ON THE CLASSIFICATION OF NON-BIG ULRICH VECTOR BUNDLES ON FOURFOLDS

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Abstract. We give an almost complete classification of non-big Ulrich vector bundles on fourfolds. This allows to classify them in the case of Picard rank one fourfolds, of Mukai fourfolds and in the case of del Pezzo n-folds for $n \leq 4$. We also classify Ulrich bundles with non-big determinant on del Pezzo and Mukai n-folds, $n \geq 2$.

1. Introduction

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible complex variety of dimension $n \geq 1$. A vector bundle $\mathcal{E}$ on X is Ulrich if $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$. We refer for example to [EiSc, B, CMRPL] and references therein for the importance of Ulrich vector bundles and the relation with properties of X.

One often useful geometrical consequence of the existence of a non-big Ulrich vector bundle $\mathcal{E}$ on X is that X is covered by linear spaces, as shown in [LS, Thm. 2]. This property gives very strong conditions on the geometry of X especially in low dimension, thus allowing the classification of non-big Ulrich vector bundles for $n \leq 3$, achieved in [Lo, LM]. On the other hand, already for $n = 4$, the classification of varieties covered by lines is far more incomplete. For example, for $n = 4$, one can have scrolls $X \to Y$ which are a projective bundle only over a proper open subset of a threefold Y. Nevertheless, denoting by H a hyperplane section of X, observe that still a lot of information on the pair (X, H) is available in dimension $n \geq 2$. We have (see [BS] and §3.1) the nef value $\tau = \tau(X, H)$ and the nef value morphism

$$\phi_\tau = \phi_\tau(X, H) := \varphi_{m(K_X+\tau H)} : X \to X'.$$

Classical adjunction theory allows to divide the pairs (X, H) into different cases, depending on the behaviour of ϕ_τ (this is especially useful for $n = 4$, see Lemma 3.1). In this paper we study this morphism in the presence of a non-big rank r Ulrich vector bundle $\mathcal{E}$, as follows. We have a morphism $\Phi : X \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ and its lift with connected fibers in the Stein factorization $\widetilde{\Phi} : X \to \widetilde{\Phi(X)}$. When $c_1(\mathcal{E})^n = 0$, we first prove a useful technical result, the Dichotomy Lemma (see Lemma 2.14), that allows to compare the fibers of Φ (or of $\widetilde{\Phi}$) and of a given morphism starting from X. A nice application of it is given in two standard cases arising in adjunction theory, a quadric fibration in Proposition 2.17 and a blow-up of a point in Proposition 2.18. Moreover when $n = 4$, applying the Dichotomy Lemma to ϕ_τ in several cases, together with the results in [LS] and [LMS] in the case $c_1(\mathcal{E})^n > 0$, we get a pretty complete classification of non-big Ulrich vector bundles, as stated below. In many cases we have a linear Ulrich triple, in the sense of Definition 2.11. The cases in the theorem below are also listed in Tables 1 and 2 at the end of the paper.

Theorem 1.

Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension 4 and let $\mathcal{E}$ be a rank r vector bundle on X. If $\mathcal{E}$ is Ulrich not big then $(X, \mathcal{O}_X(1), \mathcal{E})$ is one of the following.

If $c_1(\mathcal{E})^4 = 0$:

(i) $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1), \mathcal{O}_{\mathbb{P}^4}^{\oplus r})$.

(ii1) $(X, \mathcal{O}_X(1), \mathcal{E}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1), p^*(\mathcal{O}_{\mathbb{P}^3}(1))^{\oplus r})$, *where $p = \Phi : \mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^3$ is the second projection.*

(ii2) $(X, \mathcal{O}_X(1), \mathcal{E})$ *is a linear Ulrich triple with $p = \phi_\tau = \widetilde{\Phi}$ and $b = 1$. In particular $\mathcal{E}$ is pull-back of a twisted Ulrich vector bundle on B.*

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(iii) $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1), p^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus r})$, *where* $p : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ *is one of the two projections.*

(iv) $(\mathbb{P}^1 \times Q, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Q(1), p^*(\mathcal{S}(1))^{\oplus(\frac{r}{2})})$, *where* $p : \mathbb{P}^1 \times Q \to Q = Q_3$ *is the second projection.*

(v1) $(X, \mathcal{O}_X(1))$ *is a hyperplane section of* $\mathbb{P}^2 \times \mathbb{P}^3$ *under the Segre embedding and* $\mathcal{E} \cong q^*(\mathcal{O}_{\mathbb{P}^3}(2))^{\oplus r}$, *where* $q : X \to \mathbb{P}^3$ *is the restriction of the second projection.*

(v2) $(X, \mathcal{O}_X(1), \mathcal{E})$ *is a linear Ulrich triple with* $p = \phi_\tau = \widetilde{\Phi}$ *and* $b = 2$.

(v3) $(X, \mathcal{O}_X(1), \mathcal{E})$ *is a linear Ulrich triple with* $p = \widetilde{\Phi}, b = 3$ *and* X *has a morphism to a smooth curve with all fibers* $\mathbb{P}^1 \times \mathbb{P}^2$ *embedded by the Segre embedding.*

(vi1) $(X, \mathcal{O}_X(1))$ *is a del Pezzo fibration over a smooth curve with every fiber the blow-up of* $\mathbb{P}^3$ *in a point and* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is a linear Ulrich triple with* $p = \widetilde{\Phi} : \mathbb{P}(\mathcal{F}) \to B, b = 3, \phi_\tau = h \circ p$ *where* $h : B \to X'$ *and* $(B, \det \mathcal{F})$ *is a del Pezzo fibration over* X' *with every fiber* $\mathbb{P}^2$.

(vi2) $(X, \mathcal{O}_X(1))$ *is a del Pezzo fibration over a smooth curve with smooth fibers* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, *singular fibers* $\mathbb{P}^1 \times \mathcal{Q}$, *where* $\mathcal{Q} \subset \mathbb{P}^3$ *is a quadric cone and* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is a linear Ulrich triple with* $p = \widetilde{\Phi} : \mathbb{P}(\mathcal{F}) \to B, b = 3, \phi_\tau = h \circ p$ *where* $h : B \to X'$ *and* $(B, \det \mathcal{F})$ *is a del Pezzo fibration over* X' *with smooth fibers* $\mathbb{P}^1 \times \mathbb{P}^1$ *and singular fibers* $\mathcal{Q}$.

(vi3) $(X, \mathcal{O}_X(1))$ *is a del Pezzo fibration over a smooth curve with smooth fibers* $\mathbb{P}(T_{\mathbb{P}^2})$, *singular fibers the tautological image of* $\mathbb{P}(\mathcal{F})$, *where* $\mathcal{F} = \mathcal{O}_{\mathbb{F}_1}(C_0 + f) \oplus \mathcal{O}_{\mathbb{F}_1}(C_0 + 2f)$, *that is a hyperplane section of the Segre embedding* $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. *Also* $\phi_\tau = h \circ \widetilde{\Phi}$ *where* $h : \widetilde{\Phi(X)} \to X'$ *is a fibration with general fiber* $\mathbb{P}^2$. *In particular* $\mathcal{E}_{|\mathbb{P}(T_{\mathbb{P}^2})}$ *is pull-back of a vector bundle on* $\mathbb{P}^2$.

(vii) $(X, \mathcal{O}_X(1))$ *is a quadric fibration with equidimensional fibers over a smooth surface and* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is a linear Ulrich triple with* $p = \widetilde{\Phi}, b = 3$, ϕ_τ *factorizes through* $\widetilde{\Phi}$ *and every fiber of* ϕ_τ *is a disjoint union of linear spaces.*

(viii) $(X, \mathcal{O}_X(1))$ *is a linear* $\mathbb{P}^1$*-bundle over a smooth threefold,* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is a linear Ulrich triple with* $p = \phi_\tau = \widetilde{\Phi}$ *and* $b = 3$.

(ix) $(X, \mathcal{O}_X(1))$ *is a scroll over a normal threefold with non-equidimensional fibers and* ϕ_τ *factorizes through* $\widetilde{\Phi}$ *via a generically finite degree* 1 *map and every fiber of* ϕ_τ *is a disjoint union of linear spaces. In particular, on the general fiber* $\mathbb{P}^1$ *of* ϕ_τ, *we have that* $\mathcal{E}_{|\mathbb{P}^1}$ *is trivial.*

(x1) $(\mathbb{P}^1 \times M, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes L, p^*(\mathcal{G}(L)))$, *where* M *is a Fano* 3*-fold of index* 2, $K_M = -2L$, p *is the second projection and* $\mathcal{G}$ *is a rank* r *Ulrich vector bundle for* (M, L).

(x2) $(\mathbb{P}^1 \times \mathbb{P}(T_{\mathbb{P}^2}), \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^2})}(1), p^*(\mathcal{G} \otimes (\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^3}(3))))$, *where* $p : X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes T_{\mathbb{P}^2}) \to \mathbb{P}^1 \times \mathbb{P}^2$ *is the projection map and* $\mathcal{G}$ *is a rank* r *vector bundle on* $\mathbb{P}^1 \times \mathbb{P}^2$ *such that* $H^j(\mathcal{G} \otimes S^k(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \Omega_{\mathbb{P}^2})) = 0$ *for* $j \geq 0, 0 \leq k \leq 2$.

(x3) X *is a hyperplane section of* $\mathbb{P}^2 \times Q_3$ *under the Segre embedding,* $\mathcal{O}_X(1) = (\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{Q_3}(1))_{|X}$ *and* $\mathcal{E} \cong q^*(\mathcal{S}(2))^{\oplus(\frac{r}{2})}$, *where* $q : X \to Q_3$ *is the restriction of the second projection.*

(x4) X *is twice a hyperplane section of* $\mathbb{P}^3 \times \mathbb{P}^3$ *under the Segre embedding,* $\mathcal{O}_X(1) = (\mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1))_{|X}$ *and* $\mathcal{E} \cong q^*(\mathcal{O}_{\mathbb{P}^3}(3))^{\oplus r}$, *where* $q : X \to \mathbb{P}^3$ *is the restriction of one of the two projections.*

(x5) $X \cong \mathbb{P}(\mathcal{S})$ *where* $\mathcal{S}$ *is the spinor bundle on* Q_3, $\mathcal{O}_X(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{S})}(1) \otimes p^*(\mathcal{O}_{Q_3}(1))$ *and* $\mathcal{E} \cong p^*(\mathcal{G}(3))$, *where* $p : X \to Q_3$ *is the projection and* $\mathcal{G}$ *is a rank* r *vector bundle on* Q_3 *such that* $H^j(\mathcal{G}(-2k) \otimes S^k\mathcal{S}) = 0$ *for* $j \geq 0, 0 \leq k \leq 2$.

If $c_1(\mathcal{E})^4 > 0$:

(xi) $(Q_4, \mathcal{O}_{Q_4}(1))$ *and* $\mathcal{E} \cong \mathcal{S}', \mathcal{S}'', \mathcal{S}' \oplus \mathcal{S}''$, *where* $\mathcal{S}', \mathcal{S}''$ *are the spinor bundles on* Q_4.

(xii) $(X, \mathcal{O}_X(1))$ *is a linear* $\mathbb{P}^3$*-bundle over a smooth curve* $p : X \to B$ *and on any fiber* f *of* p, $\mathcal{E}_{|f}$ *is either* $T_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-3)}$ *or* $\Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-3)}$ *or* $\mathcal{N}(1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-2)}$, *where* $\mathcal{N}$ *is a null-correlation bundle or is a quotient of type* $0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus(r+2)} \to \mathcal{E}_{|f} \to 0$.

(xiii) $(X, \mathcal{O}_X(1))$ *is a linear* $\mathbb{P}^2$*-bundle over a smooth surface* $p : X \to B$ *and on any fiber* f *of* p, $\mathcal{E}_{|f} \cong T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus(r-2)}$.

(xiv) $(X, \mathcal{O}_X(1))$ *is a quadric fibration over a smooth curve* $p : X \to B$ *and* $\mathcal{E}$ *is a rank* 2 *relative spinor bundle, that is* $\mathcal{E}_{|f}$ *is a spinor bundle on a general fiber* f *of* p.

Vice versa all $\mathcal{E}$ *in (i)-(xi) are Ulrich not big.*

Note that almost all cases in Theorem 1 are possible, see Examples 4.1-4.14. The exceptions are case (vi1) and case (xii) with restriction $\mathcal{N}(1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-2)}$. Moreover, in some cases, namely (vi3) and (ix), the information obtained on $\mathcal{E}$ is not complete, we can only obtain a factorization of ϕ_τ through $\widetilde{\Phi}$.

As a consequence of Theorem 1, we can classify completely non-big Ulrich vector bundles in the following classes of varieties.

Corollary 1.
Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $2 \le n \le 4$ and let $\mathcal{E}$ be a rank r vector bundle on X. We have:

- (i) *If $\rho(X) = 1$, then $\mathcal{E}$ is Ulrich not big if and only if $(X, \mathcal{O}_X(1), \mathcal{E})$ is one of the following:*
  - (i1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$.
  - (i2) $(Q_n, \mathcal{O}_{Q_n}(1))$ *and $\mathcal{E}$ is one of $(\mathcal{S}')^{\oplus r}, (\mathcal{S}'')^{\oplus r}$ for $n = 2$, $\mathcal{S}$ for $n = 3$ and as in (xi) of Theorem 1 for $n = 4$.*
- (ii) *If $(X, \mathcal{O}_X(1))$ is a del Pezzo variety, then $\mathcal{E}$ is Ulrich not big if and only if $(X, \mathcal{E})$ is one of the following:*
  - (ii1) $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1), q^*((\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2))^{\oplus s} \oplus (\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus(r-s)})$ *for $0 \le s \le r$ and $q : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is one of the three projections.*
  - (ii2) $(\mathbb{P}(T_{\mathbb{P}^2}), \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^2})}(1), p^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus r})$, *where $p : \mathbb{P}(T_{\mathbb{P}^2}) \to \mathbb{P}^2$ is one of the two projections.*
  - (ii3) $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1), p^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus r})$, *where $p : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ is one of the two projections.*
- (iii) *If $(X, \mathcal{O}_X(1))$ is a Mukai variety, then $\mathcal{E}$ is Ulrich not big if and only if $(X, \mathcal{E})$ is as in (x1)-(x5) of Theorem 1.*

Moreover, when $\det \mathcal{E}$ is not big, we can classify Ulrich vector bundles on del Pezzo or Mukai n-folds.

Corollary 2.
Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 2$ and let $\mathcal{E}$ be a rank r vector bundle on X such that $\det \mathcal{E}$ is not big. We have:

- (i) *If $(X, \mathcal{O}_X(1))$ is a del Pezzo n-fold, then $\mathcal{E}$ is Ulrich if and only if $(X, \mathcal{E})$ is as in (ii1)-(ii3) of Corollary 1.*
- (ii) *If $(X, \mathcal{O}_X(1))$ is a Mukai n-fold, then $\mathcal{E}$ is Ulrich if and only if $(X, \mathcal{E})$ is either as in (x1)-(x5) of Theorem 1 or is one of the following:*
  - (ii1) $(\mathbb{P}^3 \times \mathbb{P}^3, p^*(\mathcal{O}_{\mathbb{P}^3}(3))^{\oplus r})$, *where $p : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$ is one of the two projections.*
  - (ii2) $(\mathbb{P}(T_{\mathbb{P}^3}), p^*(\mathcal{O}_{\mathbb{P}^3}(3))^{\oplus r})$, *where $p : \mathbb{P}(T_{\mathbb{P}^3}) \to \mathbb{P}^3$ is one of the two projections.*
  - (ii3) $(\mathbb{P}^2 \times Q_3, q^*(\mathcal{S}(2))^{\oplus(\frac{r}{2})})$, *where $q : \mathbb{P}^2 \times Q_3 \to Q_3$ is the second projection.*

2. Notation and standard facts about (Ulrich) vector bundles

Throughout this section we will let $X \subseteq \mathbb{P}^N$ be a smooth irreducible complex variety of dimension $n \ge 1$, degree d and H a hyperplane divisor on X.

Definition 2.1. We say that $(X, \mathcal{O}_X(1))$ as above is a *linear* $\mathbb{P}^k$*-bundle* over a smooth variety B if $(X, \mathcal{O}_X(1)) = (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$, where $\mathcal{F}$ is a very ample vector bundle on B of rank $k+1$.

We say that $(X, \mathcal{O}_X(1))$ as above is a *scroll (respectively a quadric fibration; respectively a del Pezzo fibration) over a normal variety Y of dimension m* if there exists a surjective morphism with connected fibers $\phi : X \to Y$ such that $K_X + (n-m+1)H = \phi^*\mathcal{L}$ (respectively $K_X + (n-m)H = \phi^*\mathcal{L}$; respectively $K_X + (n-m-1)H = \phi^*\mathcal{L}$), with $\mathcal{L}$ ample on Y.

Definition 2.2. For $n \ge 2$ we let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric. We let S (n odd), and S', S'' (n even), be the vector bundles on Q_n, as defined in [O, Def. 1.3]. The *spinor bundles* on Q_n are $\mathcal{S} = \mathcal{S}_n = S(1)$ if n is odd and $\mathcal{S}' = \mathcal{S}'_n = S'(1)$, $\mathcal{S}'' = \mathcal{S}''_n = S''(1)$, if n is even. They all have rank $2^{\lfloor \frac{n-1}{2} \rfloor}$.

Notation 2.3. For $k \in \mathbb{Z} : 1 \le k \le n$ we denote by $F_k(X)$ the Fano variety of k-dimensional linear subspaces of $\mathbb{P}^N$ that are contained in X. For $x \in X$, we denote by $F_k(X, x) \subset F_k(X)$ the subvariety of k-dimensional linear subspaces passing through x.

The following fact is well known (see for example [R, Prop.s 2.2.1 and 2.3.9]) and will be often used without mentioning.

Remark 2.4. Let $x \in X$ be a general point. Then $F_1(X,x)$ is smooth and $\dim_{[L]} F_1(X,x) = -K_X \cdot L - 2$ for every $[L] \in F_1(X,x)$.

Definition 2.5. Given a nef line bundle $\mathcal{L}$ on X we denote by
$$\nu(\mathcal{L}) = \max\{k \geq 0 : c_1(\mathcal{L})^k \neq 0\}$$
the *numerical dimension* of $\mathcal{L}$.

Recall that when $\mathcal{L}$ is globally generated $\nu(\mathcal{L})$ is the dimension of the image of the morphism induced by $\mathcal{L}$.

Definition 2.6. Let $\mathcal{E}$ be a rank r vector bundle on X. We denote by $c(\mathcal{E})$ its Chern polynomial and by $s(\mathcal{E})$ its Segre polynomial. We set $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}(\mathcal{E}))$ with projection map $\pi : \mathbb{P}(\mathcal{E}) \to X$ and tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. We say that $\mathcal{E}$ is *nef (big, ample, very ample)* if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef (big, ample, very ample). If $\mathcal{E}$ is nef, we define the *numerical dimension* of $\mathcal{E}$ by $\nu(\mathcal{E}) := \nu(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. When $\mathcal{E}$ is globally generated we define the map determined by $\mathcal{E}$ as
$$\Phi = \Phi_{\mathcal{E}} : X \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E})).$$
For any point $x \in X$ we will denote the fiber of Φ by
$$F_x = \Phi^{-1}(\Phi(x))$$
and we set $\phi(\mathcal{E})$ for the dimension of the general fiber of $\Phi_{\mathcal{E}}$. Moreover, we set
$$\varphi = \varphi_{\mathcal{E}} = \varphi_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)} : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathcal{E})$$
$$\Pi_y = \pi(\varphi^{-1}(y)), y \in \varphi(\mathbb{P}(\mathcal{E}))$$
and
$$P_x = \varphi(\mathbb{P}(\mathcal{E}_x)).$$

Note that $\Phi(x) = [P_x]$ is the point in $\mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ corresponding to P_x.
We recall that, considering the map
$$\lambda_{\mathcal{E}} : \Lambda^r H^0(\mathcal{E}) \to H^0(\det \mathcal{E})$$
one gets a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Phi_{\mathcal{E}}} & \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E})) \\ \downarrow{\scriptstyle \varphi_{|\mathrm{Im}\lambda_{\mathcal{E}}|}} & & \downarrow{\scriptstyle P_{\mathcal{E}}} \\ \mathbb{P}\mathrm{Im}\lambda_{\mathcal{E}} & \hookrightarrow & \mathbb{P}\Lambda^r H^0(\mathcal{E}) \end{array} \tag{2.1}$$

where $P_{\mathcal{E}}$ is the Plücker embedding. In particular this implies that if $c_1(\mathcal{E})^n = 0$, then $\dim F_x \geq 1$ for every $x \in X$. We will often use this fact without further mentioning.

Definition 2.7. Let $\mathcal{E}$ be a vector bundle on $X \subseteq \mathbb{P}^N$. We say that $\mathcal{E}$ is an *Ulrich vector bundle* if $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$.

The following properties will be often used without mentioning.

Remark 2.8. Let $\mathcal{E}$ be a rank r Ulrich vector bundle on $X \subseteq \mathbb{P}^N$ and let $d = \deg X$. Then
- (i) $\mathcal{E}$ is 0-regular in the sense of Castelnuovo-Mumford, hence $\mathcal{E}$ is globally generated (by [Laz1, Thm. 1.8.5]).
- (ii) $h^0(\mathcal{E}) = rd$ (by [EiSc, Prop. 2.1] or [B, (3.1)]).
- (iii) $\mathcal{E}$ is arithmetically Cohen-Macaulay (ACM), that is $H^i(\mathcal{E}(j)) = 0$ for $0 < i < n$ and all $j \in \mathbb{Z}$ (by [EiSc, Prop. 2.1] or [B, (3.1)]).
- (iv) $\mathcal{E}_{|Y}$ is Ulrich on a smooth hyperplane section Y of X (by [B, (3.4)]).

Remark 2.9. On $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ the only rank r Ulrich vector bundle is $\mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ by [EiSc, Prop. 2.1], [B, Thm. 2.3].

We also collect here some properties that follow by [LS, Thm. 2].

Lemma 2.10. *Let $\mathcal{E}$ be an Ulrich vector bundle on X. Then F_x is a linear space contained in $X \subseteq \mathbb{P}^N$ for every $x \in X$. Moreover if*

(2.2)
$$\begin{array}{ccc} X & \xrightarrow{\widetilde{\Phi}} & \widetilde{\Phi(X)} \\ & \searrow^{\Phi} & \downarrow g \\ & & \Phi(X) \end{array}$$

is the Stein factorization of $\Phi = \Phi_{\mathcal{E}}$, then, for every $x \in X$,

$$\widetilde{F}_x := \widetilde{\Phi}^{-1}(\widetilde{\Phi}(x)) = F_x$$

and there is a vector bundle $\mathcal{H}$ on $\widetilde{\Phi(X)}$ such that $\mathcal{E} \cong \widetilde{\Phi}^* \mathcal{H}$.

Proof. For every $x \in X$ we have that F_x is a linear space by [LS, Thm. 2]. This implies that g is bijective and $\widetilde{F}_x = F_x$. As is well known, there is a rank r vector bundle $\mathcal{U}$ on $\Phi(X)$ such that $\mathcal{E} \cong \Phi^* \mathcal{U}$ and therefore also $\mathcal{E} \cong \widetilde{\Phi}^* \mathcal{H}$ with $\mathcal{H} = g^* \mathcal{U}$. □

Definition 2.11. Let $\mathcal{E}$ be a vector bundle on X. We say that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a *linear Ulrich triple* if there are a smooth irreducible variety B of dimension $b \geq 1$, a very ample vector bundle $\mathcal{F}$ and a rank r vector bundle $\mathcal{G}$ on B such that

$$(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1), p^*(\mathcal{G}(\det \mathcal{F}))) \tag{2.3}$$

where $p : X \cong \mathbb{P}(\mathcal{F}) \to B$ is the projection and

$$H^j(\mathcal{G} \otimes S^k \mathcal{F}^*) = 0 \text{ for all } j \geq 0, 0 \leq k \leq b-1. \tag{2.4}$$

Note that when $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple, then $\mathcal{E}$ is an Ulrich vector bundle on X by [Lo, Lemma 4.1]. Moreover observe that, in the case $b = 1$, we have $H^j(\mathcal{G}) = 0$ for all $j \geq 0$, hence $\mathcal{G} \otimes \mathcal{L}$ is an Ulrich vector bundle on B for any very ample line bundle $\mathcal{L}$. Thus, in this case, $\mathcal{E}$ is pull-back of a twisted Ulrich vector bundle on B.

