Direct images of pluricanonical bundles

Mihnea Popa

University of Illinois, Chicago

Torino
June 5, 2014

Vanishing, regularity, and Fujita-type statements

Joint work with Christian Schnell – arXiv:1405.6125.

- X smooth projective variety, $\dim_{\mathbf{C}} X = n$; L ample line bundle on X.

Vanishing, regularity, and Fujita-type statements

Joint work with Christian Schnell – arXiv:1405.6125.

- X smooth projective variety, $\dim_{\mathbf{C}} X = n$; L ample line bundle on X.
- Fujita Conjecture: $\omega_X \otimes L^{\otimes m}$ is globally generated for all $m \geq n+1$.

Vanishing, regularity, and Fujita-type statements

Joint work with Christian Schnell – arXiv:1405.6125.

- X smooth projective variety, $\dim_{\mathbf{C}} X = n$; L ample line bundle on X.
- Fujita Conjecture: $\omega_X \otimes L^{\otimes m}$ is globally generated for all $m \geq n+1$.
- Known only in dimension up to four (Reider, Ein-Lazarsfeld, Kawamata), but ok when L is very ample. More generally:

Vanishing, regularity, and Fujita-type statements

Joint work with Christian Schnell – arXiv:1405.6125.

- X smooth projective variety, $\dim_{\mathbf{C}} X = n$; L ample line bundle on X.
- Fujita Conjecture: $\omega_X \otimes L^{\otimes m}$ is globally generated for all $m \geq n+1$.
- Known only in dimension up to four (Reider, Ein-Lazarsfeld, Kawamata), but ok when L is very ample. More generally:

Proposition

- *$f : X \to Y$ morphism of projective varieties, X smooth,* $\dim Y = n$*.*
- *L ample and globally generated line bundle on Y. Then*

$$R^i f_* \omega_X \otimes L^{\otimes n+1}$$

is globally generated for all $i \geq 0$.

Vanishing, regularity, and Fujita-type statements

- Kodaira Vanishing: L ample $\implies H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$.

Vanishing, regularity, and Fujita-type statements

- Kodaira Vanishing: L ample $\implies H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$.

Theorem (Kollár Vanishing)

- *$f : X \to Y$ morphism of projective varieties, X smooth*
- *L ample line bundle on Y. Then*

$$H^j(Y, R^i f_* \omega_X \otimes L) = 0 \quad \text{for all } i \text{ and all } \ j > 0.$$

Vanishing, regularity, and Fujita-type statements

- Kodaira Vanishing: L ample $\implies H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$.

Theorem (Kollár Vanishing)

- *$f : X \to Y$ morphism of projective varieties, X smooth*
- *L ample line bundle on Y. Then*

$$H^j(Y, R^i f_* \omega_X \otimes L) = 0 \quad \text{for all } i \text{ and all } \ j > 0.$$

- $\mathcal{F} \in \mathrm{Coh}(Y)$ is 0-regular w.r.t. L ample and globally generated if

$$H^i(Y, \mathcal{F} \otimes L^{\otimes -i}) = 0 \ \text{ for all } i > 0.$$

Vanishing, regularity, and Fujita-type statements

- Kodaira Vanishing: L ample $\implies H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$.

Theorem (Kollár Vanishing)

- *$f : X \to Y$ morphism of projective varieties, X smooth*
- *L ample line bundle on Y. Then*

$$H^j(Y, R^i f_* \omega_X \otimes L) = 0 \quad \text{for all } i \text{ and all } \; j > 0.$$

- $\mathcal{F} \in \mathrm{Coh}(Y)$ is 0-regular w.r.t. L ample and globally generated if

$$H^i(Y, \mathcal{F} \otimes L^{\otimes -i}) = 0 \quad \text{for all } i > 0.$$

Theorem (Castelnuovo-Mumford Lemma)

$\mathcal{F}$ 0-regular sheaf on $Y \implies \mathcal{F}$ globally generated.

Vanishing, regularity, and Fujita-type statements

- Kodaira Vanishing: L ample $\Longrightarrow$ $H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$.

