Università degli Studi di Torino
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Term-ordering free involutive bases

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Giornate di Geometria Algebrica ed argomenti correlati

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Introduction

Term-ordering free involutive bases comes from the union of two different souls:

- an involutive soul;
- a term-ordering free soul.

Let us examine properly each of them.

Riquier

Riquier interprets derivatives $\frac{1}{\alpha_1!\cdots\alpha_n!}\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}}$, as terms $\tau = x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n} \in \mathcal{T}$, transforming the problem of *solving differential partial equations* in terms of *ideal membership*.

He introduced the concept (but not the notion) of S-polynomials and proved that if the normal form (Gauss-Buchberger reduction) of each S-polynomial among the elements of the basis $\mathcal{G}$ generating the system goes to zero then

- the given basis $\mathcal{G}$ generates the related ideal and the related problem *could be solvable*;
- a *solution* of the PDE is determined (and computed) as series in terms of initial conditions, formulated in terms of a decomposition of the related *escalier* N;

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If the normal form computation produces *conflicts* among the data then the PED has no solution.

EXAMPLE

The problem $\frac{\partial u}{\partial y} = f, \frac{\partial u}{\partial x} = g$ has no solution unless $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$;

If *no conflict* arose and *not all normal forms are 0*, then, exactly as in Buchberger Algorithm, the *non-zero normal forms are included in the basis* and the procedure is repeated.

Deglex ordering induced by $x_1 > x_2 > \cdots > x_n$, + large class of term-orderings to which his theory was applicable: *characterization of all term-orderings!*

Convergency: degree-compatible term-orderings.

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Janet I.

Janet, spured on by *Hadamard*, dedicated his doctorial thesis to a reformulation of Riquier's results in terms of Hilbert's results. Given $M \subset \mathcal{T}$, $|M| < \infty$, $\forall \tau \in M$ he associates a set of *multiplicative variables* and a subset of terms in (M) (*class* or *cone*) and considered M *complete* when the cones of M are a partition of (M).

Procédé régulier pour obtenir un système complet base d'un module donné, que ne pourra se prolonger indéfiniment: enlarge M with the elements $x\tau$, $\tau \in M$, x non-multiplicative for τ, not already in the union of cones.

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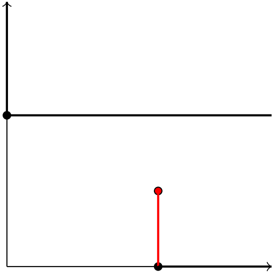
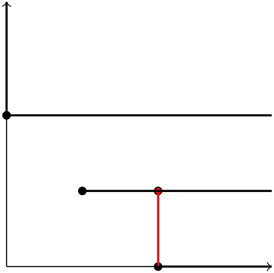


x^2, y^2



x^2, xy, y^2

JANET II.

Homogeneous case, adapting his approach to

- the solution of partial differential equation given by Cartan;
- the introduction by Delassus of the concept of *generic initial ideal* and its precise description given by Robinson and Gunther.

$I \subset k[x_1, x_2, \ldots, x_n]$ homogeneous (variables assumed generic). For each $1 \leq i \leq n$, and $p \in \mathbb{N}$:

$$\sigma_i^{(p)} := \#\{\tau \in \mathsf{N}(I), \deg(\tau) = p, \min(\tau) = i\}$$

fixes a value p and denotes $\sigma_i := \sigma_i^{(p)}$, *and* $\sigma_i' := \sigma_i^{(p+1)}$.

DEFINITION (JANET)

A finite set $E \subset \mathcal{P}$ of forms of degree at most p generating the ideal $I \subset P$, is said to be *involutive* if it satisfies the formula

$$\sum_{i=1}^{n} \sigma_i^{(p+1)} = \sum_{i=1}^{n} i\sigma_i^{(p)}. \tag{1}$$

□

The minimal degree $\bar{p}$ for which the formula is satisfied is *Castelnuovo-Mumford regularity*, and this was first noted by Malgrange.

FIRST STUDIES: THERE IS A TERM ORDER

$J \triangleleft \mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ a monomial ideal.

