p^{-1}(p(x))) = \Phi(X)$. Now for every $u \in U$ there exists $x' \in p^{-1}(p(x))$ such that $\Phi(x') = \Phi(u)$, so that $x' \in \Phi^{-1}(\Phi(u)) \subseteq p^{-1}(p(u))$. This gives that $p(u) = p(x') = p(x)$, hence $u \in p^{-1}(p(x))$. But then we get the contradiction $U \subset p^{-1}(p(x))$.

Therefore $\Phi(p^{-1}(p(x)))$ is a point and

$$p^{-1}(p(x)) \subseteq \Phi^{-1}(\Phi(x))$$

holds for every $x \in X$. Let us see that they are actually equal. Suppose that there exists $x' \in \Phi^{-1}(\Phi(x)) \setminus p^{-1}(p(x))$. Now Theorem 2 gives that $\Phi^{-1}(\Phi(x)) = \mathbb{P}^s$ for some $s \leq n - 1$. Moreover $p^{-1}(p(x')) \subseteq \Phi^{-1}(\Phi(x')) = \Phi^{-1}(\Phi(x))$, hence we have a \mathbb{P}^s containing two disjoint \mathbb{P}^{n-b} 's, namely $p^{-1}(p(x'))$ and $p^{-1}(p(x))$. But this is a contradiction since $b \leq \frac{n}{2}$.

Thus $p^{-1}(p(x)) = \Phi^{-1}(\Phi(x))$ for every $x \in X$.

Consider the Stein factorization

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\Phi}} & B_1 \\ & \searrow \Phi & \downarrow g \\ & & \Phi(X) \end{array}$$

so that $\tilde{\Phi}_* \mathcal{O}_X \cong \mathcal{O}_{B_1}$ and g is finite. Since the fibers of Φ are connected by Theorem 2, it follows that g is bijective and therefore $p^{-1}(p(x)) = \Phi^{-1}(\Phi(x)) = \tilde{\Phi}^{-1}(\tilde{\Phi}(x))$ for every $x \in X$. Hence we can apply [D, Lemma 1.15(b)] and deduce that $\tilde{\Phi}$ factorizes through p and that p factorizes through $\tilde{\Phi}$. Since they have the same fibers, it follows that $B \cong B_1$.

Now we know that $\mathcal{E} = \Phi^* \mathcal{U} = \tilde{\Phi}^*(g^* \mathcal{U})$, where \mathcal{U} is the tautological bundle on $\mathbb{G}(r - 1, \mathbb{P}H^0(\mathcal{E}))$. Thus there is a rank r vector bundle \mathcal{H} on B such that $\mathcal{E} \cong p^* \mathcal{H}$. Setting $\mathcal{G} = \mathcal{H}(-\det \mathcal{F})$ and using [Lo, Lemma 4.1] we deduce that $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (ii). \square

Remark 4.6. Let Y be a smooth irreducible variety of dimension m , let L be a very ample line bundle on Y . Let X be a hyperplane section of the Segre embedding of $\mathbb{P}^{m-1} \times Y$. As shown in [BS, Ex. 14.1.5] the restriction of the second projection $p : X \rightarrow Y$ exhibits (X, H) as a scroll over Y with general fiber a linear \mathbb{P}^{m-2} , but having some fibers equal to a linear \mathbb{P}^{m-1} . Hence X is of dimension $n = 2m - 2$ and, if (Y, L) is not covered by lines, X is not a \mathbb{P}^{n-b} -bundle over a variety of dimension b . On the other hand, choosing an Ulrich vector bundle \mathcal{G} for (Y, L) and setting $\mathcal{E} = p^*(\mathcal{G}((m-1)L))$ we get by [Lo, Lemma 4.1] an Ulrich vector bundle on X such that $c_1(\mathcal{E})^{\lfloor \frac{n}{2} \rfloor + 1} \neq 0$ and $c_1(\mathcal{E})^{\lfloor \frac{n}{2} \rfloor + 2} = 0$. There are several examples of pairs (Y, L) as above. One can take a smooth hypersurface $Y \subset \mathbb{P}^{m+1}$, which is

not covered by lines when $\deg Y \geq m+2$ and admits Ulrich vector bundles by [HUB]. See also Remark 4.2.

Proof of Corollary 5. If $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (i) or (ii) it follows by [Lo, Lemma 4.1] that \mathcal{E} is Ulrich and obviously $c_1(\mathcal{E})^{n-b+1} = 0$.

Vice versa assume that \mathcal{E} is Ulrich with $c_1(\mathcal{E})^{n-b+1} = 0$.

If $n = b$ then $c_1(\mathcal{E}) = 0$, so $\Phi_{\mathcal{E}}$ is a constant map and $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$. Hence $rd = h^0(\mathcal{E}) = r$ and we get (i). From now on assume that $n \geq b+1$.