Theorem (Kollár Vanishing)

- *$f : X \to Y$ morphism of projective varieties, X smooth*
- *L ample line bundle on Y. Then*

$$H^j(Y, R^i f_* \omega_X \otimes L) = 0 \quad \text{for all } i \text{ and all } \; j > 0.$$

- $\mathcal{F} \in \mathrm{Coh}(Y)$ is 0-regular w.r.t. L ample and globally generated if

$$H^i(Y, \mathcal{F} \otimes L^{\otimes -i}) = 0 \;\; \text{for all } i > 0.$$

Theorem (Castelnuovo-Mumford Lemma)

$\mathcal{F}$ 0-regular sheaf on Y $\Longrightarrow$ $\mathcal{F}$ globally generated.

- Kollár Vanishing $\Longrightarrow$ $R^i f_* \omega_X \otimes L^{\otimes n+1}$ is 0-regular.

Powers of canonical bundles

- Question: How about powers $\omega_X^{\otimes k}$, $k \geq 2$?

Powers of canonical bundles

- Question: How about powers $\omega_X^{\otimes k}$, $k \geq 2$?
- Motivation: Say X smooth projective, L ample on $X \overset{MMP}{\Longrightarrow}$ $\omega_X \otimes L^{\otimes n+1}$ is nef.

Powers of canonical bundles

- Question: How about powers $\omega_X^{\otimes k}$, $k \geq 2$?
- Motivation: Say X smooth projective, L ample on $X \overset{MMP}{\Longrightarrow}$ $\omega_X \otimes L^{\otimes n+1}$ is nef.
- By Kodaira Vanishing, this implies

$$H^i(X, \omega_X^{\otimes k} \otimes L^{\otimes k(n+1)-n}) = 0 \quad \text{for all} \quad i > 0.$$

since

$$kK_X + \big(k(n+1) - n\big)L = K_X + (k-1)\big(K_X + (n+1)L\big) + L.$$

Powers of canonical bundles

- Question: How about powers $\omega_X^{\otimes k}$, $k \geq 2$?
- Motivation: Say X smooth projective, L ample on $X \overset{MMP}{\Longrightarrow}$ $\omega_X \otimes L^{\otimes n+1}$ is nef.
- By Kodaira Vanishing, this implies

$$H^i(X, \omega_X^{\otimes k} \otimes L^{\otimes k(n+1)-n}) = 0 \ \text{ for all } \ i > 0.$$

since

$$kK_X + \big(k(n+1) - n\big)L = K_X + (k-1)\big(K_X + (n+1)L\big) + L.$$

- This is the type of effective vanishing statement we would like for $f_*\omega_X^{\otimes k}$.

Powers of canonical bundles

- How about effective global generation?

Powers of canonical bundles

- How about effective global generation?

Conjecture

- *$f : X \to Y$ morphism of smooth projective varieties,* $\dim Y = n$
- *L ample on Y, $k \geq 1$. Then*

$$f_* \omega_X^{\otimes k} \otimes L^{\otimes m}$$

is globally generated for $m \geq k(n+1)$.

Powers of canonical bundles

- How about effective global generation?

Conjecture

- *$f : X \to Y$ morphism of smooth projective varieties,* $\dim Y = n$
- *L ample on Y, $k \geq 1$. Then*

$$f_* \omega_X^{\otimes k} \otimes L^{\otimes m}$$

is globally generated for $m \geq k(n+1)$.

- Would follow immediately from Fujita when $f = \mathrm{Id}$.

Powers of canonical bundles

- How about effective global generation?

Conjecture

- *$f : X \to Y$ morphism of smooth projective varieties,* $\dim Y = n$
- *L ample on Y, $k \geq 1$. Then*

$$f_* \omega_X^{\otimes k} \otimes L^{\otimes m}$$

is globally generated for $m \geq k(n+1)$.

- Would follow immediately from Fujita when $f = \mathrm{Id}$.
- When $k = 1$, proved by Kawamata in dimension up to 4 when the branch locus of f is an SNC divisor.