NOTARI-SPREAFICO

Stratum $\mathcal{S}t(J, \prec)$: family of all ideals of $\mathcal{P}$ whose initial ideal w.r.t. the term order $\prec$ is J.

The homogeneous stratum is denoted $\mathcal{S}t_h(J, \prec)$.

M.ROGGERO-L.TERRACINI, 2010

$\mathcal{S}t(J, \prec)$ and $\mathcal{S}t_h(J, \prec)$ have a natural structure of *affine schemes*.

A smooth stratum is always isomorphic to an affine space; strata and homogeneous strata w.r.t. any term ordering $\prec$ of every saturated Lex-segment ideal J are smooth.

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No term-ordering, please!

Admissible *Hilbert polynomial* $p(t)$ in $\mathbb{P}^n$, $deg(p(t)) = d$.
Hilbert scheme $\mathcal{H}ilb^n_p(t)$ realized as closed subscheme of a Grassmannian $\mathbb{G}$, so "globally defined by homogeneous equations in the Plucker coordinates of $\mathbb{G}$" + "covered by open subsets (non-vanishing of a Plucker coordinate), embedded as closed subschemes of $\mathbb{A}^D$, $D = dim(\mathbb{G})$".

Too many Plucker coordinates: computations *impossible*!
→ (Bertone,Lella, Roggero, 2013) *new open cover*, marked schemes over Borel-fixed ideals: really a few!
→ constructive proofs and use a polynomial reduction process, similar to the one for Groebner bases, but are *term-ordering free*.

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The problem for strongly stable ideals

Strongly stable: monomial ideal $J \triangleleft \mathbf{k}[x_1, ..., x_n]$ s.t. $\forall \tau \in J$ and $\forall x_i,\ x_j$ s.t. $x_i | \tau$ and $x_i < x_j$, then $\frac{\tau x_j}{x_i} \in J$.

Example

$J = (x^3, y) \triangleleft \mathbf{k}[x, y]$, $x < y$:

$$\frac{x^3}{x} y = x^2 y \in J$$

Let J be a strongly stable monomial ideal in $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$: characterization of the family $\mathcal{M}f(J)$ of all homogeneous ideals $I \triangleleft \mathcal{P}$ such that the set of all terms outside J is a $\mathbf{k}$-vector basis of the quotient $\mathcal{P}/I$.

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Main Results

- $I \in Mf(J)$ if and only if it is generated by a *J-marked basis* (Cioffi-Roggero, 2013) $\rightarrow$ generalization of Groebner bases;
- *Buchberger-like criterion* for J-marked bases (Cioffi-Roggero, 2013);
- $\mathcal{M}f(J)$ can be endowed with a structure of affine scheme: *J-marked scheme* (Cioffi-Roggero, 2013);
- *superminimal reduction* (Bertone, Cioffi, Lella, Roggero, 2012) $\rightarrow$ fast!
- division algorithm which works in an *affine* context: $[J, m]-$marked bases (Bertone, Cioffi, Roggero, 2012);
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THE PROBLEM.

$J \triangleleft P$ *monomial ideal* $\rightarrow$ characterization for $\mathcal{M}f(J)$, family of all *homogeneous* ideals $I \triangleleft P$ s.t. P/I *free* A-module with *basis* $\mathrm{N}(J)$.

I s.t. $J = In_<(I)$: proper *subset* of $\mathcal{M}f(J) \Rightarrow$ overcome Groebner framework.

Whole family $\mathcal{M}f(J)$ for J *strongly stable* $\rightarrow$ limiting condition. However, they are optimal for the effective study of the Hilbert scheme.

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Riquier-Janet decomposition

We recall Janet's decomposition for terms in the semigroup ideal generated by M into disjoint classes.
Each of them contains:

1. a term $\tau \in M$;
2. the set of monomials obtained multiplying τ by products of multiplicative variables, that we call *cone* of and denote $C(\{\tau\})$.

