Block-Göttsche invariants from wall-crossing

Sara Angela Filippini

Università di Zurigo

6 giugno 2014

Outline

GW theory of GPS

Quiver representations

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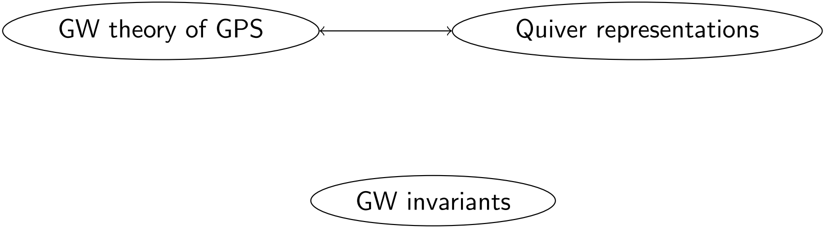

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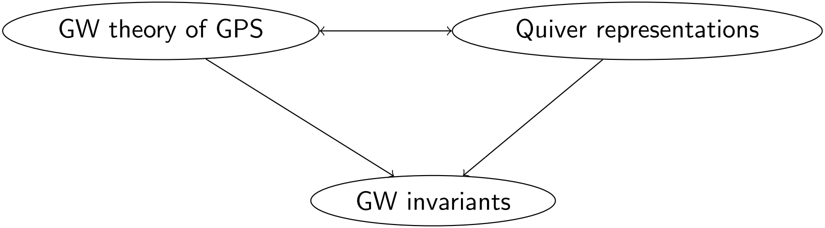

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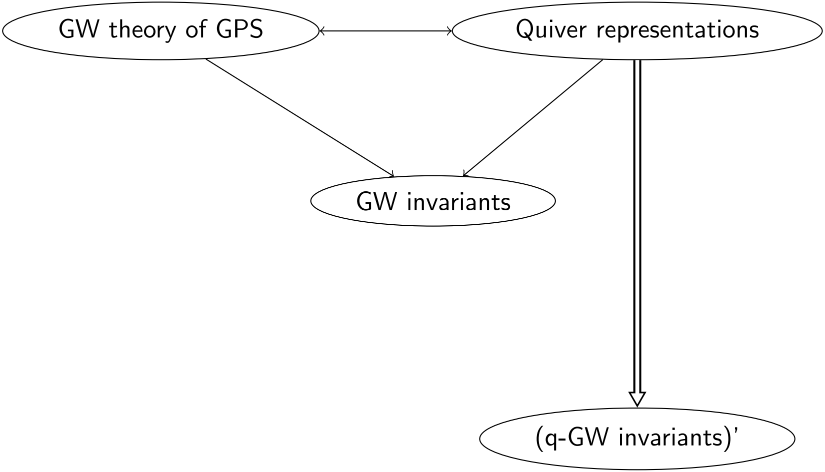
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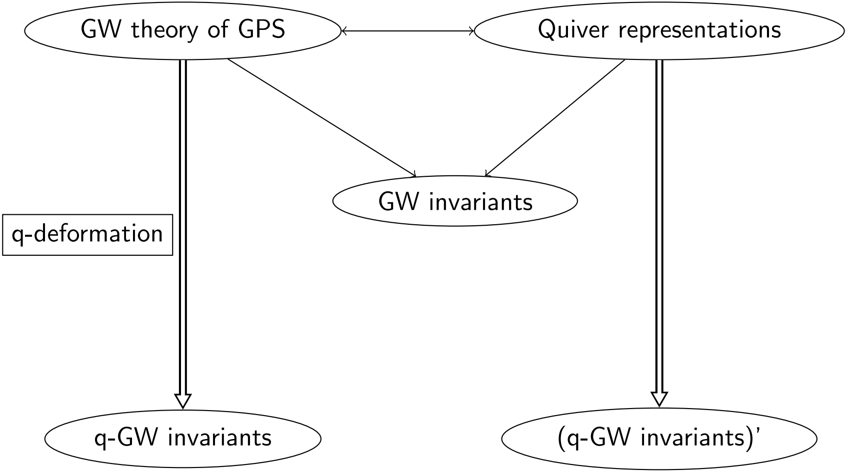
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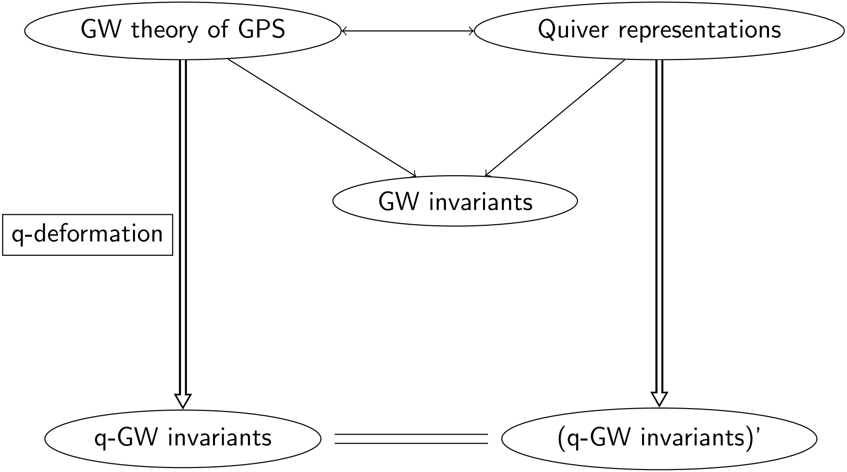
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The GW theory on weighted projective planes