Example: curves

- The conjecture holds when $Y = C =$ smooth projective curve; very special methods though.

Example: curves

- The conjecture holds when $Y = C =$ smooth projective curve; very special methods though.
- Say $f : X \to C$ surjective, C of genus g. Write

$$f_*\omega_X^{\otimes k} \otimes L^{\otimes m} \simeq f_*\omega_{X/C}^{\otimes k} \otimes \omega_C^{\otimes k} \otimes L^{\otimes m}.$$

The statement follows from the following facts:

Example: curves

- The conjecture holds when $Y = C =$ smooth projective curve; very special methods though.
- Say $f : X \to C$ surjective, C of genus g. Write

$$f_*\omega_X^{\otimes k} \otimes L^{\otimes m} \simeq f_*\omega_{X/C}^{\otimes k} \otimes \omega_C^{\otimes k} \otimes L^{\otimes m}.$$

  The statement follows from the following facts:
- Viehweg: $f_*\omega_{X/C}^{\otimes k}$ is a nef vector bundle on C for all k.

Example: curves

- The conjecture holds when $Y = C =$ smooth projective curve; very special methods though.
- Say $f : X \to C$ surjective, C of genus g. Write

$$f_*\omega_X^{\otimes k} \otimes L^{\otimes m} \simeq f_*\omega_{X/C}^{\otimes k} \otimes \omega_C^{\otimes k} \otimes L^{\otimes m}.$$

The statement follows from the following facts:

- Viehweg: $f_*\omega_{X/C}^{\otimes k}$ is a nef vector bundle on C for all k.
- Lemma: E nef vector bundle, L line bundle of degree $\geq 2g \implies E \otimes L$ globally generated.

Uses:

- Hartshorne: A vector bundle E on C is nef $\iff$ E has no line bundle quotients of negative degree.

Extension of Kollár's result for $i = 0$

Theorem

- $f : X \to Y$ *morphism of projective varieties,* X *smooth,* $\dim Y = n$.
- L *ample and globally generated line bundle on* Y, $k \geq 1$. *Then*

$$f_* \omega_X^{\otimes k} \otimes L^{\otimes m}$$

is 0*-regular, and therefore globally generated, for* $m \geq k(n+1)$.

Extension of Kollár's result for $i = 0$

Theorem

- *$f : X \to Y$ morphism of projective varieties, X smooth,* $\dim Y = n$*.*
- *L ample and globally generated line bundle on Y, $k \geq 1$. Then*

$$f_*\omega_X^{\otimes k} \otimes L^{\otimes m}$$

is 0-regular, and therefore globally generated, for $m \geq k(n+1)$.

- Effectivity of the result is crucial in applications; explained later. Also, equally useful:

Extension of Kollár's result for $i = 0$

Theorem

- *$f : X \to Y$ morphism of projective varieties, X smooth,* $\dim Y = n$.
- *L ample and globally generated line bundle on Y, $k \geq 1$. Then*

$$f_* \omega_X^{\otimes k} \otimes L^{\otimes m}$$

is 0-regular, and therefore globally generated, for $m \geq k(n+1)$.

- Effectivity of the result is crucial in applications; explained later. Also, equally useful:

Variant

The same holds if f is a fibration (i.e. its fibers are irreducible) and ω_X is replaced by $\omega_X \otimes M$, where M is a nef and f-big line bundle.

Extension to log-canonical pairs

- Important to extend to pairs; recall that (X, Δ) is log-canonical if $K_X + \Delta$ is $\mathbf{Q}$-Cartier and on a log-resolution $\mu : \tilde{X} \to X$ we have

$$K_{\tilde{X}} - \mu^*(K_X + \Delta) = P - N$$

with: • P, N effective, P exceptional, no common components.
• $N = \sum a_i E_i$ with all $a_i \leq 1$.