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We remove the finiteness condition on M.
Let $M \subset \mathcal{T}$ be a set of terms and $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be an element of M. A variable x_j is called *Janet-multiplicative* (or J-multiplicative) for τ w.r.t. M if there is no term in M of the form $\tau' = x_1^{\beta_1} \cdots x_j^{\beta_j} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$ with $\beta_j > \alpha_j$.
We denote by $M_J(\tau, M)$ the set of J-multiplicative variables for τ w.r.t. M.

The J-*cone* of τ w.r.t. M is the set

$$C(\{\tau\}) := \{\tau x_1^{\lambda_1} \cdots x_n^{\lambda_n} \,|\, \text{where } \lambda_j \neq 0 \text{ only if } x_j \in M_J(\tau, M)\}.$$

EXAMPLE (1)

Take $M = \{x_1^3, x_2^3, x_1^4x_2x_3, x_3^2\} \subseteq \mathbf{k}[x_1, x_2, x_3]$
Then: $M_J(x_1^3, M) = \{x_1\}$: no $x_1^h x_2^0 x_3^0$, $h > 3$, but we have $x_1^0 x_2^3 x_3^0$ and $x_1^4 x_2 x_3$
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Example (2)

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Observe that, by definition of multiplicative variable, the only element in $C(\{\tau\}) \cap M$ is τ itself.
Indeed, if $\tau \in M$ and also $\tau\sigma \in M$ for a non constant term σ, then $max(\sigma)$ cannot be multiplicative for τ, hence $\tau\sigma \notin C(\{\tau\})$.

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In 1924, Janet defines multiplicative variables as before and he provides both a decomposition for the semigroup ideal $\mathsf{T}(M)$ generated by a finite set of terms M and a decomposition for the complementary set $\mathsf{N}(M)$.
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Definition

A variable x_j is *Pommaret-multiplicative* or *P-multiplicative* for $\tau \in \mathcal{T}$ if and only if $x_j \leq \min(\tau)$.
The P-*cone* of τ is the set

$$C(\{\tau\}) := \{\tau x_1^{\lambda_1} \cdots x_n^{\lambda_n} \,|\, \text{where } \lambda_j \neq 0 \text{ only if } x_j \in M_P(\tau, M)\}.$$

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The two definitions of multiplicative variables appear to be very different (but they are *equivalent* in Janet's context).
In the first formulation, the set of multiplicative variables for a term in M depends on the whole set M, whereas in the second it is completely independent on the set M: the two notions are not equivalent for a general M:

Example

In $k[x_1, x_2, x_3]$ consider the ideal $I = (x_1^2 x_2, x_1 x_2^2)$ and let M be its monomial basis. Then, $M_J(x_1^2 x_2, M) = \{x_1, x_3\}$ and $M_J(x_1 x_2^2, M) = \{x_1, x_2, x_3\}$, whereas only x_1 is P-multiplicative.
Clearly also Janet and Pommaret cones do not coincide:

$C_J(x_1^2 x_2) = \{x_1^h x_2 x_3^l,\ h \geq 2,\ l \geq 0\}$
$C_P(x_1^2 x_2) = \{x_1^h x_2,\ h \geq 2\}$
$C_J(x_1 x_2^2) = \{x_1^h x_2^k x_3^l,\ h \geq 1,\ k \geq 2,\ l \geq 0\}$
$C_P(x_1 x_2^2) = \{x_1^h x_2^2,\ h \geq 1\}$

$M \subset \mathcal{T}$ is called *complete* if for every $\tau \in M$ and $x_j \notin M_J(\tau, M)$, there exists $\tau' \in M$ such that $x_j\tau \in C_J(\{\tau'\})$.

Example

All singletons are complete!

M is *stably complete* if it is complete and for every $\tau \in M$ it holds $M_J(\tau, M) = \{x_i \mid x_i \leq \min(\tau)\}$.

If M is stably complete and finite, then it is the *Pommaret basis* $\mathcal{H}(J)$ of $J = (M)$.

Example

$M = \{x^2, xy, y^2\} \subset \mathbf{k}[x, y]$, $x < y$.
$M_J(x^2, M) = M_P(x^2, M) = \{x\}$, $M_J(xy, M) = M_P(xy, M) = \{x\}$,
$M_J(y^2, M) = M_P(y^2, M) = \{x, y\}$.
Moreover, $x^2y \in C(\{xy\})$, $xy^2 \in C(\{y^2\})$.

