ON A REMARK BY DANIEL FERRAND

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ABSTRACT. Let X be a Fano threefold with index i_X and fundamental line bundle $\mathcal{O}_X(h)$. We classify μ–semistable rank two bundles $\mathcal{E}$ on X with $c_1(\mathcal{E}) = 0$, $h^0(\mathcal{E}) \neq 0$ and $h^1(\mathcal{E}(-\lceil \frac{i_X}{2} \rceil h)) = 0$.

1. INTRODUCTION

An instanton bundle on the projective space $\mathbb{P}^3$ over the complex field $\mathbb{C}$ is a rank two bundle such that $h^0(\mathcal{E}) = h^1(\mathcal{E}(-2)) = 0$ and $c_1(\mathcal{E}) = 0$.

Instanton bundles on $\mathbb{P}^3$ were first introduced in the seminal paper [ADHM] and widely studied from different viewpoints since the discovery of their connection, through the Atiyah–Penrose–Ward transformation, with the solutions of the Yang–Mills equations (see [AW]).

The projective space $\mathbb{P}^3$ is the very first example of *Fano threefold*, that is of a smooth threefold X whose anticanonical line bundle ω_X^{-1} is ample: see [IP] for a survey on the classification and properties of such threefolds. In particular, the *index of X* is the greatest $i_X \in \mathbb{Z}$ such that $\omega_X \cong \mathcal{O}_X(-i_X h)$ for an ample $\mathcal{O}_X(h) \in \operatorname{Pic}(X)$. It is well known that $1 \leq i_X \leq 4$ and that such an $\mathcal{O}_X(h)$ is uniquely determined: it is called the *fundamental line bundle* of X. Notice that $i_X = 4$ if and only if $X \cong \mathbb{P}^3$ and $i_X = 3$ if and only if $X \subset \mathbb{P}^4$ is a smooth quadric.

Because of its importance, the notion of instanton bundle has been extended to Fano threefolds. In [F, Ku] the authors defined for the first time instanton bundles on Fano threefolds X with Picard number $\varrho_X = 1$. In a second stage such a restriction has been removed (e.g. see [MMPL, CCGM]). In particular, in [ACG] (see also [AC]) the following general definition has been suggested. In order to fix the notation, if $\varepsilon \in \{0, 1\}$ we set

$$q_X^\varepsilon = \left\lceil \frac{i_X - \varepsilon}{2} \right\rceil.$$

An *instanton bundle* $\mathcal{E}$ on X is a rank two bundle $\mathcal{E}$ such that the following properties hold:

- $c_1(\mathcal{E}) = -\varepsilon h$, where $\varepsilon \in \{0, 1\}$;
- $\mathcal{E}$ is μ–semistable with respect to $\mathcal{O}_X(h)$ and $(1 - \varepsilon)h^0(\mathcal{E}) = 0$;
- $h^1(\mathcal{E}(-q_X^\varepsilon h)) = 0$.

The instanton bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = -\varepsilon h$ is called even or odd according to the parity of ε.

On the one hand, the vanishing $(1 - \varepsilon)h^0(\mathcal{E}) = 0$ is equivalent to the μ–stability of $\mathcal{E}$ when X has Picard number $\varrho_X = 1$ and $\mathcal{E}$ is even. On the other hand, if $\varrho_X = 1$ and $\mathcal{E}$ is odd, then the notions of μ–stability and μ–semistability actually

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coincide (see for example Lemma 2.2). This is no longer true if $\varrho_X \geq 2$, because many pathologies can occur in this case, as showed in [MMPL, AM].

The above discussion motivates our interest in classifying rank two bundles $\mathcal{E}$ on a Fano threefold X such that $c_1(\mathcal{E}) = 0$, $h^1(\mathcal{E}(-q_X^0 h)) = 0$, $h^0(\mathcal{E}) \neq 0$ and which are μ–semistable with respect to $\mathcal{O}_X(h)$.

Certainly $\mathcal{O}_X^{\oplus 2}$ is strictly μ–semistable, it satisfies the above restrictions and it always occurs on each Fano threefold. Hence our interest is focused on the case $\mathcal{E} \not\cong \mathcal{O}_X^{\oplus 2}$: in this case we are able to prove the result below in Section 3.

Theorem 1.1. *Let X be a smooth Fano threefold with fundamental line bundle $\mathcal{O}_X(h)$.*

Let $\mathcal{E} \not\cong \mathcal{O}_X^{\oplus 2}$ be a μ–semistable rank two bundle on X with $c_1(\mathcal{E}) = 0$, $h^0(\mathcal{E}) \neq 0$ and $h^1(\mathcal{E}(-q_X^0 h)) = 0$.

Then $i_X \leq 3$, $\mathcal{E}$ is indecomposable, $h^0(\mathcal{E}) = 1$ and each non–zero section of $\mathcal{E}$ vanishes exactly on the same locally complete intersection curve $Z \subset X$ such that $\omega_Z \cong \mathcal{O}_Z \otimes \mathcal{O}_X(-i_X h)$ Moreover, the following assertions hold:

1. *if $i_X = 3$, then the reduced structure of Z is a union of disjoint h–lines, Z is everywhere nonreduced and each connected component of Z has even h–degree;*
2. *if $i_X = 2$, then the reduced structure of Z is a union of disjoint h–lines;*
3. *if $i_X = 1$, then each connected component of Z has even h–degree and its reduced structure is a seminormal tree of smooth rational curves.*

Throughout the whole paper we work over the complex numbers. By *curve (inside a scheme X)* we mean any closed subscheme of pure dimension 1 contained in X. Moreover, for the notions of h–degree, h–line and seminormal tree of smooth rational curves we refer the reader to Definitions 3.1 and 3.2 respectively.

When $\varrho_X = 1$, an immediate corollary of the above theorem gives a complete characterization of strictly μ–semistable rank two bundles $\mathcal{E}$ on X with $c_1(\mathcal{E}) = 0$ and $h^1(\mathcal{E}(-q_X^0 h)) = 0$, because in this case it is automatically true that $h^0(\mathcal{E}) \neq 0$ (see Lemma 2.2).

In particular, the above result is well known when $X \cong \mathbb{P}^3$: see [OSS, Remark at page 136] where a draft of its proof is given in this particular case. Moreover, it is also well known that the similar problem of classifying semistable (in the sense of Gieseker) but not μ–stable rank two bundles $\mathcal{E}$ on X with $c_1(\mathcal{E}) = 0$ leads immediately to the isomorphism $\mathcal{E} \cong \mathcal{O}_X^{\oplus 2}$: see Remark 3.6.

In Section 4 we first collect several examples of rank two bundles showing that all the cases above actually occur when $i_X \geq 2$. We close that section with Example 4.4 where we deal with the case $i_X = 1$ and $\varrho_X = 1$, which needs some more care. Indeed, in this case, the picture is slightly less precise than the case $i_X \geq 2$, because, as we show, it is not possible to find sharper bounds on the degree of the reduced structure of the connected components of Z.

Finally, in Section 5, we carefully deal with strictly μ–semistable rank two bundles $\mathcal{E}$ on the smooth quadric hypersurface $X \subset \mathbb{P}^4$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E})h = 2$.

2. General results

In this section we recall some definitions and general facts which will be used later.

For the reader's convenience we recall the Riemann–Roch formula for a locally free sheaf $\mathcal{A}$ on a smooth threefold X:

$$\begin{aligned}\chi(\mathcal{A}) = \operatorname{rk}(\mathcal{A})\chi(\mathcal{O}_X) + \frac{1}{6}(c_1(\mathcal{A})^3 - 3c_1(\mathcal{A})c_2(\mathcal{A}) + 3c_3(\mathcal{A}))\\ - \frac{1}{4}(\omega_X c_1(\mathcal{A})^2 - 2\omega_X c_2(\mathcal{A})) + \frac{1}{12}(\omega_X^2 c_1(\mathcal{A}) + c_2(\Omega_X^1)c_1(\mathcal{A})).\end{aligned} \tag{2.1}$$

In what follows we recall some helpful definitions and facts concerning the stability notion of vector bundles.

Definition 2.1. Let X be a smooth irreducible variety of dimension $n \geq 1$ endowed with an ample line bundle $\mathcal{O}_X(H)$.

If $\mathcal{A}$ is any torsion–free sheaf we define the *slope of $\mathcal{A}$ (with respect to $\mathcal{O}_X(H)$)* as

$$\mu(\mathcal{A}) := \frac{c_1(\mathcal{A})H^{n-1}}{\operatorname{rk}(\mathcal{A})}.$$

The torsion–free sheaf $\mathcal{A}$ is *μ–semistable* (resp. *μ–stable*) if for all proper subsheaves $\mathcal{B}$ with $0 < \operatorname{rk}(\mathcal{B}) < \operatorname{rk}(\mathcal{A})$ we have $\mu(\mathcal{B}) \leq \mu(\mathcal{A})$ (resp. $\mu(\mathcal{B}) < \mu(\mathcal{A})$).