By hypothesis we have that $\nu(\det \mathcal{E}) \leq n - b$, hence (2.1) gives that

$$\phi(\mathcal{E}) \geq b.$$

Set $\Phi = \Phi_{\mathcal{E}}$ and, for any $x \in X$, set $f_x = p^{-1}(p(x))$ and $F_x = \Phi^{-1}(\Phi(x))$. Hence f_x is a linear subspace of \mathbb{P}^N of dimension $n - b$, while F_x is a linear subspace of \mathbb{P}^N by Theorem 2 with $\dim F_x \geq \phi(\mathcal{E})$ and equality holds for a general $x \in X$. Moreover $x \in f_x \cap F_x$.

Claim 4.7. *Either $f_x \cap F_x = \{x\}$ or $f_x = F_x$.*

Proof. Assume that $f_x \cap F_x \neq \{x\}$, hence $f_x \cap F_x$ is a positive dimensional linear subspace of \mathbb{P}^N . Now $p|_{F_x} : F_x \rightarrow B$ contracts $f_x \cap F_x$ to a point. Since any morphism from a projective space is either constant or finite, we get that $p(F_x)$ is a point and therefore $F_x \subseteq f_x$. Similarly, $\Phi|_{f_x} : f_x \rightarrow \Phi(X)$ contracts $f_x \cap F_x$ to a point, hence, as above, $\Phi(f_x)$ is a point and therefore $f_x \subseteq F_x$. This proves the Claim. \square

Moreover we have

Claim 4.8. *If $f_x \neq F_x$, then $\phi(\mathcal{E}) = b$, $F_x = \mathbb{P}^b \cong B$. Also $\Phi|_{f_x} : f_x \rightarrow \Phi(X)$ is an embedding.*

Proof. Let $z \in F_x$, so that $F_z = F_x$. If $f_z \cap F_z \neq \{z\}$ then $F_x = F_z = f_z$ by Claim 4.7. Hence $x \in f_z$ and therefore $f_x = f_z = F_x$, a contradiction. Therefore $\{z\} = f_z \cap F_z = f_z \cap F_x$, hence $p|_{F_x} : F_x \rightarrow B$ is injective. The injectivity of $\Phi|_{f_x} : f_x \rightarrow \Phi(X)$ is proved in the same way. As in the proof of Theorem 1, if $\Phi|_{f_x}$ were not an embedding, it would have a positive dimensional fiber, a contradiction. Moreover

$$b \leq \phi(\mathcal{E}) \leq \dim F_x \leq \dim B = b$$

so that $\phi(\mathcal{E}) = b$, $F_x = \mathbb{P}^b$ by Theorem 2 and $B \cong \mathbb{P}^b$. This proves the Claim. \square

Consider the Stein factorization

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\Phi}} & B_1 \\ & \searrow \Phi & \downarrow g \\ & & \Phi(X) \end{array}$$

and set $\tilde{F}_x = \tilde{\Phi}^{-1}(\tilde{\Phi}(x))$. As in the proof of Corollary 4 we see that $F_x = \tilde{F}_x$ for every $x \in X$.

We divide the proof in two cases.

Case 1: There is a point $x \in X$ such that $f_x = F_x$.

We show that $f_{x'} = F_{x'}$ for every $x' \in X$.

Assume to the contrary that there is $x' \in X$, $x' \neq x$ such that $f_{x'} \neq F_{x'}$. Then $f_{x'} \cap F_{x'} = \{x'\}$ by Claim 4.7 and $\dim F_{x'} = b$ by Claim 4.8. Therefore $f_x \cdot F_{x'} = f_{x'} \cdot F_{x'} = 1$, hence $F_x \cap F_{x'} = f_x \cap F_{x'} \neq \emptyset$ and we deduce that $F_x = F_{x'}$. But then

$$f_x \cap f_{x'} = F_{x'} \cap f_{x'} = \{x'\}$$

hence $f_x = f_{x'}$, giving the contradiction $\mathbb{P}^{n-b} = f_x = \{x'\}$.

Therefore $f_{x'} = F_{x'}$ for every $x' \in X$ and this gives that $\phi(\mathcal{E}) = n - b$ and $b \leq \frac{n}{2}$ since $\phi(\mathcal{E}) \geq b$. Now, precisely as in the proof of Corollary 4, we can apply [D, Lemma 1.15(b)] and deduce that we are in case (ii).

This concludes Case 1.

Case 2: $f_x \neq F_x$ for every $x \in X$.

Then Claim 4.8 gives that $\phi(\mathcal{E}) = b$ and, for every $x \in X$, $F_x = \mathbb{P}^b \cong B$. Also Claim 4.8 gives that $\Phi|_{f_x} : f_x \rightarrow \Phi(X)$ is an embedding and since $\dim \Phi(X) = n - b$ we get that $\Phi(X) \cong \mathbb{P}^{n-b}$.

