GIT per curve polarizzate

Fabio Felici

Dipartimento di Matematica e Fisica, Sezione Matematica
Universitá di Roma Tre

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GIT: Basic definitions

Let X be a projective variety, $G \times X \longrightarrow X$ a rational action and let L be a G-linearized ample line bundle on X. A point $p \in X$ is said to be:

- **semistable** if there is an invariant section $s \in H^0(X, L^{\otimes n})^G$ for some n such that $s(p) \neq 0$; the semistable locus is denoted by X^{ss};
- **polystable** if it is semistable and its orbit is closed in X^{ss};
- **stable** if it is polystable and its stabilizer is finite; the stable locus is denoted by X^s.

We have a rational map

$$X \dashrightarrow X/\!\!/G := \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})^G\right)$$

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Categorical quotients and Geometric quotients

The morphism

$$\pi : X^{ss} \to X /\!\!/ G$$

is a **categorical** quotient, i. e. is universal with respect to G-invariant morphisms.

If we restrict π to the stable locus we obtain a **geometric** quotient

$$\pi|_{X^s} : X^s \to \pi(X^s)$$

which is categorical quotient such that there is a one-to-one correpondence

$$\{\text{orbits of } X^s\} \longleftrightarrow \{\text{geometric points of } \pi(X^s)\}.$$

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GIT construction of $\overline{M}_g$

Fix an integer $g \geq 2$. Given d sufficiently large, we denote by

- Hilb_d the Hilbert scheme of connected curves of degree d and arithmetic genus g in $\mathbb{P}^{d-g}$;
- Chow_d the Chow scheme of 1-cycles of degree d in $\mathbb{P}^{d-g}$.

Consider the Hilbert-Chow map

$$\mathrm{Ch} : \mathrm{Hilb}_d \to \mathrm{Chow}_d,$$

which sends $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ to its 1-cycle. The linear algebraic group SL_{d-g+1} acts naturally on Hilb_d and Chow_d so that Ch is an equivariant map; moreover, these actions are naturally linearized.

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Hilbert stability VS Chow stability

There are inclusions

$$\mathrm{Ch}^{-1}(\mathrm{Chow}_d^s) \subseteq \mathrm{Hilb}_d^s \subseteq \mathrm{Hilb}_d^{ss} \subseteq \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$$

so that Ch induces a morphism

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In order to obtain a compactification of M_g, we suppose that $d = n(2g-2)$ with $n \in \mathbb{N}$ and we restrict the action of $\mathrm{SL_{d-g+1}}$ to

$$\mathrm{Hilb}_{d,\mathrm{can}} := \{[X \subset \mathbb{P}^{d-g}] \,|\, \mathcal{O}_X(1) \cong \omega_X^{\otimes n}\}$$

and $\mathrm{Chow}_{d,\mathrm{can}}$ defined as the schematic image of $\mathrm{Hilb}_{d,\mathrm{can}}$. In particular, there is a natural morphism of GIT-quotients

$$\mathrm{Hilb}_{d,\mathrm{can}}^{ss} /\!\!/ \mathrm{SL_{d-g+1}} \to \mathrm{Chow_{d,can}^{ss}} /\!\!/ \mathrm{SL_{d-g+1}}.$$

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- $n \geq 5$ (D. Mumford, D. Gieseker):

$$\begin{aligned} \mathrm{Ch}^{-1}(\mathrm{Chow}^{s}_{d,\mathrm{can}}) &= \mathrm{Ch}^{-1}(\mathrm{Chow}^{ss}_{d,\mathrm{can}}) \\ &= \{X \text{ DM-stable with } \mathcal{O}_X(1) = \omega_X^{\otimes n}\}; \end{aligned}$$

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$\overline{M}_g$ coarse moduli space of Deligne-Mumford stable curves of genus g.

We recall that a **Deligne-Mumford stable** curve is a connected nodal projective curve with finite automorphism group (i. e. each $\mathbb{P}^1$ intersects the rest of the curve in at least 3 points).