- $\mathbb{P}(a, b, 1) = \mathbb{C}^3 \backslash \{0, 0, 0\} / (\mathbb{C}^*_{a,b,1}) =$ weighted projective plane
- Assume: $gcd(a, b) = 1$.

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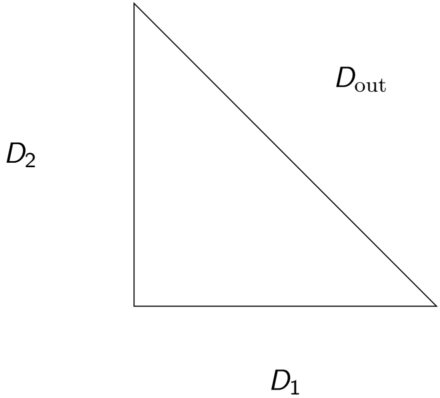

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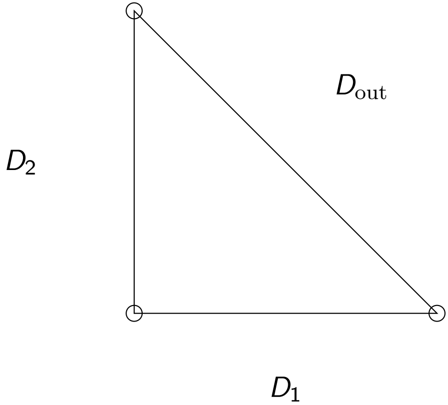

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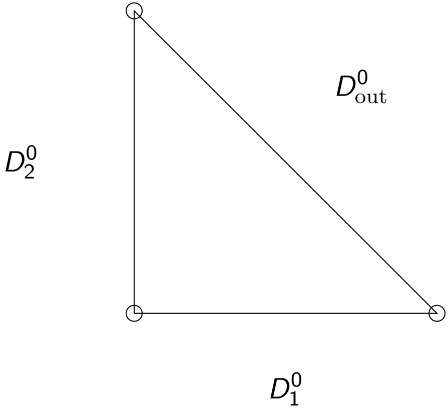