Lemma 2.12. *Let $\mathcal{E}$ be an Ulrich vector bundle on X such that $\widetilde{\Phi}$ has equidimensional fibers and suppose that $(X, \mathcal{O}_X(1)) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Then $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple with $p = \widetilde{\Phi}, B = \widetilde{\Phi(X)}$ and $b = n - \phi(\mathcal{E})$.*

Proof. This easily follows by [BS, Prop. 3.2.1], Lemma 2.10 and [Lo, Lemma 4.1]. □

In several cases we will study Ulrich vector bundles on a variety X that has some standard structure morphism. As it will be clear in the sequel, this will naturally distinguish two different cases. We will use the following

Notation 2.13. Given a morphism $h : X \to X'$ we set

$$f_x = h^{-1}(h(x)), x \in X.$$

Given an Ulrich vector bundle $\mathcal{E}$ on X, we have $x \in F_x \cap f_x$. Now

Lemma 2.14. *(Dichotomy Lemma)*
Let $\mathcal{E}$ be an Ulrich vector bundle on X such that $c_1(\mathcal{E})^n = 0$ and let $h : X \to X'$ be a morphism. Then only one of the following cases occurs for h:

(fin) $\dim F_x \cap f_x = 0$ *for every $x \in X$, or*
(fact) $F_x \subseteq f_x, h$ *factorizes through $\widetilde{\Phi}$ and f_x is a disjoint union of linear spaces, fibers of Φ, for every $x \in X$.*

Proof. Since $c_1(\mathcal{E})^n = 0$, it follows that $\dim F_x \geq 1$ for every $x \in X$. Now suppose that there is an $x_0 \in X$ such that $\dim(F_{x_0} \cap f_{x_0}) \geq 1$. Then the morphism

$$h_{|F_{x_0}} : F_{x_0} \to X'$$

has a positive dimensional fiber, namely $F_{x_0} \cap f_{x_0}$. Since $F_{x_0} = \mathbb{P}^k$ it follows that $h_{|F_{x_0}}$ is constant, that is $F_{x_0} \subseteq f_{x_0}$. Now $\widetilde{F}_{x_0} = F_{x_0} \subset f_{x_0}$, hence [D, Lemma 1.15(a)] implies that $F_x = \widetilde{F}_x \subset f_x$ for general x. Then, by semicontinuity, $\dim_x(F_x \cap f_x) \geq 1$ for every $x \in X$. Again $h_{|F_x}$ is constant, that is $\widetilde{F}_x = F_x \subseteq f_x$. Therefore h factorizes through $\widetilde{\Phi}$ by [D, Lemma 1.15(b)]. Since the fibers of $\widetilde{\Phi}$ are linear spaces, we get that f_x is a disjoint union of linear spaces for every $x \in X$. □

Definition 2.15. Let $\mathcal{E}$ be an Ulrich vector bundle on X. Given a morphism $h : X \to X'$, we define the subcase

$$\text{(emb)} \quad F_x \cap f_x = \{x\} \text{ scheme} - \text{theoretically, for every } x \in X.$$

We will say, that *case (emb) (or (fin), or (fact)) holds for h*, referring to the above or to the Dichotomy Lemma.

We now analyze how the cases (fin), (emb) or (fact) occur in some special cases.

Lemma 2.16. *Let $\mathcal{E}$ be an Ulrich vector bundle on X such that $c_1(\mathcal{E})^n = 0$ and let $h : X \to X'$ be a morphism. We have:*

(i) *If case* (fin) *holds for h and $\dim X' = 1$, then F_x is a line for every $x \in X$ and $X' \cong \mathbb{P}^1$.*
(ii) *If h is a linear $\mathbb{P}^k$-bundle or a quadric fibration over a smooth curve and case* (fin) *holds for h, then case* (emb) *holds for h.*
(iii) *Assume that case* (fact) *holds for h. If f_x is integral and $\mathrm{Pic}(f_x) \cong \mathbb{Z}$ for some $x \in X$, then $F_x = f_x$. Moreover if $h_* \mathcal{O}_X \cong \mathcal{O}_{X'}$, f_x is integral and $\mathrm{Pic}(f_x) \cong \mathbb{Z}$ for every $x \in X$, then $h = \widetilde{\Phi}$.*

Proof. Since $c_1(\mathcal{E})^n = 0$ we have that $F_x = \mathbb{P}^k, k \geq 1$ for every $x \in X$. Suppose that case (fin) holds in one of (i)-(iii). To see (i), if $\dim F_x \geq 2$, then $h_{|F_x} : F_x = \mathbb{P}^k \to X'$ is constant, contradicting case (fin). Then F_x is a line and, for every $v \in F_x$, we have that $\dim f_v \cap F_x = \dim f_v \cap F_v = 0$, so that F_x dominates X' and then $X' \cong \mathbb{P}^1$. This proves (i). Now (ii) is clear if h is a linear $\mathbb{P}^k$-bundle. If h is a quadric fibration over a smooth curve, let $x \in X$, so that F_x is a line by (i). If $F_x \cdot f_x \geq 2$, then, for a general $x' \in X$, $F_x \cdot f_{x'} = F_x \cdot f_x \geq 2$. This implies that $x \in F_x \subset \langle f_{x'} \rangle = \mathbb{P}^n$. But then $X = \mathbb{P}^n$, a contradiction. Hence $f_x \cdot F_x = 1$ for every $x \in X$ and case (emb) holds. This proves (ii). To see (iii), assume that f_x is integral and $\mathrm{Pic}(f_x) \cong \mathbb{Z}$ for some $x \in X$. Since $\Phi_{|f_x} : f_x \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ contracts F_x to a point, it must be constant, thus $f_x = F_x = \widetilde{F}_x$. Now if f_x is integral and $\mathrm{Pic}(f_x) \cong \mathbb{Z}$ for every $x \in X$, then $f_x = F_x = \widetilde{F}_x$ for every $x \in X$, hence also $\widetilde{\Phi}$ factorizes through h by [D, Lemma 1.15(b)] and we deduce that $h = \widetilde{\Phi}$. □

The following general results, applied to some standard cases arising in adjunction theory, illustrate the power of the Dichotomy Lemma.

Proposition 2.17. *Let $\mathcal{E}$ be an Ulrich vector bundle on X such that $c_1(\mathcal{E})^n = 0$ and suppose that $n \geq 4$. Let $h : X \to X'$ be a quadric fibration over a smooth curve. Then $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^1 \times Q, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Q(1), p^*(\mathcal{H}(1)))$, where $p : \mathbb{P}^1 \times Q \to Q = Q_{n-1}$ is the second projection and $\mathcal{H}$ is a direct sum of spinor bundles on Q.*

Proof. Let $H_Q \in |\mathcal{O}_Q(1)|$. By hypothesis $f_x \cong Q$ and $H_{|f_x} \cong H_Q$ for general x. Case (fact) does not hold for h, since otherwise Lemma 2.16(iii) would give the contradiction $\mathbb{P}^k = F_x = f_x$. Now the Dichotomy Lemma, Lemma 2.16(i) and (ii) give that F_x is a line for every $x \in X$ and case (emb) holds for h. It follows by Lemma 2.12 that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple with $p = \widetilde{\Phi}, B = \widetilde{\Phi(X)}$ and $b = n-1$. Moreover case (emb) implies that, for general x, there is a closed embedding $p_{|f_x} : f_x \to B$ and therefore $B \cong Q$. Hence $p_{|f_x}$ is an isomorphism and $(p^* H_Q)_{|f_x} \cong H_{|f_x}$. Set $\det \mathcal{F} = \mathcal{O}_Q(c)$ for some $c \in \mathbb{Z}$. Then

$$(1-n)H_Q = K_Q = K_{f_x} = (K_X + f_x)_{|f_x} = (-2H + (1-n+c)p^* H_Q)_{|f_x} = (c-n-1)H_Q$$

so that $c = 2$. Therefore $\mathcal{F}$ is a very ample rank 2 vector bundle on Q with $\det \mathcal{F} = 2H_Q$ and it follows that $\mathcal{F} \cong \mathcal{O}_Q(1)^{\oplus 2}$ (see for example [AW, Prop. 1.2]). Therefore $(X, \mathcal{O}_X(1)) = (\mathbb{P}^1 \times Q, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Q(1))$. Now $\mathcal{E} \cong p^*(\mathcal{G}(2))$ where $\mathcal{G}$ is a rank r vector bundle on Q such that $H^j(\mathcal{G} \otimes S^k \mathcal{F}^*) = 0$ for $j \geq 0, 0 \leq k \leq n-2$. Hence $\mathcal{H} := \mathcal{G}(1)$ is an Ulrich vector bundle on Q and we get that $\mathcal{E} \cong p^*(\mathcal{H}(1))$, where $p : \mathbb{P}^1 \times Q \to Q$ is the second projection and $\mathcal{H}$ is a direct sum of spinor bundles on Q by [LMS, Lemma 3.2(iv)]. □

Proposition 2.18. *Let $\mathcal{E}$ be an Ulrich vector bundle on X such that $c_1(\mathcal{E})^n = 0$ and suppose that $n \ge 2$. Then $(X, \mathcal{O}_X(1))$ is not a blow-up $h : X \to X_1$ of a smooth n-fold at a point with exceptional divisor E such that $\mathcal{O}_X(1)_{|E} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$.*

Proof. Assume that $(X, \mathcal{O}_X(1))$ is a blow-up as stated and apply the Dichotomy Lemma to h. Since $f_x = \{x\}$ for general x, we get that case (fact) does not hold for h, so that we are in case (fin). Moreover f_u is a linear space for every $u \in X$, hence (emb) holds for h. It follows that $\Phi_{|E} : E \to \Phi(X)$ is a closed embedding. On the other hand, we know that $\dim \Phi(X) \le n-1$ and therefore $\mathbb{P}^{n-1} \cong E \cong \Phi(E) = \Phi(X)$. Hence for every $u \in X$ we have that $F_u = F_{u_0}$ for some $u_0 \in E$. Then $E = f_{u_0}$ and $F_{u_0} \cup f_{u_0} \subset T_{u_0}X$. If $\dim F_u \ge 2$ then $\dim F_{u_0} \ge 2$ and $\dim T_{u_0}X \ge \dim\langle F_{u_0} \cup f_{u_0}\rangle \ge n+1$, contradicting the smoothness of X. Therefore F_u is a line for every $u \in X$. It follows that $\Phi : X \to \mathbb{P}^{n-1}$ has equidimensional fibers and [BS, Prop. 3.2.1] implies that Φ is a linear $\mathbb{P}^1$-bundle. Hence there is a very ample rank 2 vector bundle $\mathcal{F}$ on $\mathbb{P}^{n-1}$ such that $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)), \Phi$ is the bundle projection $p : \mathbb{P}(\mathcal{F}) \to \mathbb{P}^{n-1}$ and $\mathcal{E} \cong p^*(\mathcal{G}(\det \mathcal{F}))$ for some rank r vector bundle $\mathcal{G}$ on $\mathbb{P}^{n-1}$. Let $R = p^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and $\det \mathcal{F} = \mathcal{O}_{\mathbb{P}^{n-1}}(c)$, for some $c \in \mathbb{Z}$. Then

$$K_E = (K_X + E)_{|E} = (-2H + (c-n)R - H)_{|E}$$

that is $\mathcal{O}_{\mathbb{P}^{n-1}}(-n) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(c-n-3)$, so that $c = 3$. Therefore, since $\mathcal{F}$ is very ample, it has splitting type $(1,2)$ on any line in $\mathbb{P}^{n-1}$ and it follows by [V, Thm.] that either $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(2)$ or $n = 3$ and $\mathcal{F} \cong T_{\mathbb{P}^2}$. The latter case is excluded since $\mathbb{P}(T_{\mathbb{P}^2})$ does not contain linear $\mathbb{P}^2$'s. Therefore we are in the first case and [Lo, Lemma 4.1] gives in particular that $H^i(\mathcal{G}(-s)) = 0$ for all $i \ge 0$ and $0 \le s \le n-2$. Hence $\mathcal{G}(1)$ is an Ulrich vector bundle for $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$, so that $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\oplus r}$ by Remark 2.9. But this gives the contradiction $0 = H^{n-1}(\mathcal{G}(-n+1)) = H^{n-1}(\mathcal{O}_{\mathbb{P}^{n-1}}(-n)^{\oplus r}) \ne 0$. □

3. Non-big Ulrich vector bundles on fourfolds

In this section we will prove Theorem 1, Corollary 1 and Corollary 2.

One guide will be given by the following.

3.1. **Fourfolds and adjunction theory.** We collect some definitions and standard facts in adjunction theory, that we recall for completeness' sake.

Let X be a smooth irreducible variety such that K_X is not nef and let H be a very ample divisor. Consider the nef value of (X, H) (see [BS, Def. 1.5.3])

$$\tau = \tau(X, H) = \min\{t \in \mathbb{R} : K_X + tH \text{ is nef}\}$$

and the nef value morphism, defined for $m \gg 0$ by

$$\phi_\tau = \phi_\tau(X, H) := \varphi_{m(K_X + \tau H)} : X \to X'.$$

We recall that $(\phi_\tau)_* \mathcal{O}_X \cong \mathcal{O}_{X'}$, see [BS, Def. 1.5.3].

Lemma 3.1. *Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible fourfold covered by lines and let $H \in |\mathcal{O}_X(1)|$. Let τ be the nef value of (X, H) and let ϕ_τ be the nef value morphism. Then $(X, \mathcal{O}_X(1))$ is only one of the following:*

(a) $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$.
(b.1) $(Q_4, \mathcal{O}_{Q_4}(1))$.
(b.2) *A linear $\mathbb{P}^3$-bundle under $\phi_\tau : X \to X'$ over a smooth curve with $\tau = 4$.*
(c.1) *A del Pezzo 4-fold, that is $K_X = -3H$.*
(c.2) *A quadric fibration under $\phi_\tau : X \to X'$ over a smooth curve with $\tau = 3$.*
(c.3) *A linear $\mathbb{P}^2$-bundle under $\phi_\tau : X \to X'$ over a smooth surface with $\tau = 3$.*
(d.1) *A Mukai variety, that is $K_X = -2H$.*
(d.2) *A del Pezzo fibration under $\phi_\tau : X \to X'$ over a smooth curve with $\tau = 2$.*
(d.3) *A quadric fibration with equidimensional fibers under $\phi_\tau : X \to X'$ over a smooth surface with $\tau = 2$.*
(d.4) *A linear $\mathbb{P}^1$-bundle under $\phi_\tau : X \to X'$ over a smooth threefold with $\tau = 2$.*
(d.5) *A scroll under $\phi_\tau : X \to X'$ over a normal threefold with non-equidimensional fibers with $\tau = 2$.*
(e) *The blow-up $\phi_\tau : X \to X'$ of a smooth fourfold at $t \ge 1$ points, with exceptional divisors $E_i \cong \mathbb{P}^3$ such that $H_{|E_i} \cong \mathcal{O}_{\mathbb{P}^3}(1), 1 \le i \le t$ and $\tau = 3$.*

Proof. Let $x \in X$ be a general point and let $L \in F_1(X,x)$. Then

$$0 \le \dim_{[L]} F_1(X,x) = -K_X \cdot L - 2$$

so that $K_X \cdot L \le -2$ and $\tau \ge 2$. By [BS, Prop. 7.2.2] we have that either we are in case (a), or $\tau = 4$ and we are in cases (b.1) or (b.2) or $\tau \le 4$ and $K_X + 4H$ is big and nef. In the latter case $K_X + 4H$ is ample by [BS, Prop. 7.2.3] and $\tau \le 3$ by [BS, Prop. 7.2.4]. Moreover [BS, Prop. 7.3.2] gives that $K_X + 3H$ is ample unless $\tau = 3$ and either we are in one of the cases (c.1)-(c.3) or (e) (for (c.3) use also [SV, Thm. 0.2] and for (e) use also [SV, Thm. 0.3 and Rmk. 1]). Next if $K_X + 3H$ is ample then $\tau < 3$ and $(X, \mathcal{O}_X(1))$ is isomorphic to its first reduction (see [BS, Def. 7.3.3]). Therefore [BS, Prop. 7.3.4] implies that $\tau = 2$, so that $K_X \cdot L = -2$, hence $K_X + 2H$ is nef and not big. It follows by [BS, Prop. 7.5.3 and Thm. 14.2.3] that either we are in one of the cases (d.1)-(d.3) or $(X, \mathcal{O}_X(1))$ is a scroll under $\phi_\tau : X \to X'$ over a normal threefold. Finally in the latter case if ϕ_τ has equidimensional fibers, then we are in case (d.4) by [BS, Prop. 3.2.1], otherwise we are in case (d.5). □

3.2. **Proofs.** To this end we will use the notation in 2.13 and in Definition 2.6.

Proof of Theorem 1. Suppose that $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (i)-(xi). Then, using, in the corresponding cases, Remark 2.9, [LMS, Prop. 3.3(iii)], [Lo, Lemma 4.1], [B, (3.5)] and Remark 2.8(iv), we see that $\mathcal{E}$ is Ulrich not big.

Now assume that $\mathcal{E}$ is Ulrich not big.

It follows by [Lo, Thm. 1] that X is covered by lines, hence Lemma 3.1 gives that $(X, \mathcal{O}_X(1))$ belongs to one of the cases (a)-(e) in Lemma 3.1. We will divide the proof according to these cases.

If $(X, \mathcal{O}_X(1))$ is as in (a), we are in case (i) by Remark 2.9.

If $(X, \mathcal{O}_X(1))$ is as in (b.1), we are in case (xi) by [LMS, Prop. 3.3(iii)].

Therefore we can assume from now on that $(X, \mathcal{O}_X(1))$ is neither as in (a) nor as in (b.1).

For the rest of the proof $x \in X$ will denote a general point.

We will now divide the proof into several subcases and claims.

Case (A): $c_1(\mathcal{E})^4 > 0$.

We claim that

$$\dim F_1(X,x) \ge r + 3 - \nu(\mathcal{E}) \ge 1. \tag{3.1}$$

In fact, set $k = r + 3 - \nu(\mathcal{E})$. It follows from [LS, Cor. 2] that we can find a 1-dimensional family T of k-dimensional linear spaces $M_t, t \in T$, with $x \in M_t \subseteq X$. Consider the incidence correspondence

$$\mathcal{J} = \{([L], t) \in F_1(X,x) \times T : L \subseteq M_t\}.$$

The second projection shows that $\mathcal{J}$ is irreducible of dimension k. Next, let $B = \bigcap_{t \in T} M_t$. Since $B \subseteq M_t$ for every $t \in T$ and $\dim T = 1$, then $\dim B < k$. On the other hand, $x \in B$, hence choosing a point $x' \in M_t \setminus B$, we can find a line $L_0 = \langle x, x' \rangle$, with $[L_0] \in F_1(X,x)$ and such that $L_0 \not\subseteq B$. Therefore, since $\dim T = 1$, there are finitely many $t \in T$ such that $L_0 \subset M_t$. Thus, the first projection $p : \mathcal{J} \to F_1(X,x)$ has 0-dimensional general fibers over $p(\mathcal{J})$. Therefore $\dim F_1(X,x) \ge \dim p(\mathcal{J}) = k$. This proves (3.1).

Now we study the cases (b.2) and (c.3). To unify notation, we denote ϕ_τ by $p : X \to B$. First, observe that [LP, Thm. 1.4] and (3.1) give that

$$r + 1 \le \nu(\mathcal{E}) \le r + 2 \text{ in case (b.2) and } \nu(\mathcal{E}) = r + 2 \text{ in case (c.3).} \tag{3.2}$$

We first show that we can apply [LMS, Lemma 4.4], that we recall here for the reader's sake.

Lemma 3.2. *Let $(X, \mathcal{O}_X(1)) = (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$, where $\mathcal{F}$ is a rank $n-b+1$ very ample vector bundle over a smooth irreducible variety B of dimension b with $1 \le b \le n-1$. Let $\mathcal{E}$ be a rank r Ulrich vector bundle on X, let $p : X \to B$ be the projection morphism and suppose that*

$$p_{|\Pi_y} : \Pi_y \to B \text{ is constant for every } y \in \varphi(\mathbb{P}(\mathcal{E})). \tag{3.3}$$

Then, for every fiber f of p we have $\nu(\mathcal{E}) = b + \dim \varphi(\pi^{-1}(f)) \ge b + r - 1$. Moreover we have the following two extremal cases:

(i) *If $\nu(\mathcal{E}) = b + r - 1$ there is a rank r vector bundle $\mathcal{G}$ on B such that $\mathcal{E} \cong p^*(\mathcal{G}(\det \mathcal{F}))$ and $H^j(\mathcal{G} \otimes S^k \mathcal{F}^*) = 0$ for all $j \ge 0, 0 \le k \le b-1$.*

(ii) *If* $\nu(\mathcal{E}) = b + r$ *then either* $b = n-1$ *and* $\mathcal{E}$ *is big or* $b \le n-2$ *and* $\mathcal{E}_{|f} \cong T_{\mathbb{P}^{n-b}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-b}}^{\oplus(r-n+b)}$ *for any fiber* $f = \mathbb{P}^{n-b}$ *of* p.

Claim 3.3. *In cases (b.2) and (c.3) we have that (3.3) holds for* $p : X \to B$.

Proof. If $\nu(\mathcal{E}) = r + 1$ we are in case (b.2) and for every $y \in \varphi(\mathbb{P}(\mathcal{E}))$ we get that

$$\dim \Pi_y \ge r + 3 - \nu(\mathcal{E}) = 2.$$

Hence (3.3) holds for $p : X \to B$.

Therefore we can assume that $\nu(\mathcal{E}) = r + 2$ by (3.2).

Arguing by contradiction assume that there is an $y_0 \in \varphi(\mathbb{P}(\mathcal{E}))$ such that Π_{y_0} is not contained in a fiber of p. Then the same holds for a general $y \in \varphi(\mathbb{P}(\mathcal{E}))$: In fact, if Π_y is contained in a fiber of p, then, by specialization $\dim \Pi_{y_0} \cap f_{x_0} \ge \dim \Pi_y \cap f_x = 1$, where $y_0 \in P_{x_0}$ and $y \in P_x$. Now Π_{y_0} is a linear space of positive dimension by [LS, Thm. 2] and $p_{|\Pi_{y_0}} : \Pi_{y_0} \to B$ contracts $\Pi_{y_0} \cap f_{x_0}$ to a point, hence is constant, that is $\Pi_{y_0} \subseteq f_{x_0}$, a contradiction. Therefore $\Pi_y \not\subset f_x$ for a general $y \in \varphi(\mathbb{P}(\mathcal{E}))$. Then $F_1(X, x)$ has at least two irreducible components, namely W made of lines in f_x through x and W' made of lines of type Π_y, with $\dim W' \ge 1$ by [LS, Cor. 2]. Moreover $F_1(X, x)$ is smooth, hence $W \cap W' = \emptyset$. As is well known, $F_1(X, x) \subset \mathbb{P}^3 = \mathbb{P}(T_x X)$. In case (b.2) we have that W is a plane, thus giving a contradiction. In case (c.3) we have that W is a line and W' is a curve. We now use the fact that $F_1(X, x)$ is contained in the base locus of the linear system $|II|$ given by the second fundamental form of X at x. Observe that X is not defective by [Ei, Thm. 3.3(b)], hence [Land, Thm. 7.3(i)] gives that there is a smooth quadric $Q \in |II|$. Since $W \sqcup W' \subset Q$ we find that W' is a union of lines. But $X \subset \mathbb{P}^N$ and a line L component of W' is also a line in $\mathbb{G}(1, N)$, hence the union of the lines representing points of L gives a plane M_x such that $x \in M_x \subset X$. It follows by [Sa2, Main Thm.] that X is a linear $\mathbb{P}^k$-bundle $p' : X \to B'$ over a smooth B' so that its general fibers contain the M_x's. On the other hand, it cannot be that $k \ge 3$ for otherwise on any $\mathbb{P}^k$ we would have that $p_{|\mathbb{P}^k} : \mathbb{P}^k \to B$ is constant, thus giving the contradiction that p has a fiber of dimension k. Therefore $k = 2$ and, by construction, the $\mathbb{P}^2$-bundle structure p' is different from p. Now on any fiber f' of p' we have that $p_{|f'} : f' \to B$ cannot be constant, otherwise f' is also a fiber of p, and it follows by [Laz2, Thm. 4.1] that $B \cong \mathbb{P}^2$. Similarly, $B' \cong \mathbb{P}^2$. But then [Sa1, Thm. A] implies that $(X, \mathcal{O}_X(1)) = (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1))$, a contradiction. □

Claim 3.4. *In case (b.2) we have that* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is as in (xii).*

Proof. Note that (3.3) holds for $p : X \to B$ by Claim 3.3.

If $\nu(\mathcal{E}) = r + 1$ we can apply Lemma 3.2 and we are in case (xii).

By (3.2) it remains to study the case $\nu(\mathcal{E}) = r + 2$.