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Birational models of $\overline{M}_g$ via GIT

- $n = 3$ (D. Schubert)

$$\begin{aligned}\mathrm{Ch}^{-1}(\mathrm{Chow}^{s}_{d,\mathrm{can}}) &= \mathrm{Ch}^{-1}(\mathrm{Chow}^{ss}_{d,\mathrm{can}}) \\ &= \{X \text{ p-stable with } \mathcal{O}_X(1) = \omega_X^{\otimes 3}\};\end{aligned}$$

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$\overline{M}^{p}_{g}$: coarse moduli space of pseudo-stable curves of genus g.

We recall that a **pseudo-stable** curve is a connected projective curve with finite automorphism group, whose only singularities are nodes and cusps, and which have no elliptic tails.

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These modular birational models of GIT represent the first steps of the so called **Hassett-Keel program** (B. Hasset D. Hyeon).

$$\begin{tikzcd} \overline{M}_g \arrow[r, "\pi_1"] & \overline{M}_g^{\mathrm{p}} \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi_2"'] & & \overline{M}_g^{c}, \arrow[dl, "\pi_3"] \\ & & \overline{M}_g^{h} & \end{tikzcd}$$

where

$$\overline{M}_g^{\mathrm{p}} \cong \mathrm{Hilb}^{ss}_{3(2g-2),\mathrm{can}} /\!\!/ \mathrm{SL}_{\mathrm{d-g+1}} \cong \mathrm{Chow}^{\mathrm{ss}}_{3(2\mathrm{g}-2),\mathrm{can}} /\!\!/ \mathrm{SL}_{\mathrm{d-g+1}}$$

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Universal Jacobian over the moduli space of curves

Now consider the Hilbert-Chow map

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and the action of $\mathrm{SL_{d-g+1}}$ on the whole Hilbert and Chow schemes (for now we will consider only connected curves). There is a natural morphism of GIT-quotients

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The GIT quotients will be compactifications of the **universal jacobian**

$$J_{d,g} \longrightarrow M_g$$

where

$$J_{d,g} = \{(X, L)|, X \text{ smooth of genus } g,\ L \text{ line bundle of degree } d\}/\sim$$

The first compactification of $J_{d,g}$ was obtained by Lucia Caporaso in her PhD thesis in 1993.

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$$J_{d,g} = \{(X, L)|, X \text{ smooth of genus } g, L \text{ line bundle of degree } d\}/\sim$$

The first compactification of $J_{d,g}$ was obtained by Lucia Caporaso in her PhD thesis in 1993.

Universal Jacobian over the moduli space of curves

Now consider the Hilbert-Chow map

$$\mathrm{Ch} : \mathrm{Hilb}_d \to \mathrm{Chow}_d,$$

and the action of $\mathrm{SL_{d-g+1}}$ on the whole Hilbert and Chow schemes (for now we will consider only connected curves). There is a natural morphism of GIT-quotients

$$\mathrm{Hilb}_d^{ss} /\!\!/ \mathrm{SL_{d-g+1}} \to \mathrm{Chow_d^{ss}} /\!\!/ \mathrm{SL_{d-g+1}}.$$

The GIT quotients will be compactifications of the **universal jacobian**

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Balanced line bundles

Definition

If (X, L) is a polarized curve we say that L is **balanced** if for each subcurve Z of X the following inequality is satisfied:

$$\left|\deg_Z L - \frac{d}{2g-2}\deg_Z(\omega_X)\right| \leq \frac{|Z \cap Z^c|}{2}$$

This inequality is called **basic inequality**.

Remark

A line bundle L is balanced if and only if is slope-semistable with respect to the polarization ω_X.

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Caporaso's compactification of the universal jacobian

Theorem (Caporaso, 1994)

Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $d \geq 10(2g-2)$; assume moreover that X is connected. Then the following conditions are equivalent:

(i) $[X \subset \mathbb{P}^{d-g}]$ is semistable;

(ii) X is quasi-stable and $\mathcal{O}_X(1)$ is balanced.

In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.