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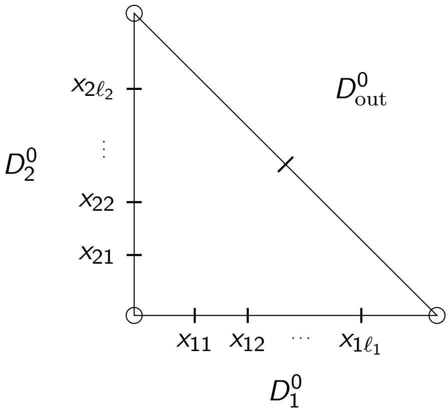

Gromov–Witten invariants

Fix *ordered* partitions $\mathbf{P}_1, \mathbf{P}_2$; $\mathbf{P}_i = (p_{ij})$, $|\mathbf{P}_i| = \sum_j p_{ij}$,

such that $\quad \operatorname{len} \mathbf{P}_i = \ell_i$.

Suppose

$$|\mathbf{P}_1| = ka, \quad |\mathbf{P}_2| = kb. \qquad\qquad (\Rightarrow gcd(|\mathbf{P}_1|, |\mathbf{P}_2|) = k)$$

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Then one can define GW invariants

$$N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] = \sharp^{vir} \left\{ \begin{array}{l} \text{rational curves intersecting the distinct} \\ \text{fixed } \ell_i \text{ points of } D_i^0 \text{ with multiplicities} \\ \text{given by the } p_{i,j} \text{ for } i = 1, 2, \\ \text{and being tangent to } D_{out}^0 \text{ of order } k \end{array} \right\}$$

Examples

(a) $N_{(1,3)}[(1, 1+1+1)] = 1$ given by

$$(u : v) \mapsto \left(u : -\frac{y_1}{x_1 x_2 x_3}(u - x_1 v)(u - x_2 v)(u - x_3 v) : v\right)$$

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(b) $N_{(1,1)}[(1+1, 1+1)] = 2$ given by

$$(u:v) \mapsto (u(u-v) : (u-2v)(u-4v) : v^2)$$

$$(u:v) \mapsto (u(u - \frac{5}{\sqrt{3}}v) : -(u - 2\sqrt{3}v)(u + \frac{4}{\sqrt{3}}v) : v^2)$$

Conjectural BPS structure

Define a series

$$N_{\mathbb{P}(a,b,1)} := \sum_{k=1}^{\infty} N_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)]\tau^k$$

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$$N_{\mathbb{P}(a,b,1)} := \sum_{k=1}^{\infty} n_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)] \sum_{d=1}^{\infty} \frac{1}{d^2} \binom{d(k-1)-1}{d-1} \tau^{dk}$$

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The $n_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)]$ are the BPS invariants underlying the GW invariants $N_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)]$.

Conjecture (GPS)

$$n_{(a,b)}[(k\mathbf{P}_1, k\mathbf{P}_2)] \in \mathbb{Z} \qquad \textit{for every } a, b, k, \mathbf{P}_1, \mathbf{P}_2.$$

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This is true for $N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)]$ in the coprime case!

Reineke–Weist Theorem

Theorem (Reineke–Weist)

If $\gcd(|\mathbf{P}_1|, |\mathbf{P}_2|) = 1$*, then*

$$N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] = \chi \underbrace{(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))}_{\substack{\textit{moduli space of stable representations} \\ \textit{of complete bipartite quiver}}}$$

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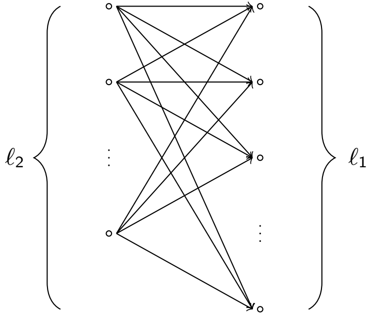

$K = K(\ell_1, \ell_2)$ quiver with set of vertices $\mathcal{Q}_0 = i_1, \ldots, i_{\ell_1}, j_1, ..., j_{\ell_2}$, and one arrow from each vertex j to each vertex i

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$$\begin{aligned} \widehat{N}'[(\mathbf{P}_1, \mathbf{P}_2)] &= \widehat{P}(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q) \\ &:= q^{-\frac{1}{2}\dim \mathcal{M}(\mathbf{P}_1, \mathbf{P}_2)} P(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q), \end{aligned}$$

where $\widehat{P}(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q)$ is the symmetrized Poincaré polynomial.