Then, Lemma 3.2 gives

$$\dim \varphi(\pi^{-1}(f)) = \nu(\mathcal{E}) - 1 = r + 1$$

for every fiber f of p. Next note that the morphism $\Phi_{|f} : f = \mathbb{P}^3 \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ cannot be constant, for otherwise $P_x = P_{x'}$ for any $x, x' \in f$, giving the contradiction $\varphi(\pi^{-1}(f)) = P_x = \mathbb{P}^{r-1}$. Hence $\Phi_{|f}$ is finite onto its image and therefore $\varphi(\pi^{-1}(f)) \subseteq \mathbb{P}H^0(\mathcal{E})$ is swept out by a family $\{P_x, x \in f\}$ of dimension 3 of linear $\mathbb{P}^{r-1}$'s. We will now show, along the lines of [Se, Lemma p. 44], that

(3.4) either $\varphi(\pi^{-1}(f)) = \mathbb{P}^{r+1}$ or $\varphi(\pi^{-1}(f)) = Q_{r+1}$, a quadric of rank at least 5.

Set $Y_f = \varphi(\pi^{-1}(f))$. Since $\dim Y_f = r + 1$, if $y \in Y_f$ is general, there is a 1-dimensional family $\{P_t, t \in T\}$ of $\mathbb{P}^{r-1}$'s through y. Let $\mathcal{C}_y = \bigcup_{t \in T} P_t$ be the corresponding cone. Then $\dim \mathcal{C}_y = r$ and set $c = \deg \mathcal{C}_y$. Let $H_1, \dots, H_{r-1}$ be general hyperplanes. Then the surface $S_f := Y_f \cap H_1 \cap \ldots \cap H_{r-1}$ is such that there are c lines contained in S_f and passing through its general point. As is well known, it follows that either $c = 1$ and S_f is a scroll or a plane or $c = 2$ and S_f is a quadric. Assume now that $Y_f \ne \mathbb{P}^{r+1}$. When $c = 1$ we get that $\mathcal{C}_y = \mathbb{P}^r$, hence Y_f is a scroll in $\mathbb{P}^r$'s. Also, when $c = 2$ we have that $Y_f = Q$ is a quadric. If Y_f is a scroll, then the composition of $\Phi_{\mathcal{E}_{|f}}$ with the Plücker embedding maps $f = \mathbb{P}^3$ to another scroll in $\mathbb{P}^r$, say Z_f, of dimension $r + 1$ over some curve Γ, whose $\mathbb{P}^r$'s are the dual of the ones in Y_f. Thus we get a map $g : f = \mathbb{P}^3 \to Z_f$ and g lifts to the normalization $\nu : \mathbb{P}(\mathcal{G}) \to Z_f$, where $\mathcal{G}$ is a vector bundle over Γ. But this gives the contradiction that $f = \mathbb{P}^3$ dominates Γ. The same argument can be applied if $\mathrm{rk} Q \le 4$ since we can see it as a scroll in $\mathbb{P}^r$ over $\mathbb{P}^1$. This proves (3.4).

We now claim that $c_1(\mathcal{E}_{|L}) > 0$ for every line $L \subset f$. To see the latter, since $\mathcal{E}$ is globally generated, we can suppose that there is a line $L_0 \subset f$ such that $c_1(\mathcal{E}_{|L_0}) = 0$. Then the same holds on any line $L \subset f$, hence $\Phi_{\mathcal{E}_{|L}}$ is constant. On the other hand, since $\dim \varphi(\pi^{-1}(f)) = r+1$, there exist $x_1, x_2 \in f$ such that $P_{x_1} \neq P_{x_2}$ hence on $L' = \langle x_1, x_2 \rangle$ we have that $\Phi_{\mathcal{E}_{|L'}}$ is not constant. This proves that $c_1(\mathcal{E}_{|L}) > 0$ for every line $L \subset f$. Now for every line $L \subset f$ we have that $H^0(\mathcal{E}) \to H^0(\mathcal{E}_{|L})$ is surjective by [LS, Lemma 3.2], hence $\varphi(\pi^{-1}(L))$ is a rational normal scroll. Then $h^0(\mathcal{E}_{|L}) \le r+3$ by (3.4) and $1 \le c_1(\mathcal{E}_{|L}) \le 3$. Furthermore, if $c_1(\mathcal{E}_{|L}) = 3$ then $\varphi(\pi^{-1}(L))$ has codimension 1 in $Q = \varphi(\pi^{-1}(f))$. Note that $r \ge 2$ since $\mathcal{E}$ is not big and $c_1(\mathcal{E})^4 > 0$, hence intersecting with general hyperplanes $H_i, 1 \le i \le r-2$ we get a surface $\varphi(\pi^{-1}(L)) \cap H_1 \cap \ldots \cap H_{r-2}$ of degree 3 inside a smooth quadric in $\mathbb{P}^4$, a contradiction. This gives that $1 \le c_1(\mathcal{E}_{|L}) \le 2$.

If $c_1(\mathcal{E}_{|L}) = 1$ then $\mathcal{E}_{|L} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)}$, hence $c_1(\mathcal{E}_{|f}) = 1$ and [Ell, Prop. IV.2.2] implies that $\mathcal{E}_{|f}$ is either $T_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-3)}, r \ge 3$ or $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-1)}$ or $\Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus(r-3)}, r \ge 3$. The first case does not occur since then (3.4) implies the contradiction $r+2 \le h^0(\mathcal{E}_{|f}) = h^0(T_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-3)}) = r+1$. In the second case we have that $\mathcal{E}_{|f}$ is big, contradicting the fact that $\dim \varphi(\pi^{-1}(f)) = r+1$. Also $c_1(\Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus(r-3)}) = r-1 \ge 2$, hence the third case does not occur. It follows that $c_1(\mathcal{E}_{|f}) = 2$ and therefore $\mathcal{E}_{|f}$ is as in (i)-(vii) of [SU, Thm. 1]. Now $r+2 \le h^0(\mathcal{E}_{|f}) \le r+3$ by (3.4), hence cases (i)-(iii) and (vii) of [SU, Thm. 1] are excluded. Thus $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (xii). □

Claim 3.5. *In case (c.3) we have that $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (xiii).*

Proof. By (3.2) and Claim 3.3 we can apply Lemma 3.2(ii) and we are in case (xiii). □

To conclude the proof of case (A) we can assume that we are neither in case (b.2) nor in case (c.3).

Again [LP, Thm. 1.4], the classification of del Pezzo 4-folds (see for example [LP, §1], [F1]) and (3.1) give that $\nu(\mathcal{E}) = r+2, \dim F_1(X,x) = 1$ and either we are in case (c.2) or in one of the following cases for $(X, \mathcal{O}_X(1))$:

(A.1) a cubic hypersurface $\mathbb{P}^5$.
(A.2) a complete intersection of two quadrics in $\mathbb{P}^6$.
(A.3) a linear section $\mathbb{G}(1,4) \cap H_1 \cap H_2$, where $\mathbb{G}(1,4) \subset \mathbb{P}^9$ in the Plücker embedding and $H_i \subset \mathbb{P}^9$ are hyperplanes $i = 1, 2$.
(A.4) $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1))$.

Moreover note that in cases (A.1)-(A.3), or in case (c.2), we have that $F_2(X,x) = \emptyset$. In fact, if not, being a closed condition, we would get that $F_2(X,x') \neq \emptyset$ holds on any point $x' \in X$. Now [Sa2, Main Thm.] gives that X is a linear $\mathbb{P}^2$-bundle over a smooth surface, hence [F2, Main Thm.] implies that we are in case (c.3), a contradiction. Therefore we can apply [LMS, Prop. 4.6] and deduce that there is a morphism $\psi : \mathbb{P}^{r-1} \to F_1(X,x)$ that is finite onto its image and

$$1 = \dim F_1(X,x) \ge r-1$$

hence $r \le 2$. On the other hand, $\mathcal{E}$ is not big and therefore $r = 2$.

Claim 3.6. *Cases (A.1), (A.2), (A.3) and (A.4) do not occur.*

Proof. Case (A.4) does not occur by [LMS, Thm. 3]. In cases (A.1) and (A.2) it is well known (see for example [R, Ex. 2.3.11]) that $F_1(X,x)$ is a smooth complete intersection of type $(2,3)$ or $(2,2)$ in $\mathbb{P}^3$, dominated by $\mathbb{P}^1$ via ψ, a contradiction.

In case (A.3) note that $\det \mathcal{E} = 2H$ by [Lo, Lemma 3.2]. Let Y be a smooth hyperplane section of X. By Remark 2.8(iv), $\mathcal{E}_{|Y}$ is Ulrich on Y and it is indecomposable since $\text{Pic}(Y) \cong \mathbb{Z}$ and there are no Ulrich line bundles on Y. It follows by [AC, Thm. 3.4] and Remark 2.8(iii) that $\mathcal{E}_{|Y} \cong S_L(l)$ or $S_C(l)$ or $S_E(l)$ for some $l \in \mathbb{Z}$ (see [AC, Ex. 3.1, 3.2, 3.3] for the definition of these sheaves). Since $\det(\mathcal{E}_{|Y}) = 2H_{|Y}$ we get that $\mathcal{E}_{|Y} \cong S_L(1)$ or $S_E(1)$. Now $h^0(\mathcal{E}_{|Y}) = 2 \deg X = 10$ while $h^0(S_L(1)) = 12$, hence this case is excluded. Therefore $\mathcal{E}_{|Y} \cong S_E(1)$ and we deduce that $c_2(\mathcal{E}) \cdot H^2 = c_2(S_E(1)) \cdot H_{|Y} = 7$. We know that $N^2(X)$ is generated by the classes of two planes M_1, M_2 with $M_1^2 = 1, M_2^2 = 2, M_1 \cdot M_2 = -1$ by [PZ, Cor. 4.7 and proof]. Hence $c_2(\mathcal{E}) = a_1[M_1] + a_2[M_2]$ for some $a_1, a_2 \in \mathbb{Z}$ and

$$7 = c_2(\mathcal{E}) \cdot H^2 = (a_1[M_1] + a_2[M_2]) \cdot H^2 = a_1 + a_2.$$

Therefore

$$s_4(\mathcal{E}^*) = c_1(\mathcal{E})^4 - 3c_1(\mathcal{E})^2 \cdot c_2(\mathcal{E}) + c_2(\mathcal{E})^2 = 5a_1^2 - 42a_1 + 94 > 0$$

giving that $\mathcal{E}$ is big, a contradiction. □

Claim 3.7. *In case (c.2) we have that $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (xiv).*

Proof. Let Q be a general fiber of ϕ_τ. Then $(\det \mathcal{E})_{|Q} = eH_{|Q}$ for some $e \geq 0$.

Subclaim 3.8. *For any line $L \subset Q$ we have that*

$$\mathcal{E}_{|L} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e). \tag{3.5}$$

Proof. We first prove that (3.5) holds for a general line $L \subset Q$.

There is a nonempty open subset $U \subseteq X$ such that $F_1(X, x)$ is smooth, $\dim F_1(X, x) = 1$ and $Q_x := \phi_\tau^{-1}(\phi_\tau(x))$ is smooth irreducible for any $x \in U$. By [LMS, Prop. 4.6] there is a 1-dimensional family of lines $Z_x := \psi(\mathbb{P}^1) \subset F_1(X, x)$, hence Z_x is an irreducible component of $F_1(X, x)$, disjoint from other components. For every $[L] \in Z_x$ we have that $\dim_{[L]} F_1(X, x) = \dim_{[L]} Z_x = 1$, hence $K_X \cdot L = -3$. Therefore $(K_X + 3H) \cdot L = 0$ and then $L \subset Q_x$. Hence $Z_x = F_1(Q_x, x)$ for every $x \in U$.

Moreover, let us see that for every $[L] \in Z_x$ property (3.5) holds. In fact, if $[L] \in Z_x$ then we have that $L = \psi(y) = \Pi_y$ for some $y \in P_x$. Since $\varphi^{-1}(y)$ is a curve contracted by φ and $x \in L = \Pi_y$, we deduce that there is a 0-dimensional subscheme Z of L of length 2 such that $H^0(\mathcal{E}) \to H^0(\mathcal{E}_{|Z})$ is not surjective. On the other hand, the restriction map $H^0(\mathcal{E}) \to H^0(\mathcal{E}_{|L})$ is surjective by [LS, Lemma 3.3], hence $H^0(\mathcal{E}_{|L}) \to H^0(\mathcal{E}_{|Z})$ is not surjective. Therefore $\mathcal{E}_{|L}$ is not ample and (3.5) holds.

Consider now the incidence correspondence

$$\mathcal{I} = \{(u, [L]) \in Q \times F_1(Q) : u \in L\}$$

together with its surjective projections

$$\begin{array}{ccc} & \mathcal{I} & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ Q & & F_1(Q) \end{array} .$$

Note that $\mathcal{I}$ is irreducible. Since $p_1^{-1}(U \cap Q)$ dominates $F_1(Q)$, there is a nonempty open subset V of $F_1(Q)$ such that $V \subset p_2(p_1^{-1}(U \cap Q))$. Now for any line $[L] \in V$ we have that $L = p_2(x, L)$ for some $(x, L) \in p_1^{-1}(U \cap Q)$, so that $x \in L \subset Q$ and $x \in U \cap Q$. Therefore $[L] \in F_1(Q, x) = F_1(Q_x, x) = Z_x$. Hence (3.5) holds for a general line $L \subset Q$. Now for any $[L] \in F_1(Q)$ we have that $\mathcal{E}_{|L} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ with $a_1 \geq a_2 \geq 0$, since $\mathcal{E}$ is globally generated. By semicontinuity we have that $h^0(\mathcal{E}_{|L}(-e)) \geq 1$ since (3.5) holds for a general line $L_t \subset Q$. Therefore we get that $a_1 \geq e$. Now $a_1 + a_2 = c_1(\mathcal{E}) \cdot L = c_1(\mathcal{E}) \cdot L_t = e$, so that $a_2 = e - a_1 \leq 0$ and therefore $a_2 = 0$ and $a_1 = e$. □

We now continue with the proof of Claim 3.7.

Note that $\mathcal{E}_{|Q}$ cannot split as $\mathcal{O}_Q \oplus \mathcal{O}_Q(e)$. In fact, if such a splitting holds, choosing a point $x \in U \cap Q$ (see proof of Subclaim 3.8) we get that $x \in L = \Pi_y$, hence there is $z \in \varphi^{-1}(y)$ such that $x = \pi(z)$. On the other hand, $z \in \varphi^{-1}(y) = \mathbb{P}(\mathcal{O}_L) \subset \mathbb{P}(\mathcal{O}_Q) \subset \mathbb{P}(\mathcal{E})$. But $\mathbb{P}(\mathcal{O}_Q)$ is contracted to a point by φ, contradicting the fact that $\varphi^{-1}(y)$ is a curve.

Now Subclaim 3.8 gives that $\mathcal{E}_{|Q}$ is a uniform rank 2 vector bundle on Q satisfying (3.5). Therefore $\mathcal{E}_{|Q}$ is indecomposable and [MOS, Cor. 6.7] gives that $\mathcal{E}_{|Q}$ is a spinor bundle, that is we are in case (xiv). This proves Claim 3.7. □

This concludes the proof of Theorem 1 in Case (A).

Case (B): $c_1(\mathcal{E})^4 = 0$.

Note that this implies that $\rho(X) \geq 2$. Recall that $(X, \mathcal{O}_X(1))$ belongs to one of the cases (a)-(e) in Lemma 3.1 and we are assuming that $(X, \mathcal{O}_X(1))$ is neither as in (a) nor as in (b.1). In many cases we will apply the notation and results of the Dichotomy Lemma with respect to the nef value morphism ϕ_τ in Lemma 3.1.

Claim 3.9. *In case (b.2), either we are in case (fin) and $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (ii1) or we are in case (fact) and $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (ii2).*

Proof. In case (fact) just apply Lemmas 2.16(iii) and 2.12 to get case (ii2). In case (fin) we are in (emb) by Lemma 2.16(ii). Now the same proof of [LS, Cor. 5, Case 2] applies and we get that $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (ii1). □

Claim 3.10. *In case (c.1) we have that* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is as in (iii).*

Proof. Since $\rho(X) \geq 2$, using the classification of del Pezzo 4-folds (see for example [LP, §1], [F1]), we see that $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. Then we are in case (iii) by [LMS, Thm. 3]. □

Claim 3.11. *In case (c.2),* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is as in (iv).*

Proof. Just apply Proposition 2.17. □

Claim 3.12. *In case (c.3), either we are in case (fact) and* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is as in (v2) with* $p = \phi_\tau = \widetilde{\Phi}$ *and* $b = 2$ *or we are in case (emb) and either* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is as in (v3) with* $p = \widetilde{\Phi}, b = 3$*, or* $(X, \mathcal{O}_X(1), \mathcal{E})$ *is as in (v1).*

Proof. Set $\phi = \phi_\tau : X \to X'$. In case (fact) just apply Lemmas 2.16(iii) and 2.12 to get $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (v2) with $p = \phi_\tau = \widetilde{\Phi}$ and $b = 2$. In case (fin) we are in (emb) by Lemma 2.16(ii). Note that it cannot be that there is a fiber $F_{x_0} = \mathbb{P}^3$, for then $\phi_{|F_{x_0}} : F_{x_0} = \mathbb{P}^3 \to X'$ is constant, hence $F_{x_0} \subset f_{x_0}$, a contradiction. It follows that if $F_x = \mathbb{P}^2$ then $F_u = \mathbb{P}^2$ for every $u \in X$, hence $\widetilde{\Phi}$ has equidimensional fibers and gives a linear $\mathbb{P}^2$-bundle over a smooth $\widetilde{\Phi(X)}$. On the other hand, $\phi_{|F_x} : F_x \to X'$ is a closed embedding, giving that $X' \cong \mathbb{P}^2$. Also $\widetilde{\Phi}_{|f_x} : f_x \to \widetilde{\Phi(X)}$ is a closed embedding, giving that $\widetilde{\Phi(X)} \cong \mathbb{P}^2$. Thus X has two different $\mathbb{P}^2$-bundle structures over $\mathbb{P}^2$ and [Sa1, Thm. A] implies that $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1))$. But then $K_X = -3H$, a contradiction since we are in case (c.3).

Therefore F_x is a line and $c_1(\mathcal{E})^3 \neq 0$.

Now we have two possibilities: either there is a fiber $F_{x_0} = \mathbb{P}^2$ or F_u is a line for every $u \in X$.

Consider the second case. We have that $\widetilde{\Phi}$ has equidimensional fibers and then $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple with $p = \widetilde{\Phi}$ and $b = 3$ by Lemma 2.12. Since we are in case (emb), we also have an unsplit family of smooth rational curves $R = \{\phi(F_u), u \in X\}$ covering X'. For each $r \in R$ let $C_r \subset X'$ be the corresponding rational curve. For every $z \in X'$ we have a morphism

$$\gamma_z : \mathbb{P}^2 \cong \phi^{-1}(z) \to R$$

defined by $\gamma_z(u) = \phi(F_u)$. Hence γ_z is either finite onto its image or constant. We claim that the first case does not occur. In fact, consider the incidence correspondence

$$\mathcal{I} = \{(z, r) \in X' \times R : z \in C_r\}$$

together with its two projections $\pi_1 : \mathcal{I} \to X'$ and $\pi_2 : \mathcal{I} \to R$. Note that $\pi_1^{-1}(z) \cong \mathrm{Im}\gamma_z$ has dimension 0 or 2 for every $z \in X'$. Now π_2 is surjective and $\pi_2^{-1}(r) \cong C_r$ for every $r \in R$, so that $\mathcal{I}$ is irreducible and $\dim \mathcal{I} = \dim R + 1$. Also π_1 is surjective and therefore $\dim R = 1 + \dim \pi_1^{-1}(z_1)$ for $z_1 \in X'$ general. If $\dim \pi_1^{-1}(z_1) = 2$, we get that $\dim R = 3$, contradicting the bend-and-break lemma (see for example [D, Prop. 3.2]). Therefore $\dim \pi_1^{-1}(z_1) = 0$ and we get that $\dim R = 1$. Now for any $z \in X'$ we have that $\dim \pi_1^{-1}(z) = \dim \mathrm{Im}\gamma_z \leq 1$, so that $\dim \pi_1^{-1}(z) = 0$ and γ_z is constant.

For every $u \in X$ let $Y_u = \phi^{-1}(\phi(F_u))$. Then $\phi_{|Y_u} : Y_u \to \mathbb{P}^1 \cong \phi(F_u)$ exhibits Y_u as a linear $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ with fibers $f_{u'}, u' \in Y_u$. Now observe that, for every $u' \in Y_u$ we have that $F_{u'} \subset Y_u$: in fact, since $u' \in Y_u$ we have that $z' := \phi(u') \in \phi(F_u)$, so that there is an $u'' \in F_u$ such that $z' = \phi(u'')$. Hence $u', u'' \in \phi^{-1}(z')$ and therefore $\gamma_{z'}(u') = \gamma_{z'}(u'')$, that is $\phi(F_{u'}) = \phi(F_{u''}) = \phi(F_u)$ since $F_{u''} = F_u$. Now for any $u_1 \in F_{u'}$ we have that $\phi(u_1) \in \phi(F_{u'}) = \phi(F_u)$ and therefore $u_1 \in Y_u$. Thus $F_{u'} \subset Y_u$ for every $u' \in Y_u$. Set $h_u := \widetilde{\Phi}_{|Y_u} : Y_u \to \widetilde{\Phi(X)}$. Then $F_{u'} = h_u^{-1}(h_u(u'))$ for every $u' \in Y_u$. This gives that $h_u(Y_u)$ has dimension 2. On the other hand for any $u' \in Y_u$ we have that $h_{u|f_{u'}}$ is a closed embedding, and therefore $h_u(Y_u) \cong \mathbb{P}^2$ and h_u exhibits Y_u as a linear $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ with fibers $F_{u'}, u' \in Y_u$. Finally observe that Y_u is smooth, since $X \cong \mathbb{P}(\mathcal{F})$ and $Y_u \cong \mathbb{P}(\mathcal{F}_{|\phi(F_u)})$. It follows by [Sa1, Thm. A] that, for every $u \in X$, $Y_u \cong \mathbb{P}^1 \times \mathbb{P}^2$ embedded by the Segre embedding in $X \subset \mathbb{P}^N$. Moreover since γ_z is constant for every $z \in X'$ it follows that there is a unique $r_z \in R$ such that $z \in C_{r_z}$. This defines a morphism $X' \to R$ with all fibers the curves C_r. Passing to the Stein factorization we get a $\mathbb{P}^1$-bundle $X' \to \widetilde{R}$ onto a smooth curve. Finally, the fibers of the composition $X \to X' \to \widetilde{R}$ are exactly the Y_u, that is $\mathbb{P}^1 \times \mathbb{P}^2$ embedded by the Segre embedding. This concludes the proof in the case that F_u is a line for every $u \in X$, and gives that $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (v3) with $p = \widetilde{\Phi}$ and $b = 3$.

Assume now that there is a fiber $F_{x_0} = \mathbb{P}^2$, so that, as above $X' \cong \mathbb{P}^2$. Recall that we have a very ample rank 3 vector bundle $\mathcal{F}$ over $\mathbb{P}^2$ such that $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$. As is well known, $F_1(X, x)$ is smooth, hence $L := F_x$ belongs to a unique irreducible component W of $F_1(X, x)$. For any line $[L'] \in W$ we have that $\det \mathcal{E} \cdot L' = \det \mathcal{E} \cdot L = 0$, hence L' is contracted by Φ, that is $L' = L$ and therefore $\dim W = 0$. Hence $0 = \dim_{[L]} F_1(X, x) = -K_X \cdot L - 2$, so that $K_X \cdot L = -2$.

Set $c_i(\mathcal{F}) = a_i H^i_{\mathbb{P}^2}$ for some $a_i \in \mathbb{Z}, i = 1, 2$. Let $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ and $R = \phi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, so that $\xi^2 \cdot R^2 = 1$ and $R^3 = 0$. We claim that

$$\xi^4 = a_1^2 - a_2 \text{ and } \xi^3 \cdot R = a_1.$$

In fact, from the standard relation

$$\sum_{i=0}^{3} (-1)^i \phi^*(c_i(\mathcal{F}))\xi^{3-i} = 0$$

we get

$$\xi^3 = a_1 \xi^2 \cdot R - a_2 \xi \cdot R^2$$

hence $\xi^3 \cdot R = a_1$ and $\xi^4 = a_1 \xi^3 \cdot R - a_2 \xi^2 \cdot R^2 = a_1^2 - a_2$.