Definition

A **quasi-stable** curve is a connected nodal projective curve such that for each connected subcurve E of genus 0,

- $|E \cap E^c| \geq 2$;
- if $|E \cap E^c| = 2$, then $E \cong \mathbb{P}^1$ (i. e. there are no chains of $\mathbb{P}^1$).

Caporaso's compactification of the universal jacobian

Problem

Describe the GIT quotient for the Hilbert and Chow scheme of curves of genus g and degree d in $\mathbb{P}^{d-g}$, as d decreases with respect to g.

Theorem (G. Bini, -, M. Melo, F. Viviani)

Caporaso's description holds under the hypothesis $d > 4(2g-2)$. Indeed, the following conditions are equivalent:

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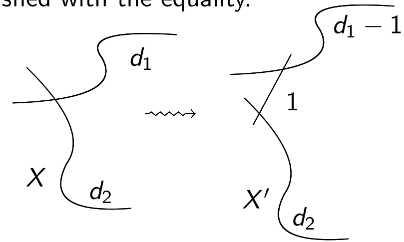
Orbits identifications

Let us consider the quotient

$$\overline{J}_{d,g} \cong \mathrm{Hilb}_d^{ss} /\!\!/ \mathrm{SL}_{d-g+1} \cong \mathrm{Chow}_d^{ss} /\!\!/ \mathrm{SL}_{d-g+1}$$

for $d > 4(2g-2)$.
In general this is a categorical quotient. There are orbits identifications whenever there exists a curve X such that for some subcurve Z the basic inequality is satisfied with the equality.



Orbits identifications

One can prove that if $\gcd(d-g+1, 2g-2) = 1$, then the basic inequality is never satisfied with the equality for each semistable curve.

Theorem (Caporaso)

$\overline{J}_{d,g}$ is a geometric quotient if and only if $\gcd(d-g+1, 2g-2) = 1$.

Definition

Let $\overline{\mathcal{J}}_{d,g}$ be the category fibered in groupoids over the category of k-schemes whose sections over a k-scheme S are pairs $(f : \mathcal{X} \to S, \mathcal{L})$ where f is a family of quasi-stable curves of genus g and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ of relative degree d that is properly balanced on the geometric fibers of f.

Theorem (M. Melo)

If $d > 4(2g-2)$ then $\overline{\mathcal{J}}_{d,g} \cong [\mathrm{Hilb}_d/\mathrm{GL}_{d-g+1}]$.

Lemma

For any integer n, there is a natural isomorphism $\overline{\mathcal{J}}_{d,g} \cong \overline{\mathcal{J}}_{d+n(2g-2),g}$ of categories fibered in groupoids:

$$(f, \mathcal{L}) \mapsto (f, \mathcal{L} \otimes \omega_{X/S}^{\otimes n})$$

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Corollary

Let $g \geq 2$ and $d \in \mathbb{Z}$.

1. $\overline{\mathcal{J}}_{d,g}$ is a smooth, irreducible and universally closed Artin stack of finite type over k and of dimension $4g-4$, containing $\mathcal{J}_{d,g}$ as a dense open substack.
2. $\overline{\mathcal{J}}_{d,g}$ admits a moduli space $\overline{J}_{d,g}$, which is a normal integral projective variety of dimension $4g-3$ containing $J_{d,g}$ as a dense open subvariety.
3. We have the following commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{J}}_{d,g} \arrow[r] \arrow[d, "\Psi^{s}"'] & \overline{J}_{d,g} \arrow[d, "\Phi^{s}"] \\
\overline{\mathcal{M}}_g \arrow[r] & \overline{M}_g
\end{tikzcd}$$

where Ψ^{s} is universally closed and surjective and Φ^{s} is projective and surjective.

Canonical compactified Jacobians

Theorem

Consider the natural map $\Phi^{\mathrm{st}} : \overline{J}_{d,g} \longrightarrow \overline{M}_g$. Then

$$(\Phi^{\mathrm{st}})^{-1}(X) \cong \overline{\mathrm{Jac_d}}(X)/\mathrm{Aut}(X)$$

for any $X \in \overline{M}_g$ where $\overline{\mathrm{Jac_d}}(X)$ is the **canonical compactified Jacobian** of X in degree d, i. e. the moduli space of of rank-1, torsion-free sheaves on X of degree d that are slope-semistable with respect to ω_X.

