Giornate di Geometria Algebrica ed Argomenti Correlati XII (Torino, 2014)

Numeri di Chern fra topologia e geometria birazionale

(Work in progress with P. Cascini)

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Hirzebruch (1954)

Which linear combinations of Chern numbers are topologically invariant?

The main problem

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- $\dim X = 3$. By Hirzebruch-Riemann-Roch we have

$$\left|\frac{1}{24}c_1c_2\right| = |\chi(\mathcal{O}_X)| = |1-h^{1,0}+h^{2,0}-h^{3,0}| \leq 1+b_1+b_2+b_3.$$

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Question (Kotshick)

Does $K_X^3 = -c_1^3$ take only finitely many values on projective algebraic structures with the same underlying 6-manifold?

The volume

If X is a variety of dimension n, then the volume of X is defined as

$$\mathrm{vol}(X) := \limsup_{m \to +\infty} \frac{n! h^0(X, mK_X)}{m^n},$$

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Theorem 1

Let X be a smooth projective 3-fold of general type. Then

$$\mathrm{vol}(X) \leq 64(b_1(X) + b_3(X) + b_2(X)).$$

The volume takes finitely many values on projective algebraic structures of general type with the same underlying 6-manifold.

MMP

Let X be a smooth 3-fold of general type. Then there exists a sequence of birational maps

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m = Y$$

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Y is called a minimal model of X. Note that

$$\mathrm{vol}(K_Y) = K_Y^3.$$

Proof of Theorem 1 I

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- B.M.Y. inequality for minimal model of general type (Tian-Wang 2011):

$$K_Y^{n-2}.\left(K_Y^2 - 2\frac{n+1}{n}c_2(Y)\right) \leq 0 \ \Rightarrow \ K_Y^3 \leq \frac{8}{3}K_Y.c_2(Y)$$

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- Riemann-Roch for terminal 3-folds (Kawamata, Reid):

$$\chi(Y, \mathcal{O}_Y) = -\frac{1}{24}K_Y.c_2(Y) + \sum_{p\in\mathcal{B}(Y)} \frac{r(p)^2-1}{24r(p)}$$

Proof of Theorem 1 II

- Hence

$$\mathrm{vol}(X) = \mathrm{vol}(Y) = K_Y^3 \leq \frac{8}{3} K_Y . c_2(Y)$$
$$= \frac{8}{3}\left(-\chi(Y, \mathcal{O}_Y) + \sum_{p \in \mathcal{B}(Y)} \frac{r(p)^2 - 1}{24 r(p)}\right).$$

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- $-\chi(Y, \mathcal{O}_Y) = -\chi(X, \mathcal{O}_X) \leq b_1(X) + b_3(X)$.

Proof of Theorem 1 II

- Hence

$$\begin{aligned}\mathrm{vol}(X) = \mathrm{vol}(Y) = K_Y^3 &\leq \frac{8}{3} K_Y.c_2(Y) \\ &= \frac{8}{3}\left(-\chi(Y,\mathcal{O}_Y) + \sum_{p\in\mathcal{B}(Y)} \frac{r(p)^2-1}{24r(p)}\right).\end{aligned}$$

- $-\chi(Y,\mathcal{O}_Y) = -\chi(X,\mathcal{O}_X) \leq b_1(X) + b_3(X)$.
- Topological bound on the singularities of Y (Cascini-Zhang, 2012):

$$\sum_{p\in\mathcal{B}(Y)} \frac{r(p)^2-1}{r(p)} \leq 2b_2(X).$$

The main theorem

To any threefold X we can associate an integral cubic form $F_X \in \mathbb{Z}[x_1, \ldots, x_b]$, which comes from the trilinear intersection form

$$H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}.$$

Denote by Δ_{F_X} the discriminant of F_X.

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Theorem 2

Let X be a smooth 3-fold of general type. Assume that $\Delta_{F_X} \neq 0$ and that there is an MMP for X composed only by divisorial contractions to points and blow-downs to smooth curves. Then there exists a topological invariant D_X such that

$$|K_X^3| \leq D_X.$$

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  with a topological invariant of X.
- The number of steps in bounded by b_2. (With flips it is bounded by $2b_2$.)

A quick remark

- $Z = \mathbb{P}^3$ and C a smooth rational curve of degree d.
- $\pi : W \to Z$ be blow-up along C.
- $b_1(W) = b_3(W) = b_5(W) = 0$, $b_2(W) = b_4(W) = 2$.