Tropical vertex group

Fix integers a, b and a function $f_{(a,b)} \in \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$ of the form

$$f_{(a,b)} = 1 + t\, x^a y^b \underbrace{g(x^a y^b, t)}_{g \in \mathbb{C}[z][[t]]}$$

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Define $\theta_{(a,b),f_{(a,b)}} \in \mathrm{Aut}_{\mathbb{C}[[t]]}\, \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$ by

$$\begin{cases} \theta_{(a,b),f_{(a,b)}}(x) = x\, f_{(a,b)}^{-b}, \\ \theta_{(a,b),f_{(a,b)}}(y) = y\, f_{(a,b)}^{a}. \end{cases}$$

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Definition (KS, GS)

The tropical vertex group $\mathbb{V} \subset \mathrm{Aut}_{\mathbb{C}[[t]]} \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$ *is the* (t)*-adic completion of the subgroup generated by all* $\theta_{(a,b), f_{(a,b)}}$.

Remark. Elements of $\mathbb{V}$ are formal 1-parameter families of holomorphic symplectomorphisms of $\mathbb{C}^* \times \mathbb{C}^*$:
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Example

Fix $\ell_1, \ell_2 \in \mathbb{N}$.

$$\begin{cases} \theta_{(1,0),(1+tx)^{\ell_1}}(x) & = & x, \\ \theta_{(1,0),(1+tx)^{\ell_1}}(y) & = & y(1+tx)^{\ell_1}. \end{cases}$$

$$\begin{cases} \theta_{(0,1),(1+ty)^{\ell_2}}(x) & = & x(1+ty)^{-\ell_2}, \\ \theta_{(0,1),(1+ty)^{\ell_2}}(y) & = & y. \end{cases}$$

Basic question: compute commutators in $\mathbb{V}$. More precisely, compute

$$[\theta_{(a,b),f}, \theta_{(a',b'),f'}] = \theta^{-1}_{(a',b'),f'}\theta_{(a,b),f}\theta_{(a',b'),f'}\theta^{-1}_{(a,b),f}$$

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Suppose that $a, b, a', b' \geqslant 0$, and that $\mu(a,b) \leqslant \mu(a',b')$
$((a,b)$ follows (a',b') in clockwise order).
Then $\exists!$ collection $a'', b'' > 0$, and attached $f_{(a'',b'')}$ such that

$$[\theta_{(a,b),f}, \theta_{(a',b'),f'}] = \underbrace{\overrightarrow{\prod_{(a'',b'')}} \theta_{(a'',b''),f_{(a'',b'')}}}_{\substack{\text{decreasing slopes of rays} \\ \text{(from L to R)}}},$$

with $gcd(a'', b'') = 1$.

Example

For $\ell_1 = \ell_2 = 2$ a closed formula is known:

$$[\theta_{(1,0),(1+tx)^2}, \theta_{(0,1),(1+ty)^2}] = \prod_{k}^{\rightarrow} \theta_{(k,k+1),f_{(k,k+1)}} \cdot \theta_{(1,1),f_{(1,1)}} \cdot \theta_{(k+1,k),f_{(k+1,k)}},$$

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where

$$\begin{cases} f_{1,1} &= (1-t^2xy)^{-4} \\ f_{k,k+1} &= (1+t^{2k+1}x^k y^{k+1})^2 \\ f_{k+1,k} &= (1+t^{2k+1}x^{k+1}y^k)^2. \end{cases}$$