The case $2(2g-2) < d < \frac{7}{2}(2g-2)$

Theorem (-)

If $2(2g-2) < d < \frac{7}{2}(2g-2)$ and $g \geq 3$ then the following conditions are equivalent:

(i) $[X \subset \mathbb{P}^{d-g}]$ is semistable;

(ii) $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ is semistable;

(iii) X is quasi-pseudo-stable and $\mathcal{O}_X(1)$ is balanced.

Definition

A **quasi-pseudo-stable** curve X is projective curve whose only singularities are nodes, cusps and tacnodes with lines, such that

- X has no elliptic tails;
- $|E \cap E^c| \geq 2$ for each connected subcurve E of genus 0;
- if $|E \cap E^c| = 2$ for some connected subcurve E of genus 0, then $E \cong \mathbb{P}^1$ (i. e. there are no chains of $\mathbb{P}^1$).

Orbits identifications

Let us consider the quotient

$$\overline{J}_{d,g}^{\mathrm{ps}} \cong \mathrm{Hilb}_d^{ss} /\!\!/ \mathrm{SL}_{\mathrm{d-g+1}}$$

for $2(2g-2) < d < \frac{7}{2}(2g-2)$.

Theorem (-)

$\overline{J}_{d,g}^{\mathrm{ps}}$ is a geometric quotient if and only if $\gcd(d-g+1, 2g-2) = 1$.

Definition

Let $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ be the category fibered in groupoids over the category of k-schemes whose sections over a k-scheme S are pairs $(f : \mathcal{X} \to S, \mathcal{L})$ where f is a family of quasi-pseudo-stable curves of genus g and $\mathcal{L}$ is line bundle on $\mathcal{X}$ of relative degree d that is properly balanced on the geometric fibers of f.

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If $2(2g-2) < d \leq \frac{7}{2}(2g-2)$ then $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} \cong [\mathrm{Hilb}_d/\mathrm{GL}_{\mathrm{d-g+1}}]$.

Lemma

For any integer n, there is a natural isomorphism $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} \cong \overline{\mathcal{J}}_{d+n(2g-2),g}^{\mathrm{ps}}$ of categories fibered in groupoids:

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Corollary (-)

Let $g \geq 3$ and $d \in \mathbb{Z}$.

1. $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ is a smooth, irreducible and universally closed Artin stack of finite type over k and of dimension $4g-4$, containing $\mathcal{J}_{d,g}$ as a dense open substack.
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$$\begin{tikzcd}
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where Ψ^{ps} is universally closed and surjective and Φ^{ps} is projective and surjective.

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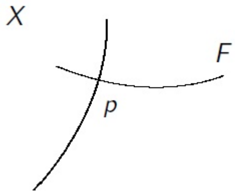
Elliptic tails



Given a line bundle L on X, we can write

$$L_{|F} = \mathcal{O}_F((d_F - 1)p + q)$$

where $d_F = \deg_F L$ denotes the degree of L on F, for a uniquely determined smooth point q of F.

We say that F is

- **special** with respect to L if $q = p$;
- **non-special** otherwise.

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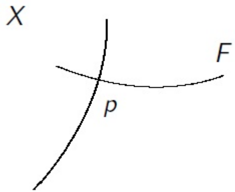

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The case $\frac{7}{2}(2g-2) < d < 4(2g-2)$

Theorem (-)

If $\frac{7}{2}(2g-2) < d < 4(2g-2)$ and $g \geq 3$ then the following conditions are equivalent:

(i) $[X \subset \mathbb{P}^{d-g}]$ is semistable;

(ii) $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ is semistable;

(iii) X is quasi-weakly-pseudo-stable without tacnodes or special elliptic tails (with respect to $\mathcal{O}_X(1)$) and $\mathcal{O}_X(1)$ is balanced.