Example

Let M be the set of terms $\{x, y^2\}$ in $k[x, y]$, with $x < y$.
The multiplicative variables for every term in M are those lower than or equal to its minimal one:

$$M_J(x, M) = \{x\}$$
$$M_J(y^2, M) = \{x, y\}.$$

However, M is *not complete* since yx does not belong to the J-cone of any term in M.

Let M be a set of terms (possibly infinite).
If $\tau, \tau' \in M$ and $\tau \neq \tau'$, then $C(\{\tau\}) \cap C_J(\{\tau'\}) = \emptyset$.
If, moreover, M is complete and $\mathsf{T}(M)$ is the semigroup ideal it generates, then $\forall \gamma \in \mathsf{T}(M)$, $\exists \tau \in M$ such that $\gamma \in C_J(\{\tau\})$.
Hence, the J-cones of the elements in M give a *partition* of $\mathsf{T}(M)$.

Each term in $\mathsf{T}(M)$ can be written in a *unique way* as a product of

1. an element $\tau \in M$;
2. a term $x^\eta = x_i^{\eta_i} \cdots x_j^{\eta_j}$, with $x_i, ..., x_j \in M_J(\tau, M)$.

DEFINITION

Let M be a complete system of terms. The *star decomposition* of every term $\gamma \in (M)$ w.r.t. M, is the *unique couple* of terms (τ, η), with $\tau \in M$, such that $\gamma = \tau\eta$ and $\gamma \in C_J(\{\tau\})$. If (τ, η) is the star decomposition of γ w.r.t. M, we will write $\gamma = \tau *_M \eta$.

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The star set

Given a monomial ideal $J \triangleleft P$ we define the *star set* as

$$\mathcal{F}(J) := \{x^\alpha \in \mathcal{T} \setminus \mathsf{N}(J) \mid \frac{x^\alpha}{\min(x^\alpha)} \in \mathsf{N}(J)\}.$$

For every monomial ideal J, the star set $\mathcal{F}(J)$ is the *unique stably complete* system of generators of J. Hence, if M is stably complete, $M = \mathcal{F}((M))$.

In this context we have

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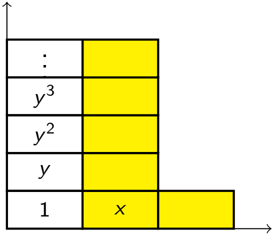
For an arbitrary monomial ideal J, $\mathcal{F}(J)$ can be *infinite*.
For example, if $J = (x) \lhd k[x, y]$, $x < y$, then
$\mathcal{F}(J) = \{xy^n \mid n \in \mathbb{N}\}$.



Not all the complete systems turn out to be of the form of a *star set*.

For example, the complete system $M = \{x^h y,\ h \geq 1\} \subseteq k[x, y]$ is not the star set of the ideal $J := (M)$.

Indeed, $\mathrm{N}(J) = \{x^m,\ m \geq 0\} \cup \{y^l,\ l > 0\}$ and all the terms of the form xy^k, $k > 1$, do not belong to M, even if $\frac{xy^k}{\min(xy^k)} = y^k \in \mathrm{N}(M)$.

Moreover, for $h > 1$, $\frac{x^h y}{x} = x^{h-1} y \in M$, so $x^h y \notin \mathcal{F}(J)$.

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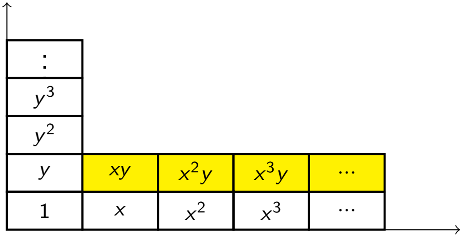

A monomial ideal J is

1. *stable* if $\tau \in J,\ x_j > \min(\tau) \Rightarrow \frac{x_j \tau}{\min(\tau)} \in J$
2. *quasi stable* if $\tau \in J,\ x_j > \min(\tau) \Rightarrow \exists t \geq 0 :\ \frac{x_j^t \tau}{\min(\tau)} \in J.$