The problem of determining when a vector bundle $\mathcal{E}$ on a variety X is μ–stable, or even only μ–semistable, is in general hard without any further assumptions on X and $\mathcal{E}$. On the other hand, when $\varrho_X = 1$, we have the following well known result.

Lemma 2.2. *Let X be a smooth irreducible variety of dimension $n \geq 1$ and let $\mathcal{O}_X(H)$ be an ample line bundle which generates* $\operatorname{Pic}(X)$*.*

If $\mathcal{E}$ is a rank two bundle on X, then the following assertion hold:

1. *if $c_1(\mathcal{E}) \in \{\ 0, -H\ \}$, then $\mathcal{E}$ is μ–stable if and only if $h^0(\mathcal{E}) = 0$;*
2. *if $c_1(\mathcal{E}) = 0$, then $\mathcal{E}$ is μ–semistable if and only if $h^0(\mathcal{E}(-H)) = 0$;*
3. *if $c_1(\mathcal{E}) = -H$, then $\mathcal{E}$ is μ–stable if and only if it is μ–semistable.*

Proof. See [OSS, Lemmas II.1.2.3 and II.1.2.5]: though the results are stated therein for $X = \mathbb{P}^n$, it is easy to check that the proofs hold for every X and H as in the statement, when H is effective. Passing to $mH, m \gg 0$, one gets the general case of H ample. The lemma also follows easily by [JMPSE, Corollary 4]. □

We close this section recalling what we need about the relation between subschemes of codimension 2 and rank two bundles. To this purpose we observe that for each subscheme $Z \subset X$ there exists the exact sequence

$$0 \longrightarrow \mathcal{I}_{Z/X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0. \tag{2.2}$$

Let $\mathcal{F}$ be a rank two bundle on a smooth irreducible variety X of dimension $n \geq 2$ and let $0 \neq s \in H^0(\mathcal{F})$. In general, its *zero–locus* $(s)_0 \subset X$ is either empty or its codimension is at most 2. We can always write

$$(s)_0 = Y \cup Z \tag{2.3}$$

where Z has pure codimension 2 (or it is empty) and Y has pure codimension 1 (or it is empty). In particular $\mathcal{F}(-Y)$ has a non–zero section vanishing on Z, thus we can consider the following exact sequence induced by its Koszul complex

$$0 \longrightarrow \mathcal{O}_X(Y) \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_{Z/X}(-Y + c_1(\mathcal{F})) \longrightarrow 0. \tag{2.4}$$

Tensoring (2.4) by $\mathcal{O}_Z$ yields $\mathcal{I}_{Z/X}/\mathcal{I}^2_{Z/X} \cong \mathcal{F}^\vee(Y)\otimes\mathcal{O}_Z$, whence the normal bundle of Z inside X satisfies $\mathcal{N}_{Z/X} \cong \mathcal{F}(-Y) \otimes \mathcal{O}_Z$. If $Y = 0$, then Z is locally complete intersection inside X, because $\operatorname{rk}(\mathcal{F}) = 2$. In particular, it has no embedded components and the adjunction formula

$$\omega_Z \cong \mathcal{O}_Z \otimes \omega_X(c_1(\mathcal{F})) \tag{2.5}$$

holds for such a scheme as well (see [H1, Proof of Proposition III.7.2, page 180]).

The Serre correspondence allows us to revert the above construction as follows.

Theorem 2.3. *Let X be a smooth irreducible variety X of dimension $n \geq 2$ and let $Z \subset X$ be a locally complete intersection subscheme of pure codimension 2.*

If $\det(\mathcal{N}_{Z/X}) \cong \mathcal{O}_Z \otimes \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Pic}(X)$ such that $H^2(\mathcal{L}^\vee) = 0$, then there exists a rank two bundle $\mathcal{F}$ on X such that:

(1) $\det(\mathcal{F}) \cong \mathcal{L}$;

(2) $\mathcal{F}$ has a global section s such that Z coincides with the zero locus $(s)_0$ of s.

Moreover, if $H^1(\mathcal{L}^\vee) = 0$, the above two conditions determine $\mathcal{F}$ up to isomorphism.

Proof. See [A, Theorem 1.1]. □

For further notation and all the other necessary results not explicitly mentioned in the paper, unless otherwise stated we tacitly refer to [H2].

3. The proof of Theorem 1.1

This section contains some preliminary results needed in the next sections. We begin with the following definitions.

Definition 3.1. Let X be a smooth variety and let $\mathcal{O}_X(H)$ be an ample line bundle on X.

The *H–degree* of a curve $Z \subset X$ is HZ. An *H–line* is a smooth irreducible rational curve $L \subset X$ such that $HL = 1$.

Definition 3.2. Let C be a reduced connected curve.

We say that C is a *seminormal tree of smooth rational curves* if its irreducible components are smooth rational curves, the singularities are given by smooth branches with independent tangents (that is they are seminormal singularities) and the dual graph is a tree.

Seminormal trees of smooth rational curves are characterized as the reduced connected curves with arithmetic genus 0 in [C, Proposition 1.8].

Lemma 3.3. *Let X be a smooth Fano threefold with fundamental line bundle $\mathcal{O}_X(h)$.*

Let $\mathcal{E}$ be a rank two bundle on X such that $c_1(\mathcal{E}) = \varepsilon h$ and $h^1(\mathcal{E}(-eh)) = 0$ and $\mathcal{E} \not\cong \mathcal{O}_X \oplus \mathcal{O}_X(\varepsilon h)$, for some $\varepsilon, e \in \mathbb{Z}$ with $1 \leq 1 + \varepsilon \leq e \leq i_X$. Assume that there is $s \in H^0(\mathcal{E})$ such that $Z = (s)_0 \subset X$ is a curve.

Then one of the following assertions hold:

(1) $e = i_X - 1$ and Z_{red} is a union of disjoint h–lines;

(2) $e = i_X$ and each connected component of Z_{red} is a seminormal tree of smooth rational curves.

Proof. Since Z is a curve, it follows that the sequence (2.4) corresponding to Z becomes

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z/X}(\varepsilon h) \longrightarrow 0, \tag{3.1}$$

where either $Z = \emptyset$ or $Z \subset X$ is a locally complete intersection curve and (2.5) gives

$$\omega_Z \cong \mathcal{O}_Z \otimes \mathcal{O}_X((\varepsilon - i_X)h). \tag{3.2}$$

If $Z = \emptyset$, then (3.1) splits, because $h^1(\mathcal{O}_X(-\varepsilon h)) = 0$ for each Fano threefold by Kodaira vanishing, since $\varepsilon \geq 0$. Hence $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{O}_X(\varepsilon h)$, contradicting our hypothesis.

It remains to handle the case $Z \neq \emptyset$: we will deal with it in what follows. Since $e \geq 1$ and $\varepsilon - e \leq -1$, we have that $h^i(\mathcal{O}_X(-eh)) = h^i(\mathcal{O}_X((\varepsilon - e)h)) = 0$ for $i \leq 2$.

Hence the cohomologies of (2.2) tensored by $\mathcal{O}_X((\varepsilon - e)h)$ and (3.1) tensored by $\mathcal{O}_X(-eh)$ imply

$$h^0(\mathcal{O}_Z \otimes \mathcal{O}_X((\varepsilon - e)h)) = h^1(\mathcal{I}_{Z/X}((\varepsilon - e)h)) = h^1(\mathcal{E}(-eh)) = 0. \tag{3.3}$$

Each connected component of Z still satisfies the same properties of the whole Z: we restrict to a fixed connected component $\overline{Z}$ of Z from now on.