For now we restrict to the simplest case:

$$[\theta_{(1,0),(1+tx)^{\ell_1}}, \theta_{(0,1),(1+ty)^{\ell_2}}] = \prod_{(a,b)}^{\longrightarrow} \theta_{(a,b),f_{(a,b)}}.$$

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However, there are very interesting theoretical results on computing $\{(a,b), f_{(a,b)}\}$:

Theorem A (GPS '10)

Consider the formal power series

$$\log f_{(a,b)} = \sum_{k\geqslant 0} c_k^{(a,b)} (tx)^{ak} (ty)^{bk}.$$

Then

$$c_k^{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} N_{(a,b)}[(\mathbf{P}_a, \mathbf{P}_b)],$$

where $\mathbf{P}_a, \mathbf{P}_b =$ *ordered partitions, and* $\operatorname{len} \mathbf{P}_a = \ell_1$, $\operatorname{len} \mathbf{P}_b = \ell_2$.

Tropical significance

The GPS Theorem is based on a tropical computation together with some nice correspondence results. The tropical technique leads to:

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Theorem A' (GPS '10)

$$c_k^{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} \sum_{\mathbf{w}} \prod_{i=1}^{2} \frac{R_{\mathbf{P}_i|\mathbf{w}_i}}{|\operatorname{Aut}(\mathbf{w}_i)|} N_{(a,b)}^{\mathrm{trop}}(\mathbf{w}),$$

where $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ *is a pair of weight vectors of arbitrary length parametrizing a family of tropical counts* $\{N_{(a,b)}^{\mathrm{trop}}(\mathbf{w})\}$*.*

$R_{\mathbf{P}_i|\mathbf{w}_i}$, $|\operatorname{Aut}(\mathbf{w}_i)|$ are some ramification and automorphism factors.

Geometric meaning: rational plane tropical curves with $|\mathbf{w}_1| + |\mathbf{w}_2|$ incoming ends and a single outgoing end.

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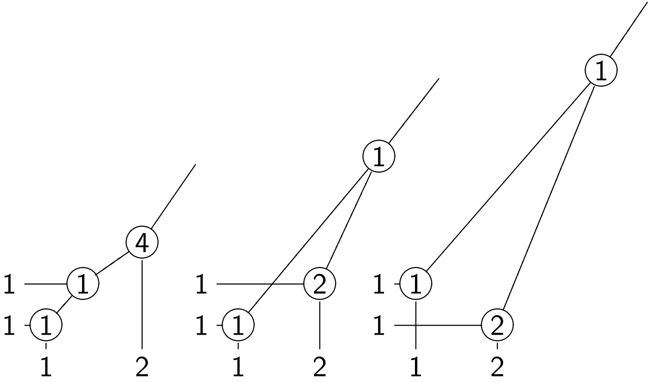
$N^{\text{trop}}((1,1),(1,2)) = 8$



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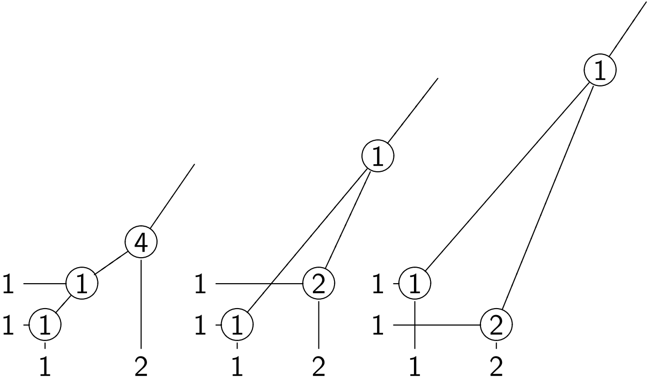

Fact: These counts are well-defined, and depend only on $\mathbf{w}$.