Extension to log-canonical pairs

- Important to extend to pairs; recall that (X, Δ) is log-canonical if $K_X + \Delta$ is $\mathbf{Q}$-Cartier and on a log-resolution $\mu : \tilde{X} \to X$ we have

$$K_{\tilde{X}} - \mu^*(K_X + \Delta) = P - N$$

with: • P, N effective, P exceptional, no common components.
• $N = \sum a_i E_i$ with all $a_i \leq 1$.

- Extension of Kollár vanishing:

Theorem (Ambro-Fujino Vanishing)

- *Same setting; let (X, Δ) be a log-canonical pair such that Δ is a $\mathbf{Q}$-divisor with SNC support*
- *B line bundle on X such that $B \sim_{\mathbf{Q}} K_X + \Delta + f^*H$, with H ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on Y. Then*

$$H^j(Y, R^i f_* B) = 0 \text{ for all } i \text{ and all } j > 0.$$

- The main technical result is a vanishing theorem partially extending Ambro-Fujino vanishing in the case $i = 0$.

- The main technical result is a vanishing theorem partially extending Ambro-Fujino vanishing in the case $i = 0$.

Theorem

- *$f : X \to Y$ morphism of projective varieties, X normal, $\dim Y = n$.*
- *(X, Δ) log-canonical $\mathbf{Q}$-pair on X.*
- *B line bundle on X such that $B \sim_{\mathbf{Q}} k(K_X + \Delta + f^*H)$ for some $k \geq 1$, H ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on Y.*
- *L ample and globally generated line bundle on Y. Then:*

$$H^i(Y, f_*B \otimes L^{\otimes m}) = 0 \ \text{ for all } i > 0 \ \text{ and } \ m \geq (k-1)(n+1-t) - t + 1,$$

where $t := \sup \{ s \in \mathbf{Q} \mid H - sL \text{ is ample} \}$.

- The main technical result is a vanishing theorem partially extending Ambro-Fujino vanishing in the case $i = 0$.

Theorem

- *$f: X \to Y$ morphism of projective varieties, X normal,* $\dim Y = n$.
- *(X, Δ) log-canonical $\mathbf{Q}$-pair on X.*
- *B line bundle on X such that $B \sim_{\mathbf{Q}} k(K_X + \Delta + f^*H)$ for some $k \geq 1$, H ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on Y.*
- *L ample and globally generated line bundle on Y. Then:*

$$H^i(Y, f_*B \otimes L^{\otimes m}) = 0 \text{ for all } i > 0 \text{ and } m \geq (k-1)(n+1-t) - t + 1,$$

where $t := \sup \{s \in \mathbf{Q} \mid H - sL \text{ is ample}\}$.

- Special case: If $k(K_X + \Delta)$ is Cartier, can take $H = L$ and $t = 1$, so:

$$H^i(Y, f_*\mathcal{O}_X(k(K_X + \Delta)) \otimes L^{\otimes m}) = 0 \text{ for } m \geq k(n+1) - n.$$

Main idea

- Theorem implies the main global generation result (and an extension to log-canonical pairs) via 0-regularity.

Main idea

- Theorem implies the main global generation result (and an extension to log-canonical pairs) via 0-regularity.
- Idea of proof: a combination of Viehweg-style methods towards weak positivity and the use of Kollár and Ambro-Fujino vanishing. Recall:

Main idea

- Theorem implies the main global generation result (and an extension to log-canonical pairs) via 0-regularity.
- Idea of proof: a combination of Viehweg-style methods towards weak positivity and the use of Kollár and Ambro-Fujino vanishing. Recall:
- $B \sim_{\mathbf{Q}} k(K_X + \Delta + f^*H), \quad k \geq 1, \ (X, \Delta)$ log-canonical, $\ f : X \to Y$.

Main idea

- Theorem implies the main global generation result (and an extension to log-canonical pairs) via 0-regularity.
- Idea of proof: a combination of Viehweg-style methods towards weak positivity and the use of Kollár and Ambro-Fujino vanishing. Recall:

- $B \sim_{\mathbf{Q}} k(K_X + \Delta + f^*H)$, $k \geq 1$, (X, Δ) log-canonical, $f : X \to Y$.
- Consider adjunction morphism

$$f^* f_* B \to B$$

Log-resolution arguments $\Longrightarrow$ reduce to X smooth, the image is $B \otimes \mathcal{O}_X(-E)$, and $E + \Delta$ divisor with SNC support.