Now (3.2) and (3.3) give

$$h^1(\mathcal{O}_Z \otimes \mathcal{O}_X((e - i_X)h) = h^0(\omega_Z \otimes \mathcal{O}_X((-e + i_X)h) = h^0(\mathcal{O}_Z \otimes \mathcal{O}_X((\varepsilon - e)h)) = 0.$$

Then the exact sequence

$$0 \longrightarrow \mathcal{I}_{\overline{Z}/Z} \otimes \mathcal{O}_X((e - i_X)h) \longrightarrow \mathcal{O}_Z \otimes \mathcal{O}_X((e - i_X)h) \longrightarrow \mathcal{O}_{\overline{Z}} \otimes \mathcal{O}_X((e - i_X)h) \longrightarrow 0$$

implies that

$$h^1(\mathcal{O}_{\overline{Z}} \otimes \mathcal{O}_X((e - i_X)h)) = 0. \tag{3.4}$$

If $\overline{Z}_{red}$ is the reduced scheme structure on $\overline{Z}$, then

$$h^0(\mathcal{O}_{\overline{Z}_{red}}) = 1, \qquad p_a(\overline{Z}_{red}) \geq 0. \tag{3.5}$$

Let $\overline{Z}^{(1)}_{red}, \dots, \overline{Z}^{(r)}_{red}$ be the irreducible components of $\overline{Z}_{red}$. Then, for any $1 \leq i \leq r$ and $t \geq 1$, we have that $\mathcal{O}_{\overline{Z}^{(i)}_{red}} \otimes \mathcal{O}_X(th)$ is ample, hence $H^0(\mathcal{O}_{\overline{Z}^{(i)}_{red}} \otimes \mathcal{O}_X(-th)) = 0$ (see for example [Mum, Proposition 1]). Hence the inclusion $\mathcal{O}_{\overline{Z}_{red}} \subseteq \bigoplus_{i=1}^r \mathcal{O}_{\overline{Z}^{(i)}_{red}}$ yields

$$H^0(\mathcal{O}_{\overline{Z}_{red}} \otimes \mathcal{O}_X(th)) = 0 \text{ for } t \leq -1. \tag{3.6}$$

Moreover the exact sequence

$$0 \longrightarrow \mathcal{I}_{\overline{Z}_{red}/\overline{Z}} \otimes \mathcal{O}_X((e - i_X)h) \longrightarrow \mathcal{O}_{\overline{Z}} \otimes \mathcal{O}_X((e - i_X)h) \longrightarrow \mathcal{O}_{\overline{Z}_{red}} \otimes \mathcal{O}_X((e - i_X)h) \longrightarrow 0$$

and (3.4) show that $h^1(\mathcal{O}_{\overline{Z}_{red}} \otimes \mathcal{O}_X((e - i_X)h)) = 0$. Hence, applying the Riemann–Roch theorem [L, Theorem 7.3.26] on $\overline{Z}_{red}$, we get

$$\begin{aligned} h^0(\mathcal{O}_{\overline{Z}_{red}} \otimes \mathcal{O}_X((e - i_X)h)) = \chi(\mathcal{O}_{\overline{Z}_{red}} \otimes \mathcal{O}_X((e - i_X)h)) = \\ = (e - i_X)\overline{Z}_{red}h + 1 - p_a(\overline{Z}_{red}). \end{aligned} \tag{3.7}$$

If $e = i_X$, then (3.5) yields $p_a(\overline{Z}_{red}) = 0$, hence $\overline{Z}_{red}$ is a seminormal tree of smooth rational curves by [C, Proposition 1.8]. This gives (2).

If $e \leq i_X - 1$, then (3.5), (3.6) and (3.7) imply

$$0 \leq p_a(\overline{Z}_{red}) = (e - i_X)\overline{Z}_{red}h + 1 \leq 0,$$

hence $p_a(\overline{Z}_{red}) = 0, i_X = e + 1$ and $\overline{Z}_{red}$ is an h–line. This gives (1). □

Remark 3.4. The cases listed in Lemma 3.3 are not mutually exclusive, because we have not assumed anywhere that e is the minimal integer such that $h^1(\mathcal{E}(-eh)) = 0$.

We are now ready to prove Theorem 1.1 stated in the Introduction.

Proof of Theorem 1.1.

Recall that we are assuming $\mathcal{E} \not\cong \mathcal{O}_X^{\oplus 2}$. We first show that $\mathcal{E}$ is indecomposable. In fact, assume to the contrary that $\mathcal{E}$ is decomposable. Since $c_1(\mathcal{E}) = 0$ and $h^0(\mathcal{E}) \neq 0$, there is a line bundle $\mathcal{L}$ on X such that $\mathcal{E} \cong \mathcal{L} \oplus \mathcal{L}^{-1}$ and $h^0(\mathcal{L}) \neq 0$. Now, if $D \in |\mathcal{L}|$, we get by the μ–semistability of $\mathcal{E}$ that $Dh^2 = \mu(\mathcal{O}_X(D)) \leq \mu(\mathcal{E}) = 0$. But D is effective and h is ample, hence D must be trivial and therefore $\mathcal{E} \cong \mathcal{O}_X^{\oplus 2}$, a contradiction. This proves that $\mathcal{E}$ is indecomposable.

Let $s \neq 0$ be a section in $H^0(\mathcal{E})$ and write $(s)_0 = Y \cup Z$ as in (2.3). If $Y \neq \emptyset$, then (2.4) gives an inclusion $\mathcal{O}_X(Y) \subset \mathcal{E}$ and $\mu(\mathcal{O}_X(Y)) = Yh^2 > 0 = \mu(\mathcal{E})$, which contradicts the μ–semistability of $\mathcal{E}$.

We deduce that $Z = (s)_0$ and either $Z = \emptyset$ or $Z \subset X$ is a locally complete intersection curve. Hence the sequence (2.4) associated to s becomes

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z/X} \longrightarrow 0. \tag{3.8}$$

If $Z = \emptyset$, we have that $\mathcal{I}_{Z/X} \cong \mathcal{O}_X$ and the sequence (3.8) splits, since $h^1(\mathcal{O}_X) = 0$. But then $\mathcal{E} \cong \mathcal{O}_X^{\oplus 2}$, a contradiction. Thus Z is a locally complete intersection curve.

The cohomology of (3.8) yields $h^0(\mathcal{E}) = 1$, hence the zero locus of each non–zero section in $H^0(\mathcal{E})$ is Z. Moreover (2.5) gives

$$\omega_Z \cong \mathcal{O}_Z \otimes \mathcal{O}_X(-i_X h). \tag{3.9}$$

Let $i_X = 1$. In this case $q_X^0 = i_X$, hence the connected components of the reduced scheme structure on Z are seminormal trees of smooth rational curves by assertion (2) of Lemma 3.3.

Let $i_X \geq 2$. In this case we have $q_X^0 = i_X - 1$, hence again Lemma 3.3 implies that the reduced structure of Z is a union of disjoint h–lines.

Moreover, if $i_X = 3$, then (3.9) induces a similar isomorphism for the dualizing sheaf of each connected component of Z. Thus each such component must be nonreduced.

As we already pointed out in the proof of Lemma 3.3, each connected component $\overline{Z} \subset Z$ satisfies the same properties of Z. In particular, it is a local complete intersection inside X. Moreover, its normal bundle is the restriction of the normal bundle of Z, whence $\det(\mathcal{N}_{\overline{Z}/X}) \cong \mathcal{O}_{\overline{Z}}$.

Thus Theorem 2.3 yields the existence of a rank two bundle $\overline{\mathcal{E}}$ with $c_1(\overline{\mathcal{E}}) = 0$ fitting into an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \overline{\mathcal{E}} \longrightarrow \mathcal{I}_{\overline{Z}/X} \longrightarrow 0.$$

Then (2.1) yields

$$\chi(\overline{\mathcal{E}}) = 2 - \frac{i_X}{2} c_2(\overline{\mathcal{E}})h,$$

hence, if i_X is odd, $\overline{Z}h = c_2(\overline{\mathcal{E}})h$ is necessarily even. □

Remark 3.5. If $i_X = 1$ and $\overline{Z} = \overline{Z}_{red}$, then $\overline{Z}h = 2$. Indeed $\omega_{\overline{Z}} \cong \mathcal{O}_{\overline{Z}} \otimes \mathcal{O}_X(-h)$ by (3.9), hence

$$-\overline{Z}h + 1 = \chi(\omega_{\overline{Z}}) = -h^1(\omega_{\overline{Z}}) = -h^0(\mathcal{O}_{\overline{Z}}) = -1,$$

thanks to the Riemann–Roch theorem on $\overline{Z}$, the connectedness of $\overline{Z}$ and $p_a(\overline{Z}) = 0$.

Remark 3.6. The same argument used in [OSS, p.89] proves that if X is a smooth variety endowed with an ample line bundle $\mathcal{O}_X(h)$ which generates $\mathrm{Pic}(X)$, $n \geq 2$ and $h^1(\mathcal{O}_X) = 0$, then every rank 2 bundle $\mathcal{E}$ which is semistable with respect to $\mathcal{O}_X(h)$ is either μ–stable or isomorphic to $\mathcal{O}_X^{\oplus 2}$.

Remark 3.7. Recall that a *Del Pezzo variety* is a pair $(X, \mathcal{O}_X(H))$ where X is a smooth variety of dimension $n \geq 1$ and $\mathcal{O}_X(H)$ is an ample line bundle such that $\omega_X^{-1} \cong \mathcal{O}_X((n-1)H)$ (see [IP, Section 3.2]).