Refinement

We can actually work over $\mathbb{C}[[s_1, \ldots, s_{\ell_1}, t_1, \ldots, t_{\ell_2}]]$, and consider

$$[\prod_{i=1}^{\ell_1} \theta_{(1,0),1+s_i x}, \prod_{j=1}^{\ell_2} \theta_{(0,1),1+t_j y}].$$

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where

$$\log f_{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)] s^{\mathbf{P}_1} t^{\mathbf{P}_2} x^{ka} y^{kb}.$$

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Corollary

The invariants $N_{(a,b)}[(\mathbf{P}_1, \mathbf{P}_2)]$ are determined by the factorization $(\star)$.

Natural q-deformation

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Let $(\Gamma, \langle -, - \rangle)$ be a lattice with antisymmetric, bilinear form. Consider the Lie algebra

$$\mathfrak{g} \quad \text{generated by} \quad e_\alpha, \quad \alpha \in \Gamma,$$

with

$$[e_\alpha, e_\beta] = \langle \alpha, \beta \rangle \, e_{\alpha+\beta},$$

$$e_\alpha e_\beta = e_{\alpha+\beta}.$$

$\Rightarrow$ Poisson algebra.

The tropical vertex group revisited

Let R be a complete local or Artin $\mathbb{C}$-algebra, and

$$\widehat{\mathfrak{g}} = \mathfrak{g}\,\widehat{\otimes}_{\mathbb{C}} R = \varinjlim \mathfrak{g} \otimes_{\mathbb{C}} R/\mathfrak{m}_R^k.$$

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Let $f_\alpha \in \widehat{\mathfrak{g}}$ be an element of the form

$$f_\alpha \in 1 + \mathfrak{m}_R[e_\alpha]e_\alpha. \tag{1.1}$$

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Let $f_\alpha \in \widehat{\mathfrak{g}}$ be an element of the form

$$f_\alpha \in 1 + \mathfrak{m}_R[e_\alpha]e_\alpha. \tag{1.1}$$

Then we introduce θ_{α,f_α} automorphisms of the R-algebra $\widehat{\mathfrak{g}}$ by

$$\theta_{\alpha,f_\alpha}(e_\beta) = e_\beta f_\alpha^{\langle\alpha,\beta\rangle}.$$

Write: $\theta^{\Omega}_{\alpha,f_\alpha} = \theta_{\alpha,f^{\Omega}_\alpha}$ for every $\Omega \in \mathbb{Q}$.

The tropical vertex group revisited

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Definition

The tropical vertex group $\mathbb{V}_{\Gamma,R}$ is the completion with respect to $\mathfrak{m}_R \subset R$ of the subgroup of $\mathrm{Aut}_R(\widehat{\mathfrak{g}})$ generated by all the transformations θ_{α,f_α}.

The wall-crossing group

Elements of $\mathbb{V}_{\Gamma,R}$ of the form $\theta_{\alpha,1+\sigma e_\alpha}$ with $\sigma \in \mathfrak{m}_R$ play a special role.

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Fact:

$$\theta_{\alpha,1+\sigma e_{m\alpha}} = \exp\left(\frac{1}{m}\,\mathrm{ad}(\mathrm{Li}_2(-\sigma e_{m\alpha}))\right).$$

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Fixing a local complete or Artin $\mathbb{C}$-algebra R as usual, we define

$$\hat{\mathfrak{g}}_q = \mathfrak{g}_q \,\hat{\otimes}_{\mathbb{C}} R.$$

(fundamental case: $\mathfrak{g}_q[[t]]$, where t is a central variable.)