Main idea

- Theorem implies the main global generation result (and an extension to log-canonical pairs) via 0-regularity.
- Idea of proof: a combination of Viehweg-style methods towards weak positivity and the use of Kollár and Ambro-Fujino vanishing. Recall:

- $B \sim_{\mathbf{Q}} k(K_X + \Delta + f^*H)$, $k \geq 1$, (X, Δ) log-canonical, $f : X \to Y$.
- Consider adjunction morphism

$$f^* f_* B \to B$$

Log-resolution arguments $\Longrightarrow$ reduce to X smooth, the image is $B \otimes \mathcal{O}_X(-E)$, and $E + \Delta$ divisor with SNC support.

- Consider smallest $p \geq 0$ such that $f_* B \otimes L^{\otimes p}$ globally generated

Idea of proof:

- Obtain

$$B + pf^*L \sim k(K_X + \Delta + f^*H) + pf^*L \sim D + E$$

with D smooth and transverse to the support of $E + \Delta$.

Idea of proof:

- Obtain

$$B + pf^*L \sim k(K_X + \Delta + f^*H) + pf^*L \sim D + E$$

with D smooth and transverse to the support of $E + \Delta$.

- Interesting reduction leads to

$$B - E' + mf^*L \sim_{\mathbf{Q}} K_X + \Delta' + f^*H',$$

where Δ' is log-canonical with SNC support, E' is contained in the relative base locus of B, and

$$H' \text{ ample} \iff m + t - \frac{k-1}{k} \cdot p > 0.$$

Idea of proof:

- Obtain

$$B + pf^*L \sim k(K_X + \Delta + f^*H) + pf^*L \sim D + E$$

with D smooth and transverse to the support of $E + \Delta$.

- Interesting reduction leads to

$$B - E' + mf^*L \sim_{\mathbf{Q}} K_X + \Delta' + f^*H',$$

where Δ' is log-canonical with SNC support, E' is contained in the relative base locus of B, and

$$H' \text{ ample} \iff m + t - \frac{k-1}{k} \cdot p > 0.$$

- Ambro-Fujino Vanishing then implies in this range:

$$H^i(Y, f_*B \otimes L^{\otimes m}) = 0 \text{ for all } i > 0.$$

Idea of proof:

- Get that $f_*B \otimes L^{\otimes m}$ is 0-regular, hence globally generated, for

$$m > \frac{k-1}{k} \cdot p - t + n.$$

Idea of proof:

- Get that $f_*B \otimes L^{\otimes m}$ is 0-regular, hence globally generated, for

$$m > \frac{k-1}{k} \cdot p - t + n.$$

- But we've chosen p minimal with this same property, which then implies all the effective inequalities we're looking for:

$$m \leq k(n+1) - n \ \text{ and } \ p \leq k(n+1).$$

Applications

The effective statements above govern different types of applications:

Applications

The effective statements above govern different types of applications:

- Vanishing theorems for direct images of pluricanonical bundles.

Applications

The effective statements above govern different types of applications:

- Vanishing theorems for direct images of pluricanonical bundles.
- (Effective) weak positivity, and subadditivity of Iitaka dimension.

Applications

The effective statements above govern different types of applications:

- Vanishing theorems for direct images of pluricanonical bundles.
- (Effective) weak positivity, and subadditivity of Iitaka dimension.
- Generic vanishing for direct images of pluricanonical bundles.