J monomial ideal,TFAE:

I) J stable

II) $\mathcal{F}(J) = \mathrm{G}(J)$

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B) $|\mathcal{F}(J)| < \infty$

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In $k[x, y, z]$ with $x < y < z$:

- considered $J = (z, y^2)$, we get $M = \mathcal{F}(J) = \mathrm{G}(J) = \{z, y^2\}$, since J is *stable*;
- taken the ideal $J' = (z^2, y)$, we get $M = \mathcal{F}(J) = \{z^2, yz, y\} \supset \mathrm{G}(J)$.
  In fact, J is *quasi stable*, but it is not stable;
- given $J = (y)$, the star set is $M = \mathcal{F}(J) = \{z^k y \mid k \geq 0\}$, and it holds $|\mathcal{F}(J)| = \infty$, since J is *not stable*.

We generalize the notions of J-marked polynomial, J-marked basis and J-marked family given for J strongly stable.

DEFINITION

Let M be a complete system of terms and J be the ideal it generates.

- A M-*marked set* is a set $\mathcal{G}$, not necessarily finite, containing, $\forall x^\alpha \in M$, a homogeneous (monic) marked polynomial $f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma$, with $\mathrm{Ht}(f_\alpha) = x^\alpha$ and $\mathrm{Supp}(f_\alpha - x^\alpha) \subset \mathsf{N}(J)$, so that $|\mathrm{Supp}(f) \cap J| = 1$.
- A M-*marked basis* $\mathcal{G}$ is a M-marked set such that $\mathsf{N}(J)$ is a basis of $P/(\mathcal{G})$ as A-module, i.e. $P = (\mathcal{G}) \oplus \langle \mathsf{N}(J) \rangle$ as an A-module.
- The M-*marked family* $\mathcal{M}f(M)$ is the set of all homogeneous ideals I that are generated by a M-marked basis.

Defining the reduction

Definition

Let M be a *complete* system and $\mathcal{G}$ a M-marked set. $\xrightarrow{\mathcal{G}}$ transitive closure of the relation $h \xrightarrow{\mathcal{G}} h - cf_\alpha x^\eta$, where $x^\alpha x^\eta = x^\alpha *_M x^\eta$ is a term appearing in h with a non-zero coefficient c.
$\xrightarrow{\mathcal{G}}$ *noetherian* if the length r of any sequence

$$h = h_0 \xrightarrow{\mathcal{G}} h_1 \xrightarrow{\mathcal{G}} \dots \xrightarrow{\mathcal{G}} h_r$$

is bounded by an integer number $m = m(h)$ (*NOT in general*).
Equivalently, if we continue rewriting terms in this way we obtain, after a finite number of reductions, a polynomial with support in $\mathrm{N}(J)$.

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The relation $\xrightarrow{\mathcal{G}}$ generalizes to a term-ordering free context, the concept of *involutive polynomial reduction* by Blinkov and Gerdt.

Let $M := \{xz, yz, y^2\}$ a set of terms in $k[x, y, z]$ with $x < y < z$. We find the following sets of multiplicative variables:

- $M_J(xz, M) = \{x, z\}$
- $M_J(y^2, M) = \{x, y\}$
- $M_J(yz, M) = \{x, y, z\}$

and one can check that M is complete.
Let $\mathcal{G}$ the M-marked set $\{f_{xz} = xz - xy,\ f_{yz} = yz - z^2,\ f_{y^2} = y^2\}$.
Then we have the *infinite sequence of reductions*:

$$xz^2 = xz *_M z \xrightarrow{G} xz^2 - f_{xz}z = xyz = yz *_M x \xrightarrow{\mathcal{G}} xyz - f_{yz}x = xz^2$$

QUEST FOR NOETHERIANITY

We define the following special subset of the ideal $(\mathcal{G})$ in order to prove that the reduction $\xrightarrow{\mathcal{G}}$ is always noetherian if $\mathcal{G}$ is marked on a stably complete system.