If X is a Fano threefold with $i_X = 2$ with fundamental line bundle $\mathcal{O}_X(h)$, then $(X, \mathcal{O}_X(h))$ is Del Pezzo. Conversely, if $(X, \mathcal{O}_X(H))$ is a Del Pezzo threefold, then either X is a Fano threefold with $i_X = 2$ and $\mathcal{O}_X(H)$ is its fundamental line bundle or $X \cong \mathbb{P}^3$ and $\mathcal{O}_X(H) \cong \mathcal{O}_{\mathbb{P}^3}(2)$ (see [IP, Section 3.3]).

On the one hand, a vector bundle $\mathcal{E}$ on $X \cong \mathbb{P}^3$ is μ–(semi)stable with respect to $\mathcal{O}_{\mathbb{P}^3}(1)$ if and only if it is μ–(semi)stable with respect to $\mathcal{O}_X(H) \cong \mathcal{O}_{\mathbb{P}^3}(2)$. Thus, the only strictly μ–semistable rank two bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = 0$ on $X \cong \mathbb{P}^3$ such that $h^1(\mathcal{E}(-H)) = h^1(\mathcal{E}(-2)) = 0$ is $\mathcal{O}_X^{\oplus 2}$. On the other hand, there are no

H–lines on X. We deduce that Theorem 1.1 holds also in the case $X \cong \mathbb{P}^3$ and $\mathcal{O}_X(H) \cong \mathcal{O}_{\mathbb{P}^3}(2)$.

Remark 3.8. The hypothesis $h^0(\mathcal{E}) \neq 0$ is crucial in the proof of Theorem 1.1 (and in Lemma 3.3). Indeed without this restriction, the classification of strictly μ–semistable rank two bundles $\mathcal{E}$ with $c_1(\mathcal{E}) = 0$ and $h^1(\mathcal{E}(-q_X^0 h))$ on a Fano variety X with $\varrho_X \geq 2$ is certainly much richer than the one described in Theorem 1.1.

As an example, consider the general hyperplane section $X \subset \mathbb{P}^7$ of Segre product $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$. The projections from $\mathbb{P}^2 \times \mathbb{P}^2$ onto the two factors induce maps $p_i \colon X \to \mathbb{P}^2$. It is well known that $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(h_i) := p_i^*\mathcal{O}_{\mathbb{P}^2}(1)$ for $i = 1, 2$ and $\mathcal{O}_X(h) \cong \mathcal{O}_X(h_1 + h_2)$. For each pair of positive integers a_1, a_2 such that $|a_1 - a_2| \leq 1$, the extension

$$0 \longrightarrow \mathcal{O}_X(a_1h_1 - a_2h_2) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(-a_1h_1 + a_2h_2) \longrightarrow 0$$

gives a strictly μ–semistable rank two bundle $\mathcal{E}$ such that $c_1(\mathcal{E}) = 0$ and, thanks to [CFM, Proposition 2.5], $h^0(\mathcal{E}) = h^1(\mathcal{E}(-h)) = 0$: see also [MMPL, Proposition 3.5] for further details about this example.

4. Examples

In the examples below we show that Lemma 3.3 and, consequently, Theorem 1.1 are both sharp and cannot be improved without any further assumption on the Fano variety X. More precisely, we prove the existence of a strictly μ–semistable rank two bundle $\mathcal{E} \ncong \mathcal{O}_X^{\oplus 2}$ satisfying the hypothesis of Lemma 3.3 for each admissible pair (e, ε), that is $e = i_X, i_X - 1$ and $0 \leq \varepsilon \leq e - 1$, and such that

$$h^1(\mathcal{E}(-(e-1)h)) \neq 0. \tag{4.1}$$

As pointed out in Lemma 2.2, such bundles are μ–semistable when $\varepsilon = 0$ and $\mathrm{Pic}(X)$ is generated by the fundamental line bundle $\mathcal{O}_X(h)$.

To this purpose we will make obviously use of Theorem 2.3, starting from curves as in Theorem 1.1. In particular, in order to show the sharpness of the assertions (1) and (2) of the aforementioned theorem, we must construct examples starting from curves whose reduced structure is necessarily a disjoint union of h–lines. The existence of such subschemes is obvious when $i_X \geq 3$ and we refer the interested reader to [IP, Sections 3.4, 3.5, 4.2, 4.4 and 4.5] for the case $i_X \leq 2$.

Example 4.1 (The cases $(e, \varepsilon) = (i_X - 1, i_X - 2)$ and $(i_X, i_X - 1)$)**.** This example is a generalization of the well–known construction in $\mathbb{P}^3$ described in [H3, Examples 3.1.1 and 3.1.2].

Let $Z_r \subset X$ be the union of $k \geq 2$ pairwise disjoint smooth irreducible rational curves of h–degree r where $\max\{\ 1, 3 - i_X\ \} \leq r \leq 2$. By construction we have $\omega_{Z_r} \cong \mathcal{O}_{Z_r} \otimes \mathcal{O}_X((r-3)h)$, hence

$$\det(\mathcal{N}_{Z_r/X}) \cong \omega_{Z_r} \otimes \mathcal{O}_X(i_X h) \cong \mathcal{O}_{Z_r} \otimes \mathcal{O}_X((r + i_X - 3)h).$$

Thus Theorem 2.3 yields the existence of a rank two bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = \varepsilon h$ fitting into (3.1), where $\varepsilon = r + i_X - 3 \in \{\ i_X - 2, i_X - 1\ \}$.

Trivially $h^0(\mathcal{E}) \neq 0$. The cohomology of (3.1) tensored by $\mathcal{O}_X(-h)$ returns $h^0(\mathcal{E}(-h)) = h^0(\mathcal{I}_{Z/X}((r + i_X - 4)h))$. If either $i_X \leq 2$ or $i_X = 3$ and $r = 1$, then the latter dimension is trivially 0 because $Z_r \neq \emptyset$. If $i_X = 3$ and $r = 2$, then $r + i_X - 4 = 1$ and $X \subset \mathbb{P}^4$ is a smooth quadric hypersurface. It is easy to check that there are no hyperplanes in $\mathbb{P}^4$ containing two disjoint conics. Similarly if $i_X = 4$, then $r + i_X - 4 = r$ and $X \cong \mathbb{P}^3$. Again there are no hypersurfaces of degree r in $\mathbb{P}^3$ containing two disjoint integral subschemes of degree r when $r = 1$ or $r = 2$ and $k \geq 3$. When $(i_X, r, k) \neq (4, 2, 2)$ we conclude that $h^0(\mathcal{I}_{Z/X}((r + i_X - 4)h)) = 0$

again, hence $h^0(\mathcal{E}(-h)) = 0$: in particular, if $\varrho_X = 1$ and $\varepsilon = 0$, then $\mathcal{E}$ is μ–semistable by Lemma 2.2.

Let $e := 1+\varepsilon = r+i_X-2$. The cohomology of (3.1) tensored by $\mathcal{O}_X(-eh)$ returns $h^1(\mathcal{E}(-eh)) = h^1(\mathcal{I}_{Z/X}(-h))$. The cohomology of (2.2) tensored by $\mathcal{O}_X(-h)$ and the definition of Z_r imply that the latter dimension is zero. The same argument also yields $h^1(\mathcal{E}(-(e-1)h)) = h^1(\mathcal{I}_{Z/X}) = k-1 \geq 1$.

Thus, when $(i_X, r, k) \neq (4,2,2)$, $\mathcal{E}$ is a rank two bundle satisfying (4.1) and the hypotheses of Lemma 3.3 with $(e,\varepsilon) \in \{\ (i_X-1, i_X-2), (i_X, i_X-1)\ \}$ and $c_2(\mathcal{E})h = rk$.

Example 4.2 (The cases $(e,\varepsilon) = (i_X, i_X-2)$, (i_X, i_X-3) and (i_X, i_X-4))**.** If $i_X = 1$, then there are no further cases besides the ones handled in Example 4.1. Thus we will assume $i_X \geq 2$ in what follows. We also assume that X is not a Del Pezzo threefold of degree 1 and 2, hence, as is well–known, $\mathcal{O}_X(h)$ induces an aCM embedding $X \subset \mathbb{P}^N$.

Consider smooth pairwise disjoint conics $C_1, \dots, C_k \subset X$. It is easy to check that $\mathcal{I}_{C_i/\mathbb{P}^N}(i_X)$ is 0–regular, hence the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_{X/\mathbb{P}^N} \longrightarrow \mathcal{I}_{C_i/\mathbb{P}^N} \longrightarrow \mathcal{I}_{C_i/X} \longrightarrow 0,$$

suitably tensored implies that the same holds for $\mathcal{I}_{C_i/X}(i_X h)$. Since C_i is smooth and $i_X \geq 2$, it follows the existence of a smooth surface $Y_i \in |i_X h|$ through C_i.