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Now we can define q-dilogarithm:

$$\mathbf{E}(\sigma \hat{e}_\alpha) = \sum_{n \geqslant 0} \frac{(-q^{\frac{1}{2}} \sigma \hat{e}_\alpha)^n}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

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The q-wall-crossing group $\mathbb{U}_{\Gamma,R}$ is the completion of the subgroup of $\mathrm{Aut}_{\mathbb{C}(q^{\pm\frac{1}{2}})\otimes_{\mathbb{C}}R}\,\widehat{\mathfrak{g}}_q$ generated by the $\hat{\theta}^\Omega[(-q^{\frac{1}{2}})^n\sigma\hat{e}_\alpha)]$ for $\alpha \in \Gamma$, $\sigma \in \mathfrak{m}_R$, $\Omega \in \mathbb{Q}$, $n \in \mathbb{Z}$.

q-Factorization

The factorization $[\theta_{(1,0),(1+tx)^{\ell_1}}, \theta_{(0,1),(1+ty)^{\ell_2}}] = \prod^{\rightarrow}_{(a,b)} \theta_{(a,b),f_{(a,b)}}$

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Suppose that α_1 follows α_2 in clockwise order.

Lemma

Let $\Gamma \simeq \mathbb{Z}^2$. Then $\exists!\ \Omega_n(k\alpha) \in \mathbb{Q}$ such that

$$[\hat{\theta}^{\ell_1}[\sigma_1 \hat{\mathrm{e}}_{\alpha_1}], \hat{\theta}^{\ell_2}[\sigma_2 \hat{\mathrm{e}}_{\alpha_2}]] = \prod^{\rightarrow}_{\gamma} \prod_{k \geqslant 1} \prod_{n \in \mathbb{Z}} \hat{\theta}^{(-1)^n \Omega_n(k\gamma)}[(-q^{\frac{1}{2}})^n \sigma^{k\gamma} \hat{\mathrm{e}}_{k\gamma}],$$

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Factorization problem: find

$$\hat{\theta}_\gamma = \prod_{k \geqslant 1} \prod_{n \in \mathbb{Z}} \hat{\theta}^{(-1)^n \Omega_n(k\gamma)}[(-q^{\frac{1}{2}})^n \sigma^{k\gamma} \hat{e}_{k\gamma}].$$

q-analogue of Theorem A'

Refinement: As in the numerical case, we can work over $\mathbb{C}[[s_1, \ldots, s_{\ell_1}, t_1, \ldots, t_{\ell_2}]]$, and look at $\prod_i \hat{\theta}[s_i \hat{e}_{\alpha_1}] \prod_j \hat{\theta}[t_j \hat{e}_{\alpha_2}]$.

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Then

Lemma (Stoppa-F.)

$$\hat{\theta}_{a_1\alpha_1 + a_2\alpha_2} = \mathrm{Ad}\exp\Big(\sum_{|\mathbf{P}_1| = ka_1} \sum_{|\mathbf{P}_2| = ka_2} \sum_{\mathbf{w}} \prod_{i=1}^{2} \frac{\widehat{R}_{\mathbf{P}_i|\mathbf{w}_i}}{|\mathrm{Aut}(\mathbf{w}_i)|} \widehat{N}^{\mathrm{trop}}_{(\alpha_1,\alpha_2)}(\mathbf{w}) \\ s^{\mathbf{P}_1} t^{\mathbf{P}_2} \frac{\hat{e}_{k(a_1\alpha_1 + a_2\alpha_2)}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \Big).$$

where

$$\widehat{R}_{\mathbf{P}_i|\mathbf{w}_i,q} = \prod_j \frac{(-1)^{w_{ij}-1}}{w_{ij}[w_{ij}]_q} \#\{I_{i,\bullet}, \mathbf{P}_i|\mathbf{w}_i\},$$

$\widehat{N}^{\mathrm{trop}}_{(\alpha_1,\alpha_2)}(\mathbf{w})$= *Block-Göttsche invariant (replace m_V with $[m_V]_q$).*

Main theorem

Corollary

A natural candidate for the q-deformed GW invariant is

$$\widehat{N}[(\mathbf{P}_1, \mathbf{P}_2)] = \sum_{\mathbf{w}} \prod_{i=1}^{2} \frac{\widehat{R}_{\mathbf{P}_i|\mathbf{w}_i}}{|\operatorname{Aut}(\mathbf{w}_i)|} \widehat{N}^{\mathrm{trop}}_{(\alpha_1,\alpha_2)}(\mathbf{w}).$$

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RW $\Rightarrow$ $$\widehat{N}'[(\mathbf{P}_1, \mathbf{P}_2)] = \widehat{P}(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q).$$

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Theorem (Stoppa-F.)