Now there are $a, b \in Z$ such that

$$\det \mathcal{E} = a\xi + bR.$$

Note that it cannot be that $a = 0$, for otherwise we would get that $c_1(\mathcal{E})^3 = 0$, a contradiction. Also, the condition $c_1(\mathcal{E})^4 = 0$ gives

$$6a^2b^2 + 4a^3ba_1 + a^4(a_1^2 - a_2) = 0. \tag{3.6}$$

Now $\xi \cdot L = 1$ and $\det \mathcal{E} \cdot L = 0$, giving

$$a + bR \cdot L = 0. \tag{3.7}$$

Therefore $K_X \cdot L = -2$ gives

$$(-3\xi + \phi^*(K_{\mathbb{P}^2} + \det \mathcal{F})) \cdot L = -2$$

that is

$$(a_1 - 3)R \cdot L = 1$$

and therefore

$$a_1 = 4 \text{ and } R \cdot L = 1.$$

We get from (3.7) that $b = -a$ and from (3.6) that $a_2 = 6$.

Let M be any line in $\mathbb{P}^2$. Then $\mathcal{F}_{|M}$ is ample and $c_1(\mathcal{F}_{|M}) = c_1(\mathcal{F}) \cdot M = 4$. This implies that $\mathcal{F}_{|M} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. It follows by [Ele, Prop. 5.1] that $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$ or $T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. The first case is excluded since it has $a_2 = 5$. Therefore $\mathcal{F} \cong T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ and $\det \mathcal{E} = a(\xi - R)$. Since $|\xi - R|$ is base-point free and defines a morphism $q : X \to \mathbb{P}^3$, it follows by [Lo, Lemma 5.1] that there is a rank r vector bundle $\mathcal{G}$ on $\mathbb{P}^3$ such that $\mathcal{E} \cong q^*\mathcal{G}$. We now claim that $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus r}$. To this end observe that we can see X as a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^3$ under the Segre embedding given by $L = \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)$. Moreover $\mathcal{O}_X(1) = L_{|X}$ and $q = p_{2|X} : X \to \mathbb{P}^3$, where $p_2 : \mathbb{P}^2 \times \mathbb{P}^3 \to \mathbb{P}^3$ is the second projection. Consider the exact sequence

$$0 \to (p_2^*\mathcal{G})(-(p+1)L) \to (p_2^*\mathcal{G})(-pL) \to \mathcal{E}(-pH) \to 0.$$

Since $H^i(\mathcal{E}(-pH)) = 0$ for all $i \geq 0$ and $1 \leq p \leq 4$ we deduce that

$$H^i((p_2^*\mathcal{G})(-(p+1)L)) \cong H^i((p_2^*\mathcal{G})(-pL)) \text{ for all } i \geq 0 \text{ and } 1 \leq p \leq 4.$$

Now the Künneth formula gives that

$$h^0(\mathcal{O}_{\mathbb{P}^2}(p-2))h^{i-2}(\mathcal{G}(-p-1)) = h^0(\mathcal{O}_{\mathbb{P}^2}(p-3))h^{i-2}(\mathcal{G}(-p)) \text{ for } i \geq 2 \text{ and } 1 \leq p \leq 4$$

and one easily sees that this implies that $H^j(\mathcal{G}(-2)(-s)) = 0$ for all $j \geq 0$ and $1 \leq s \leq 3$. But then $\mathcal{G}(-2)$ is an Ulrich vector bundle for $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and therefore $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus r}$ by Remark 2.9. Thus $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (v1). □

In the sequel we will use the standard notation V_7 for the blow up of $\mathbb{P}^3$ in a point.

In order to study cases (d.2)-(d.5) we first observe the following.

Claim 3.13. *In cases (d.2), (d.3), (d.4) and (d.5), case* (fin) *does not hold for ϕ_τ.*

Proof. Suppose that case (fin) holds for ϕ_τ and let L be any line such that $x \in L \subseteq F_x$. Since $\tau = 2$ we have that $(K_X + 2H) \cdot L \geq 0$, that is $K_X \cdot L \geq -2$. On the other hand, we have that

$$0 \leq \dim_{[L]} F_1(X, x) = -K_X \cdot L - 2$$

so that $K_X \cdot L = -2$ and therefore $\phi_\tau(L)$ is a point, so that $L \subseteq f_x \cap F_x$, a contradiction. □

Claim 3.14. *In case (d.2) we have that every smooth fiber is isomorphic to only one of $V_7, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}(T_{\mathbb{P}^2})$.*

Proof. By Claim 3.13 we are in case (fact) for ϕ_τ, hence ϕ_τ factorizes through $\widetilde{\Phi}$. Let f_u be a smooth fiber. If $\mathrm{Pic}(f_u) \cong \mathbb{Z}$, Lemma 2.16(iii) gives that $f_u = F_u$, hence $f_u \cong \mathbb{P}^3$ and $\mathcal{O}_X(1)_{|f_u} \cong \mathcal{O}_X(1)_{|F_u} \cong \mathcal{O}_{\mathbb{P}^3}(1)$. But $\phi_\tau : X \to X'$ is a del Pezzo fibration for $(X, \mathcal{O}_X(1))$, so that $K_{X|f_u} = -2H_{|f_u}$, giving the contradiction

$$-4H_{\mathbb{P}^3} = K_{f_u} = K_{X|f_u} = -2H_{|f_u} = -2H_{\mathbb{P}^3}.$$

Now the classification of del Pezzo 3-folds (see for example [LP, §1], [F1]) implies that f_u is either $V_7, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}(T_{\mathbb{P}^2})$. In the first case f_u is a del Pezzo 3-fold of degree 7 and then $f_u \cong V_7$. In the other two cases observe that $\rho(f_u) = \rho(f_x)$ by [JR, Thm. 1.4] and therefore $f_u \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ when $f_x \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $f_u \cong \mathbb{P}(T_{\mathbb{P}^2})$ when $f_x \cong \mathbb{P}(T_{\mathbb{P}^2})$. □

Claim 3.15. *In case (d.2), if $f_x \cong V_7$, then $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (vi1).*

Proof. By [F3, (4.7)] we have that $f_u \cong V_7$ for every $u \in X$. We claim that F_u is a line for every $u \in X$. Assume to the contrary that there is an $u \in X$ such that F_u is not a line. We know that $F_u \subset f_u$. Then it cannot be that $F_u = \mathbb{P}^3$, for otherwise, $\mathbb{P}^3 \cong V_7$, a contradiction. Therefore $F_u = \mathbb{P}^2$. Let E be the exceptional divisor of the blow-up $\varepsilon : f_u \cong V_7 \to \mathbb{P}^3$ in a point and let $\widetilde{H}$ be the pull back of a plane. Observe that since $K_{f_u} = (K_X + f_u)_{|f_u} = K_{X|f_u} = -2H_{|f_u}$ and, as is well known, $\mathrm{Pic}(V_7)$ has no torsion, we get that $H_{|f_u} = 2\widetilde{H} - E$. Since E is the only linear plane contained in V_7 (with respect to $2\widetilde{H} - E$), we deduce that $F_u = E$. Let now $u' \in f_u \setminus E$. Then $F_{u'}$ is not contained in E, and therefore, by what we have just proved, $F_{u'}$ is a line. As above $F_{u'} \cap E \neq \emptyset$, for otherwise $\mathcal{O}_{\mathbb{P}^1}(1) = (2\widetilde{H} - E)_{|F_{u'}} = 2\widetilde{H}_{|F_{u'}}$, a contradiction. But then $F_{u'} \cap F_u = F_{u'} \cap E \neq \emptyset$ and therefore $F_{u'} = F_u = E$, a contradiction. Hence F_u is a line for every $u \in X$ and Lemma 2.12 implies that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple with $p = \widetilde{\Phi} : X \cong \mathbb{P}(\mathcal{F}) \to B$ and $b = 3$. As we know, $\phi_\tau = h \circ p$ where $h : B \to X'$. Also there is an ample line bundle $\mathcal{L}$ on X' such that

$$mp^*(K_B + \det \mathcal{F}) = m(K_X + 2H) = \phi_\tau^* \mathcal{L} = p^*(h^* \mathcal{L})$$

and therefore $m(K_B + \det \mathcal{F}) = h^* \mathcal{L}$, so that $(B, \det \mathcal{F})$ is a del Pezzo fibration over X'. Also, for every $x' \in X'$, we have that $V_7 \cong \mathbb{P}(\mathcal{F}_{|h^{-1}(x')})$, hence $h^{-1}(x')$ is a smooth surface and we have a surjective morphism $p_{|V_7} : V_7 \to h^{-1}(x')$. Now E is a plane and the fibers $F_{u'}, u' \in V_7$ of $p_{|V_7}$ are lines. As above $F_{u'} \cap E \neq \emptyset$. On the other hand, it cannot be that $F_{u'} \subset E$, for otherwise the morphism $\Phi_{|E} : \mathbb{P}^2 \cong E \to \Phi(X)$, that contracts $F_{u'}$ to a point, would be constant, therefore implying the contradiction $F_{u'} = E$. Hence $F_{u'} \cap E$ is a point for every $u' \in V_7$ and $p_{|E} : E \to h^{-1}(x')$ is a closed embedding, thus giving that $h^{-1}(x') \cong \mathbb{P}^2$. □

Claim 3.16. *In case (d.2), if $f_x \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, then $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (vi2), while if $f_x \cong \mathbb{P}(T_{\mathbb{P}^2})$, then $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (vi3).*

Proof. By Claim 3.13 we are in case (fact) and, for every $u \in X$, f_u is a disjoint union of fibers F_v of Φ. Also, $K_{f_x} = -2H_{|f_x}$, hence $H^3 \cdot f_u = H^3 \cdot f_x = 6$.

Subclaim 3.17. *f_u is normal for every $u \in X$.*

Proof. Note that f_u is integral by [F3, (4.6)], hence there is no linear $\mathbb{P}^3$ contained in f_u. Moreover, f_u is not a cone by [F3, §1, p. 232]. Assume that f_u is not normal. It follows by [F4, Thm. 2.1(a)] and [F5, Thm. II] that f_u is the projection $\pi_O : \Sigma \to f_u$ of a rational normal threefold scroll $\Sigma \subset \mathbb{P}^8$

of degree 6 from a point $O \notin \Sigma$. In particular a general curve section C of f_u is a rational curve of degree 6 in $\mathbb{P}^5$ and with arithmetic genus 1, because so is the arithmetic genus of a general curve section of f_x. Therefore C has a unique double point and this implies that $\mathrm{Sing}(f_u)$ is a plane. Then there is a quadric $Q \subset \Sigma$ such that $\pi_O(Q) = \mathrm{Sing}(f_u)$ and it is easily checked that then Σ is the embedding of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ by the tautological line bundle and Q is the embedding of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Let $p : \Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ be the projection map. Then any plane contained in Σ must be a fiber of p. Also, since $p_{|Q} : Q \to \mathbb{P}^1$ is surjective, it follows that the planes $M_z := p^{-1}(z), z \in \mathbb{P}^1$ intersect Q, and therefore $L_z := M_z \cap Q$ is an effective divisor on Q. On the other hand, $M_z \not\subset \langle Q \rangle = \mathbb{P}^3$, for otherwise L_z would be a hyperplane section of Q and then, for $z \neq z' \in \mathbb{P}^1$ we would get the contradiction $\emptyset \neq L_z \cap L_{z'} \subset M_z \cap M_{z'} = \emptyset$. Therefore, since $L_z \subset M_z \cap \langle Q \rangle$, it follows that L_z is a line for every $z \in \mathbb{P}^1$. Now $\pi_O(L_z)$ and $\pi_O(L_{z'})$ are two lines in the plane $\mathrm{Sing}(f_u)$, hence they intersect. Therefore $\pi_O(M_z) \cap \pi_O(M_{z'}) \neq \emptyset$ for any $z, z' \in \mathbb{P}^1$.

Let $u' \in f_u$ be a general point. If $F_{u'} = \mathbb{P}^2$ then, for general $u'' \in f_u$, there are two planes $M_{z'}, M_{z''} \subset \Sigma$ such that $\pi_O(M_{z'}) = F_{u'}, \pi_O(M_{z''}) = F_{u''}$. But this gives the contradiction

$$\emptyset = F_{u'} \cap F_{u''} = \pi_O(M_{z'}) \cap \pi_O(M_{z''}) \neq \emptyset.$$

Assume now that $F_{u'}$ is a line. Observe that $F_{u'} \not\subset \pi_O(M_z)$ for any $z \in \mathbb{P}^1$, for otherwise $\Phi_{|\pi_O(M_z)} : \mathbb{P}^2 = \pi_O(M_z) \to \Phi(X)$ must be constant, since it contracts the line $F_{u'}$ to a point. But then $F_{u'} = \pi_O(M_z)$, a contradiction. Hence there is a line $L \subset \Sigma$ such that $F_{u'} = \pi_O(L)$ and $L \not\subset Q$. But lines in Σ not contained in Q must be contained in a plane M_z, thus giving a contradiction. □

We assume from now on that

$$V := f_u \text{ is a singular fiber of } \phi_\tau.$$

Note that $K_V = -2H_{|V}$ and $V \subset \mathbb{P}^7 = \mathbb{P}H^0(H_{|V})$ is of degree 6 by [F3, (1.5)].

Let us recall some notation. Let $a_s \geq \ldots \geq a_0 \geq 0$ be integers and let $S = S(a_0, \ldots, a_s) = \varphi_\xi(\mathbb{P}(\mathcal{G})) \subset \mathbb{P}^{\sum_{i=0}^s a_i + s}$ be the rational normal scroll, where $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_s)$, $p : \mathbb{P}(\mathcal{G}) \to \mathbb{P}^1$ and ξ is the tautological line bundle. We denote, for every $t \in \mathbb{P}^1$, by $R = R_t$ a ruling on $\mathbb{P}(\mathcal{G})$ and by $G = G_t$ its image on S. When some a_i are zero, $S(a_0, \ldots, a_s)$ is a cone with vertex S_0 and inverse image W_0 on $\mathbb{P}(\mathcal{G})$. If $a_k = 1$ for some k, let $l(S) = \max\{k \geq 0 : a_k = 1\}$, let $S_1 = S(a_0, \ldots, a_{l(S)})$, $W_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{l(S)}))$ with tautological line bundle ξ_1 and ruling R_1. We recall that if a line $L \subset S$ is not contained in a ruling, then $L \subset S_1$. For every subvariety $Y \subset S$ such that $Y \not\subseteq S_0$ we denote its strict transform on $\mathbb{P}(\mathcal{G})$ by $\widetilde{Y} := \overline{\varphi_\xi^{-1}(Y \setminus S_0)}$.

Subclaim 3.18. *Let $S_V \subset \mathbb{P}^7$ be one of $S(0,0,1,3), S(0,0,2,2)$ or $S(0,1,1,2)$ and let $\mathcal{G}_V$ be one of $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3), \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}$ or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. Then $V \subset S_V, \widetilde{V} \sim 2\xi - 2R$ on $\mathbb{P}(\mathcal{G}_V), W_0 \subset \widetilde{V}, S_0 \subset V$ and $\varphi_\xi^{-1}(V) = \widetilde{V}$.*

Proof. Let $v_0 \in \mathrm{Sing}(V)$, let $\pi_{v_0} : V \dashrightarrow \mathbb{P}^6$ be the projection with center v_0 and let $V' \subset \mathbb{P}^6$ be its image. Since V is not a cone by [F3, p. 232], it follows that V' is an irreducible non-degenerate threefold of degree at most 4. Hence $\deg V' = 4$ and [EH, Thm. 1] implies that either V' is a rational normal scroll or a cone over the Veronese surface $S \subset \mathbb{P}^5$. In the second case, we have that $V \subset \mathcal{C}$ the cone with vertex a line L over $S \subset \mathbb{P}^5$. Let $\pi_L : \mathbb{P}^7 \dashrightarrow \mathbb{P}^5$ be the projection with center L. Note that any plane $M \subset \mathcal{C}$ is of type $\langle p, L \rangle, p \in S$. Now, for any point $z \in M \setminus L$ we have that $p = \pi_L(z) \in S$ and $M = \langle p, L \rangle$. Also, any line $L' \subset \mathcal{C}$ must intersect L, for otherwise $\pi_L(L')$ is a line in S, a contradiction. If there is a point $v \in V$ such that F_v is a plane, then $L \subset F_v$ and picking any $v' \in V \setminus F_v$ we find the contradiction

$$\emptyset \neq F_{v'} \cap L \subseteq F_{v'} \cap F_v = \emptyset.$$

Therefore F_v is a line for every $v \in V$. This means that $\dim \Phi(V) = 2$ and V is disjoint union of the 2-dimensional family of lines $\{\Phi^{-1}(y), y \in \Phi(V)\}$. Since they all intersect L, it follows that there is point $v' \in L$ contained in infinitely many such lines, a contradiction.

Therefore V' is a rational normal scroll of degree 4, hence it can be one of $S(0,1,3), S(0,2,2), S(0,0,4)$ or $S(1,1,2)$ and, as a consequence, $V \subset S_V \subset \mathbb{P}^7$, where S_V is one of $S(0,0,1,3), S(0,0,2,2), S(0,0,0,4)$ or $S(0,1,1,2)$. Let C be a general curve section of V so that C is a smooth irreducible elliptic curve of degree 6. If $S_V = S(0,0,0,4)$ then $C \subset S(0,4)$, which is not possible since $S(0,4)$ does not contain an elliptic curve of degree 6. Thus S_V is one of $S(0,0,1,3), S(0,0,2,2)$ or $S(0,1,1,2)$. Hence the vertex of S_V is a point or a line and we can consider $\widetilde{V} \sim a\xi + bR$ on $\mathbb{P}(\mathcal{G}_V)$. It follows that V is a Weil divisor linearly equivalent to $aH + bG$ on S_V and $C \sim aH_{|Y} + bG_{|Y}$, where Y, the general surface section of S_V, is a non-degenerate smooth irreducible surface of degree 4 in $\mathbb{P}^5$, that is $Y = S(1,3)$ or $S(2,2)$. Then this gives that $a = 2$ and $b = -2$. Now W_0 is covered by curves Γ contracted by φ_ξ, hence $\Gamma \equiv c(\xi^3 - 4\xi^2 R), c > 0$. Therefore $\Gamma \cdot \widetilde{V} = c(\xi^3 - 4\xi^2 R) \cdot (2\xi - 2R) = -2c < 0$, hence $\Gamma \subset \widetilde{V}$ and it follows that $W_0 \subset \widetilde{V}, S_0 \subset V$ and $\widetilde{V} = \varphi_\xi^{-1}(V)$. □

Subclaim 3.19. *Let $S_V = S(0,1,1,2)$. Then $V \cap S_1 = M_1 \cup M_2$, where M_1 and M_2 are two planes. Moreover either $S_0 \subset M_1 = M_2$ or $M_1 \cap M_2 = S_0$ and both $M_1 \cap G_t$ and $M_2 \cap G_t$ are lines for every $t \in \mathbb{P}^1$. Furthermore $\langle S(1,2), M_1\rangle = \langle S(1,2), M_2\rangle = \mathbb{P}^7$.*

Proof. We have $S_V = \varphi_\xi(\mathcal{G})$ where $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. It is easily follows that $V \cap S_1 = \varphi_\xi(\widetilde{V} \cap W_1)$. Moreover $W_0 \subset \widetilde{V} \cap W_1$ by Subclaim 3.18, hence $S_0 \subset V \cap S_1$. Now let $D = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. Note that $D \sim \xi$ and $D \cap W_0 = \emptyset$, hence $\varphi_\xi(D)$ is a hyperplane section of S not passing through the vertex S_0. Let $Q = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$, with tautological line bundle ξ_Q and fiber R_Q. Note that $Q \subseteq D \cap W_1$. Also, since $W_1 \sim \xi - 2R$, we have that $\xi^2 \cdot D \cdot W_1 = \xi^3 \cdot (\xi - 2R) = 2 = \xi^2 \cdot Q$ and therefore $Q = D \cap W_1$. Now $\widetilde{V} \cap W_1 \cap D = \widetilde{V} \cap Q \sim 2\xi_Q - 2R_Q$, hence $\widetilde{V} \cap W_1 \cap D$ is a curve of type $(0,2)$ (or $(2,0)$) on Q, that is the union of two, possibly coincident, lines L_1 and L_2 on Q. Also note that $\widetilde{V} \cap W_1 \sim 2\xi_1 - 2R_1$ on W_1 and $H^0(\xi_1 - \widetilde{V} \cap W_1) = H^0(-\xi_1 + 2R_1) = 0$ and therefore $V \cap S_1 = \varphi_\xi(\widetilde{V} \cap W_1)$ is a non-degenerate degree 2 surface in $\mathbb{P}^4 = \mathbb{P}(H^0(\xi_1))$. This implies that $V \cap S_1$ is reducible, hence $V \cap S_1 = M_1 \cup M_2$, where M_1 and M_2 are two, possibly coincident, planes. Also, in case $M_1 \neq M_2$, they must intersect in a point, hence in S_0. Now assume that $M_1 \subset G_t$. Let $\widetilde{M_i}, i = 1,2$ be the strict transforms, so that $\widetilde{V} \cap W_1 = \widetilde{M_1} \cup \widetilde{M_2}$ and $\widetilde{M_1} \subset R_t$. Then $\widetilde{M_1} = W_1 \cap R_t$, hence $\widetilde{M_1} \sim R_1$ and therefore $\widetilde{M_2} \sim 2\xi_1 - 3R_1$, a contradiction since $H^0(2\xi_1 - 3R_1) = 0$. Hence $\widetilde{M_1} \not\subset R_t$ for every $t \in \mathbb{P}^1$ and then $\pi_{|\widetilde{M_1}} : \widetilde{M_1} \to \mathbb{P}^1$ is surjective. Hence $\widetilde{M_1} \cap R_t \neq \emptyset$ for every $t \in \mathbb{P}^1$ and then it is a divisor on $\widetilde{M_1}$. It follows that both $M_1 \cap G_t$ and $M_2 \cap G_t$ are lines for every $t \in \mathbb{P}^1$. Also let $\widetilde{M_i} \sim a_i\xi_1 + b_iR_1, i = 1,2$, so that $a_i \geq 0, a_1 + a_2 = 2$ and $1 = \xi_1^2 \cdot (a_1\xi_1 + b_1R_1) = 2a_1 + b_1$. If $a_1 = 0$ then $b_1 = 1$ and we get the same contradiction as above. Similarly if $a_2 = 0$. Therefore $\widetilde{M_1} \sim \xi_1 - R_1$. Since $\langle S(1,2)\rangle = \mathbb{P}^4$, to prove that $\langle S(1,2), M_1\rangle = \mathbb{P}^7$, we just need to prove that $H^0(\mathcal{I}_{\widetilde{M_1} \cup W_{12}/\mathbb{P}(\mathcal{G})}(\xi)) = 0$, were $W_{12} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \subset \mathbb{P}(\mathcal{G})$. If not, there is a $D_1 \in |\xi|$ such that $D_1 \cap W_1 = \widetilde{M_1} + W_1 \cap W_{12}$. But then $W_1 \cap W_{12} \sim R_1$, hence it maps to a point in $\mathbb{P}^1$. But this is a contradiction since $W_1 \cap W_{12}$ contains $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1))$. □

Subclaim 3.20. *Let $t \in \mathbb{P}^1$ and let $\Sigma_t = V \cap G_t$. Then Σ_t is a smooth quadric in $\mathbb{P}^3 = G_t$ such that $S_0 \subset \Sigma_t$. In particular,* $\dim(\widetilde{\Phi}(V)) = 2$.