Since F_x is a linear space of dimension b for every $x \in X$, we get that $\Phi : X \rightarrow \mathbb{P}^{n-b}$ is a linear \mathbb{P}^b -bundle. Now $f_x \neq F_x$ for every $x \in X$, hence the two linear bundle structures given by $p : X \rightarrow B \cong \mathbb{P}^b$ and $\Phi : X \rightarrow \Phi(X) \cong \mathbb{P}^{n-b}$ are different, that is that there is no isomorphism $h : \Phi(X) \rightarrow B$ such that $p = h \circ \Phi$. Then it follows by [S1, Thm. A] that $(X, H) \cong (\mathbb{P}^b \times \mathbb{P}^{n-b}, \mathcal{O}_{\mathbb{P}^b}(1) \boxtimes \mathcal{O}_{\mathbb{P}^{n-b}}(1))$. Since $\mathcal{E} = \Phi^*\mathcal{U}$ where \mathcal{U} is the tautological bundle on $\mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$, [Lo, Lemma 4.1] together with [ES, Prop. 2.1] (or [Be, Thm. 2.3]) give that $\mathcal{E} \cong q^*(\mathcal{O}_{\mathbb{P}^{n-b}}(b))^{\oplus r}$, where q is the second projection. Then we are in case (i). This concludes Case 2. \square

With the same methods we can also classify Ulrich vector bundles with non-big determinant on the Segre product.

Corollary 4.9. *Let \mathcal{E} be a rank r vector bundle on $\mathbb{P}^b \times \mathbb{P}^{n-b} \subset \mathbb{P}^N$ embedded with $\mathcal{O}_{\mathbb{P}^b}(1) \boxtimes \mathcal{O}_{\mathbb{P}^{n-b}}(1)$. Then \mathcal{E} is Ulrich with $\det \mathcal{E}$ not big if and only if \mathcal{E} is either $p^*(\mathcal{O}_{\mathbb{P}^b}(n-b))^{\oplus r}$ or $q^*(\mathcal{O}_{\mathbb{P}^{n-b}}(b))^{\oplus r}$ where p and q are the projections.*

Proof. If \mathcal{E} is either $p^*(\mathcal{O}_{\mathbb{P}^b}(n-b))^{\oplus r}$ or $q^*(\mathcal{O}_{\mathbb{P}^{n-b}}(b))^{\oplus r}$ then it is Ulrich by [Lo, Lemma 4.1] and clearly $\det \mathcal{E}$ is not big.

Vice versa assume that \mathcal{E} is Ulrich with $\det \mathcal{E}$ not big.

Set $\Phi = \Phi_{\mathcal{E}}$. For any $x \in X = \mathbb{P}^b \times \mathbb{P}^{n-b}$, let $f_x = p^{-1}(p(x))$, $g_x = q^{-1}(q(x))$ and $F_x = \Phi^{-1}(\Phi(x))$. We know by Theorem 2 that F_x is a linear subspace contained in X and passing through x . But it is well known that, on the Segre embedding, any linear subspace must be contained in a fiber. Then either $F_x \subseteq f_x$ or $F_x \subseteq g_x$. Assume for example that $F_x \subseteq f_x$ for some $x \in X$. Since $\det \mathcal{E}$ is not big we get $\dim F_x \geq 1$ by (2.1). Moreover $\Phi(F_x) = \Phi(x)$, hence, as in the proof of Claim 4.7, $\Phi(f_x) = \Phi(x)$ and therefore $F_x = f_x$. We prove now that $F_{x'} = f_{x'}$ for any $x' \in X$. In fact, otherwise, $F_{x'} = g_{x'}$ for some $x \neq x' \in X$ and we get that $F_x \cap F_{x'} = f_x \cap g_{x'} = \{(p(x), q(x'))\}$. But then $F_x = F_{x'}$ giving the contradiction $F_x = \{(p(x), q(x'))\}$.

Then we get $\mathcal{E} = q^*(\mathcal{O}_{\mathbb{P}^{n-b}}(b))^{\oplus r}$ as in the proof of Corollary 5. The same works if $F_x \subseteq g_x$ for some $x \in X$.

Alternatively, we can prove the corollary as follows.

Set $L = p^*\mathcal{O}_{\mathbb{P}^b}(1)$ and $M = q^*\mathcal{O}_{\mathbb{P}^{n-b}}(1)$. Then we can write $\det \mathcal{E} = aL + cM$ for some $a, c \in \mathbb{Z}$ and

$$0 = (aL + cM)^n = \binom{n}{b} a^b c^{n-b}$$

implies that either $a = 0$ or $c = 0$. It follows by [Lo, Lemmas 5.1 and 4.1] that \mathcal{E} is as claimed. \square

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