DEFINITION

Let $\mathcal{G}$ be a M-marked set on a complete system of terms M and let $J := (M)$. For each degree s, we denote by $\mathcal{G}^{(s)}$ the set of homogeneous polynomial

$$\mathcal{G}^{(s)} := \{f_\alpha x^\eta \mid x^\alpha *_M x^\eta \in (M)_s\}$$

marked on the terms of J_s in the natural way $\mathrm{Ht}(f_\alpha x^\eta) = x^\alpha x^\eta$.

Let $\mathcal{G}$ be a complete M-marked set on the stably system of terms $M = \mathcal{F}(J)$.

1. Every term in $\mathrm{Supp}(x^\beta x^\epsilon - f_\beta x^\epsilon)$ either belongs to $\mathrm{N}((M))$ or is of the type $x^\alpha *_M x^\eta$ with $x^\eta <_{Lex} x^\epsilon$.
2. If $f_\beta \in \mathcal{G}$, then all the polynomials $f_{\alpha_i} x^{\eta_i} \in \mathcal{G}^{(s)}$ used in the reduction of $x^\beta x^\epsilon$ (except $f_\beta x^\epsilon$ if it belongs to $\mathcal{G}^{(s)}$) are such that $x^\epsilon >_{Lex} x^{\eta_i}$.
3. If $g = \sum_{i=1}^m c_i f_{\alpha_i} x^{\eta_i}$, with $c_i \in k - \{0\}$ and $f_{\alpha_i} x^{\eta_i} \in \mathcal{G}^{(s)}$ pairwise different, then $g \neq 0$ and its support contains some term of the ideal J.

The reduction theorem

Let $\mathcal{G}$ be a M-marked set on a *stably complete* system of terms M and let J be the ideal generated by M.

Then the reduction process $\xrightarrow{\mathcal{G}}$ is *noetherian* and, for every integer s, $P_s = \langle \mathcal{G}^{(s)} \rangle \oplus \langle \mathsf{N}(J)_s \rangle$.

Indeed, for every $h \in P_s$

$h = f + g$ with $f \in \langle \mathcal{G}^{(s)} \rangle$ and $g \in \langle \mathsf{N}(J)_s \rangle \Longleftrightarrow h \xrightarrow{\mathcal{G}}_* g$ and $f = h - g$

Marked Basis

Theorem

Let $\mathcal{G}$ be a $\mathcal{F}(J)$-marked set. Then:

$$(\mathcal{G}) \in \mathcal{M}f(J) \Longleftrightarrow \forall f_\beta \in \mathcal{G},\ \forall x_i > \min(x^\beta) :\ f_\beta x_i \xrightarrow{\mathcal{G}}_* 0$$

This is a term-ordering free generalization of the concept of *local involutivity*, defined by Blinkov and Gerdt $\rightarrow$ general theory for involutivity.

First reduction step of $f_\beta x_i$: rewrite $x^\beta x_i$ throughout $f_\alpha x^\eta$ with $x^\beta x_i = x^\alpha x^\eta \in C_J(\{x^\alpha\})$.
We get $f_\beta x_i \xrightarrow{\mathcal{G}} f_\beta x_i - f_\alpha x^\eta$, the *S-polynomial*
$S(f_\beta, f_\alpha) := \frac{lcm(x^\beta, x^\alpha)}{x^\beta} f_\beta - \frac{lcm(x^\beta, x^\alpha)}{x^\alpha} f_\alpha$.
The reduction theorem becomes

$$(\mathcal{G}) \in \mathcal{M}f(J) \Longleftrightarrow \forall f_\alpha, f_\beta \in \mathcal{G} : \ S(f_\alpha, f_\beta) \xrightarrow{\mathcal{G}}_* 0.$$

But it is sufficient to check a special subset of the S-polynomials.
If J is quasi stable ($|\mathcal{F}(J)| < \infty$) this subset corresponds to the basis for the first syzygies of the terms in $\mathcal{F}(J)$.
The maximal degree of these special S-polynomials cannot exceed $1 + \max\{\deg(x^\alpha) \mid x^\alpha \in \mathcal{F}(J)\}$.
Indeed, if J is quasi stable, $reg(J) = \max\{deg(\tau), \tau \in \mathcal{F}(J)\}$.

