La proprietà di Betti Weak Lefschetz

Favacchio Giuseppe

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Joint work with:
A. Ragusa and G. Zappalà : *Linear quotients of Artinian Weak Lefschetz Algebras*.

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$$A = R/I = \bigoplus_{i=0}^{s} A_i$$

be an Artinian standard graded algebra.

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$$\operatorname{rank}(\times\ell) = \min\{\dim_k A_i, \dim_k A_{i+1}\}.$$

Definition

We say that A has the **Weak Lefschetz property (WLP)** if there is an element $\ell \in R_1$ (called a Lefschetz element) such that the linear map

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An algebra with the WLP will be call **Weak Lefschetz algebra**.

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An sequence $(1, h_1, ..., h_s)$, with $h_s \neq 0$ is called unimodal if there is an integer t such that

$$h_1 \leq \ldots \leq h_t \geq h_{t+1} \geq \ldots \geq h_s.$$

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Theorem (Stanley 1980)

Let $R = k[x_1, \dots, x_n]$. Let I be an Artinian monomial complete intersection, i.e.

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- Do all complete intersections have WLP?

Theorem (Harima - Migliore - Nagel - J. Watanabe 2003)

Let $R = k[x, y, z]$. *Let* $I = (F_1, F_2, F_3)$ *be a complete intersection ideal. Then* R/I *has WLP.*

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- do all Artinian Gorenstein algebras in codimension 3 have WLP?

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$\varphi_{\ell,i}$ injective $\Rightarrow H_{A/\ell A}(i+1) = \Delta H_A(i+1)$.
$\varphi_{\ell,i}$ surjective $\Rightarrow H_{A/\ell A}(i+1) = 0$ and
$\Delta H_A(i+1) = -\dim_k \operatorname{Ker} \varphi_{\ell,i}$.

Proposition

Let A be an Artinian standard graded algebra. The following are equivalent

- *A has the WLP;*
- *there is an element $\ell \in R_1$ such that $H_{A/\ell A} = \Delta H_A^+$.*

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- *A has the WLP;*
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A has the WLP if and only if its Hilbert function has a *good behavior* with respect to a generic linear form.

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Now we define

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There exists u such that S_{H_u} contains a non empty open subset $U \subseteq \mathbb{P}_k(R_1)$.

Definition

With the above notation we say that $A/\ell A$ has the generic Hilbert function with respect to A iff $[\ell] \in S_{H_u}$. In this case $H_u = H_{A/\ell A}$ will be called the Hilbert function of the generic linear section of A and will be denoted by H_A^{gen}.

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There exists v such that Z_{β_v} contains a nonempty open subset V of S^{gen}.

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With the above notation we say that $A/\ell A$ has the generic Betti sequence with respect to A iff $[\ell] \in Z_{\beta_v}$. In this case $\beta(A/\ell A)$ will be called the Betti sequence of the generic linear section of A and will be denoted by β_A^{gen}.

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The linear forms $\ell_0 = x + y + z$, $\ell_1 = y + z$ and $\ell_2 = z$ are WL forms for A;we have that

$$H_{A/\ell_0 A} = H_{A/\ell_1 A} = H_{A/\ell_0 A} = \Delta H^+_{R/I} = H^{gen} = (1, 2, 3, 4, 2, 0)$$

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Consequently $\beta_A^{gen} = ((4^3), (6^2))$.
Moreover one can see that $\mathcal{B}_A$ contains only these three Betti sequences.

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$$H_A^{gen} \geq \Delta H_A^{+} \qquad \beta_A^{gen} \geq ?$$

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$$\big(R/I\big)_t \xrightarrow{\psi} \big(R/(I_{\leq t})\big)_{t+1}$$

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Theorem

$$\overline{\beta}_{0\ t+1}(\overline{A}) - \beta_{0\ t+1}(A) = \dim_k(\operatorname{Ker}\psi) - \dim_k(\operatorname{Ker}\varphi).$$

Corollary

With the above notation

i) ψ *is surjective iff* $\overline{\beta}_{0\ t+1}(\overline{A}) = 0$;

ii) ψ *is injective iff* $\overline{\beta}_{0\ t+1}(\overline{A}) = \beta_{0\ t+1}(A) + \Delta H_A(t+1)$.