Definition

A **quasi-weakly-pseudo-stable** curve X is a projective curve whose only singularities are nodes, cusps, tacnodes with lines, such that for each connected subcurve E of genus 0,

- $|E \cap E^c| \geq 2$;
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Let us consider the quotient

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for $\frac{7}{2}(2g-2) < d < 4(2g-2)$.

Theorem

$\overline{J}_{d,g}^{\mathrm{wp}}$ is a geometric quotient if and only if $\gcd(d-g+1, 2g-2) = 1$.

Definition

Let $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$ be the category fibered in groupoids over the category of k-schemes whose sections over a k-scheme S are pairs $(f : \mathcal{X} \to S, \mathcal{L})$ where f is a family of quasi-weakly-pseudo-stable curves of genus g and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ of relative degree d that is properly balanced on the geometric fibers of f and such that the geometric fibers of f do not contain tacnodes with a line nor special elliptic tails relative to $\mathcal{L}$.

Theorem (-)

If $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$ then $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} \cong [\mathrm{Hilb}^{ss}_d/\mathrm{GL}_{\mathrm{d-g+1}}]$.

Lemma

1. For any integer n, there is a natural isomorphism $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} \cong \overline{\mathcal{J}}^{\mathrm{wp}}_{d+n(2g-2),g}$ of categories fibered in groupoids:.
2. There is natural isomorphism $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} \cong \overline{\mathcal{J}}^{\mathrm{wp}}_{-d,g}$ of categories fibered in groupoids: $(f, \mathcal{L}) \mapsto (f, \mathcal{L}^{-1})$.

Corollary (-)

Let $g \geq 3$ and $d \in \mathbb{Z}$.

1. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$ is a smooth, irreducible and universally closed Artin stack of finite type over k and of dimension $4g-4$, containing $\mathcal{J}_{d,g}$ as a dense open substack.
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where Ψ^{wp} is universally closed and surjective and Φ^{wp} is projective and surjective.

Summary of semistable singularities

- $d > 4(2g-2)$: nodes
- $d = 4(2g-2)$: $\begin{cases} \text{cusps} & \text{IN}, \\ \text{special elliptic tails} & \text{OUT}. \end{cases}$
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The case $d = 4(2g-2)$

Theorem (-)

Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $d = 4(2g-2)$ and $g \geq 3$; assume moreover that X is connected. Then it holds that

(i) $[X \subset \mathbb{P}^{d-g}]$ is semistable if and only if X is quasi-wp-stable without tacnodes nor special elliptic tails (with respect to $\mathcal{O}_X(1)$) and $\mathcal{O}_X(1)$ is balanced (same description as in the interval $\frac{7}{2}(2g-2) < d < 4(2g-2)$).

(ii) $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ is semistable if and only if X is quasi-wp-stable without tacnodes and $\mathcal{O}_X(1)$ is balanced.

In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.

In this case the semistable loci Hilb_d^{ss} and Chow_d^{ss} are different: indeed special elliptic tails are Chow semistable, but Hilbert unstable.

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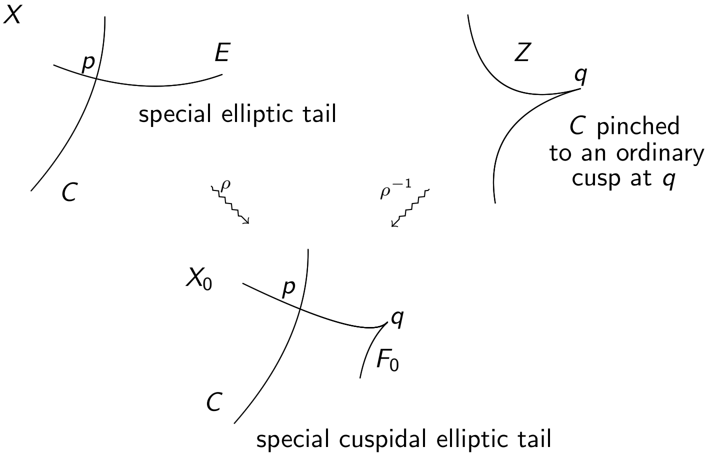