Thus we know that C_i is contained in a $K3$ surface in $Y_i \in |-K_X|$ without restrictions on $i_X \geq 2$. We set $\mathcal{O}_{Y_i}(h_{Y_i}) := \mathcal{O}_{Y_i} \otimes \mathcal{O}_X(h)$, hence the degree of Y_i is $h_{Y_i}^2 = i_X h^3$ and $h_{Y_i} C_i = 2$.

Recall that $\omega_{Y_i} \cong \mathcal{O}_{Y_i}$, hence the adjunction formula on Y_i implies $C_i^2 = -2$. Let $2 \leq r \leq i_X$ and consider the divisor $Z_{i,r} := rC_i \subset Y_i$. The sequence (2.2) for the pair $Z_{i,r} \subset Y_i$ and the Riemann–Roch theorem on Y_i imply that $\chi(\mathcal{O}_{Z_{i,r}}) = r^2$, whence $p_a(Z_{i,r}) = 1 - r^2 \leq -3$.

Since $(h_{Y_i} + C_i)C_i = 0$ and $(h_{Y_i} + C_i)^2 = h_{Y_i}^2 + 2 > 0$, it follows that $h_{Y_i} + C_i$ is big and nef, hence, for every $t \geq 1$ we have

$$h^1(\mathcal{O}_{Y_i}(t(h_{Y_i} + C_i))) = 0 \tag{4.2}$$

by the Kawamata–Viehweg vanishing theorem.

Moreover, the Riemann–Roch theorem on Y_i implies

$$h^0\big(\mathcal{O}_{Y_i}(h + C_i)\big) \geq \frac{h_{Y_i}^2}{2} + 3 > \frac{h_{Y_i}^2}{2} + 2 = h^0\big(\mathcal{O}_{Y_i}(h)\big),$$

hence C_i is not in the fixed locus of $|h_{Y_i} + C_i|$ and the general element in $|h_{Y_i} + C_i|$ does not intersect $Z_{i,r}$. Thus the same is true for $|r(h_{Y_i} + C_i)|$ and the adjunction formula on Y_i gives

$$\omega_{Z_{i,r}} \cong \det(\mathcal{N}_{Z_{i,r}/Y_i}) \cong \mathcal{O}_{Z_{i,r}} \otimes \mathcal{O}_{Y_i}(Z_{i,r}) \cong \mathcal{O}_{Z_{i,r}} \otimes \mathcal{O}_{Y_i}(-rh_{Y_i}) \cong \mathcal{O}_{Z_{i,r}} \otimes \mathcal{O}_X(-rh).$$

If $Z_r := \bigcup_{i=1}^k Z_{i,r}$, then the adjunction formula on X returns

$$\det(\mathcal{N}_{Z_r/X}) \cong \mathcal{O}_{Z_r} \otimes \mathcal{O}_X((i_X - r)h).$$

because the subschemes $Z_{i,r}$ are pairwise disjoint. Theorem 2.3 yields the existence of a rank two bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = \varepsilon h$ fitting into (3.1) where $\varepsilon = i_X - r$.

As usual $h^0(\mathcal{E}) \neq 0$. We have $h^0(\mathcal{E}(-h)) = h^0(\mathcal{I}_{Z_r/X}((i_X-r-1)h))$ and we claim that the latter dimension is zero. Indeed, if $i_X = 4$ and $r = 2$, then $i_X - r - 1 = 1$. Since $p_a(Z_{i,r}) \leq -3$, it follows that $Z_{i,r}$ is not contained in any plane inside $X \cong \mathbb{P}^3$, hence the same is true for Z_r. If either $i_X \neq 4$ or $r \neq 2$, then $i_X - r - 1 \leq 0$. We conclude also in this case that $h^0(\mathcal{I}_{Z_r/X}((i_X - r - 1)h)) = 0$ because $Z_r \neq \emptyset$. Thus the claim is proved, hence $h^0(\mathcal{E}(-h)) = 0$: in particular, if $\varrho_X = 1$ and $\varepsilon = 0$, then $\mathcal{E}$ is μ–semistable by Lemma 2.2.

We claim that $h^1(\mathcal{E}(-i_Xh)) = 0$ and $h^1(\mathcal{E}((1-i_X)h)) \neq 0$. To prove such a claim, we notice that the cohomology of (3.1) tensored by $\mathcal{O}_X(-th)$ yields

$$h^1(\mathcal{E}(-th)) = h^1(\mathcal{I}_{Z_r/X}((i_X - r - t)h))$$

where $t \in \mathbb{Z}$. If $t \geq i_X - 1$, then $i_X - r - t \leq -1$, hence the cohomologies of the sequence (2.2) for X tensored by $\mathcal{O}_X((i_X - r - t)h)$ and for Y_i tensored by $\mathcal{O}_{Y_i}((i_X - r - t)h_{Y_i})$ imply

$$\begin{aligned} h^1(\mathcal{I}_{Z_r/X}((i_X - r - t)h)) &= h^0(\mathcal{O}_{Z_r} \otimes \mathcal{O}_X((i_X - r - t)h)) = \\ &= \sum_{i=1}^{k} h^0(\mathcal{O}_{Z_{i,r}} \otimes \mathcal{O}_X((i_X - r - t)h)) = \\ &= \sum_{i=1}^{k} h^1(\mathcal{O}_{Y_i}((i_X - r - t)h_{Y_i} - rC_i)). \end{aligned}$$

Taking $t = i_X$, the vanishing $h^1(\mathcal{O}_{Y_i}(r(h_{Y_i} + C_i))) = 0$ in (4.2) and the Serre duality theorem on Y_i, yield $h^1(\mathcal{E}(-i_Xh)) = 0$.

Now take $t = i_X - 1$. On the one hand, the Riemann–Roch theorem on Y_i yields

$$\chi(\mathcal{O}_{Y_i}((1-r)h_{Y_i} - rC_i)) = \frac{(r-1)^2(h_{Y_i}^2 + 2)}{2} + 1.$$

On the other hand, $((r-1)h_{Y_i} + rC_i)C_i = -2$, hence C_i is a fixed component of $|(r-1)h_{Y_i} + rC_i|$. Thanks to the Serre duality theorem, the Riemann–Roch theorem on Y_i and (4.2), it follows that

$$\begin{aligned} h^2(\mathcal{O}_{Y_i}((1-r)h_{Y_i} - rC_i)) &= h^0(\mathcal{O}_{Y_i}((r-1)h_{Y_i} + rC_i)) = \\ &= h^0(\mathcal{O}_{Y_i}((r-1)(h_{Y_i} + C_i))) = \\ &= \chi(\mathcal{O}_{Y_i}((r-1)(h_{Y_i} + C_i)) = \frac{(r-1)^2(h_{Y_i}^2 + 2)}{2} + 2. \end{aligned}$$

Thus

$$h^1(\mathcal{O}_{Y_i}((1-r)h_{Y_i} - rC_i)) = -\chi(\mathcal{O}_{Y_i}((1-r)h_{Y_i} + rC_i)) + h^2(\mathcal{O}_{Y_i}((1-r)h_{Y_i} - rC_i)) = 1,$$

whence $h^1(\mathcal{E}((1-i_X)h)) = \sum_{i=1}^k h^1(\mathcal{O}_{Y_i}((1-r)h_{Y_i} - rC_i)) = k$.

In particular $\mathcal{E}$ is a rank two bundle satisfying (4.1) and the hypotheses of Lemma 3.3 with $(e, \varepsilon) \in \{\ (i_X, i_X - 4), (i_X, i_X - 3), (i_X, i_X - 2)\ \}$ and $c_2(\mathcal{E})h = 2rk$.

Example 4.3 (The cases $(e, \varepsilon) = (i_X - 1, i_X - 3)$ and $(i_X - 1, i_X - 4)$)**.** If $i_X \leq 2$, then all the admissible cases are covered by Examples 4.1 and 4.2. Thus $i_X \geq 3$ from now on, hence $\varrho_X = 1$.

Consider smooth pairwise disjoint lines $L_1, \dots, L_k \subset X$. If $i_X = 4$, let $Y_i \subset \mathbb{P}^3$ be a smooth cubic surface through L_i. If $i_X = 3$ by combining the cohomologies of the twists of sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2) \longrightarrow \mathcal{I}_{L_i/\mathbb{P}^4} \longrightarrow \mathcal{I}_{L_i/X} \longrightarrow 0$$

and of the Koszul complex resolving $\mathcal{I}_{L_i/\mathbb{P}^4}$, we deduce that $\mathcal{I}_{L_i/X}(2h)$ is 0–regular, hence there exists a smooth surface $Y_i \in |2h|$ through L_i.