Let us consider a graded minimal free resolution of A as a R-module

$$F_\bullet : 0 \to F_{c-1} \to \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_0 \to R \to A \to 0$$

and a graded minimal free resolution of $\overline{A}$ as a $\overline{R}$-module

$$G_\bullet : 0 \to G_{c-2} \to \cdots \to G_i \xrightarrow{d_i'} G_{i-1} \to \cdots \to G_0 \to \overline{R} \to \overline{A} \to 0.$$

Let $\pi_\bullet : F_\bullet \to G_\bullet$ be a lifting of the natural map of R-modules $\pi : A \to \overline{A}$.

Theorem

With the above notation, for every $i \geq 0$, *let*

$$\{\gamma_{i1}, \ldots, \gamma_{i\beta_i}\}, \quad \deg \gamma_{i1} \leq \ldots \leq \deg \gamma_{i\beta_i},$$

be a minimal set of generators for $\operatorname{Im} d_i$, *and* $u_i := |\{j \mid \deg \gamma_{ij} \leq t + i\}|$. *If* $u_i > 0$ *then* $\{\pi_{i-1}(\gamma_{i1}), \ldots, \pi_{i-1}(\gamma_{iu_i})\}$ *can be completed to a minimal set of generators for* $\operatorname{Im} d'_i$ *with elements of degree* $\geq t + i$.

Theorem 4.3 give a description of the graded Betti numbers of $\overline{A}$.

Corollary

$$\overline{\beta}_{ih}(\overline{A}) = \beta_{ih}(A), \text{ for } i \geq 0 \text{ and } h - i \leq t - 1.$$

Moreover $\overline{\beta}_{i\ t+i}(\overline{A}) \geq \beta_{i\ t+i}(A)$.

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Moreover $\overline{\beta}_{i\ t+i}(\overline{A}) \geq \beta_{i\ t+i}(A)$.

Corollary

$\beta_{n-1\ h}(A) = 0$ *for all* $h \leq t + n - 1$.

$$\beta(A) = \begin{pmatrix} \beta_{01} & \beta_{12} & \cdots & \beta_{n-1,n} \\ \beta_{02} & \beta_{13} & \cdots & \beta_{n-1,n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{0,t-1} & \beta_{1,t} & \cdots & \beta_{n-1,n+t-2} \\ \beta_{0,t} & \beta_{1,t+1} & \cdots & \beta_{n-1,n+t-1} \\ \beta_{0,t+1} & \beta_{1,t+2} & \cdots & \beta_{n-1,n+t} \\ \beta_{0,t+2} & \beta_{1,t+3} & \cdots & \beta_{n-1,n+t+1} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$\overline{\beta}(\overline{A}) = \begin{pmatrix} \beta_{01} & \overline{\beta}_{12} & \cdots & \overline{\beta}_{n-1,n} = 0 \\ \beta_{02} & \overline{\beta}_{13} & \cdots & \overline{\beta}_{n-1,n+1} = 0 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{0,t-1} & \overline{\beta}_{1,t} & \cdots & \overline{\beta}_{n-1,n+t-2} = 0 \\ \beta_{0,t} & \overline{\beta}_{1,t+1} & \cdots & \overline{\beta}_{n-1,n+t-1} = 0 \\ \overline{\beta}_{0,t+1} & \overline{\beta}_{1,t+2} & \cdots & \overline{\beta}_{n-1,n+t} = 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

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Corollary

$$\overline{\beta}_{1\ t+1}(\overline{A}) = \beta_{1\ t+1}(A) + \dim_k(\operatorname{Ker}\psi).$$

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$\overline{\beta}_{1\ t+1}(\overline{A}) = \beta_{1\ t+1}(A) + \dim_k(\operatorname{Ker}\psi)$.

Proposition

$\overline{\beta}_{i\ t+i}(\overline{A}) = \beta_{i\ t+i}(A)$ *for every* $i \geq 0$ *iff* ψ *is injective.*

$$\beta(A) = \begin{pmatrix} \beta_{01} & \beta_{12} & \cdots & \beta_{n-1,n} = 0 \\ \beta_{02} & \beta_{13} & \cdots & \beta_{n-1,n+1} = 0 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{0,t-1} & \beta_{1,t} & \cdots & \beta_{n-1,n+t-2} = 0 \\ \beta_{0,t} & \beta_{1,t+1} & \cdots & \beta_{n-1,n+t-1} = 0 \\ \beta_{0,t+1} & \beta_{1,t+2} & \cdots & \beta_{n-1,n+t} \\ \beta_{0,t+2} & \beta_{1,t+3} & \cdots & \beta_{n-1,n+t+1} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