The case $d = \frac{7}{2}(2g-2)$

Theorem (-)

Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $d = \frac{7}{2}(2g-2)$ and $g \geq 3$; assume moreover that X is connected. Then it holds that

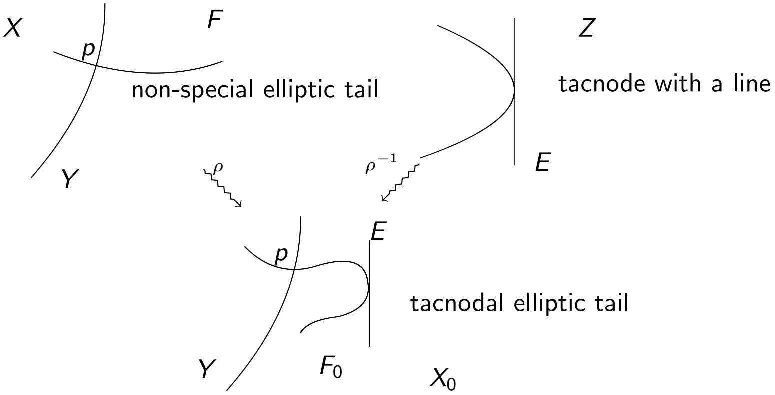
(i) $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d^{ss}$ is semistable if and only if X is quasi-p-stable and $\mathcal{O}_X(1)$ is balanced (same description as in the interval $2(2g-2) < d < \frac{7}{2}(2g-2)$).

(ii) $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}]) \in \mathrm{Chow}_d^{ss}$ if and only if X is quasi-wp-stable without special elliptic tails (with respect to $\mathcal{O}_X(1)$) and $\mathcal{O}_X(1)$ is balanced.

In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.

Also in this case the semistable loci Hilb_d^{ss} and Chow_d^{ss} are different.

The case $d = \frac{7}{2}(2g-2)$



Extra components of the GIT quotient

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There is a commutative diagram

$$\begin{tikzcd}
\pi_0(\mathrm{Hilb}_d^{ss}) \arrow[rr, "\phi"] \arrow[d, "\eta"'] & & \{\text{partitions of } \gcd(d, g-1)\} \\
\pi_0(\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})) \arrow[urr, "\phi'"'] & &
\end{tikzcd}$$

where all the maps are one-to-one correspondences and η is induced by the inclusion $\mathrm{Hilb}_d^{ss} \subseteq \mathrm{Ch}^{-1}(\mathrm{Hilb}_d^{ss})$.

Corollary

In particular, if $\gcd(d, g-1) = 1$ then Hilb_d^{ss} and Chow_d^{ss} are connected and consist only of connected curves.

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Open problems

Problem 1

Describe the (semi-,poly-)stable points of Hilb_d and Chow_d in the case $2(2g-2) < d \leq 4(2g-2)$ for $g = 2$.

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(i) Describe the (semi-,poly-)stable points of Hilb_d and Chow_d in the case $d = 2(2g-2)$.

(ii) Describe the (semi-,poly-)stable points of Hilb_d and Chow_d in the case $d = 2(2g-2) - \epsilon$ (for small ϵ).

(iii) What is the next critical value of $\frac{d}{2g-2} < 2$ at which the GIT quotients change?

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Problem 3

Describe the birational maps fitting into the following commutative diagram

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\overline{\mathcal{M}}_g \arrow[r, hook] & \overline{\mathcal{M}}_g^{\mathrm{wp}} & \overline{\mathcal{M}}_g^{\mathrm{p}} \arrow[l, hook']
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More generally, one would like to set up a Hassett-Keel program for the Caporaso's compactified universal Jacobian stack $\overline{\mathcal{J}}_{d,g}$ and give an interpretation of the alternative compactifications $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$ and $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ of $\mathcal{J}_{d,g}$ as the first steps in this program.

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THANK YOU!