If $\mathcal{G}$ M-marked set, but not M-marked basis, then $\exists f_\alpha, f_\beta \in \mathcal{G}$, s.t. $S(f_\alpha, f_\beta) = x^\eta f_\alpha - x^\gamma f_\beta \xrightarrow{\mathcal{G}}_* h \neq 0$.
Take $x^\eta f_\alpha$, 2 different *terminating reduction processes*, leading to:

1. the reduction $x^\eta f_\alpha \xrightarrow{f_\alpha} 0$, w.r.t. the polynomial f_α, different from our reduction procedure;
2. the reduction process described above $x^\eta f_\alpha \xrightarrow{\mathcal{G}} x^\eta f_\alpha - x^\gamma f_\beta \xrightarrow{\mathcal{G}}_* h \neq 0$.

On the other hand, if $\mathcal{G}$ is a M-marked basis, $\forall f \in \mathcal{P}$, $\exists! h \in \langle \mathrm{N}(J) \rangle$, such that $f - h \in (\mathcal{G})$. Any reduction process, applied to f, *either* gives h as outcome *or* it does *not terminate*.

Not all Marked bases are Groebner bases!!

Let J be the monomial ideal (x^3, xy, y^3) in $k[x, y]$ with $x < y$. Its star set is $\mathcal{F}(J) = \{x^3, xy, xy^2, y^3\}$.
The $\mathcal{F}(J)$-marked set
$\mathcal{G} := \{f_1 := \mathbf{x^3},\ f_2 := \mathbf{xy} - x^2 - y^2,\ f_3 := \mathbf{xy^2},\ f_4 = \mathbf{y^3}\}$ is a $\mathcal{F}(J)$-market basis:

- $yf_1 = xf_1 + x^2 f_2 + xf_3 \xrightarrow{\mathcal{G}}_* 0$,
- $yf_2 = f_1 - xf_2 - f_4 \xrightarrow{\mathcal{G}}_* 0$
- $yf_3 = xf_4 \xrightarrow{\mathcal{G}}_* 0$.

This is a simple example of a marked basis which is not a Gröbner basis. In fact, it is obvious that $\mathrm{Ht}(f_2) = xy$ cannot be the leading term of f_2 with respect to any term-ordering and, more generally, that J cannot be the initial ideal of the ideal $(\mathcal{G})$, even though $(\mathcal{G}) \oplus \mathrm{N}(J) = k[x, y]$.

Our bases are involutive!

With the notation due to Janet, if J is a quasi stable monomial ideal, then

$$\sum_{i=1}^{n} \sigma_i^{(p+1)}(J) = \sum_{i=1}^{n} i\sigma_i^{(p)}(J).$$

The same equality holds if I is a homogeneous ideal generated by a J-marked basis $\mathcal{G}$ with J quasi stable.
Therefore $\mathcal{G}$ is an involutive basis.

Note that for an ideal I generated by a J-marked set $\mathcal{G}$ which is not a marked basis, only the inequality $\sum_{i=1}^{n} \sigma_i^{(p+1)} \leq \sum_{i=1}^{n} i\sigma_i^{(p)}$ holds true.

Our bases are involutive!

With the notation due to Janet, if J is a quasi stable monomial ideal, then

$$\sum_{i=1}^{n} \sigma_i^{(p+1)}(J) = \sum_{i=1}^{n} i\sigma_i^{(p)}(J).$$

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Therefore $\mathcal{G}$ is an involutive basis.

Note that for an ideal I generated by a J-marked set $\mathcal{G}$ which is not a marked basis, only the inequality $\sum_{i=1}^{n} \sigma_i^{(p+1)} \leq \sum_{i=1}^{n} i\sigma_i^{(p)}$ holds true.