By adjunction on X we know that Y_i is a Del Pezzo surface of degree $d = 7 - i_X$. In what follows we recall some facts about Y_i: we refer to [Man, Sections 24 and 25] for their proofs. The surface Y_i is isomorphic to the blow up of $\mathbb{P}^2$ at $9-d$ general points. As usual we denote by ℓ the pull–back of a general line in $\mathbb{P}^2$ via the blowing up map and by e_j the exceptional divisors for $1 \leq j \leq 9 - d$. Moreover we can always assume that $L_i = e_1$ for each i. If we set $\mathcal{O}_{Y_i}(h_{Y_i}) := \mathcal{O}_{Y_i} \otimes \mathcal{O}_X(h)$, then $h_{Y_i} = 3\ell - \sum_{j=1}^{9-d} e_j$.

The subscheme $Z_{i,r} := rL_i \subset Y_i \subset X$ with $2 \le r \le i_X - 1$ is fixed and does not intersect the general element in $|3\ell - \sum_{j=2}^{9-d} e_j|$, hence

$$\det(\mathcal{N}_{Z_{i,r}/Y_i}) \cong \mathcal{O}_{Z_{i,r}} \otimes \mathcal{O}_{Y_i}(rL_i) \cong \mathcal{O}_{Z_{i,r}} \otimes \mathcal{O}_{Y_i}(-rh_{Y_i}).$$

Thus the adjunction formula on Y_i yields

$$\omega_{Z_{i,r}} \cong \mathcal{O}_{Z_{i,r}} \otimes \mathcal{O}_{Y_i}(-(r+1)h_{Y_i}) \cong \mathcal{O}_{Z_{i,r}} \otimes \mathcal{O}_X(-(r+1)h).$$

As in the previous example we set $Z_r := \bigcup_{i=1}^k Z_{i,r}$ and notice that

$$\det(\mathcal{N}_{Z_r/X}) \cong \mathcal{O}_{Z_r} \otimes \mathcal{O}_X((i_X - r - 1)h).$$

because the subschemes $Z_{i,r}$ are pairwise disjoint. Theorem 2.3 yields the existence of a rank two bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = \varepsilon h$ fitting into (3.1) and such that $\varepsilon = i_X - r - 1 \in \{\ i_X - 4, i_X - 3\ \}$.

Trivially $h^0(\mathcal{E}) \ne 0$. The cohomology of (3.1) tensored by $\mathcal{O}_X(-h)$ yields $h^0(\mathcal{E}(-h)) = h^0(\mathcal{I}_{Z_r/X}((i_X - r - 2)h)) = 0$, because $i_X - r - 2 \le 0$ and $Z_r \ne \emptyset$. Thus $h^0(\mathcal{E}(-h)) = 0$, hence $\mathcal{E}$ is μ–semistable by Lemma 2.2 if $\varepsilon = 0$ because $\varrho_X = 1$.

The cohomology of (3.1) tensored by $\mathcal{O}_X((1 - i_X)h)$ yields

$$h^1(\mathcal{E}((1 - i_X)h)) = h^1(\mathcal{I}_{Z_r/X}(-rh)). \tag{4.3}$$

The cohomology of the sequence (2.2) for X tensored by $\mathcal{O}_X(-rh)$ and for Y_i tensored by $\mathcal{O}_Y(-rh_{Y_i})$ imply

$$h^1(\mathcal{I}_{Z_r/X}(-rh)) = h^0(\mathcal{O}_{Z_r} \otimes \mathcal{O}_X(-rh)) = \sum_{i=1}^k h^1(\mathcal{O}_{Y_i}(-r(h_{Y_i} + L_i))). \tag{4.4}$$

Since $(h_{Y_i} + L_i)L_i = 0$ and $(h_{Y_i} + L_i)^2 = 1 + i_X > 0$, it follows that $h_{Y_i} + L_i$ is big and nef, hence $h^1(\mathcal{O}_{Y_i}(-r(h_{Y_i} + L_i))) = 0$ by the Kawamata–Viehweg vanishing theorem. Therefore (4.3) and (4.4) give

$$h^1(\mathcal{E}((1 - i_X)h)) = \sum_{i=1}^k h^1(\mathcal{O}_{Y_i}(-r(h_{Y_i} + L_i))) = 0.$$

We have $h^1(\mathcal{O}_{Y_i}((r-2)h_{Y_i} + (r-1)L_i)) = h^1(K_{Y_i} + (r-1)(h_{Y_i} + L_i)) = 0$ by the Kawamata–Viehweg vanishing theorem. Moreover the Serre duality theorem gives that $h^2(\mathcal{O}_{Y_i}((r-2)h_{Y_i} + (r-1)L_i)) = h^0(-(r-1)(h_{Y_i} + L_i)) = 0$. Hence the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{Y_i}((r-2)h_{Y_i} + (r-1)L_i) \longrightarrow \mathcal{O}_{Y_i}((r-2)h_{Y_i} + rL_i) \longrightarrow \\ \longrightarrow \mathcal{O}_{L_i} \otimes \mathcal{O}_{Y_i}((r-2)h_{Y_i} + rL_i) \longrightarrow 0 \end{aligned}$$

implies that $h^1(\mathcal{O}_{Y_i}((1-r)h_{Y_i} - rL_i)) = h^1(\mathcal{O}_{L_i} \otimes \mathcal{O}_{Y_i}((r-2)h_{Y_i} + rL_i)) = h^1(\mathcal{O}_{\mathbb{P}^1}(-2)) = 1$. Thus a similar computation as in (4.3) and (4.4) yields

$$h^1(\mathcal{E}((2 - i_X)h)) = \sum_{I=1}^k h^1(\mathcal{O}_{Y_i}((1-r)h_{Y_i} - rL_i)) = k.$$

In particular $\mathcal{E}$ is a rank two bundle satisfying (4.1) and the hypotheses of Lemma 3.3 with $(e, \varepsilon) \in \{\ (i_X - 1, i_X - 4), (i_X - 1, i_X - 3)\ \}$ and $c_2(\mathcal{E})h = rk$.

We now give an example showing that, rational curves of every even degree at least 4 can occur in assertion (3) of Theorem 1.1 when $\mathcal{O}_X(h)$ is very ample.

Example 4.4 (The case $(e, \varepsilon, i_X) = (1, 0, 1)$)**.** If X is a Fano threefold with $i_X = 1$, then (2.1) implies that h^3 is even and we call *genus* of X the number $g := \frac{h^3}{2} + 1$. The threefold X is called *prime* if $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(h)$ and *of the principal series* if $\mathcal{O}_X(h)$ is very ample. We have $3 \le g \le 12$, $g \ne 11$ for prime Fano threefolds of the principal series (see [I], [Muk, Theorem 1.10], [CLM, Theorems 6 and 7]).

A polarized K3 surface is a pair $(S, \mathcal{L})$ where S is a K3 surface and $\mathcal{L}$ is an ample line bundle on S. The polarized K3 surface $(S, \mathcal{L})$ is called *BN–general* if $h^0(\mathcal{A})h^0(\mathcal{B}) < h^0(\mathcal{L})$ for every non–trivial decomposition $\mathcal{L} \cong \mathcal{A} \otimes \mathcal{B}$. We will use below the fact, proved in [Muk, (3.9) and Theorems 4.7, 5.5], that a BN–general polarized K3 surface $S \subset \mathbb{P}^g$ can be realized as a hyperplane section of a prime Fano threefold $X \subset \mathbb{P}^{g+1}$ of degree $2g - 2$.

Let $r \geq 1$ be an integer. We claim that for every g as above, there are

$$C \subset S \subset X \subset \mathbb{P}^{g+1}$$

where X is an anticanonically embedded prime Fano threefold of the principal series of genus g, S is a smooth hyperplane section of X (hence a K3 surface) and C is a smooth irreducible rational curve of degree $2r$.

To this end observe that, by [Kn1, Theorem 1.1(iv)], there is a smooth K3 surface $S \subset \mathbb{P}^g$ containing a smooth irreducible rational curve C of degree $2r$ and such that $\mathrm{Pic}(S)$ is freely generated by $\mathcal{O}_X(H)$ and $\mathcal{O}_X(C)$. Now [Kn2, Lemma 3.7(i)] implies that (S, H) is a BN-general polarized K3 surface, hence, using the fact mentioned above, we can realize S as a hyperplane section of a prime Fano threefold $X \subset \mathbb{P}^{g+1}$. This proves the claim.