Vanishing theorems

- We have seen that the key result is a partial extension of Ambro-Fujino. It implies:

Vanishing theorems

- We have seen that the key result is a partial extension of Ambro-Fujino. It implies:

Corollary

- *$f : X \to Y$ morphism of projective varieties, X smooth,* $\dim Y = n$
- *L ample and globally generated on Y, $k \geq 1$. Then*

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes L^{\otimes m}) = 0 \text{ for all } i > 0 \text{ and } m \geq k(n+1) - n.$$

Vanishing theorems

- We have seen that the key result is a partial extension of Ambro-Fujino. It implies:

Corollary

- *$f : X \to Y$ morphism of projective varieties, X smooth,* $\dim Y = n$
- *L ample and globally generated on Y, $k \geq 1$. Then*

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes L^{\otimes m}) = 0 \;\text{ for all } i > 0 \;\text{ and }\; m \geq k(n+1) - n.$$

- Relative Fujita: Case $k = 1$ of the main conjecture says that $f_*\omega_X \otimes L^{\otimes m}$ is globally generated for $m \geq n+1$, L ample.

Vanishing theorems

- We have seen that the key result is a partial extension of Ambro-Fujino. It implies:

Corollary

- *$f : X \to Y$ morphism of projective varieties, X smooth,* $\dim Y = n$
- *L ample and globally generated on Y, $k \geq 1$. Then*

$$H^i(Y, f_*\omega_X^{\otimes k} \otimes L^{\otimes m}) = 0 \text{ for all } i > 0 \text{ and } m \geq k(n+1) - n.$$

- Relative Fujita: Case $k = 1$ of the main conjecture says that $f_*\omega_X \otimes L^{\otimes m}$ is globally generated for $m \geq n + 1$, L ample.

Corollary

If Relative Fujita holds, then the Corollary above holds with L only assumed to be ample.

Weak positivity

- Fundamental notion introduced by Viehweg:

  Definition: A torsion-free $\mathcal{F}$ on X projective is weakly positive on a non-empty open set $U \subseteq X$ if for every ample A on X and $a \in \mathbf{N}$, the sheaf $S^{[ab]}\mathcal{F} \otimes A^{\otimes b}$ is generated by global sections over U for $b \gg 0$. ($S^{[p]}\mathcal{F} :=$ reflexive hull of $S^p\mathcal{F}$.)

Weak positivity

- Fundamental notion introduced by Viehweg:

  Definition: A torsion-free $\mathcal{F}$ on X projective is weakly positive on a non-empty open set $U \subseteq X$ if for every ample A on X and $a \in \mathbf{N}$, the sheaf $S^{[ab]}\mathcal{F} \otimes A^{\otimes b}$ is generated by global sections over U for $b \gg 0$. ($S^{[p]}\mathcal{F} :=$ reflexive hull of $S^p\mathcal{F}$.)

- Intuition: higher rank generalization of pseudo-effective line bundles; very roughly, there exists a fixed line bundle A such that $\mathcal{F}^{\otimes a} \otimes A$ is globally generated over a fixed open set U, for all $a \geq 0$.

Weak positivity

- Fundamental notion introduced by Viehweg:

  Definition: A torsion-free $\mathcal{F}$ on X projective is weakly positive on a non-empty open set $U \subseteq X$ if for every ample A on X and $a \in \mathbf{N}$, the sheaf $S^{[ab]}\mathcal{F} \otimes A^{\otimes b}$ is generated by global sections over U for $b \gg 0$. ($S^{[p]}\mathcal{F} :=$ reflexive hull of $S^p\mathcal{F}$.)

- Intuition: higher rank generalization of pseudo-effective line bundles; very roughly, there exists a fixed line bundle A such that $\mathcal{F}^{\otimes a} \otimes A$ is globally generated over a fixed open set U, for all $a \geq 0$.

Theorem (Viehweg)

If $f : X \to Y$ is a surjective morphism of smooth projective varieties, then $f_*\omega_{X/Y}^{\otimes k}$ is weakly positive for every $k \geq 1$.

Weak positivity

- Case $k = 1$ typically uses Hodge theory (Fujita, Kawamata) – however Kollár provided effective version using vanishing theorems.

Weak positivity

- Case $k = 1$ typically uses Hodge theory (Fujita, Kawamata) – however Kollár provided effective version using vanishing theorems.
- Results above allow us to do the same for $k > 1$.