Proof. Note that $\Sigma_t = \varphi_\xi(\widetilde{V} \cap R_t)$ and, by Subclaim 3.18, $\deg \Sigma_t = \xi^2 \cdot (2\xi - 2R) \cdot R = 2$, so that Σ_t is a quadric in $\mathbb{P}^3 = G_t$. Also $S_0 \subset V$ by Subclaim 3.18, hence $S_0 \subset \Sigma_t$. To prove that Σ_t is smooth, let $Z = S_1$ when $S_V = S(0,0,1,3)$ or $S(0,1,1,2)$ and $Z = S_0$ when $S_V = S(0,0,2,2)$. Observe that $V \not\subset Z$, for otherwise we would have that either $V = Z = S(0,0,1) = \mathbb{P}^3$ or $V = Z = S(0,1,1)$ a quadric cone in $\mathbb{P}^4$, a contradiction. Let $\tilde{v} \in \widetilde{V} \setminus \varphi_\xi^{-1}(Z)$ and $v = \varphi_\xi(\tilde{v})$. Note that $v \notin S_0$, for otherwise $\varphi_\xi(\tilde{v}) = v \in S_0 \subseteq Z$. Let L be any line such that $v \in L \subseteq F_v$. Then $L \not\subset S_0$ and $\tilde{v} \in \widetilde{L}$, because $\tilde{v} \notin W_0$. Since $L \not\subset S_1$ we get that $L \subset G_{\pi(\tilde{v})}$ and therefore $F_v \subset G_{\pi(\tilde{v})}$. Hence $\dim_v F_v \cap G_{\pi(\tilde{v})} = \dim_v F_v \geq 1$ and it follows by semicontinuity that $\dim_v F_v \cap G_{\pi(\tilde{v})} \geq 1$ for every $\tilde{v} \in \widetilde{V}$. We have that

$$\Sigma_t = \bigsqcup_{y \in \Phi(V)} \Phi^{-1}(y) \cap G_t$$

and if $v \in \Phi^{-1}(y) \cap G_t$, then $\Phi^{-1}(y) = F_v$ and there is $\tilde{v} \in R_t$ such that $v = \varphi_\xi(\tilde{v})$, so that $t = \pi(\tilde{v})$ and $\dim \Phi^{-1}(y) \cap G_t = \dim F_v \cap G_{\pi(\tilde{v})} \geq 1$. Since $\Phi^{-1}(y) = \mathbb{P}^k, k = 1,2$, two cases are possible. If there is an $y \in \Phi(V)$ such that $\dim \Phi^{-1}(y) \cap G_t = 2$ then $\Phi^{-1}(y) \cap G_t = \Phi^{-1}(y) = \mathbb{P}^2$ and Σ_t is either reducible or a double plane. Since $\Sigma_t \subset G_t = \mathbb{P}^3$ and $\Sigma_t \subset X$, we can apply the same method in the proof of Claim 3.25 and get a contradiction. Therefore $\Phi^{-1}(y) \cap G_t$ is a line for every $y \in \Phi(V)$ such that $\Phi^{-1}(y) \cap G_t \neq \emptyset$. Hence Σ_t is covered by a family of disjoint lines, so that Σ_t is smooth. Moreover note that a general fiber $F_v, v \in V$ cannot be a plane, for otherwise the argument above would show that $\dim_v F_v \cap G_{\pi(\tilde{v})} \geq 2$ and then we would have an $y \in \Phi(V)$ such that $\dim \Phi^{-1}(y) \cap G_t = 2$, a contradiction. Therefore a general F_v is a line and $\dim \Phi(V) = 2$. Since $\widetilde{\Phi}$ and Φ have the same fibers, we get that $\dim(\widetilde{\Phi}(V)) = 2$. □

Subclaim 3.21. *The case $S_V = S(0,0,1,3)$ does not occur. In the other cases we have that $V = \varphi_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}(\mathbb{P}(\mathcal{F}))$, where $\mathcal{F} = \mathcal{O}_{\mathbb{F}_0}(f) \oplus \mathcal{O}_{\mathbb{F}_0}(2C_0 + f)$ if $S_V = S(0,0,2,2)$, while $\mathcal{F} = \mathcal{O}_{\mathbb{F}_1}(C_0 + f) \oplus \mathcal{O}_{\mathbb{F}_1}(C_0 + 2f)$ if $S_V = S(0,1,1,2)$.*

Proof. Let S_V be $S(0,0,1,3), S(0,0,2,2)$ or $S(0,1,1,2)$ and let Y be respectively $S(1,3), S(2,2)$ or $S(1,2)$. We have that $S_V \subset \mathbb{P}^7$, and, in the first two cases, S_V is a cone with vertex the line S_0 over $Y \subset \mathbb{P}^5$, while in the third case $Y \subset \mathbb{P}^4$ and $\langle Y \rangle \cap M_1 = \emptyset$, where $M_1 \subset V$ is the plane given in Subclaim 3.19. Set $N = S_0$ in the first two cases and $N = M_1$ in the third case. Let $z \in Y$ and let $t \in \mathbb{P}^1$ be such that $z \in G_t$. Consider the plane $M_z = \langle z, S_0 \rangle$ in the first two cases and $M_z = \langle z, L_1 \rangle$ in the third case, where $L_1 = M_1 \cap G_t$. Since $M_z \subset G_t$, we get by Subclaim 3.20 that $M_z \cap \Sigma_t$ is the union of S_0 (or L_1) and a line L_z, meeting S_0 (or L_1, hence also M_1) in a point. This defines a morphism $Y \to \mathbb{G}(1,7)$ which in turn gives rise to a rank 2 globally generated vector bundle $\mathcal{F}$ on Y such that

$$V = \bigcup_{z \in Y} L_z = \varphi_{\xi_{\mathcal{F}}}(\mathbb{P}(\mathcal{F})).$$

Moreover note that there is a unique such line passing through the general point of V, for the lines L_z are the fibers of the projection $\pi_N : V \dashrightarrow Y$. In particular $\varphi_{\xi_{\mathcal{F}}}$ is birational and we find that

$$h^0(\xi_{\mathcal{F}}) = 8 \text{ and } \xi_{\mathcal{F}}^3 = \deg V = 6. \tag{3.8}$$

Also, since the lines L_z meet N in a point, it follows that N gives rise to a section $\mathbb{P}(\mathcal{L}_0) \subset \mathbb{P}(\mathcal{F})$ of $\mathbb{P}(\mathcal{F}) \to Y$, where $\mathcal{L}_0$ is a line bundle quotient of $\mathcal{F}$. Hence $\mathcal{L}_0$ is globally generated and defines a morphism $\varphi_{\mathcal{L}_0} : Y \to N$. Thus we have, for some integers a and b, an exact sequence

$$0 \to \mathcal{O}_Y(aC_0 + bf) \to \mathcal{F} \to \mathcal{L}_0 \to 0. \tag{3.9}$$

Now, when $S_V = S(0,0,2,2)$ we have that $Y \cong \mathbb{F}_0$ and $\mathcal{L}_0$ must be $\mathcal{O}_{\mathbb{F}_0}(f)$. By (3.8) we get that $a \geq 0$ and $b \geq 0$, for otherwise (3.9) gives that $8 = h^0(\xi_{\mathcal{F}}) = h^0(\mathcal{F}) \leq 2$. Hence $H^1(\mathcal{O}_{\mathbb{F}_0}(aC_0 + bf)) = 0$ and (3.9) gives that

$$8 = h^0(\mathcal{F}) = (a+1)(b+1) + 2.$$

Moreover, using the well-known fact that $\xi_{\mathcal{F}}^3 = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, we get from (3.9) that

$$a(2b+1) = 6$$

and it follows that $a = 2, b = 1$. Since $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{F}_0}(f), \mathcal{O}_{\mathbb{F}_0}(2C_0 + f)) \cong H^1(\mathcal{O}_{\mathbb{F}_0}(2C_0)) = 0$, we get that (3.9) splits and $\mathcal{F} \cong \mathcal{O}_{\mathbb{F}_0}(f) \oplus \mathcal{O}_{\mathbb{F}_0}(2C_0 + f)$.

Next we exclude the case $S_V = S(0,0,1,3)$. We have that $Y \cong \mathbb{F}_2$ and it is easily seen that $\mathcal{L}_0 \cong \mathcal{O}_{\mathbb{F}_2}(f)$. Then (3.9) gives an exact sequence

$$0 \to \mathcal{O}_f(a) \to \mathcal{F}_{|f} \to \mathcal{O}_f \to 0$$

with $a = c_1(\mathcal{F}) \cdot f \geq 0$, since $\mathcal{F}$ is globally generated. But then the sequence splits and we find that $\mathcal{F}_{|f} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)$. On the other hand if we let $G_t = \langle f, S_0 \rangle$ we see that the quadric $\Sigma_t = \varphi_{\xi_{\mathcal{F}_{|f}}}(\mathbb{P}(\mathcal{F}_{|f}))$ is a cone, contradicting Subclaim 3.20. Thus this case does not occur.

Finally assume that $S_V = S(0,1,1,2)$. Then $Y \cong \mathbb{F}_1$ and it is easily seen that $\mathcal{L}_0$ must be $\mathcal{O}_{\mathbb{F}_1}(C_0 + f)$. By (3.8) we get that $a \geq 0$ and $b \geq 0$, for otherwise $8 = h^0(\xi_{\mathcal{F}}) = h^0(\mathcal{F}) \leq 3$. Now $\xi_{\mathcal{F}}^3 = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$ becomes $a^2 - 2ab - b + 5 = 0$, that can be rewritten as

$$(2a+1)(4b - 2a + 1) = 21.$$

The possible integer solutions are $(a,b) = (0,5), (1,2), (3,2)$ and $(10,5)$. As it is easily seen, the cases $(0,5)$ and $(3,2)$ have $H^1(\mathcal{O}_{\mathbb{F}_1}(aC_0 + bf)) = 0$ and $h^0(\mathcal{O}_{\mathbb{F}_1}(aC_0 + bf)) \neq 5$, while the case $(10,5)$ has $h^0(\mathcal{O}_{\mathbb{F}_1}(10C_0 + 5f)) > 8$, so they all contradict (3.9). Thus we have that $a = 1, b = 2$. Since $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{F}_1}(C_0 + f), \mathcal{O}_{\mathbb{F}_1}(C_0 + 2f)) \cong H^1(\mathcal{O}_{\mathbb{F}_1}(f)) = 0$, we get that (3.9) splits and $\mathcal{F} = \mathcal{O}_{\mathbb{F}_1}(C_0 + f) \oplus \mathcal{O}_{\mathbb{F}_1}(C_0 + 2f)$. □

Subclaim 3.22. *Let $\mathcal{Q} \subset \mathbb{P}^3$ be the quadric cone. In the case $S_V = S(0,0,2,2)$, we have that $V \cong \mathbb{P}^1 \times \mathcal{Q} \subset \mathbb{P}^7$ embedded by $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathcal{Q}}(1)$.*

Proof. By Subclaim 3.21 we have that $V = \varphi_{\xi_{\mathcal{F}}}(\mathbb{P}(\mathcal{F}))$, where $\mathcal{F} = \mathcal{O}_{\mathbb{F}_0}(f) \oplus \mathcal{O}_{\mathbb{F}_0}(2C_0 + f)$. Let us first consider the exceptional locus $\mathrm{Exc}(\varphi_{\xi_{\mathcal{F}}})$ of $\varphi_{\xi_{\mathcal{F}}}$. We claim that

$$\mathrm{Exc}(\varphi_{\xi_{\mathcal{F}}}) = \mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f)). \tag{3.10}$$

Let $p : \mathbb{P}(\mathcal{F}) \to \mathbb{F}_0$ be the projection map. From Grothendieck's relation

$$\xi_{\mathcal{F}}^2 = \xi_{\mathcal{F}} p^* c_1(\mathcal{F}) - p^* c_2(\mathcal{F})$$

we deduce, setting $R = p^*(C_0) p^*(f)$, that

$$\xi_{\mathcal{F}}^2 = 2\xi_{\mathcal{F}} p^* C_0 + 2\xi_{\mathcal{F}} p^* f - 2R \tag{3.11}$$

and therefore that $\xi_{\mathcal{F}}^2 p^* C_0 = \xi_{\mathcal{F}}^2 p^* f = 2$. Also (3.11) gives that $N^2(\mathbb{P}(\mathcal{F}))$ is generated by $\xi_{\mathcal{F}} p^* C_0, \xi_{\mathcal{F}} p^* f$ and R. Let $C \subset \mathbb{P}(\mathcal{F})$ be an irreducible curve contracted by $\varphi_{\xi_{\mathcal{F}}}$. Then $C \equiv a\xi_{\mathcal{F}} p^* C_0 + b\xi_{\mathcal{F}} p^* f + cR$ for some integers a, b, c. Since $p^* C_0$ and $p^* f$ are nef, we get that $0 \le C \cdot p^* C_0 = b$ and $0 \le C \cdot p^* f = a$. Now

$$0 = \xi_{\mathcal{F}} \cdot C = 2a + 2b + c$$

gives that $c = -2a - 2b$, hence either $a > 0$ or $b > 0$. Also $\mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f)) \sim \xi_{\mathcal{F}} - 2p^* C_0 - p^* f$ and therefore

$$C \cdot \mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f)) = [a\xi_{\mathcal{F}} p^* C_0 + b\xi_{\mathcal{F}} p^* f - (2a + 2b)R] \cdot (\xi_{\mathcal{F}} - 2p^* C_0 - p^* f) = -a - 2b < 0$$

so that $C \subset \mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f))$. On the other hand $H^0(\mathcal{F}) \cong H^0(\xi_{\mathcal{F}}) \to H^0(\xi_{\mathcal{O}_{\mathbb{F}_0}(f)}) \cong H^0(\mathcal{O}_{\mathbb{F}_0}(f))$ is surjective, hence $\varphi_{\xi_{\mathcal{F}}}$ restricts to $\varphi_{\mathcal{O}_{\mathbb{F}_0}(f)}$ on $\mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f)) \cong \mathbb{F}_0$, a morphism that contracts all curves $C \sim f$. Since V is normal by Subclaim 3.17, this proves (3.10) and moreover the proof of Subclaim 3.21 shows that $\mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f))$ is contracted to the line S_0. Now for every $t \in \mathbb{P}^1$ let f_t be the corresponding line on $\mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f)) \cong \mathbb{F}_0$. We have an exact sequence

$$0 \to \mathcal{F}(-f_t) \to \mathcal{F} \to \mathcal{F}_{|f_t} \to 0$$

with $H^1(\mathcal{F}(-f_t)) = H^1(\mathcal{O}_{\mathbb{F}_0} \oplus \mathcal{O}_{\mathbb{F}_0}(2C_0)) = 0$, showing that $H^0(\mathcal{F}) \cong H^0(\xi_{\mathcal{F}}) \to H^0(\xi_{\mathcal{F}_{|f_t}}) \cong H^0(\mathcal{F}_{|f_t})$ is surjective. Since $\mathcal{F}_{|f_t} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ we deduce that $\varphi_{\xi_{\mathcal{F}}}$ maps $\mathbb{P}(\mathcal{F}_{|f_t})$ onto a quadric cone $\mathcal{Q}_t \subset \mathbb{P}^3$. On the other hand $\mathbb{P}(\mathcal{F}_{|f_t}) \cap \mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f))$ is a curve on $\mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f))$ isomorphic to $f_t \subset \mathbb{F}_0$ and this curve is contracted to the point in S_0 corresponding to t. Now any point $v \in V \setminus S_0$ belongs to a unique cone $\mathcal{Q}_t$. On the other hand, if $v \in S_0$ then v is the vertex of the cone $\mathcal{Q}_t$ where $t \in \mathbb{P}^1$ is the image of f_t on $\mathbb{P}(\mathcal{O}_{\mathbb{F}_0}(f))$. This clearly gives an isomorphism $V \cong \mathbb{P}^1 \times \mathcal{Q}$. □

Subclaim 3.23. *If $S_V = S(0,1,1,2)$, then $\rho(V) = 2$ and in fact V is a hyperplane section of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$.*

Proof. By Subclaim 3.21 we have that $V = \varphi_{\xi_{\mathcal{F}}}(\mathbb{P}(\mathcal{F}))$, where $\mathcal{F} = \mathcal{O}_{\mathbb{F}_1}(C_0 + f) \oplus \mathcal{O}_{\mathbb{F}_1}(C_0 + 2f)$. Consider the isomorphism $\mathbb{P}(\mathcal{O}_{\mathbb{F}_1}(C_0 + f)) \cong \mathbb{F}_1$ and let $\widetilde{C}_0$ be the curve on $\mathbb{P}(\mathcal{O}_{\mathbb{F}_1}(C_0 + f))$ isomorphic to $C_0 \subset \mathbb{F}_1$. We first claim that $\widetilde{C}_0$ is the unique curve contracted by $\varphi_{\xi_{\mathcal{F}}}$. To see the latter, let $p : \mathbb{P}(\mathcal{F}) \to \mathbb{F}_1$ be the projection map. From Grothendieck's relation

$$\xi_{\mathcal{F}}^2 = \xi_{\mathcal{F}} p^* c_1(\mathcal{F}) - p^* c_2(\mathcal{F})$$

we deduce, setting $R = p^*(C_0) p^*(f)$, that

$$\xi_{\mathcal{F}}^2 = 2\xi_{\mathcal{F}} p^* C_0 + 3\xi_{\mathcal{F}} p^* f - 2R \tag{3.12}$$

and therefore that $\xi_{\mathcal{F}}^2 p^* C_0 = 1, \xi_{\mathcal{F}}^2 p^* f = 2$. Also (3.12) gives that $N^2(\mathbb{P}(\mathcal{F}))$ is generated by $\xi_{\mathcal{F}} p^* C_0, \xi_{\mathcal{F}} p^* f$ and R. Let $C \subset \mathbb{P}(\mathcal{F})$ be an irreducible curve contracted by $\varphi_{\xi_{\mathcal{F}}}$. Then $C \equiv a\xi_{\mathcal{F}} p^* C_0 + b\xi_{\mathcal{F}} p^* f + cR$ for some integers a, b, c. Since $p^*(C_0 + f)$ and $p^* f$ are nef, we get that $0 \le C \cdot p^*(C_0 + f) = b$ and $0 \le C \cdot p^* f = a$. Now

$$0 = \xi_{\mathcal{F}} \cdot C = a + 2b + c$$

gives that $c = -a - 2b$, hence either $a > 0$ or $b > 0$. Also $\mathbb{P}(\mathcal{O}_{\mathbb{F}_1}(C_0 + f)) \sim \xi_{\mathcal{F}} - p^* C_0 - 2p^* f$ and therefore

$$C \cdot \mathbb{P}(\mathcal{O}_{\mathbb{F}_1}(C_0 + f)) = [a\xi_{\mathcal{F}} p^* C_0 + b\xi_{\mathcal{F}} p^* f - (a + 2b)R] \cdot (\xi_{\mathcal{F}} - p^* C_0 - 2p^* f) = -a - b < 0$$

so that $C \subset \mathbb{P}(\mathcal{O}_{\mathbb{F}_1}(C_0+f))$. On the other hand, via the isomorphism $\mathbb{P}(\mathcal{O}_{\mathbb{F}_1}(C_0+f)) \cong \mathbb{F}_1$ we have that

$$(\xi_{\mathcal{F}})_{|\mathbb{P}(\mathcal{O}_{\mathbb{F}_1}(C_0+f))} = \xi_{\mathcal{O}_{\mathbb{F}_1}(C_0+f)} \cong C_0+f$$

so that

$$0 = \xi_{\mathcal{F}} \cdot C = (C_0+f) \cdot p(C)$$

and therefore $p(C) = C_0$, hence $C = \widetilde{C}_0$. This proves the above claim.

Since V is normal by Subclaim 3.17, we have that $\operatorname{Exc}(\varphi_{\xi_{\mathcal{F}}}) = \widetilde{C}_0$ and therefore $\varphi_{\xi_{\mathcal{F}}}$ is an isomorphism outside $\widetilde{C}_0$ and contracts the latter to a singular point of V, namely to S_0. It follows that $(\varphi_{\xi_{\mathcal{F}}})_*\mathcal{O}_{\mathbb{P}(\mathcal{F})} \cong \mathcal{O}_V$ and therefore $(\varphi_{\xi_{\mathcal{F}}})_*\mathcal{O}^*_{\mathbb{P}(\mathcal{F})} \cong \mathcal{O}^*_V$. Moreover $R^1(\varphi_{\xi_{\mathcal{F}}})_*\mathcal{O}^*_{\mathbb{P}(\mathcal{F})}$ is a skyscraper sheaf supported on the point S_0 and on S_0 it is $H^1(\widetilde{C}_0, \mathcal{O}^*_{\widetilde{C}_0})$, so that

$$H^0(V, R^1(\varphi_{\xi_{\mathcal{F}}})_*\mathcal{O}^*_{\mathbb{P}(\mathcal{F})}) \cong H^1(\widetilde{C}_0, \mathcal{O}^*_{\widetilde{C}_0}) \cong H^1(\mathbb{P}^1, \mathcal{O}^*_{\mathbb{P}^1}) \cong \mathbb{Z}.$$

Now the Leray spectral sequence gives rise to the exact sequence

$$0 \to H^1(V, \mathcal{O}^*_V) \to H^1(\mathbb{P}(\mathcal{F}), \mathcal{O}^*_{\mathbb{P}(\mathcal{F})}) \to H^0(V, R^1(\varphi_{\xi_{\mathcal{F}}})_*\mathcal{O}^*_{\mathbb{P}(\mathcal{F})}) \to 0.$$

Since $H^1(\mathbb{P}(\mathcal{F}), \mathcal{O}^*_{\mathbb{P}(\mathcal{F})}) \cong \mathbb{Z}^3$, we deduce that $\operatorname{Pic}(V)$ has rank 2.

Now let us see that V is a hyperplane section of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. Note that this also proves, by Lefschetz's theorem, that $\rho(V) = 2$.

Under the morphism $\varphi_{\xi_{\mathcal{F}}}$ we see that $\mathbb{P}(\mathcal{O}_{\mathbb{F}_1}(C_0+f))$ gets mapped onto $\mathbb{P}^2$ with $\widetilde{C}_0$ contracted to a point $P \in \mathbb{P}^2$, while $\mathbb{P}(\mathcal{O}_{\mathbb{F}_1}(C_0+2f))$ gets mapped isomorphically onto the rational normal surface scroll $S(1,2) \subset \mathbb{P}^4$. Moreover the fibers of p not meeting $\widetilde{C}_0$, give rise to a family of disjoint lines meeting $S(1,2)$ and $\mathbb{P}^2$ in a point and giving an isomorphism $S(1,2) \setminus S(1) \cong \mathbb{P}^2 - \{P\}$, while the fibers meeting $\widetilde{C}_0$, give rise to lines meeting $S(1)$ and passing through P. One can put coordinates so that V, that is the union of these lines, is a hyperplane section of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. To see this let $P = (1:0:0) \in \mathbb{P}^2$, consider the line $(0:s:t)$ and parametrize the lines through P with coordinates $(a:b)$, so that a point in $\mathbb{P}^2$ has coordinates $(a:bs:bt)$. Parametrize the points of $S(1,2)$, join of the line $(s:0:0:t:0:0)$ and the conic $(0:s^2:st:0:st:t^2)$, by $(as:bs^2:bst:at:bst:bt^2)$, inside the hyperplane $Z_2 - Z_4 = 0$ in $\mathbb{P}^5$. Now a point in a line joining a point of $S(1,2)$ and of $\mathbb{P}^2$ has coordinates $(ma:mbs:mbt:nas:nbs^2:nbst:nat:nbst:nbt^2)$. It follows that V, that is the locus of these points, is a hyperplane section of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$: if we have coordinates $(X_0:X_1:X_2)$ and $(Y_0:Y_1:Y_2)$ this can be seen by setting $m = X_0, ns = X_1, nt = X_2, a = Y_0, bs = Y_1$ and $bt = Y_2$. □

Subclaim 3.24. *If* $S_V = S(0,1,1,2)$ *then* $f_x \cong \mathbb{P}(T_{\mathbb{P}^2})$, *while if* $S_V = S(0,0,2,2)$ *then* $f_x \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Let $V = f_u$ be such that $S_V = S(0,1,1,2)$. Then $V = \varphi_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}(\mathbb{P}(\mathcal{F}))$ by Subclaim 3.21 and therefore $\rho(V) = 2$ by Subclaim 3.23. Also the only singular point of V is an ordinary double point by [F4, Thm. 2.9], hence in particular is terminal. It follows by [JR, Thm. 1.4] that $\rho(f_x) = 2$ and therefore that $f_x \cong \mathbb{P}(T_{\mathbb{P}^2})$.