Suppose $(|\mathbf{P}_1|, |\mathbf{P}_2|)$ *coprime. Then the two choices of quantization coincide:*

$$\widehat{N}'[(\mathbf{P}_1, \mathbf{P}_2)] = \widehat{N}[(\mathbf{P}_1, \mathbf{P}_2)].$$

Sketch of the Proof I

Refinement $(k^1, k^2) \vdash (\mathbf{P}_1, \mathbf{P}_2)$ = sets of integers $(k^1, k^2) = (\{k^1_{w,i}\}, \{k^2_{w,j}\})$ s.t. for $i = 1, \ldots, \ell_1$ and $j = 1, \ldots, \ell_2$ $p_{1i} = \sum_w w k^1_{w,i}$, $p_{2j} = \sum_w w k^2_{w,j}$.

A fixed refinement k^i induces a weight vector $\mathbf{w}(k^i) = (w_{i1}, \ldots, w_{it_i})$ of length $t_i = \sum_w m_w(k^i)$, by

$$w_{ij} = w \text{ for all } j = \sum_{r=1}^{w-1} m_r(k^i) + 1, \ldots, \sum_{r=1}^{w} m_r(k^i).$$

By combinatorial argument rearrange as

$$\widehat{N}[(\mathbf{P}_1, \mathbf{P}_2)] = \sum_{(k_1,k_2)\vdash(\mathbf{P}_1,\mathbf{P}_2)} \prod_{i=1}^{2} \prod_{j=1}^{\ell_i} \prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}!\, w^{k^i_{w,j}} [w]_q^{k^i_{w,j}}} \widehat{N}^{\mathrm{trop}}_{(\alpha_1,\alpha_2)}(\mathbf{w}(k^1), \mathbf{w}(k^2)).$$

Sketch of the proof II

Introduce infinite bipartite quiver $\mathcal{N}$, with
$\mathcal{N}_0 = \{i_{(w,m)} \mid (w,m) \in \mathbb{N}^2\} \cup \{j_{(w,m)} \mid (w,m) \in \mathbb{N}^2\}$ and
$\mathcal{N}_1 = \{\alpha_1, \ldots, \alpha_{w \cdot w'} : i_{(w,m)} \to j_{(w',m')}, \forall\, w, w', m, m' \in \mathbb{N}\}$.
Fact: (k^1, k^2) induces a *thin* (i.e. type one) dimension vector $d(k^1, k^2)$ for $\mathcal{N}$ $\Rightarrow$ moduli space of stable *abelian* representations $\mathcal{M}_{d(k^1,k^2)}(\mathcal{N})$.
MPS formula for Poincaré polynomials can be expressed as

$$\widehat{P}(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q) = \sum_{(k^1,k^2) \vdash (\mathbf{P}_1, \mathbf{P}_2)} \prod_{i=1}^{2} \prod_{j=1}^{\ell_i} \prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}!\, w^{k^i_{w,j}} [w]_q^{k^i_{w,j}}} \widehat{P}(\mathcal{M}_{d(k^1,k^2)}(\mathcal{N}))(q).$$

Claim follows from

$$\widehat{P}(\mathcal{M}_{d(k^1,k^2)}(\mathcal{N}))(q) = \widehat{N}^{\mathrm{trop}}_{(\alpha_1,\alpha_2)}(\mathbf{w}(k^1), \mathbf{w}(k^2)). \tag{1.2}$$

Grazie!