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- $\pi : W \to Z$ be blow-up along C.
- $b_1(W) = b_3(W) = b_5(W) = 0$, $b_2(W) = b_4(W) = 2$.
- $K_W^3 = K_Z^3 - 2K_Z.C + 2 - 2g(C) = -62 + 8d$
- The Betti numbers are in general not enough to bound K_X^3.

Divisorial contractions to curves I

Let $f : W \to Z$ be a blow-down to a smooth curve with exceptional divisor E.

- $K_W = f^* K_Z + E$ and $H^2(W, \mathbb{Z}) \cong \mathbb{Z}[E] \bigoplus H^2(Z, \mathbb{Z})$.

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- Let $E_1, \dots, E_n$ be the pull-back of a basis of $H^2(Z, \mathbb{Z})$.
- $E.E_i.E_j = E_{i|E}.E_{j|E} = 0$ and $K_W^3 - K_Z^3 = -2E^3 + 6 - 6g(C)$.

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- $E.E_i.E_j = E_{i|E}.E_{j|E} = 0$ and $K_W^3 - K_Z^3 = -2E^3 + 6 - 6g(C)$.
- Let $x_0, \ldots, x_n$ be coordinates on $H^2(W,\mathbb{Z})$ with respect to $E, E_1, \ldots, E_n$. Then

$$F_W(x_0, \ldots, x_n) = ax_0^3 + 3x_0^2(\sum_{i=1}^{n} b_i x_i) + F_Z(x_1, \ldots, x_n),$$

where $a = E^3$ and $b_i \in \mathbb{Z}$.

An arithmetic result

Theorem 3

Let $F \in \mathbb{Z}[x_0, \dots, x_n]$ be a cubic form such that $\Delta_F \neq 0$. Then, modulo the action of $GL(\mathbb{Z}, n)$ on $(x_1, \dots, x_n)$, there are only finitely many triples $(a, (b_1, \dots, b_n), G)$ such that $a, b_i \in \mathbb{Z}$, $G(x_1, \dots, x_n)$ is a cubic form and F can be written as

$$F = ax_0^3 + (\sum b_i x_i)x_0^2 + G(x_1, \dots, x_n).$$

Moreover $\Delta_G \neq 0$.

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Define the Skansen number of X as

$S_X := \sup\{|a| : F_X$ may be written in reduced form w.r.t. $(a, b, G)\}$.

Divisorial contractions to curves II

Let $f : W \to Z$ be a blow-down to a smooth curve C of genus g.

- $\chi(W \setminus E) = \chi(Z \setminus C) \;\Rightarrow\; b_3(W) = b_3(Z) + 2g$.

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- The cubic form on Z is determined (up to finitely many possibilities) by the cubic form on W.

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- $K_W^3 - K_Z^3 = d^3 E^3$
- Using Kawakita's classification, we can prove that

$$0 < K_W^3 - K_Z^3 \leq 2^8 b_2^{2b_2},$$

where $b_2 = b_2(X)$.

Divisorial contractions to points II

$f : W \to Z$ divisorial contraction to a point.

- $H^2(W, \mathbb{Q}) \cong \mathbb{Q}[E] \bigoplus H^2(Z, \mathbb{Q})$, but $H^2(W, \mathbb{Z})/H^2(Z, \mathbb{Z})$ may have torsion.

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- This torsion depends on the singularities of W and Z.
- Admitting rational coefficients with bounded denominators we have something like

$$F_W(x_0, \dots, x_n) = ax_0^3 + F_Z(x_1, \dots, x_n).$$

Reduced forms

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2. To prove our result for binary and ternary cubics we need Siegel and Faltings theorems on the finiteness of integral and rational points on algebraic curves.

A basic case I

Let $F \in \mathbb{Z}[x, y, z]$ be a ternary cubic such that $\Delta \neq 0$. The algebra of the invariants of F (under the action of $SL(3, \mathbb{Z})$) is generated by two polynomials S and T in the coefficients of F and

$$\Delta = T^2 - 64S^3.$$

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$$F = Ax^3 + (B_1 y + C_1 z)x^2 + G(y,z).$$
3. We need to prove that there are only finitely many $a,b,c \in K$ such that
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6. Since $p_g(C) = 3$, by Faltings theorem we have only a finite number of K-rational points $[a, b, c, 1]$ on C.

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- What happens to the cubic form?