Weak positivity

- Case $k = 1$ typically uses Hodge theory (Fujita, Kawamata) – however Kollár provided effective version using vanishing theorems.
- Results above allow us to do the same for $k > 1$.

Theorem

- *$f : X \to Y$ surjective "mild" morphism of smooth projective varieties,*
- *L ample and globally generated on Y, $A := \omega_Y \otimes L^{\otimes n+1}$, $s \geq 1$. Then*

$$f_*(\omega_{X/Y}^{\otimes k})^{[\otimes s]} \otimes A^{\otimes k}$$

is globally generated on fixed open set U containing the smooth locus of f.

Weak positivity

- Case $k = 1$ typically uses Hodge theory (Fujita, Kawamata) – however Kollár provided effective version using vanishing theorems.
- Results above allow us to do the same for $k > 1$.

Theorem

- *$f : X \to Y$ surjective "mild" morphism of smooth projective varieties,*
- *L ample and globally generated on Y, $A := \omega_Y \otimes L^{\otimes n+1}$, $s \geq 1$. Then*

$$f_*(\omega_{X/Y}^{\otimes k})^{[\otimes s]} \otimes A^{\otimes k}$$

is globally generated on fixed open set U containing the smooth locus of f.

- Implies Viehweg's result via semistable reduction.

Weak positivity

- Another advantage: vanishing theorems method extends the picture to adjoint bundles.

Weak positivity

- Another advantage: vanishing theorems method extends the picture to adjoint bundles.

Theorem

$f : X \to Y$ fibration between smooth projective varieties, M nef and f-big line bundle on X $\implies$ $f_*(\omega_{X/Y} \otimes M)^{\otimes k}$ is weakly positive for every $k \geq 1$.

Weak positivity

- Another advantage: vanishing theorems method extends the picture to adjoint bundles.

Theorem

$f : X \to Y$ fibration between smooth projective varieties, M nef and f-big line bundle on X $\implies$ $f_*(\omega_{X/Y} \otimes M)^{\otimes k}$ is weakly positive for every $k \geq 1$.

- Using argument of Viehweg, get subadditivity of Iitaka dimension over a base of general type:

Corollary

In the situation of the Theorem, denote by F the general fiber of f, and by M_F the restriction of M to F. If Y is of general type, then

$$\kappa(\omega_X \otimes M) = \kappa(\omega_F \otimes M_F) + \dim Y.$$

Generic vanishing

- Definition: A abelian variety, $\mathcal{F} \in \mathrm{Coh}(A) \implies \mathcal{F}$ is a *GV*-sheaf if for all $i \geq 0$:

$$\mathrm{codim}_{\mathrm{Pic}^0(A)}\{\alpha \in \mathrm{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes \alpha) \neq 0\} \geq i$$

Generic vanishing

- Definition: A abelian variety, $\mathcal{F} \in \mathrm{Coh}(A) \implies \mathcal{F}$ is a GV-sheaf if for all $i \geq 0$:

$$\mathrm{codim}_{\mathrm{Pic}^0(A)}\{\alpha \in \mathrm{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes \alpha) \neq 0\} \geq i$$

- Generic vanishing theorems address this property, especially for ω_X; crucial for studying the birational geometry of X with $b_1(X) \neq 0$.

Generic vanishing

- Definition: A abelian variety, $\mathcal{F} \in \mathrm{Coh}(A) \implies \mathcal{F}$ is a GV-sheaf if for all $i \geq 0$:

$$\mathrm{codim}_{\mathrm{Pic}^0(A)}\{\alpha \in \mathrm{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes \alpha) \neq 0\} \geq i$$

- Generic vanishing theorems address this property, especially for ω_X; crucial for studying the birational geometry of X with $b_1(X) \neq 0$.

- Green-Lazarsfeld: If $f : X \to A$ is generically finite onto its image, then $f_*\omega_X$ is a GV-sheaf.