THE STRUCTURE OF SCHEME

Let $M = \{x_1^\alpha, \ldots, x_s^\alpha\}$ and consider $B := A[C]$, where C is a compact notation for the set of variables $C_{i,\beta}$ $i = 1, \ldots, s$ and $x^\beta \in \mathrm{N}(J)_{|\alpha_i|}$.
M-marked set in $B[x_1, \ldots, x_n]$

$$\mathcal{G} := \{f_{\alpha_i} := x^{\alpha_i} + \sum C_{i,\beta} x^\beta \mid x^\beta \in \mathrm{N}(J)_{|\alpha_i|}, \mathrm{Ht}(f_{\alpha_i}) = x^{\alpha_i}\}.$$

Each M-marked set can be obtained specializing $\mathcal{G}$, as $\phi(\mathcal{G})$ for a suitable morphism of A-algebras $\phi : A[C] \to A$.
By the uniqueness of the M-marked basis generating each ideal in $\mathcal{M}f(J)$, $\forall I \in \mathcal{M}f(J)$, $\exists! \phi$ s.t. $(\phi(\mathcal{G})) = I$.
Construct a set of polynomials $\mathcal{R}$ that will define the scheme we associate to M. If $g \in B[x_1, \ldots, x_n]$, $\mathrm{coeff}_x(g)$ is the set of coefficients of g w.r.t. $x_1, \ldots, x_n$; hence $\mathrm{coeff}_x(g) \subset B = A[C]$ is a set of polynomials in the variables C.

The structure of scheme

Let $M = \{x_1^\alpha, ..., x_s^\alpha\}$ and consider $B := A[C]$, where C is a compact notation for the set of variables $C_{i,\beta}$ $i = 1, \ldots, s$ and $x^\beta \in \mathrm{N}(J)_{|\alpha_i|}$.
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$\forall x^{\alpha_i} \in M$ and $x_j > \min(x^{\alpha_i})$, let $g_{\alpha_i,j} \in B[x_1, \ldots, x_n]$ be such that $f_{\alpha_i} x_j \xrightarrow{\mathcal{G}}_* g_{\alpha_i,j}$.

Definition

Let M be a stably complete system in $\mathcal{T}$, A be any ring, and $\mathcal{R}$ be the union of $\mathrm{coeff}_x(g_{\alpha_i,j})$ for every $x^{\alpha_i} \in M$ and $x_j > \min(x^{\alpha_i})$. We will call M-*marked scheme* over the ring A, and denote with $\mathbf{Mf}_M(A)$ the affine scheme $\mathrm{Spec}(A[C]/(\mathcal{R}))$.

Every M-marked set in $A[x_1, \ldots, x_n]$ is a M-marked basis if and only if the coefficients of the terms in the tails satisfy the conditions given by $\mathcal{R}$.

In particular, if $A = k$ is an algebraically closed field, then the closed points of $\mathbf{Mf}_M(A)$ correspond to the ideals in the marked family $\mathcal{M}f(J)$ where J is the ideal in $k[x_1, \ldots, x_n]$ generated by M.

The above construction of $\mathcal{R}$ is in fact *independent* from the fixed commutative ring A, in the sense that it is preserved by extension of scalars. We can first choose $\mathbb{Z}$ as the coefficient ring and then apply the standard map $\mathbb{Z} \to A$.
More formally, for every stably complete set of terms M we can define a functor between the category of rings to the category of sets

$$\underline{\mathbf{Mf}}_M : \underline{Rings} \to \underline{\mathrm{Set}}$$

that associates to any ring A the set
$\underline{\mathbf{Mf}}_M(A) := \mathcal{M}f(MA[x_1, \ldots, x_n])$ and to any morphism $\phi : A \to B$ the map

$$\begin{array}{rccc} \underline{\mathbf{Mf}}_J(\phi) : & \underline{\mathbf{Mf}}_M(A) & \longrightarrow & \underline{\mathbf{Mf}}_M(B) \\ & \mathsf{I} & \longmapsto & I \otimes_A B. \end{array}$$

Moreover, it is possible to prove that $\underline{\mathbf{Mf}}_M$ is a representable functor represented by the scheme $\mathbf{Mf}_M(\mathbb{Z}) = \mathrm{Spec}(\mathbb{Z}[C]/(\mathcal{R}))$.

Thanks for your attention!