Now let Z_r be the divisor rC on S. We first show that

$$H^1(\mathcal{O}_{Z_r}(-C)) = 0 \tag{4.5}$$

for each $r \geq 1$. In fact, since C is connected and $H^1(\mathcal{O}_S) = 0$, we have that $H^1(\mathcal{O}_S(-C)) = 0$. Now, from the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-(r+1)C) \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_{Z_r}(-C) \longrightarrow 0. \tag{4.6}$$

the map

$$H^2(\mathcal{O}_S(-(r+1)C)) \longrightarrow H^2(\mathcal{O}_S(-C))$$

is dual to

$$H^0(\mathcal{O}_S(C)) \longrightarrow H^0(\mathcal{O}_S((r+1)C))$$

and this is an isomorphism because C is a base component of $|(r+1)C|$, since $C^2 = -2$. Therefore the cohomology of (4.6) gives (4.5). Next we claim that, for $r \geq 2$, we have

$$\mathcal{O}_{Z_r} \otimes \mathcal{O}_S(h_S + rC) \cong \mathcal{O}_{Z_r}, \tag{4.7}$$

where $\mathcal{O}_S(h_S) := \mathcal{O}_S \otimes \mathcal{O}_X(h)$. To see this, first observe that we have an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_S(-rC) & \longrightarrow & \mathcal{O}_S(-C) & \longrightarrow & \mathcal{O}_{Z_{r-1}}(-C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_S & \xrightarrow{\mathrm{id}} & \mathcal{O}_S & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{Z_r} & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

so that the snake lemma gives the exact sequence

$$0 \longrightarrow \mathcal{O}_{Z_{r-1}}(-C) \longrightarrow \mathcal{O}_{Z_r} \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Now, exactly as in [BE, Proof of Proposition 4.1], the above gives rise to an exact sequence

$$0 \longrightarrow \mathcal{O}_{Z_{r-1}}(-C) \longrightarrow \mathcal{O}^*_{Z_r} \longrightarrow \mathcal{O}^*_C \longrightarrow 1$$

and applying (4.5) we get that the restriction of line bundles gives an isomorphism

$$\mathrm{Pic}(Z_r) = H^1(\mathcal{O}^*_{Z_r}) \to H^1(\mathcal{O}^*_C) = \mathrm{Pic}(C) \cong \mathbb{Z}.$$

Thus $\mathcal{O}_{Z_r} \otimes \mathcal{O}_S(h_S + rC) \cong \mathcal{O}_{Z_r}$, because $(h_S + rC)C = 0$, hence (4.7) is proved.

Now the adjunction formula on S and (4.7) give

$$\omega_{Z_r} \cong \mathcal{O}_{Z_r} \otimes \mathcal{O}_S(rC) \cong \mathcal{O}_{Z_r} \otimes \mathcal{O}_S(-h_S) \cong \mathcal{O}_{Z_r} \otimes \mathcal{O}_X(-h).$$

Since Z_r is locally complete intersection inside X, we deduce that

$$\det(\mathcal{N}_{Z_r/X}) \cong \mathcal{O}_{Z_r}.$$

Also $H^1(\mathcal{O}_X) = 0$, hence Theorem 2.3 gives rise to a rank 2 vector bundle $\mathcal{E}$ on X with $c_1(\mathcal{E}) = 0$, having a section s with $Z_r = (s)_0$ and sitting in an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z_r/X} \longrightarrow 0. \tag{4.8}$$

We have $H^i(\mathcal{O}_X(-h)) = 0$ for $i = 0, 1$ and $H^0(\mathcal{I}_{Z_r/X}(-h)) = 0$. Thus $h^0(\mathcal{E}(-h)) = 0$, hence $\mathcal{E}$ is strictly μ–semistable by Lemma 2.2. Finally, to check $H^1(\mathcal{E}(-h)) = 0$, using (4.8), it remains to show that $H^1(\mathcal{I}_{Z_r/X}(-h)) = 0$. To this end consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2h) \longrightarrow \mathcal{I}_{Z_r/X}(-h) \longrightarrow \mathcal{I}_{Z_r/S}(-h) \longrightarrow 0. \tag{4.9}$$

We have $(h_S + rC)C = 0$ and $(h_S + rC)^2 = h^3 + 2r^2 > 0$, hence $h_S + rC$ is big and nef and then

$$H^1(\mathcal{I}_{Z_r/S}(-h)) = H^1(\mathcal{O}_S(-h - rC)) = 0$$

by the Kawamata–Viehweg vanishing theorem. Also $H^1(\mathcal{O}_X(-2h)) = 0$, hence (4.9) shows that $H^1(\mathcal{I}_{Z_r/X}(-h)) = 0$.

5. The case of the quadric

We now focus our attention on the case $i_X = 3$. Thus, from now on X will denote the smooth quadric in $\mathbb{P}^4$ and $\mathcal{O}_X(h)$ its fundamental divisor.

If $\mathcal{E}$ is a semistable rank two bundle on X with $c_1(\mathcal{E}) = 0$, then it is either μ–stable or $\mathcal{E} \cong \mathcal{O}_X^{\oplus 2}$ (see Remark 3.6), hence we will focus on strictly μ–semistable bundles.

It follows by Theorem 1.1 that such an $\mathcal{E}$ satisfies $c_2(\mathcal{E})h \geq 2$ and we will deal only with the case $c_2(\mathcal{E})h = 2$ in what follows. In particular we will show, in Proposition 5.3 that each such bundle is obtained via the construction in Example 4.3.

In order to prove it, we denote by $\mathcal{R}$ the open subscheme of the Hilbert scheme $\mathcal{H}ilb^{2t+3}(X)$ whose points represent locally Cohen-Macaulay curves $Z \subset X$ with Hilbert polynomial $2t + 3$.

Lemma 5.1. *Let $\mathcal{E}$ be a strictly μ–semistable rank two bundle on X with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E})h = 2$.*

Then $h^0(\mathcal{E}(-h)) = 0$, $h^0(\mathcal{E}) = 1$ and each non–zero section in $H^0(\mathcal{E})$ vanishes exactly on the same $Z \in \mathcal{R}$.

Proof. The μ–semistability of $\mathcal{E}$ implies that $h^0(\mathcal{E}(-h)) = 0$ thanks to Lemma 2.2. The equality $h^0(\mathcal{E}) = 1$ and the existence and uniqueness of Z follow as in the first lines of the proof of Theorem 1.1. Moreover, $\chi(\mathcal{O}_Z) = 2 - \chi(\mathcal{E}) = 3$ thanks to the sequences (3.8), (2.2) for the inclusion $Z \subset X$ and (2.1), hence $p_a(Z) = -2$, that is $Z \in \mathcal{R}$. □

In the following proposition we collect some helpful results on the schemes in $\mathcal{R}$.

Proposition 5.2. *Let $Z \subseteq X$ be a subscheme.*

Then $Z \in \mathcal{R}$ if and only if there is smooth a Del Pezzo quartic surface $Y \subset X$ and a line $L \subset Y$ such that Z is the divisor $2L$ inside Y.

Moreover, Z is locally complete intersection inside X, $\omega_Z \cong \mathcal{O}_Z \otimes \mathcal{O}_X(-3h)$, $h^0(\mathcal{I}_{Z/X}(h)) = 0$ and

$$h^1(\mathcal{I}_{Z/X}(th)) = \begin{cases} 1 & \text{if } t=-1, \\ 2 & \text{if } t=0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Proof. If $Y \subset X$ is a smooth Del Pezzo quartic and $Z = 2L$ on Y for some line $L \subset Y$, then $\deg(Z) = 2$ and the adjunction formula on Y yields $p_a(Z) = -2$, hence $Z \in \mathcal{R}$.

Conversely, let $Z \in \mathcal{R}$. Thus $Zh = 2$ and $p_a(Z) = -2$, hence Z is a double structure on a line $L \subset X$ necessarily. Up to a proper choice of the homogeneous coordinates $x_0, \dots, x_4$ in $\mathbb{P}^4$, we can assume that the homogeneous ideal of L inside $\mathbb{C}[x_0, \dots, x_4]$ is $I_L = (x_0, x_1, x_2)$.