Assume now that $S_V = S(0,0,2,2)$ so that we know by Subclaim 3.22 that $V = \mathbb{P}^1 \times \mathcal{Q} \subset \mathbb{P}^7$. Moreover $T := \widetilde{\Phi}(V)$ is a surface by Subclaim 3.20. We now show that $T \cong \mathcal{Q}$. In fact, let U be the open subset of $\widetilde{\Phi(X)}$ such that the fibers of $\widetilde{\Phi}$ have dimension 1. For every $v \in V$ we know that $F_v \subset f_v = f_u = V$, hence F_v is a line, since $\mathbb{P}^1 \times \mathcal{Q}$ does not contain planes. Then $V \subset \widetilde{\Phi}^{-1}(U)$ and, similarly, $f_x \subset \widetilde{\Phi}^{-1}(U)$. Now there is a rank 2 vector bundle $\mathcal{G}$ on U such that $\widetilde{\Phi}^{-1}(U) \cong \mathbb{P}(\mathcal{G})$ and $\mathcal{O}_X(1)_{|\mathbb{P}(\mathcal{G})} \cong \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$. Therefore $V \cong \mathbb{P}(\mathcal{G}_{|T})$ and, since V is normal by Subclaim 3.17, so must be T. But it is easily seen that $\mathbb{P}^1 \times \mathcal{Q}$ has only one $\mathbb{P}^1$-bundle structure over a normal surface, namely the second projection $\mathbb{P}^1 \times \mathcal{Q} \to \mathcal{Q}$ and therefore $T \cong \mathcal{Q}$. Now (2.2) gives rise to the commutative diagram

$$\begin{array}{ccc} \widetilde{\Phi}^{-1}(U) \cong \mathbb{P}(\mathcal{G}) & \xrightarrow{\widetilde{\Phi}_{|\mathbb{P}(\mathcal{G})}} & U \\ & \underset{\phi_{\tau|\mathbb{P}(\mathcal{G})}}{\searrow} & \downarrow{\scriptstyle g_{|U}} \\ & & X' \end{array} .$$

In particular $g_{|U}$ gives a deformation of $\mathcal{Q}$ to $T_x := \widetilde{\Phi}(f_x)$. Also we know that there is an ample line bundle $\mathcal{L}$ on X' such that $K_X + 2H = \phi_\tau^* \mathcal{L}$ and therefore, restricting to $\widetilde{\Phi}^{-1}(U) \cong \mathbb{P}(\mathcal{G})$ we get, setting $p = \widetilde{\Phi}_{|\mathbb{P}(\mathcal{G})}$, that $p^*((g_{|U})^*\mathcal{L}) = p^*(K_U + \det \mathcal{G})$, so that $K_U + \det \mathcal{G} = (g_{|U})^*\mathcal{L}$. In particular this shows that both $K_{\mathcal{Q}}$ and K_{T_x} are restrictions of a line bundle on U. But then $K_{T_x}^2 = K_{\mathcal{Q}}^2 = 8$. Since $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}(T_{\mathbb{P}^2})$ have as only $\mathbb{P}^1$-bundle structures over a smooth surface the projections to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{P}^2$, we have that T_x is either $\mathbb{P}^1 \times \mathbb{P}^1$ when $f_x \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ when $f_x \cong \mathbb{P}(T_{\mathbb{P}^2})$. Thus $f_x \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. □

To finish the proof of Claim 3.16 observe that we already know the smooth fibers of ϕ_τ by Claim 3.14. On the other hand, if $V = f_u$ is a singular fiber, Subclaims 3.18, 3.21 and Subclaim 3.24 imply that all singular fibers are either all of type $S(0,0,2,2)$ or all of type $S(0,1,1,2)$. Hence, when $f_x \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we get that all singular fibers are $\mathbb{P}^1 \times \mathcal{Q}$ by Subclaims 3.24 and 3.22. In particular F_u is a line for every $u \in X$ and it follows by Lemma 2.12 that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple with $p = \widetilde{\Phi}$ and $b = 3$. This gives case (vi2). Finally if $f_x \cong \mathbb{P}(T_{\mathbb{P}^2})$ then we are in case (vi3) by Subclaim 3.21. This completes the proof of Claim 3.16. □

Claim 3.25. *In case (d.3), $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (vii).*

Proof. By Claim 3.13 we are in case (fact) for ϕ_τ, hence ϕ_τ factorizes through $\widetilde{\Phi}$. Suppose first that there is an $x_0 \in X$ such that F_{x_0} is a linear $\mathbb{P}^k$ with $2 \le k \le 3$. Since $F_{x_0} \subset f_{x_0}$ and $\dim f_{x_0} = 2$, it follows that $\mathbb{P}^2 = F_{x_0} \subset f_{x_0}$, so that f_{x_0} is a reducible quadric. If $f_{x_0} = F_{x_0} \cup M$ with M a plane distinct from F_{x_0}, note that f_{x_0} spans a $\mathbb{P}^3$: If not, then $F_{x_0} \cap M$ is a point but then a general hyperplane section of X gives a conic fibration over a curve with a fiber union of two disjoint lines, contradicting the fact that all fibers must have arithmetic genus 0. Now $\Phi_{|M} : M \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ contracts the line $F_{x_0} \cap M$ to a point, hence it is constant, so that $\Phi(F_{x_0}) = \Phi(M)$ and therefore $M \subset F_{x_0}$, a contradiction. Hence f_{x_0} is a double plane with $(f_{x_0})_{red} = F_{x_0}$. Again f_{x_0} spans a $\mathbb{P}^3$: In fact, a general hyperplane section of X gives a conic fibration over a curve with a fiber which is a double line with arithmetic genus 0. But such a double line spans a plane, hence f_{x_0} spans a $\mathbb{P}^3$. Since F_{x_0} is a fiber of Φ we have that $(\det \mathcal{E})_{|F_{x_0}} \cong \mathcal{O}_{\mathbb{P}^2}$. Now consider the exact sequence (see for example [BE, Proof of Prop. 4.1])

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{f_{x_0}}^* \to \mathcal{O}_{F_{x_0}}^* \to 1.$$

Since $H^1(\mathcal{O}_{\mathbb{P}^2}(-1)) = 0$ we get that the restriction map

$$\operatorname{Pic}(f_{x_0}) = H^1(\mathcal{O}_{f_{x_0}}^*) \to H^1(\mathcal{O}_{F_{x_0}}^*) = \operatorname{Pic}(F_{x_0})$$

is injective, hence $(\det \mathcal{E})_{|f_{x_0}} \cong \mathcal{O}_{f_{x_0}}$. Now f_{x_0} is a quadric in $\mathbb{P}^3$, hence $h^0(\mathcal{O}_{f_{x_0}}) = 1$. Therefore $\Phi(f_{x_0})$ is a point and this gives that, scheme-theoretically, $f_{x_0} = F_{x_0}$. But this contradicts [LS, Thm. 2]. It follows that $\dim F_u = 1$ for every $u \in X$ and then we can apply Lemma 2.12. □

Claim 3.26. *In case (d.4), $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (viii).*

Proof. By Claim 3.13 we are in case (fact) for ϕ_τ, hence $F_u = f_u$ for every $u \in X$. Thus $\phi_\tau = \widetilde{\Phi}$ and we just apply Lemma 2.12. □

Claim 3.27. *In case (d.5), $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (ix).*

Proof. By Claim 3.13 we are in case (fact) for ϕ_τ, hence there is a morphism $\psi : \widetilde{\Phi(X)} \to X'$ such that $\phi_\tau = \psi \circ \widetilde{\Phi}$. Now F_x is a line, hence $f_x = F_x = \widetilde{F}_x$. Therefore the general fiber of ψ must be a point, that is ψ is generically finite of degree 1. □

Claim 3.28. *Case (e) does not occur.*

Proof. This follows by Proposition 2.18. □

We are therefore left with case (d.1) in which $(X, \mathcal{O}_X(1))$ is a Mukai fourfold, that is $K_X = -2H$. We will use Mukai's classification [M, Thm. 7], [W1, Table 0.3].

We remark that we can assume that $c_1(\mathcal{E})^3 \ne 0$, for otherwise [LS, Cor. 4] implies that $(X, \mathcal{O}_X(1))$ is a linear $\mathbb{P}^{4-b}$-bundle over a smooth variety of dimension $b \le 2$, contradicting $K_X = -2H$.

We will divide the proof in two subcases of case (d.1):

(B.1) there exists $x_0 \in X$ such that $\dim F_{x_0} \ge 2$.
(B.2) $\dim F_u = 1$ for every $u \in X$.

In the second case we have the following general fact.

Claim 3.29. *In case (B.2) we have that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple with $p = \widetilde{\Phi}, B = \widetilde{\Phi(X)}$ and $b = 3$. Moreover the $\mathbb{P}^1$-bundle structure $p : X \cong \mathbb{P}(\mathcal{F}) \to B$ occurs in the following cases:*

(i) *$X = \mathbb{P}^1 \times M$ and either $\mathcal{F} \cong L^{\oplus 2}$ where $K_M = -2L$, or $B = \mathbb{P}^1 \times \mathbb{P}^2$ and either $M = V_7$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes (\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ or $M = \mathbb{P}(T_{\mathbb{P}^2})$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes T_{\mathbb{P}^2}$.*
(ii) *$(X, \mathcal{O}_X(1))$ is as in Examples 6, 8 or 9 in Mukai's classification and the $\mathbb{P}^1$-bundle structure is the one given in* [M, Thm. 7 and Ex. 1-9] *(see also* [W1, Table 0.3]*).*

Proof. The first fact follows by Lemma 2.12. As for the $\mathbb{P}^1$-bundle structure, this is a well-known fact that can be easily proved either by following the proof of [W2, Thm. 0.1] or simply using the fact that the morphism p is given by a globally generated line bundle $\mathcal{L}$ on X with $\mathcal{L}^4 = 0$ (most calculations of this type are done in the course of the proofs). □

Claim 3.30. *If X is a Mukai fourfold of product type, then $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (x1) or (x2).*

Proof. We have that $X \cong \mathbb{P}^1 \times M$ with M a Fano threefold of even index, hence $\mathcal{O}_X(1) \cong \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes L$ where $K_M = -2L$. Let $p_i, i = 1, 2$ be the two projections. Then $\det \mathcal{E} = p_1^*(\mathcal{O}_{\mathbb{P}^1}(a)) + p_2^* N$ for some $a \in \mathbb{Z}$ and some line bundle N on M. Now

$$0 = c_1(\mathcal{E})^4 = 4aN^3.$$

If $a = 0$, then N is globally generated and [Lo, Lemma 5.1] gives that there is a vector bundle $\mathcal{H}$ on M such that $\mathcal{E} \cong p_2^*\mathcal{H}$. Hence we are in case (x1) by [Lo, Lemma 4.1]. Suppose now that $N^3 = 0$. Then $\rho(M) \ge 2$, for otherwise we have that then $N \cong \mathcal{O}_M$ and therefore $c_1(\mathcal{E})^2 = 0$, a contradiction. Hence the only possibility is that either $M = V_7, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or a hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$.

First, assume that we are in case in case (B.1).

We have $\mathbb{P}^k = F_{x_0} \subset X$ for $k = 2$ or 3. Since $p_{1|F_{x_0}} : \mathbb{P}^k \to \mathbb{P}^1$ must be constant, it follows that $F_{x_0} = \{y\} \times Z$, where $y \in \mathbb{P}^1$ and Z is a linear $\mathbb{P}^k$ contained in M. In most cases we will write this as $F_{x_0} \subset M$. Therefore $k = 2$ and $L_{|F_{x_0}} = \mathcal{O}_{\mathbb{P}^2}(1)$.

If $M = V_7$, let $\varepsilon : V_7 \to \mathbb{P}^3$ be the blow-up map with exceptional divisor E. Let $\widetilde{H}$ be the pull back of a plane, so that $L = 2\widetilde{H} - E$. Note that it cannot be that $E \cap Z = \emptyset$, for otherwise $\mathcal{O}_{\mathbb{P}^2}(1) = L_{|Z} = 2\widetilde{H}_{|Z}$. Therefore $\dim E \cap Z \ge 1$ and the morphism $\varepsilon_{|Z} : \mathbb{P}^2 = Z \to \mathbb{P}^3$ contracts $E \cap Z$ to a point, hence it is constant, so that $Z = E$. Now $h = \mathrm{id}_{\mathbb{P}^1} \times \varepsilon : X \cong \mathbb{P}^1 \times V_7 \to \mathbb{P}^1 \times \mathbb{P}^3$ contracts F_{x_0} to a point. But then the Dichotomy Lemma implies that $F_x \subset h^{-1}(h(x)) = \{x\}$, a contradiction.

If $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ observe that if $p_i, 1 \le i \le 3$ is a projection, then $p_{i|F_{x_0}} : \mathbb{P}^k = F_{x_0} \to \mathbb{P}^1$ must be constant, giving a contradiction.

If M a hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$, then, as is well known, $F_{x_0} = \mathbb{P}^2 \times \{y\}$ or $\{z\} \times \mathbb{P}^2$. In the first case, if $p_2 : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ is the second projection, then $p_{2|M}(F_{x_0}) = \{y\}$, hence $F_{x_0} \subset (p_{2|M})^{-1}(y)$. But this is a contradiction since $\dim(p_{2|M})^{-1}(y) = 1$, as one can see using the isomorphism $M \cong \mathbb{P}(T_{\mathbb{P}^2})$ and that $p_{2|M} : \mathbb{P}(T_{\mathbb{P}^2}) \to \mathbb{P}^2$ is the projection map. In the second case a similar contradiction can be obtained.

Next assume that we are in case in case (B.2), so that we can apply Claim 3.29.

If $\mathcal{F} \cong L^{\oplus 2}$, setting $\mathcal{G}' = \mathcal{G}(L)$ we get that $\mathcal{E} \cong p^*(\mathcal{G}'(L))$ and this gives again case (x1).

Now when $M = \mathbb{P}(T_{\mathbb{P}^2})$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes T_{\mathbb{P}^2}$ we are in case (x2) by (2.4).

Finally consider the case $M = V_7$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes (\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ on $B = \mathbb{P}^1 \times \mathbb{P}^2$. For ease of notation we will set, for any $c, d \in \mathbb{Z}$, $\mathcal{O}_B(c, d) = \mathcal{O}_{\mathbb{P}^1}(c) \boxtimes \mathcal{O}_{\mathbb{P}^2}(d), \mathcal{O}_B(c) = \mathcal{O}_B(c, c)$ and $\mathcal{H}(c, d) = \mathcal{H} \otimes \mathcal{O}_B(c, d)$ for any sheaf $\mathcal{H}$ on B. Now (2.4) gives in particular the vanishings

$$H^j(\mathcal{G}(-s)) = H^j(\mathcal{G}(-1, -2)) = H^j(\mathcal{G}(-2, -3)) = 0 \text{ for } j \ge 0, 0 \le s \le 2.$$

Since the same vanishings hold for any direct summand of $\mathcal{G}$, we can assume that $\mathcal{G}$ is indecomposable. Note that the vanishings give that $\mathcal{G}(1)$ is an Ulrich bundle for $(B, \mathcal{O}_B(1))$, hence it is also ACM by

Remark 2.8(iii). Then [FMS, Thm. B] gives that either $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^1} \boxtimes \Omega_{\mathbb{P}^2}(1)$ or $\mathcal{G}$ fits into an exact sequence
$$0 \to \mathcal{O}_B(-1,0)^{\oplus a} \to \mathcal{G} \to \mathcal{O}_B(1,-1)^{\oplus b} \to 0.$$
In the first case we find the contradiction
$$0 = H^3(\mathcal{G}(-2,-3)) = H^3(\mathcal{O}_{\mathbb{P}^1}(-2) \boxtimes \Omega_{\mathbb{P}^2}(-2)) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \otimes H^2(\Omega_{\mathbb{P}^2}(-2)) \neq 0.$$
In the second case we have an exact sequence
$$0 \to \mathcal{O}_B(-2)^{\oplus a} \to \mathcal{G}(-1,-2) \to \mathcal{O}_B(0,-3)^{\oplus b} \to 0.$$
But $H^2(\mathcal{G}(-1,-2)) = H^3(\mathcal{O}_B(-2)) = 0$ giving, for $b \neq 0$, the contradiction $H^2(\mathcal{O}_B(0,-3)) = 0$. Therefore $b = 0$ and $\mathcal{G} = \mathcal{O}_B(-1,0)^{\oplus a}$, giving the contradiction $0 = H^3(\mathcal{G}(-2,-3)) = H^3(\mathcal{O}_B(-3))^{\oplus a} \neq 0$. Thus this case does not occur. This proves Claim 3.30. □

To finish the proof of Theorem 1, it remains to consider the Mukai fourfolds not of product type, which, by [M, Thm. 7], are linear sections of the varieties listed in [M, Ex. 1-9].

Claim 3.31. *If X is a Mukai fourfold not of product type, then $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (x3)-(x5).*

Proof. We refer to [M, Thm. 7], [W1, Table 0.3].

In example 1 we have that X is a double cover $f : X \to \mathbb{P}^2 \times \mathbb{P}^2$ ramified along a divisor of type $(2,2)$ and $\mathcal{O}_X(1) = f^*L, L = \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$. For $i = 1,2$ let $p_i : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the projections and set $q_i = f \circ p_i : X \to \mathbb{P}^2$. Let Y be a smooth hyperplane section of X and set $h_i = q_{i|Y} : Y \to \mathbb{P}^2$. By Lefschetz's theorem we know that the restriction map $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ is an isomorphism. On the other hand, Y is just the Fano threefold listed as No. 6b in Mori-Mukai's list [MM1, Table 2] and it follows from [MM2, Thm. 5.1 and proof of Thm. 1.7] that $\operatorname{Pic}(Y)$ is generated by $h_i^*(\mathcal{O}_{\mathbb{P}^2}(1)), i = 1,2$. Therefore $\operatorname{Pic}(X)$ is generated by $A = q_1^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $B = q_2^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Now $\det \mathcal{E} = aA + bB$, for some $a, b \in \mathbb{Z}$, hence
$$0 = c_1(\mathcal{E})^4 = 12a^2b^2.$$
Therefore either $a = 0$ or $b = 0$, giving the contradiction $c_1(\mathcal{E})^3 = 0$. Thus example 1 is excluded.

In examples 2 and 4, setting $Q = Q_3$, we have that X is the hyperplane section of $\mathbb{P}^2 \times Z$, where $Z = \mathbb{P}^3$ (respectively Q) under the Segre embedding given by $L = \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes M$, with $M = \mathcal{O}_{\mathbb{P}^3}(2)$ (resp. $M = \mathcal{O}_Q(1)$) and $\mathcal{O}_X(1) = (\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1))_{|X}$ (resp. $\mathcal{O}_X(1) = L_{|X}$). Let $p_i, i = 1,2$ be the two projections on $\mathbb{P}^2 \times Z$ and let $q = p_{2|X}$. By Lefschetz's theorem we know that $\operatorname{Pic}(X)$ is generated by $A_{|X}$ and $B_{|X}$ where $A = p_1^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $B = p_2^*(\mathcal{O}_{\mathbb{P}^3}(1))$ (resp. $B = p_2^*(\mathcal{O}_Q(1))$). Now $\det \mathcal{E} = aA_{|X} + bB_{|X}$, for some $a, b \in \mathbb{Z}$. Hence, in both cases,
$$0 = c_1(\mathcal{E})^4 = \sum_{j=0}^{4} \binom{4}{j} a^j b^{4-j} A^j \cdot B^{4-j} X = 4ab^2(b + 3a).$$
If $a \neq 0$, then either $b = 0$ and $\det \mathcal{E} = aA$, hence $c_1(\mathcal{E})^3 = 0$, a contradiction, or $b = -3a$ and $\det \mathcal{E} = a(A - 3B)$. But this is not nef: if $a > 0$, let $y \in \mathbb{P}^2$ and choose a curve C in the surface $(\{y\} \times Z) \cap X$. Then $A \cdot C = 0, B \cdot C = \frac{1}{2} L \cdot C > 0$ (resp. $B \cdot C = L \cdot C > 0$), hence $\det \mathcal{E} \cdot C = -3aB \cdot C < 0$; if $a < 0$, let $z \in Z$ and let $C = (\mathbb{P}^2 \times \{z\}) \cap X$. Then $B \cdot C = 0, A \cdot C = L \cdot C > 0$, hence $\det \mathcal{E} \cdot C = aL \cdot C < 0$. Since $\det \mathcal{E}$ is globally generated, this is a contradiction. Therefore $a = 0, \det \mathcal{E} = q^*(\mathcal{O}_Z(b))$ and it follows by [Lo, Lemma 5.1] that there is a rank r vector bundle $\mathcal{G}$ on Z such that $\mathcal{E} \cong q^*\mathcal{G}$.

When $Z = Q$, exactly as in the case of the hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^3$ (proof of Claim 3.12), we see that $H^i(\mathcal{E}(-pH)) = 0$ for all $i \geq 0$ and $1 \leq p \leq 4$, together with the Künneth formula implies that $H^j(\mathcal{G}(-2)(-s)) = 0$ for all $j \geq 0$ and $1 \leq s \leq 3$. But then $\mathcal{G}(-2)$ is an Ulrich vector bundle for $(Q, \mathcal{O}_Q(1))$, hence r is even and $\mathcal{G} \cong \mathcal{S}(2)^{\oplus(\frac{r}{2})}$ by [LMS, Lemma 3.2(iv)], so that we are in case (x3). Hence we are done with example 4.

When $Z = \mathbb{P}^3$ we will reach a contradiction. To this end set $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)$ and consider the exact sequence
$$0 \to (p_2^*\mathcal{G})(-p\mathcal{L} - X) \to (p_2^*\mathcal{G})(-p\mathcal{L}) \to \mathcal{E}(-pH) \to 0.$$
Since $H^i(\mathcal{E}(-pH)) = 0$ for all $i \geq 0$ and $1 \leq p \leq 4$ we deduce that
$$H^i((p_2^*\mathcal{G})(-p\mathcal{L} - X)) \cong H^i((p_2^*\mathcal{G})(-p\mathcal{L})) \text{ for all } i \geq 0 \text{ and } 1 \leq p \leq 4.$$

Now the Künneth formula gives that

$$h^0(\mathcal{O}_{\mathbb{P}^2}(p-2))h^{i-2}(\mathcal{G}(-p-2)) = h^0(\mathcal{O}_{\mathbb{P}^2}(p-3))h^{i-2}(\mathcal{G}(-p)) \text{ for } i \ge 2 \text{ and } 1 \le p \le 4$$

and one easily sees that this implies that $H^j(\mathcal{G}(-4)) = H^j(\mathcal{G}(-6)) = 0$ and $h^j(\mathcal{G}(-3)) = 3h^j(\mathcal{G}(-5))$ for all $j \ge 0$. Setting $\mathcal{H} = \mathcal{G}(-3)$ we find

$$H^j(\mathcal{H}(-1)) = H^j(\mathcal{H}(-3)) = 0 \text{ and } h^j(\mathcal{H}) = 3h^j(\mathcal{H}(-2)) \text{ for all } j \ge 0. \tag{3.13}$$

Now let M be a plane in $\mathbb{P}^3$ and consider, for $l \in \mathbb{Z}$, the exact sequences

$$0 \to \mathcal{H}(-l-1) \to \mathcal{H}(-l) \to \mathcal{H}(-l)_{|M} \to 0. \tag{3.14}$$

We have $H^0(\mathcal{H}(-1)) = 0$ by (3.13), hence also $H^0(\mathcal{H}(-2)) = 0$ and then (3.13) gives that

$$h^0(\mathcal{H}) = 3h^0(\mathcal{H}(-2)) = 0. \tag{3.15}$$

Now $H^1(\mathcal{H}(-1)) = 0$ by (3.13), hence (3.14) with $l = 0$ implies that $H^0(\mathcal{H}_{|M}) = 0$, hence also $H^0(\mathcal{H}(-1)_{|M}) = 0$. Setting $l = 1$ in (3.14) and using again $H^1(\mathcal{H}(-1)) = 0$, we deduce that $H^1(\mathcal{H}(-2)) = 0$. We have $H^3(\mathcal{H}(-3)) = 0$ by (3.13), hence also $H^3(\mathcal{H}(-2)) = 0$. Since $H^2(\mathcal{H}(-1)) = 0$ by (3.13), then (3.14) with $l = 1$ implies that $H^2(\mathcal{H}(-1)_{|M}) = 0$, hence also $H^2(\mathcal{H}_{|M}) = 0$. Therefore (3.14) with $l = 0$ and (3.13) imply that $H^2(\mathcal{H}) = 0$, whence also $H^2(\mathcal{H}(-2)) = 0$ by (3.13). Thus we have proved that $H^j(\mathcal{H}(-2)) = 0$ for $j \ge 0$ and together with (3.14) this implies that $\mathcal{H}$ is an Ulrich vector bundle on $\mathbb{P}^3$. But this contradicts (3.15).

Thus we are done with example 2.