ψ is injective iff

$$\overline{\beta}(\overline{A}) = \left(\begin{array}{llll} \beta_{01} & \beta_{12} & \cdots & \beta_{n-1,n} = 0 \\ \beta_{02} & \beta_{13} & \cdots & \beta_{n-1,n+1} = 0 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{0,t-1} & \beta_{1,t} & \cdots & \beta_{n-1,n+t-2} = 0 \\ \hline \beta_{0,t} & \beta_{1,t+1} & \cdots & \beta_{n-1,n+t-1} = 0 \\ \hline \overline{\beta}_{0\ t+1} & \overline{\beta}_{1,t+2} & \cdots & \overline{\beta}_{n-1,n+t} = 0 \\ 0 & 0 & \cdots & 0 \end{array}\right)$$

Collecting the previous results we can give a description of the graded Betti numbers of $\overline{A}$.

$$\overline{\beta}_{ij}(\overline{A}) = \begin{cases} \beta_{ij}(A) & \text{if } j \leq t+i-1 \\ \beta_{ij}(A) + m_i & \text{if } j = t+i \\ \sum_{h \geq i+1} (-1)^{h+i+1} \beta_{h\ j}(A) + (-1)^{i+1} \Delta^{c-1} \Delta H_A^+(j) + m_{i+1} & \text{if } j = t+i+1 \\ 0 & \text{if } j > t+i+1 \end{cases} \quad (1)$$

where $m_i \geq 0$ and in particular $m_0 = 0$ and $m_1 = \dim_k \ker \psi$.

Definition

We say that $A = R/I$ has the *Betti Weak Lefschetz Property*, briefly β-WLP, if there exists $\ell \in R_1$ such that

1. ℓ is a Weak Lefschetz form for A;
2. ψ_ℓ is injective, i.e. $\times\ell : (R/I)_t \to (R/(I_{\leq t}))_{t+1}$ is injective.

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Proposition

Let A be a standard graded R-algebra. The following are equivalent

1. *A has the β-WLP and ℓ is a β-WL form;*
2. *The graded Betti numbers of $\overline{A}$ are determined by (1) with $m_i = 0$ for every i.*

It is known from Harima Migliore Nagel Watanabe, (2003), that if H is a Weak Lefschetz sequence then the set

$$\mathcal{B}_H^{\mathrm{WL}} = \{\beta_A \mid H_A = H \text{ and } A \text{ has the WLP}\}$$

admits exactly one maximal element, say, β^H.

Proposition

Let H be a Weak Lefschetz sequence and let $A = R/I$ be an Artinian algebra with $H_A = H$ such that A has the WLP. If $\beta_{0\ t+1}(A) = \beta^H_{0\ t+1}$ then A has the β-WLP.

Let $R = k[x_1, \ldots, x_n]$. Let $A = R/I$ be a WL Artinian standard graded algebra, where $I = (g_1, \ldots, g_n)$ is a complete intersection ideal and $\deg g_i \leq \deg g_{i+1}$ for $1 \leq i \leq n-1$.

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Proposition

Let A be as above then

1. *If $\deg g_n > t$ then A has the β-WLP.*
2. *If $\deg g_n \leq t$ and $\Delta H_A(t+1) = 0$ then A has the β-WLP.*
3. *If $\deg g_n \leq t$ and $\Delta H_A(t+1) < 0$ then A has not the β-WLP.*

Let $R = k[x_1, \dots, x_n]$. Let $A = R/I$ be a WL Artinian standard graded algebra, where $I = (g_1, \dots, g_n)$ is a complete intersection ideal and $\deg g_i \leq \deg g_{i+1}$ for $1 \leq i \leq n-1$.

Proposition

Let A be as above then

1. *If* $\deg g_n > t$ *then A has the* β*-WLP.*
2. *If* $\deg g_n \leq t$ *and* $\Delta H_A(t+1) = 0$ *then A has the* β*-WLP.*
3. *If* $\deg g_n \leq t$ *and* $\Delta H_A(t+1) < 0$ *then A has not the* β*-WLP.*

The item 3 of the previous proposition in particular says that ψ_ℓ is not injective but still it has maximal rank.

Definition

We say that $A = R/I$ has the *generators Weak Lefschetz Property*, briefly β_0-WLP, if there exists $\ell \in R_1$ such that

1. ℓ is a Weak Lefschetz form for A;
2. ψ_ℓ has maximal rank.

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Note that for complete intersections the WLP and the β_0-WLP are equivalent.

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$$\beta\text{-WLP} \not\Leftarrow \beta_0\text{-WLP} \not\Leftarrow \text{WLP}.$$