Generic vanishing

- Definition: A abelian variety, $\mathcal{F} \in \mathrm{Coh}(A) \implies \mathcal{F}$ is a GV-sheaf if for all $i \geq 0$:

$$\mathrm{codim}_{\mathrm{Pic}^0(A)}\{\alpha \in \mathrm{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes \alpha) \neq 0\} \geq i$$

- Generic vanishing theorems address this property, especially for ω_X; crucial for studying the birational geometry of X with $b_1(X) \neq 0$.
- Green-Lazarsfeld: If $f : X \to A$ is generically finite onto its image, then $f_*\omega_X$ is a GV-sheaf.

  Statement in fact stronger, but anyway generalized as follows:
- Hacon: If $f : X \to A$ arbitrary morphism, then $R^i f_*\omega_X$ is a GV-sheaf, for all i.

Generic vanishing

Theorem

Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. Then $f_*\omega_X^{\otimes k}$ is a GV-sheaf for every $k \geq 1$.

Generic vanishing

Theorem

Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. Then $f_*\omega_X^{\otimes k}$ is a GV-sheaf for every $k \geq 1$.

- Idea: Depends on the fact that via pullback by multiplication maps

$$\cdot m : A \longrightarrow A$$

$f_*\omega_X^{\otimes k}$ remains of the same form, while $(\cdot m)^* L \equiv L^{\otimes m^2}$.

Generic vanishing

Theorem

Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. Then $f_*\omega_X^{\otimes k}$ is a GV-sheaf for every $k \geq 1$.

- Idea: Depends on the fact that via pullback by multiplication maps

$$\cdot m : A \longrightarrow A$$

  $f_*\omega_X^{\otimes k}$ remains of the same form, while $(\cdot m)^* L \equiv L^{\otimes m^2}$.

- For $m \gg 0$, apply the effective vanishing theorems discussed above + criterion of Hacon.

Higher direct images?

- The original statements for $k = 1$ (e.g. Kollár or Ambro-Fujino vanishing, Hacon's generic vanishing) hold for higher direct images as well. However, the Viehweg-style methods do not.

Higher direct images?

- The original statements for $k = 1$ (e.g. Kollár or Ambro-Fujino vanishing, Hacon's generic vanishing) hold for higher direct images as well. However, the Viehweg-style methods do not.
- Question: Are there analogues of these effective results for $R^i f_* \omega_X^{\otimes k}$ with $i > 0$?

Higher direct images?

- The original statements for $k = 1$ (e.g. Kollár or Ambro-Fujino vanishing, Hacon's generic vanishing) hold for higher direct images as well. However, the Viehweg-style methods do not.
- Question: Are there analogues of these effective results for $R^i f_* \omega_X^{\otimes k}$ with $i > 0$?

  For instance, for all i and k:
- Is $R^i f_* \omega_X^{\otimes k} \otimes L^{k(n+1)}$ globally generated?

Higher direct images?

- The original statements for $k = 1$ (e.g. Kollár or Ambro-Fujino vanishing, Hacon's generic vanishing) hold for higher direct images as well. However, the Viehweg-style methods do not.
- Question: Are there analogues of these effective results for $R^i f_* \omega_X^{\otimes k}$ with $i > 0$?

  For instance, for all i and k:

- Is $R^i f_* \omega_X^{\otimes k} \otimes L^{k(n+1)}$ globally generated?
- Is $R^i f_* \omega_X^{\otimes k}$ a GV-sheaf?
- etc...

Higher direct images?

- The original statements for $k = 1$ (e.g. Kollár or Ambro-Fujino vanishing, Hacon's generic vanishing) hold for higher direct images as well. However, the Viehweg-style methods do not.

- Question: Are there analogues of these effective results for $R^i f_* \omega_X^{\otimes k}$ with $i > 0$?

  For instance, for all i and k:

- Is $R^i f_* \omega_X^{\otimes k} \otimes L^{k(n+1)}$ globally generated?
- Is $R^i f_* \omega_X^{\otimes k}$ a GV-sheaf?
- etc...

  No obvious reason why these shouldn't hold, but would require an interesting new idea!

Thank you! Grazie!