In example 3 we have that X is twice a hyperplane section of $\mathbb{P}^3 \times \mathbb{P}^3$ under the Segre embedding given by $L = \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)$ and $\mathcal{O}_X(1) = L_{|X}$. Let $p_i, i = 1, 2$ be the two projections on $\mathbb{P}^3 \times \mathbb{P}^3$ and let $p = p_{1|X}$ and $q = p_{2|X}$. By Lefschetz's theorem we know that $\operatorname{Pic}(X)$ is generated by $A = p^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $B = q^*(\mathcal{O}_{\mathbb{P}^3}(1))$ with $A^4 = B^4 = 0, A \cdot B^3 = A^3 \cdot B = 1$ and $A^2 \cdot B^2 = 2$. Now $\det \mathcal{E} = aA + bB$, for some $a, b \in \mathbb{Z}$, hence

$$0 = c_1(\mathcal{E})^4 = 4ab(b^2 + 3ab + a^2).$$

and, the case $b = 0$ being completely similar, we can assume that $a = 0$ and $\det \mathcal{E} = q^*(\mathcal{O}_{\mathbb{P}^3}(b))$. It follows by [Lo, Lemma 5.1] that there is a rank r vector bundle $\mathcal{G}$ on $\mathbb{P}^3$ such that $\mathcal{E} \cong q^*\mathcal{G}$. We now claim that $\mathcal{G}(-3)$ is an Ulrich vector bundle for $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. To this end observe that, as is well known, we can see X as the hyperplane section of $\mathbb{P}(T_{\mathbb{P}^3})$ embedded by the tautological line bundle ξ and $q := (p_2)_{|\mathbb{P}(T_{\mathbb{P}^3})} : \mathbb{P}(T_{\mathbb{P}^3}) \to \mathbb{P}^3$ is just its projection map. Thus $\mathcal{E} \cong (q^*\mathcal{G})_{|X}$ and we have an exact sequence

$$0 \to (q^*\mathcal{G})(-(s+1)\xi) \to (q^*\mathcal{G})(-s\xi) \to \mathcal{E}(-sH) \to 0.$$

Since $H^i(\mathcal{E}(-sH)) = 0$ for all $i \ge 0$ and $1 \le s \le 4$ we deduce that

$$H^i((q^*\mathcal{G})(-(s+1)\xi)) \cong H^i((q^*\mathcal{G})(-s\xi)) \text{ for all } i \ge 0 \text{ and } 1 \le s \le 4.$$

Setting $j = i - 2$, the Leray spectral sequence implies that

$$H^j(\mathcal{G} \otimes R^2q_*(-(s+1)\xi)) \cong H^j(\mathcal{G} \otimes R^2q_*(-s\xi)) \text{ for all } j \ge 0 \text{ and } 1 \le s \le 4.$$

Since $R^2q_*(-2\xi) = 0$ we deduce that $H^j(\mathcal{G} \otimes R^2q_*(-h\xi)) = 0$ for all $j \ge 0$ and $3 \le h \le 5$. On the other hand, $R^2q_*(-h\xi) \cong (S^{h-3}\Omega_{\mathbb{P}^3})(-4)$ and therefore we have that $H^j(\mathcal{G}(-4) \otimes S^k\Omega_{\mathbb{P}^3}) = 0$ for all $j \ge 0$ and $0 \le k \le 2$. But this is condition (6.4) in [Lo] (applied to $\mathcal{G}(-4)$), and it is proved there that this implies that $\mathcal{G}(-3)$ is an Ulrich vector bundle for $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Hence $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^3}(3)^{\oplus r}$ by Remark 2.9 and we are in case (x4). Thus we are done with example 3.

In example 5 we have that X is the blow-up of $Q = Q_4$ along a conic. Let $\varepsilon : X \to Q$ be the blow-up map with exceptional divisor E and let $\widetilde{H} = \varepsilon^*(\mathcal{O}_Q(1))$. We have that $\det \mathcal{E} = a\widetilde{H} + bE$, for some $a, b \in \mathbb{Z}$. It is easily checked that $E^4 = 6, \widetilde{H} \cdot E^3 = 2, \widetilde{H}^i \cdot E^{4-i} = 0$ for $i = 2, 3$ and $\widetilde{H}^4 = 2$. But then

$$0 = c_1(\mathcal{E})^4 = 6b^4 + 8ab^3 + 2a^4$$

implies that either $a = b = 0$, that is $c_1(\mathcal{E}) = 0$, a contradiction, or $b = -a$, and in this case $\det \mathcal{E} = a(\widetilde{H} - E)$, hence $c_1(\mathcal{E})^3 = 0$, again a contradiction. Thus example 5 is excluded.

In example 6 we have, by [W1, Table 0.3 and Thm. 1.1(iii)], [SW, Cor. page 206] (or [MOS, Thm. 6.5]), that $(X, \mathcal{O}_X(1))$ has two linear $\mathbb{P}^1$-bundle structures $p : X \cong \mathbb{P}(\mathcal{F}) \to B$ with $\mathcal{F} \cong \mathcal{N}(2)$ or $\mathcal{F} \cong \mathcal{S}(1)$ where $\mathcal{N}$ is a null-correlation bundle on $B = \mathbb{P}^3$ and $\mathcal{S}$ is the spinor bundle on $B = Q_3$. We first prove that case (B.1) does not occur. In fact, the Dichotomy Lemma applied to p in the case $B = Q_3$ and Lemma 2.16(ii) give that we are in case (emb). But this implies that $p_{|F_{x_0}} : \mathbb{P}^k = F_{x_0} \to Q_3$ is an embedding, hence that $k = 2$. Therefore $p_{|F_{x_0}}$ is the composition of the Veronese embedding v_s of $\mathbb{P}^2$ with an isomorphic projection. By Severi's theorem [Se] on projection of smooth surfaces it follows that $s = 2$. But it is well known that the Veronese surface in $\mathbb{P}^4$ is not contained in a smooth quadric. Therefore we are in case (B.2) and Claim 3.29 gives that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple, so that $\mathcal{E} \cong p^*(\mathcal{G}(4))$, in the case $B = \mathbb{P}^3$, or $p^*(\mathcal{G}(3))$, in the case $B = Q_3$, where $\mathcal{G}$ is a rank r vector bundle on B satisfying (2.4). When $B = Q_3$ we get case (x5). We now prove that, still in case (B.2), $B = \mathbb{P}^3$ does not occur. In fact, we have that $X \cong \mathbb{P}(\mathcal{N})$ and setting $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{N})}(1), R = p^*(\mathcal{O}_{\mathbb{P}^3}(1))$, then $\mathcal{O}_X(1) = \xi + 2R$. It is easily verified that $\xi^4 = \xi^2 \cdot R^2 = R^4 = 0, \xi^3 \cdot R = -1, \xi \cdot R^3 = 1$ and $H^4 = 24$. Now consider the surface S complete intersection of two general divisors in $|H|$. Then S is a K3 surface and $\mathcal{E}_{|S}$ is an Ulrich vector bundle for $(S, H_{|S})$ by Remark 2.8(iv). We can write $\det \mathcal{E} = c_1 R$ for some $c_1 \in \mathbb{Z}$. It follows by [C, Prop. 2.1] that

$$c_1(\mathcal{E}) \cdot H^3 = c_1(\mathcal{E}_{|S}) \cdot H_{|S} = \frac{3r}{2} H_{|S}^2 = 36r \tag{3.16}$$

and

$$c_2(\mathcal{E}_{|S}) = \frac{1}{2} c_1(\mathcal{E}_{|S})^2 - r(H_{|S}^2 - \chi(\mathcal{O}_S)) = \frac{1}{2} c_1(\mathcal{E}_{|S})^2 - 22r. \tag{3.17}$$

From (3.16) we get

$$36r = c_1 R \cdot (\xi + 2R)^3 = 11 c_1$$

giving $c_1 = \frac{36r}{11}$. But $\mathcal{E}_{|S}$ is μ-semistable by [CH, Thm. 2.9], hence, using (3.17), Bogomolov's inequality gives

$$0 \le 2r c_2(\mathcal{E}_{|S}) - (r-1) c_1(\mathcal{E}_{|S})^2 = 4c_1^2 - 44r^2 = -\frac{140r^2}{121}$$

a contradiction. Thus the case $X \cong \mathbb{P}(\mathcal{N})$ does not occur and we are done with example 6.

To see example 7, consider $\varepsilon : Y \to \mathbb{P}^5$ the blow-up along a line with exceptional divisor E and let $\widetilde{H} = \varepsilon^*(\mathcal{O}_{\mathbb{P}^5}(1))$. Set $L = 2\widetilde{H} - E$. Then X is a smooth divisor in $|L|$ and $\mathcal{O}_X(1) = L_{|X}$. By Lefschetz's theorem we know that $\mathrm{Pic}(X)$ is generated by $\widetilde{H}_{|X}$ and $E_{|X}$. It is easily checked that $E^5 = -4, \widetilde{H} \cdot E^4 = -1, \widetilde{H}^5 = 1$ and $\widetilde{H}^i \cdot E^{5-i} = 0$ for $2 \le i \le 4$. Now $\det \mathcal{E} = a\widetilde{H}_{|X} + bE_{|X}$, for some $a, b \in \mathbb{Z}$ and

$$0 = c_1(\mathcal{E})^4 = \sum_{j=0}^{4} \binom{4}{j} a^j b^{4-j} \widetilde{H}^j \cdot E^{4-j} (2\widetilde{H} - E) = 2b^4 + 2a^4 + 4ab^3.$$

If $a = 0$ we get that $b = 0$, that is $c_1(\mathcal{E}) = 0$, a contradiction. If $a \neq 0$ then $b = -a$ and $\det \mathcal{E} = a(\widetilde{H} - E)_{|X}$. Now observe that $Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$ and the bundle morphism $q : Y \to \mathbb{P}^3$ is just the morphism defined by $|\widetilde{H} - E|$. Moreover $L = \xi + q^*(\mathcal{O}_{\mathbb{P}^3}(1))$, where ξ is the tautological line bundle. If we set $h = q_{|X} : X \to \mathbb{P}^3$ we deduce from $\det \mathcal{E} = a(\widetilde{H} - E)_{|X}$ and [Lo, Lemma 5.1] that there is a rank r vector bundle $\mathcal{G}$ on $\mathbb{P}^3$ such that $\mathcal{E} \cong h^*\mathcal{G}$. Hence $\mathcal{E} \cong (q^*\mathcal{G})_{|X}$ and we have an exact sequence

$$0 \to (q^*\mathcal{G})(-(p+1)L) \to (q^*\mathcal{G})(-pL) \to \mathcal{E}(-pH) \to 0.$$

Since $H^i(\mathcal{E}(-pH)) = 0$ for all $i \ge 0$ and $1 \le p \le 4$ we deduce that

$$H^i((q^*\mathcal{G})(-(p+1)L)) \cong H^i((q^*\mathcal{G})(-pL)) \text{ for all } i \ge 0 \text{ and } 1 \le p \le 4.$$

Setting $j = i - 2$, the Leray spectral sequence implies that

$$H^j(\mathcal{G}(-p-1) \otimes R^2 q_*(-(p+1)\xi)) \cong H^j(\mathcal{G}(-p) \otimes R^2 q_*(-p\xi)) \text{ for all } j \ge 0 \text{ and } 1 \le p \le 4.$$

Since $R^2 q_*(-2\xi) = 0$ we deduce that $H^j(\mathcal{G}(-h) \otimes R^2 q_*(-h\xi)) = 0$ for all $j \ge 0$ and $3 \le h \le 5$. On the other hand, $R^2 q_*(-h\xi) \cong (S^{h-3}(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1))(-1)$ and therefore we have that $H^j(\mathcal{G}(-s)) = 0$ for all $j \ge 0$ and $4 \le s \le 8$. In particular $\mathcal{G}(-3)$ is an Ulrich vector bundle for $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, hence $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^3}(3)^{\oplus r}$ by Remark 2.9. But this gives the contradiction

$$0 = H^3(\mathcal{G}(-7)) = H^3(\mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus r}) \neq 0.$$

Thus example 7 is excluded.

In examples 8 and 9 we first claim that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple. In fact, we already know that $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$ where $\mathcal{F} = \mathcal{O}_B(1) \oplus \mathcal{O}_B(m)$, with $B = \mathbb{P}^3$ (respectively $B = Q_3$), $m = 3$ (resp. $m = 2$). Now let $p : X \cong \mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B(m-1)) \to B$ be the projection map, let ξ be the tautological line bundle and let $R = p^*(\mathcal{O}_B(1))$. Set $\det \mathcal{E} = a\xi + bR$, for some $a, b \in \mathbb{Z}$. If $a = 0$ we get that $\det \mathcal{E} = p^*(\mathcal{O}_B(b))$ and it follows by [Lo, Lemmas 4.1 and 5.1] that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple. Assume that $a \neq 0$. In example 8 we have that $\xi^i \cdot R^{4-i} = 2$ for $1 \le i \le 4$ and $R^4 = 0$. Hence

$$0 = c_1(\mathcal{E})^4 = 2a^4 + 8a^3b + 12a^2b^2 + 8ab^3$$

so that $b = -\frac{1}{2}a$ and $2 \det \mathcal{E} = a(2\xi - R)$. Let f be a fiber of p, so that $0 \le c_1(\mathcal{E}) \cdot f = 2a$, hence $a > 0$. But $Q_3 =: Q \cong \mathbb{P}(\mathcal{O}_Q) \subset X$ and $\xi_{|\mathbb{P}(\mathcal{O}_Q)} \cong \mathcal{O}_Q$, while $R_{|\mathbb{P}(\mathcal{O}_Q)} \cong \mathcal{O}_Q(1)$, hence $(2 \det \mathcal{E})_{|\mathbb{P}(\mathcal{O}_Q)} = \mathcal{O}_Q(-a)$, a contradiction since $\det \mathcal{E}$ is globally generated. In example 9 we have that $\xi^4 = 8, \xi^3 \cdot R = 4, \xi^2 \cdot R^2 = 2, \xi \cdot R^3 = 1$ and $R^4 = 0$. Now

$$0 = c_1(\mathcal{E})^4 = 8a^4 + 16a^3b + 12a^2b^2 + 4ab^3$$

hence $b = -a$ and $\det \mathcal{E} = a(\xi - R)$. But $\mathbb{P}^3 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^3}) \subset X$ and $\xi_{|\mathbb{P}(\mathcal{O}_{\mathbb{P}^3})} \cong \mathcal{O}_{\mathbb{P}^3}$, while $R_{|\mathbb{P}(\mathcal{O}_{\mathbb{P}^3})} \cong \mathcal{O}_{\mathbb{P}^3}(1)$, hence $(\det \mathcal{E})_{|\mathbb{P}(\mathcal{O}_{\mathbb{P}^3})} = \mathcal{O}_{\mathbb{P}^3}(-1)$ is not globally generated, a contradiction. Thus we have proved that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple. Hence $\mathcal{E} \cong p^*(\mathcal{G}(m+1))$, where $\mathcal{G}$ is a rank r vector bundle on B such that $H^j(\mathcal{G} \otimes S^k\mathcal{F}^*) = 0$ for $j \ge 0, 0 \le k \le 2$. This gives in particular that $H^j(\mathcal{G}(-s)) = 0$ for $j \ge 0$ and $0 \le s \le 3$. Hence $\mathcal{G}(1)$ is an Ulrich vector bundle for $(B, \mathcal{O}_B(1))$. When $B = \mathbb{P}^3$ we get by Remark 2.9 that $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus r}$. But this gives the contradiction

$$0 = H^3(\mathcal{G}(-3)) = H^3(\mathcal{O}_{\mathbb{P}^3}(-4))^{\oplus r} \neq 0.$$

Instead, when $B = Q_3$, we get by [LMS, Lemma 3.2(iv)] that $\mathcal{G} \cong \mathcal{S}(-1)^{\oplus \binom{r}{2}} \cong S^{\oplus \binom{r}{2}}$. But this gives the contradiction

$$0 = h^3(\mathcal{G}(-3)) = h^3(S(-3))^{\oplus \binom{r}{2}} = h^0(S^*)^{\oplus \binom{r}{2}} = h^0(\mathcal{S})^{\oplus \binom{r}{2}} \neq 0.$$

Thus examples 8 and 9 are excluded.

This proves Claim 3.31. □

This concludes the proof of Theorem 1 in Case (B). Therefore the proof of Theorem 1 is complete. □

Finally, our last two results.

Proof of Corollary 1. If $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (i) or as in (iii), it follows by Remark 2.9 and [LMS, Thm. 1] and 1 that $\mathcal{E}$ is Ulrich not big. If $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (ii1)-(ii3), it follows by [Lo, Prop. 6.1 and 6.2] for (ii1) and (ii2) and by [B, (3.5)] for (ii3), that $\mathcal{E}$ is Ulrich not big.

Vice versa assume that $\mathcal{E}$ is Ulrich not big.

Suppose first that $\rho(X) = 1$.

If $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ we are in case (i1) by Remark 2.9. If $(X, \mathcal{O}_X(1)) \not\cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ then $c_1(\mathcal{E})^n > 0$ by [Lo, Lemma 3.2] and we are in case (i2) by [LM, Thms. 1 and 2] and Theorem 1. This proves (i).

Now suppose that $(X, \mathcal{O}_X(1))$ is a del Pezzo variety.

If $n = 2$, since del Pezzo surfaces are not covered by lines, there is no such $\mathcal{E}$.

If $n = 3$ it follows by [LM, Thm. 3] that $c_1(\mathcal{E})^3 = 0, c_1(\mathcal{E})^2 \neq 0$. According to the classification of del Pezzo 3-folds (see for example [LP, §1], [F1]), we get that X is either $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}(T_{\mathbb{P}^2})$ or V_7. In the first two instances we are in cases (ii1) and (ii2) by [Lo, Prop. 6.1 and 6.2]. Now consider the third one, that is we have $\varepsilon : X = V_7 \to \mathbb{P}^3$ the blow-up map with exceptional divisor E and let $\widetilde{H} = \varepsilon^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Hence $\det \mathcal{E} = a\widetilde{H} + bE$, for some $a, b \in \mathbb{Z}$. It is easily checked that $\widetilde{H}^3 = E^3 = 1, \widetilde{H}^i \cdot E^{3-i} = 0$ for $i = 1, 2$. But then

$$0 = c_1(\mathcal{E})^3 = a^3 + b^3$$

implies that $b = -a$, that is $\det \mathcal{E} = a(\widetilde{H} - E)$. As is well known, we have that $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ and the morphism induced by $|\widetilde{H} - E|$ is just the projection map $p : X \to \mathbb{P}^2$. We deduce by [Lo, Lemma 5.1] that there is a rank r vector bundle $\mathcal{G}$ on $\mathbb{P}^2$ such that $\mathcal{E} \cong p^*(\mathcal{G}(2))$. But now [Lo, Lemma 4.1] gives that $H^i(\mathcal{G}(-s)) = 0$ for all $i \ge 0$ and $1 \le s \le 3$. In particular $\mathcal{G}$ is an Ulrich vector bundle for $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, hence $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^2}^{\oplus r}$ by Remark 2.9. But this gives the contradiction $0 = H^2(\mathcal{G}(-3)) = H^2(\mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus r}) \neq 0$. Thus this case is excluded.

If $n = 4$, using the classification of del Pezzo 4-folds (see for example [LP, §1], [F1]) and case (i), we see that $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. Then we are in case (ii3) by [LMS, Thm. 3]. This proves (ii).

Now suppose that $(X, \mathcal{O}_X(1))$ is a Mukai variety.

We have that $n = 4$ since the Mukai n-folds are not covered by lines for $n \leq 3$. Therefore we are in cases (x1)-(x5) of Theorem 1. This proves (iii). □

Proof of Corollary 2. If $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (i) or (ii), it follows by Corollary 1, Theorem 1, [B, (3.5)] and [Lo, Prop. 6.2] that $\mathcal{E}$ is Ulrich with $\det \mathcal{E}$ not big.

Vice versa assume that $\mathcal{E}$ is Ulrich. Since $\mathcal{E}$ is globally generated and $\det \mathcal{E}$ is not big, we have that $c_1(\mathcal{E})^n = 0$, hence $\rho(X) \geq 2$ and $\mathcal{E}$ is not big.

If $(X, \mathcal{O}_X(1))$ is a del Pezzo n-fold, then either $n \leq 4$ and we get cases (ii1)-(ii3) of Corollary 1 by the same corollary, or $n \geq 5$. But in the latter case the classification of del Pezzo n-folds (see for example [LP, §1], [F1]) gives that $\rho(X) = 1$, hence this case does not occur. This proves (i).

Now suppose that $(X, \mathcal{O}_X(1))$ is a Mukai n-fold.

If $n \leq 4$ we get cases (x1)-(x5) of Theorem 1 by Corollary 1(iii).

If $n \geq 5$, Mukai's classification [M, Thm. 7] gives that X is as in examples 3, 4 and 7 in [M, Ex. 2].

In example 3 we have that either $X = \mathbb{P}^3 \times \mathbb{P}^3$ or $X = \mathbb{P}(T_{\mathbb{P}^3})$. In the second case $\mathcal{E}$ is as in (ii2) by [Lo, Prop. 6.2]. In the first case $c_1(\mathcal{E})^6 = 0$ immediately gives that $\det \mathcal{E} = p^*(\mathcal{O}_{\mathbb{P}^3}(a))$, for some $a \in \mathbb{Z}$, where $p : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$ is one of the two projections. It follows by [Lo, Lemmas 4.1 and 5.1] and Remark 2.9 that $\mathcal{E}$ is as in (ii1).

In example 4 we have that $X = \mathbb{P}^2 \times Q_3$ and $c_1(\mathcal{E})^5 = 0$ immediately gives that either $\det \mathcal{E} = p_1^*(\mathcal{O}_{\mathbb{P}^2}(a))$ or $p_2^*(\mathcal{O}_{Q_3}(a))$, for some $a \in \mathbb{Z}$, where $p_i, i = 1, 2$ are the projections. In the second case it follows by [Lo, Lemmas 4.1 and 5.1] and [LMS, Lemma 3.2(iv)] that $\mathcal{E}$ is as in (ii3). In the first case it follows by [Lo, Lemma 5.1] that there is a rank r vector bundle $\mathcal{G}$ on $\mathbb{P}^2$ such that $\mathcal{E} \cong p_1^*\mathcal{G}$. Now the vanishings $H^i(\mathcal{E}(-pH)) = 0$ for all $i \geq 0$ and $1 \leq p \leq 5$, together with the Künneth formula imply that $H^j(\mathcal{G}(-2)(-s)) = 0$ for all $j \geq 0$ and $1 \leq s \leq 3$. But then $\mathcal{G}(-2)$ is an Ulrich vector bundle for $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and Remark 2.9 gives that $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^2}(2)^{\oplus r}$, thus giving the contradiction $0 = H^2(\mathcal{G}(-5)) = H^2(\mathcal{O}_{\mathbb{P}^2}(-3)) \neq 0$.

In example 7 we have that X is the blow-up $\varepsilon : X \to \mathbb{P}^5$ along a line with exceptional divisor E and let $\widetilde{H} = \varepsilon^*(\mathcal{O}_{\mathbb{P}^5}(1))$. As in the proof of Theorem 1 (see Claim 3.31) we have that $E^5 = -4, \widetilde{H} \cdot E^4 = -1, \widetilde{H}^5 = 1$ and $\widetilde{H}^j \cdot E^{5-j} = 0$ for $2 \leq j \leq 4$. Now $\det \mathcal{E} = a\widetilde{H} + bE$, for some $a, b \in \mathbb{Z}$ and

$$0 = c_1(\mathcal{E})^5 = \sum_{j=0}^{5} \binom{5}{j} a^j b^{5-j} \widetilde{H}^j \cdot E^{5-j} = a^5 - 5ab^4 - 4b^5.$$

If $a = 0$ we get that $b = 0$, that is $c_1(\mathcal{E}) = 0$, a contradiction. If $a \neq 0$ then $b = -a$ and $\det \mathcal{E} = a(\widetilde{H} - E)$. Again as in the proof of Theorem 1 (see Claim 3.31), using the morphism $p : X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(1)) \to \mathbb{P}^3$ we deduce by [Lo, Lemma 5.1] that there is a rank r vector bundle $\mathcal{G}$ on $\mathbb{P}^3$ such that $\mathcal{E} \cong p^*(\mathcal{G}(3))$. Now [Lo, Lemma 4.1] gives that $H^i(\mathcal{G}(-s)) = 0$ for all $i \geq 0$ and $1 \leq s \leq 4$. In particular $\mathcal{G}$ is an Ulrich vector bundle for $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, hence $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ by Remark 2.9. But this gives the contradiction $0 = H^3(\mathcal{G}(-4)) = H^3(\mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus r}) \neq 0$. Thus example 7 is excluded.

This proves (ii). □

4. Examples

In this section we will give some examples that are significant both with respect to Theorem 1 (for the statement but also for the method of proof) and to the fact that they do not appear in lower dimension.

Example 4.1. (cfr. cases (v2) and (v3) in Theorem 1)

Let C be a smooth curve, let L be a very ample line bundle on C, let $\mathcal{G}$ be an Ulrich vector bundle for (C, L) and let $X = \mathbb{P}^2 \times C \times \mathbb{P}^1$ and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes L \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$. For case (v3), let $p : X \to \mathbb{P}^2 \times C$ be the projection and let $\mathcal{E} = p^*(\mathcal{O}_{\mathbb{P}^2}(2) \boxtimes \mathcal{G}(L))$. Then $\mathcal{E}$ is Ulrich on X by [B, (3.5)] and $c_1(\mathcal{E})^4 = 0$. Similarly, for case (v2), let $q : X \to C \times \mathbb{P}^2$ be the projection and let $\mathcal{E} = q^*(\mathcal{G}(2L) \boxtimes \mathcal{O}_{\mathbb{P}^1}(3))$.

Example 4.2. (cfr. case (vi2) in Theorem 1)

Let B be a smooth irreducible curve and let L be a very ample line bundle on B such that K_B+2L is very ample. Let $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$. Let $X = B \times Y$, $\mathcal{O}_X(1) = L \boxtimes \mathcal{L}$ and let $p_1 : X \to B$ be the first projection. Then $K_X + 2H = p_1^*(K_B + 2L)$, hence $p_1 = \varphi_{K_X+2H}$. Thus φ_{K_X+2H} gives a del Pezzo fibration on X over B with all fibers $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{F}$ be any Ulrich vector bundle for (B, L) and let $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{S}'$ on Y. Then $\mathcal{E} := \mathcal{F}(3L) \boxtimes \mathcal{G}$ is an Ulrich vector bundle on X by [B, (3.5)] with $c_1(\mathcal{E})^4 = 0, c_1(\mathcal{E})^3 \neq 0$.

Example 4.3. (cfr. case (vi2) in Theorem 1)

Let Y be a smooth irreducible divisor of type $(1, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^3$ and let $\mathcal{O}_Y(1) = (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1))_{|Y}$. Then $\pi_{1|Y} : Y \to \mathbb{P}^1$ is a quadric fibration whose fibers are either smooth or a cone in $\mathbb{P}^3$ and there are 12 such cones by [Lant, §2]. Let $X = \mathbb{P}^1 \times Y$ and $\mathcal{O}_X(1) = (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Y(1))_{|X}$. Let $\mathcal{G}$ be an Ulrich vector bundle for $(Y, \mathcal{O}_Y(1))$ and let $\mathcal{E} = q^*(\mathcal{G}(1))$, where $q : X \to Y$ is the restriction of the second projection. It follows by [Lo, Lemma 4.1] that $\mathcal{E}$ is an Ulrich vector bundle on X and we are in case (vi2) in Theorem 1.

Example 4.4. (cfr. case (vi3) in Theorem 1)

Let $Y = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ and let $L = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$. Let $X \in |L|$ be smooth and irreducible and let $\mathcal{O}_X(1) = L_{|X}$. Let $p_1 : X \to \mathbb{P}^1$ be the restriction of the first projection on Y. Then $K_X + 2H = p_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$, hence $p_1 = \varphi_{K_X+2H}$. A general fiber F of p_1 is a hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$, that is $F \cong \mathbb{P}(T_{\mathbb{P}^2})$. Thus φ_{K_X+2H} gives a del Pezzo fibration on X over $\mathbb{P}^1$ with general fiber $\mathbb{P}(T_{\mathbb{P}^2})$. To construct $\mathcal{E}$ observe that if $p : Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}^{\oplus 3}) \to \mathbb{P}^1 \times \mathbb{P}^2$ with tautological line bundle ξ, then $L = \xi + p^*M$, where $M = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$. Let $\mathcal{E}' = p^*(\mathcal{O}_{\mathbb{P}^1}(4) \boxtimes \mathcal{O}_{\mathbb{P}^2}(2))$. Then $\mathcal{E}'$ is an Ulrich line bundle for (Y, L) by [B, (3.5)] and [Lo, Lemma 4.1(ii)]. Hence $\mathcal{E} = \mathcal{E}'_{|X}$ is an Ulrich line bundle for $(X, \mathcal{O}_X(1))$ with $c_1(\mathcal{E})^4 = 0, c_1(\mathcal{E})^3 \neq 0$. Moreover Φ is just the composition of $p_{|X} : X \to \mathbb{P}^1 \times \mathbb{P}^2$ with the embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ by $\mathcal{O}_{\mathbb{P}^1}(4) \boxtimes \mathcal{O}_{\mathbb{P}^2}(2)$. Finally, $p_{|X}$ has $M^3 = 3$ fibers that are a linear $\mathbb{P}^2$ by [BS, Ex. 14.1.5], hence so does Φ.

We do not know if there is an example of a triple $(X, \mathcal{O}_X(1), \mathcal{E})$ with $c_1(\mathcal{E})^4 = 0, c_1(\mathcal{E})^3 \neq 0$ and with the del Pezzo fibration having as fibers the blow-up of $\mathbb{P}^3$ in a point.

Example 4.5. (cfr. case (vii) in Theorem 1)

Let $X = \mathbb{P}^2 \times Q$, where $Q = Q_2$, let $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^2}(2) \boxtimes \mathcal{O}_Q(1)$ and let $p_1 : X \to \mathbb{P}^2$ be the first projection. Then $K_X + 2H = p_1^*(\mathcal{O}_{\mathbb{P}^2}(1))$, hence $p_1 = \varphi_{K_X+2H}$ is a quadric fibration over $\mathbb{P}^2$. Let $\mathcal{F}$ be an Ulrich vector bundle on $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and let $\mathcal{E} = \mathcal{F}(2) \boxtimes \mathcal{S}'$. Then $\mathcal{E}$ is an Ulrich vector bundle for $(X, \mathcal{O}_X(1))$ by [B, (3.5)]. Moreover $c_1(\mathcal{E})^4 = 0$ and $c_1(\mathcal{E})^3 \neq 0$.

Example 4.6. (cfr. case (ix) in Theorem 1)

Let Y be a smooth irreducible threefold and let M be a very ample line bundle on Y such that $K_Y + 3M$ is very ample. Let $Z = \mathbb{P}^2 \times Y$ and let $L = \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes M$. Let $X \in |L|$ be smooth and irreducible and let $\mathcal{O}_X(1) = L_{|X}$. Let $p_2 : Z \to Y$ be the second projection and let $q = p_{2|X}$. Then $K_X + 2H = q^*(K_Y + 3M)$. Hence $q = \varphi_{K_X+2H}$ gives a structure of scroll over Y. Let $\mathcal{G}$ be an Ulrich vector bundle for (Y, M) and let $\mathcal{E}' = p_2^*(\mathcal{G}(2M))$. Then $\mathcal{E}'$ is an Ulrich vector bundle for (Y, L) by [Lo, Lemma 4.1]. Hence $\mathcal{E} = \mathcal{E}'_{|X}$ is an Ulrich vector bundle for $(X, \mathcal{O}_X(1))$ by Remark 2.8(iv). Since $\mathcal{E} = q^*(\mathcal{G}(2M))$ it follows that Φ factorizes through $q : X \to Y$. Finally, q has $M^3 > 0$ fibers that are a linear $\mathbb{P}^2$ by [BS, Ex. 14.1.5]. Therefore $c_1(\mathcal{E})^4 = 0, c_1(\mathcal{E})^3 \neq 0$ and Φ has M^3 fibers that are a linear $\mathbb{P}^2$.

Example 4.7. (cfr. case (x1) in Theorem 1)

On a Fano 3-fold M of index 2, there are several examples of Ulrich vector bundles for (M, L), where $K_M = -2L$. See for instance [B, CMRPL].

Example 4.8. (cfr. case (x2) in Theorem 1)

Let $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes T_{\mathbb{P}^2}, (X, \mathcal{O}_X(1)) = (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$ and let $p : X \to \mathbb{P}^1 \times \mathbb{P}^2$ be the projection map. It is easily verified that $\mathcal{E} = p^*(\mathcal{O}_{\mathbb{P}^1}(3) \boxtimes \mathcal{O}_{\mathbb{P}^2}(2))$ is an Ulrich bundle as in (x2) of Theorem 1.

Example 4.9. (cfr. case (x5) in Theorem 1)

The following example was kindly suggested to us by D. Faenzi, whom we thank.

Let $Q \subset \mathbb{P}^4 = \mathbb{P}(V)$ be a smooth quadric. We have a natural identification

$$\Lambda^2 V \cong H^0(\Omega_{\mathbb{P}^4}(2)) \cong H^0(\Omega_Q(2)) \subset H^0(\Omega_{\mathbb{P}^4}(2)_{|Q}). \tag{4.1}$$

We can define a vector bundle $\mathcal{F}$ by setting

$$\mathcal{H}^* = \mathcal{H}om(\Omega_{\mathbb{P}^4}(2)_{|Q}, \mathcal{O}_Q) \otimes H^0(\Omega_{\mathbb{P}^4}(2))$$

and considering the exact sequence

$$0 \to \mathcal{F} \to \mathcal{H}^* \to \mathcal{O}_Q \to 0$$

where the morphism $\psi : \mathcal{H}^* \to \mathcal{O}_Q$ is defined locally by sending $\phi \otimes \sigma$ to $\phi(\sigma_{|Q})$.

Now set $\mathcal{G} = \mathcal{F}^*(-1)$ so that we have an exact sequence

$$0 \to \mathcal{O}_Q \to \mathcal{H} \to \mathcal{G}(1) \to 0 \tag{4.2}$$

defining a rank 39 vector bundle $\mathcal{G}$ on Q. We have

Claim 4.10. *$H^j(\mathcal{G}(-2k) \otimes S^k\mathcal{S}) = 0$ for all $j \geq 0, 0 \leq k \leq 2$.*

Proof. To see this set $\mathcal{H}_1 = \Omega_{\mathbb{P}^4}(2)_{|Q}$ and consider the twisted Euler sequence, for every $l \in \mathbb{Z}$,

$$0 \to \mathcal{H}_1(l) \to H^0(\mathcal{O}_Q(1)) \otimes \mathcal{O}_Q(l+1) \to \mathcal{O}_Q(l+2) \to 0. \tag{4.3}$$

For $l = -1$ we see that $H^j(\mathcal{H}_1(-1)) = 0$ for $j \geq 0$, hence also $H^j(\mathcal{H}(-1)) = 0$ for $j \geq 0$. Then (4.2) tensored by $\mathcal{O}_Q(-1)$ gives that

$$H^j(\mathcal{G}) = 0 \text{ for } j \geq 0.$$

Choosing $l = -3$ in (4.3) and tensoring with $\mathcal{S}$, we get the exact sequence

$$0 \to \mathcal{H}_1(-3) \otimes \mathcal{S} \to \mathcal{S}(-2)^{\oplus 5} \to \mathcal{S}(-1) \to 0.$$

Using the fact that $\mathcal{S}$ is Ulrich, we deduce that $H^j(\mathcal{H}_1(-3) \otimes \mathcal{S}) = 0$ for $j \geq 0$, hence also $H^j(\mathcal{H}(-3) \otimes \mathcal{S}) = 0$ for $j \geq 0$. Then (4.2) tensored by $\mathcal{S}(-3)$ gives that

$$H^j(\mathcal{G}(-2) \otimes \mathcal{S}) = 0 \text{ for } j \geq 0.$$

To finish the proof of (4.10) it remains to prove that

$$H^j(\mathcal{G}(-4) \otimes S^2\mathcal{S}) = 0 \text{ for } j \geq 0. \tag{4.4}$$

To this end we collect some well-known vanishings, that can be easily obtained using [O] and restricting to the hyperplane section.

Subclaim 4.11.

(i) $H^0((S^2\mathcal{S})(l)) = 0$ *for* $l \leq -2$.
(ii) $H^1((S^2\mathcal{S})(l)) = 0$ *for* $l \neq -2$.
(iii) $H^2((S^2\mathcal{S})(l)) = 0$ *for* $l \neq -3$.
(iv) $h^2((S^2\mathcal{S})(-3)) = 1$.
(v) $H^3((S^2\mathcal{S})(-4)) = 0$.
(vi) $h^3((S^2\mathcal{S})(-5)) = 10$.

Setting $l = -5$ in (4.3) and tensoring with $S^2\mathcal{S}$, we get the exact sequence

$$0 \to \mathcal{H}_1(-5) \otimes S^2\mathcal{S} \to (S^2\mathcal{S})(-4)^{\oplus 5} \to (S^2\mathcal{S})(-3) \to 0 \tag{4.5}$$

and we deduce by Subclaim 4.11(i), (ii) and (iii) that $H^j(\mathcal{H}_1(-5) \otimes S^2\mathcal{S}) = 0$ for $j = 0, 1, 2$, hence also

$$H^j(\mathcal{H}(-5) \otimes S^2\mathcal{S}) = 0 \text{ for } j = 0, 1, 2. \tag{4.6}$$

Now tensoring (4.2) with $(S^2\mathcal{S})(-5)$ we get the exact sequence

$$0 \to (S^2\mathcal{S})(-5) \to \mathcal{H}(-5) \otimes S^2\mathcal{S} \to \mathcal{G}(-4) \otimes S^2\mathcal{S} \to 0 \tag{4.7}$$

and applying (4.6) and Subclaim 4.11(ii) and (iii) we get that

$$H^j(\mathcal{G}(-4) \otimes S^2\mathcal{S}) = 0 \text{ for } j = 0, 1.$$

Moreover (4.6) and (4.7) give rise to the exact sequence

$$0 \to H^2(\mathcal{G}(-4) \otimes S^2\mathcal{S}) \to H^3((S^2\mathcal{S})(-5)) \to H^3(\mathcal{H}(-5) \otimes S^2\mathcal{S}) \to H^3(\mathcal{G}(-4) \otimes S^2\mathcal{S}) \to 0. \tag{4.8}$$

Now note that from (4.5) we have, applying Subclaim 4.11(iii), (iv) and (v), that $h^3(\mathcal{H}_1(-5) \otimes S^2\mathcal{S}) = h^2((S^2\mathcal{S})(-3)) = 1$, hence
$$h^3(\mathcal{H}(-5) \otimes S^2\mathcal{S})) = 10.$$
Since $h^3((S^2\mathcal{S})(-5)) = 10$ by Subclaim 4.11(vi), to complete the proof it remains to show that the morphism
$$H^3((S^2\mathcal{S})(-5)) \to H^3(\mathcal{H}(-5) \otimes S^2\mathcal{S})$$
is injective, or, by Serre's duality, that
$$H^0(\mathcal{H}^* \otimes S^2\mathcal{S}) \to H^0(S^2\mathcal{S})$$
is surjective.

To see the latter, using the isomorphism
$$\mathcal{H}om(\Omega_{\mathbb{P}^4}(2)_{|Q}, \mathcal{O}_Q) \otimes \Omega_Q(2) \cong \mathcal{H}om(\Omega_{\mathbb{P}^4}(2)_{|Q}, \Omega_Q(2))$$
and recalling that $S^2\mathcal{S} \cong \Omega_Q(2)$ by [O, Ex. 1.5], we have that the map $H^0(\mathcal{H}^* \otimes S^2\mathcal{S}) \to H^0(S^2\mathcal{S})$ can be identified, with our choices, with the map
$$\mathrm{Hom}(\Omega_{\mathbb{P}^4 |Q}(2), \Omega_Q(2)) \otimes H^0(\Omega_{\mathbb{P}^4}(2)) \to H^0(\Omega_Q(2))$$
which sends $\phi \otimes \sigma$ to $\phi(\sigma_{|Q})$. This map is clearly surjective by (4.1). Hence it is an isomorphism and therefore (4.8) gives that
$$H^j(\mathcal{G}(-4) \otimes S^2\mathcal{S}) = 0 \text{ for } j = 2, 3.$$
This proves (4.4) and the claim. □

Example 4.12. (cfr. cases (xii)-(xiii) in Theorem 1)

Let B be a smooth irreducible variety of dimension $b = 1, 2$ and let $\mathcal{F}$ be a rank $5 - b$ very ample vector bundle on B. Let $X = \mathbb{P}(\mathcal{F})$ with tautological line bundle $\mathcal{O}_X(1)$ and projection $p : X \to B$. Let M be a line bundle on B such that $H^i(M) = 0$ for all $i \geq 0$ and set $\mathcal{E} = \Omega_{X/B}(2H + p^*M)$. When $b = 2$ suppose also that $H^i(\mathcal{F}(M - \det \mathcal{F})) = 0$ for all $i \geq 0$. Then $\mathcal{E}$ is an Ulrich vector bundle for $(X, \mathcal{O}_X(1))$, $\mathcal{E}$ it is not big, $\mathcal{E}_{|f} \cong \Omega_f(2)$ and, in many cases, $c_1(\mathcal{E})^4 > 0$. This is shown for $b = 1$ in [LM, Lemma 4.1]. With the same method it can be shown for $b = 2$. An example for $b = 2$ can be obtained by picking a very ample line bundle L on B, $\mathcal{F} = L^{\oplus 3}$ and $M = \mathcal{L}(-2L)$ where $\mathcal{L}$ is an Ulrich line bundle for $(B, 2L)$. Explicitly one can take $B = \mathbb{P}^1 \times \mathbb{P}^1, L = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ and $M = \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$. Note that for $b = 2$ we have by (3.1) that $\nu(\mathcal{E}) = r + 2$. Moreover, in order to get restrictions $\mathcal{E}_{|f}$ with trivial summands, one can add to $\mathcal{E}$ direct summands of type $p^*(\mathcal{L}(\det \mathcal{F}))$, where $\mathcal{L}$ is a line bundle on B such that $H^j(\mathcal{L} \otimes S^k\mathcal{F}^*) = 0$ for $j \geq 0, 0 \leq k \leq b - 1$.

Example 4.13. (cfr. case (xii) in Theorem 1)

Let $X = \mathbb{P}^1 \times \mathbb{P}^3$ and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)$. It is easily seen that the vector bundle
$$\mathcal{E} = [\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes (T_{\mathbb{P}^3}(-1))^{\oplus 2}] \oplus [\mathcal{O}_{\mathbb{P}^1}(3) \boxtimes \mathcal{O}_{\mathbb{P}^3}]^{\oplus (r-6)}$$
is Ulrich, $c_1(\mathcal{E})^4 > 0, \nu(\mathcal{E}) = r + 2$ and $\mathcal{E}_{|f} = T_{\mathbb{P}^3}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus (r-6)}$. This is the last possible case in Theorem 1(xii), as in [SU, Thm. 1(v)].

We do not know if the case with restriction $\mathcal{N}(1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus (r-2)}$ actually occurs in Theorem 1(xii).

Example 4.14. (cfr. case (xiv) in Theorem 1)

Let B be a smooth irreducible curve, let L be a very ample line bundle on B and let $Q = Q_3$. Let $X = B \times Q$ and let $\mathcal{O}_X(1) = L \boxtimes \mathcal{O}_Q(1)$. Then the first projection $p_1 : X \to B$ is a quadric fibration associated to $K_X + 3H$. Let $\mathcal{L}$ be an Ulrich line bundle for (B, L) and let $\mathcal{E} = \mathcal{L}(3L) \boxtimes \mathcal{S}$. Then $\mathcal{E}$ is an Ulrich rank 2 relative spinor bundle for $(X, \mathcal{O}_X(1))$ by [B, (3.5)] and [LMS, Lemma 3.2(iii)]. Note that $c_1(\mathcal{E})^4 > 0$. Moreover $\mathcal{E}$ is not big by [LMS, Prop. 3.3(iii)] and [LM, Lemma 2.4].

TABLE 1.

Cases in Theorem 1 with $c_1(\mathcal{E})^4 = 0$

Case	X	$\mathcal{O}_X(1)$	$\mathcal{E}$	B (if linear Ulrich triple)	case Lemma 3.1, ϕ_τ	p (Def. 2.3) or q	Example
(i)	$\mathbb{P}^4$	$\mathcal{O}_{\mathbb{P}^4}(1)$	$\mathcal{O}_{\mathbb{P}^4}^{\oplus r}$		(a)		
(ii1)	$\mathbb{P}^1 \times \mathbb{P}^3$	$\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)$	$p^*(\mathcal{O}_{\mathbb{P}^3}(1))^{\oplus r}$	$\mathbb{P}^3$	$(b.2), \mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^1$	$\mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^3$	
(ii2)	$\mathbb{P}(\mathcal{F})$	$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$	$p^*(\mathcal{G}(\det \mathcal{F}))$	curve	$(b.2), p$	bundle map	
(iii)	$\mathbb{P}^2 \times \mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$	$p^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus r}$	$\mathbb{P}^2$	$(c.1)$	$\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$	
(iv)	$\mathbb{P}^1 \times Q_3$	$\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Q_3}(1)$	$p^*(\mathcal{S}(1))^{\oplus(\frac{r}{2})}$	Q_3	$(c.2), \mathbb{P}^1 \times Q_3 \to \mathbb{P}^1$	$\mathbb{P}^1 \times Q_3 \to Q_3$	
(v1)	$(\mathbb{P}^2 \times \mathbb{P}^3) \cap H$	$(\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1))_{\vert X}$	$q^*(\mathcal{O}_{\mathbb{P}^3}(2))^{\oplus r}$		$(c.3), X \to \mathbb{P}^2$ projection linear $\mathbb{P}^2$-bundle	$X \to \mathbb{P}^3$ projection	
(v2)	$\mathbb{P}(\mathcal{F})$	$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$	$p^*(\mathcal{G}(\det \mathcal{F}))$	surface	$(c.3), p$	bundle map	4.1
(v3)	$\mathbb{P}(\mathcal{F})$	$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$	$p^*(\mathcal{G}(\det \mathcal{F}))$	3-fold	$(c.3)$, linear $\mathbb{P}^2$-bundle over surface	bundle map	4.1
(vi1)	$\mathbb{P}(\mathcal{F})$	$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$	$p^*(\mathcal{G}(\det \mathcal{F}))$	3-fold, del Pezzo fibration, fibers $\mathbb{P}^2$	$(d.2)$, del Pezzo fibration, fibers V_7	bundle map	
(vi2)	$\mathbb{P}(\mathcal{F})$	$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$	$p^*(\mathcal{G}(\det \mathcal{F}))$	3-fold, del Pezzo fibration, smooth fibers $\mathbb{P}^1 \times \mathbb{P}^1$	$(d.2)$, del Pezzo fibration, smooth fibers $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	bundle map	4.2, 4.3
(vi3)	del Pezzo fibration, smooth fibers $\mathbb{P}(T_{\mathbb{P}^2})$	del Pezzo fibration	$\mathcal{E}_{\vert \mathbb{P}(T_{\mathbb{P}^2})}$ pull-back from $\mathbb{P}^2$		$(d.2)$, del Pezzo fibration, smooth fibers $\mathbb{P}(T_{\mathbb{P}^2})$		4.4
(vii)	$\mathbb{P}(\mathcal{F})$	$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$	$p^*(\mathcal{G}(\det \mathcal{F}))$	3-fold	$(d.3)$, quadric fibration over surface	bundle map	4.5
(viii)	$\mathbb{P}(\mathcal{F})$	$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$	$p^*(\mathcal{G}(\det \mathcal{F}))$	3-fold	$(d.4), p$	bundle map	
(ix)	scroll over a normal threefold	scroll	$\mathcal{E}_{\vert \mathbb{P}^1}$ trivial		$(d.5)$, scroll over a normal threefold		4.6
(x1)	$\mathbb{P}^1 \times M$, $K_M = -2L$	$\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes L$	$p^*(\mathcal{G}(L))$, $\mathcal{G}$ Ulrich for (M, L)	M	$(d.1)$	$\mathbb{P}^1 \times M \to M$	4.7
(x2)	$\mathbb{P}^1 \times \mathbb{P}(T_{\mathbb{P}^2})$	$\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^2})}(1)$	$p^*(\mathcal{G} \otimes (\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^3}(3)))$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(d.1)$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes T_{\mathbb{P}^2}) \to \mathbb{P}^1 \times \mathbb{P}^2$	4.8
(x3)	$(\mathbb{P}^2 \times Q_3) \cap H$	$(\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{Q_3}(1))_{\vert X}$	$q^*(\mathcal{S}(2))^{\oplus(\frac{r}{2})}$		$(d.1)$	$X \to Q_3$ projection	
(x4)	$(\mathbb{P}^3 \times \mathbb{P}^3) \cap H \cap H'$	$(\mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1))_{\vert X}$	$q^*(\mathcal{O}_{\mathbb{P}^3}(3))^{\oplus r}$		$(d.1)$	$X \to \mathbb{P}^3$ projection	
(x5)	$\mathbb{P}(\mathcal{S})$	$\mathcal{O}_{\mathbb{P}(\mathcal{S})}(1) \otimes p^*(\mathcal{O}_{Q_3}(1))$	$p^*(\mathcal{G}(3))$	Q_3	$(d.1)$	bundle map	4.9

TABLE 2.

Cases in Theorem 1 with $c_1(\mathcal{E})^4 > 0$							
Case	X	$\mathcal{O}_X(1)$	$\mathcal{E}$	B	case Lemma 3.1, ϕ_τ	p	Example
(xi)	Q_4	$\mathcal{O}_{Q_4}(1)$	$\mathcal{S}', \mathcal{S}'', \mathcal{S}' \oplus \mathcal{S}''$		$(b.1)$		
(xii)	$\mathbb{P}(\mathcal{F})$	$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$	$\mathcal{E}_{\|f} = T_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-3)}$, $\Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-3)}$, $\mathcal{N}(1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-2)}$, or quotient $0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus(r+2)} \to \mathcal{E}_{\|f} \to 0$	curve	$(b.2), p$	bundle map	4.12, 4.13
(xiii)	$\mathbb{P}(\mathcal{F})$	$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$	$\mathcal{E}_{\|f} \cong T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus(r-2)}$	surface	$(c.3), p$	bundle map	4.12
(xiv)	quadric fibration	quadric fibration	$\mathcal{E}_{\|f}$ spinor bundle on general f	curve	$(c.2), p$		4.14

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