As pointed out in [NNS, Theorem 2.4] the homogeneous ideal $I_Z \subseteq \mathbb{C}[x_0, \dots, x_4]$ of Z has the form $I_Z = I_L^2 + (x_0, x_1, x_2)B$ where β_1 and β_2 are non–negative integers and $B := (b_{i,j})_{\substack{1\le i\le 3\\ 1\le j\le 2}}$ is a 3×2 matrix with coefficients $b_{i,j} \in \mathbb{C}[x_3, x_4]$ that are homogeneous polynomials of degree β_j and whose 2×2 minors define a codimension 2 subscheme in $\mathbb{P}^4$. Since $\mathcal{I}_{L/Z}^2 = 0$, it follows that $\mathcal{I}_{L/Z}$ is a line bundle on $L \cong \mathbb{P}^1$. Thus the sequence (2.2) for the inclusion $L \subset Z$ becomes,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(c-1) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_L \longrightarrow 0$$

for some integer c. The equality $p_a(Z) = -2$ implies $\chi(\mathcal{O}_{\mathbb{P}^1}(c-1)) = 2$, whence $c = 2$, hence $\beta_1 + \beta_2 = 2$ thanks to [NNS, Lemma 2.6]. In particular $I_Z = I_L^2 + (F_1, F_2)$ where, for $j = 1, 2$, F_j is the product of (x_0, x_1, x_2) times the j^{th} column of B, hence it is a form of degree β_j.

If $\beta_1 = 0$, then the homogeneous part of degree 2 of I_Z would be generated by b_1x_3, b_1x_4, x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2 where $b_1 := b_{1,1}x_0 + b_{2,1}x_1 + b_{3,1}x_2$: it follows that Z would be not contained in any smooth quadric, contradicting the definition of $\mathcal{R}$.

We deduce that $\beta_1 = \beta_2 = 1$, hence I_Z does not contain linear forms. The cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_{X/\mathbb{P}^4} \longrightarrow \mathcal{I}_{Z/\mathbb{P}^4} \longrightarrow \mathcal{I}_{Z/X} \longrightarrow 0$$

tensored by $\mathcal{O}_{\mathbb{P}^4}(t)$ and the isomorphism $\mathcal{I}_{X/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^4}(-2)$ yield $h^0(\mathcal{I}_{Z/X}(h)) = 0$ and (5.1) by [NNS, Corollary 3.2].

Moreover, F_1 and F_2 are quadratic forms such that (F_1, F_2) is the ideal of a scheme Y of dimension 2 which is smooth along L (see [NNS, Theorem 2.4]). In particular, I_Z is generated by the quadrics through Z, hence the base locus of the linear system of the quadrics through Z is exactly L. It follows that two general quadrics through Z are transversal outside Z: since F_1 and F_2 are transversal along L, the same is true for each general pair of quadrics through Z, hence we can assume that Y is everywhere smooth. By adjunction on $\mathbb{P}^4$ it follows that Y is a Del Pezzo surface. Since Y is smooth, the tangent space at every point of Z has dimension at most 2. On the other hand X is smooth and contains Z, hence there must be another smooth quadric, containing Z and transversal to X. Therefore we can assume that $Y \subset X$. As in Example 4.3 we can assume that Y is the blow up of $\mathbb{P}^2$ at 5 general points. Let ℓ be the pull–back of a general line in $\mathbb{P}^2$ and let e_j be the exceptional divisors for $1 \le j \le 5$: we can assume that $Z \subset Y$ is the Cartier divisor $2e_1$, hence it is locally complete intersection inside Q because

Y is smooth. As pointed out in Example 4.3 the adjunction formula on Y yields $\omega_Z \cong \mathcal{O}_Z \otimes \mathcal{O}_X(-3h)$. □

We can now prove the announced result.

Proposition 5.3. *There is a bijective correspondence between the following sets:*

(1) the set of strictly μ–semistable rank two bundles $\mathcal{E}$ on X with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E})h = 2$, up to isomorphism.

(2) the set of divisors of type $2L$ such that $L \subset Y \subset X$ where L is a line and Y is a smooth Del Pezzo quartic surface.

Proof. Let $L \subset X$ be a line and let $Y \subset X$ be a smooth Del Pezzo quartic surface containing L. Then, as proved in Example 4.3, Theorem 2.3 gives a rank two bundle $\mathcal{E}$ on X, unique up to isomorphism, with $c_1(\mathcal{E}) = 0, c_2(\mathcal{E})h = 2Lh = 2$ and $h^0(\mathcal{E}(-h)) = 0$. It follows that $\mathcal{E}$ is strictly μ–semistable by Lemma 2.2. Thus $\mathcal{E}$ is as in (1).

Vice versa let $\mathcal{E}$ be as in (1). It follows by Lemma 5.1 that $\mathcal{E}$ satisfies $h^0(\mathcal{E}(-h)) = 0$, $h^0(\mathcal{E}) = 1$ and each non–zero section in $H^0(\mathcal{E})$ vanishes exactly on the same $Z \in \mathcal{R}$. Hence Proposition 5.2 shows that there is a smooth Del Pezzo quartic surface $Y \subset X$ and a line $L \subset Y$ such that Z is the divisor $2L$ inside Y. □

By Proposition 5.2 we obtain the following corollaries.

Corollary 5.4. *Let $\mathcal{E}$ be a strictly μ–semistable rank two bundle on X with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E})h = 2$.*

Then the cohomology table of $\mathcal{E}$ is as follows in the range $-3 \le j \le 2$

$i = 3$	1	0	0	0	0	0
$i = 2$	2	1	0	0	0	0
$i = 1$	0	0	1	2	0	0
$i = 0$	0	0	0	1	5	21
	$j = -3$	$j = -2$	$j = -1$	$j = 0$	$j = 1$	$j = 2$

Table 1: The values of $h^i(\mathcal{E}(jh))$.

Proof. We already know that $h^0(\mathcal{E}) = 1, h^0(\mathcal{E}(-h)) = 0$ and $Z \in \mathcal{R}$ by Lemma 5.1. Hence $h^0(\mathcal{E}(jh)) = 0$ for $j \le -1$. Also we can apply Proposition 5.2 and then (5.1) yields $h^1(\mathcal{E}(jh)) = 0$ for $j \ne -1, 0$. Then the rest of the values in the table follow by the Serre duality theorem, that gives $h^i(\mathcal{E}(jh)) = h^{3-i}(\mathcal{E}(-(3+j)h))$, and (2.1). □

Remark 5.5. Let $\mathcal{E}$ be a μ–semistable rank two bundle on X with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E})h = 2$.

If $\mathcal{E}$ is strictly μ–semistable its cohomology table is given in Corollary 5.4. On the other hand, if $\mathcal{E}$ is μ–stable, it follows by Lemma 2.2 that $h^0(\mathcal{E}(jh)) = 0$ for all $j \le -1$ and by [OS, Corollary 2.4] that $h^1(\mathcal{E}(jh)) = 0$ for all $j \le -2$. It follows by the Serre duality theorem that $h^2(\mathcal{E}) = h^3(\mathcal{E}) = 0$.

Hence, in both cases, $h^0(\mathcal{E}) = h^1(\mathcal{E}) - 1$ by (2.1). Therefore $\mathcal{E}$ is μ–stable (resp. strictly μ–semistable) if and only if $h^0(\mathcal{E}) = 0$ (resp. 1), if and only if $h^1(\mathcal{E}) = 1$ (resp. 2).

Moreover, $\mathcal{E}$ is 2–regular, hence there is a smooth curve $C \subset X$ which is the zero–locus of a section of $H^0(\mathcal{E}(2h))$. Hence, as in (2.4), we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(2h) \longrightarrow \mathcal{I}_{C/X}(4h) \longrightarrow 0.$$

Notice that $Ch = c_2(\mathcal{E}(2h))h = 10$ and $\omega_C \cong \mathcal{O}_C \otimes \mathcal{O}_X(h)$. The cohomologies of the above sequence tensored by $\mathcal{O}_X(-4h)$, of the sequence (2.2) for the inclusion $C \subset X$ return

$$h^0(\mathcal{O}_C) = h^1(\mathcal{I}_{C/X}) + 1 = h^1(\mathcal{E}(-2h)) + 1 = 1,$$

hence C is connected. We deduce that C is the linear projection, from an outside point, of a canonical curve of genus 6 contained in $\mathbb{P}^5$ onto a hyperplane.

We have $h^0(\mathcal{I}_{C/X}(2h)) = h^0(\mathcal{E})$, hence C corresponds, via the Serre correspondence, to a strictly μ–semistable bundle if and only if it is contained in a single quadric.

Corollary 5.6. *Let $Z \in \mathcal{R}$.*
Then $h^0(\mathcal{N}_{Z/X}) = 7$ and $h^1(\mathcal{N}_{Z/X}) = 0$.

Proof. Since Z is locally complete intersection, we have an exact sequence

$$0 \longrightarrow \mathcal{N}_{Z/Y} \longrightarrow \mathcal{N}_{Z/X} \longrightarrow \mathcal{O}_Z \otimes \mathcal{N}_{Y/X} \longrightarrow 0. \tag{5.2}$$

We have $\mathcal{N}_{Z/Y} \cong \mathcal{O}_Z \otimes \mathcal{O}_Y(2e_1)$, $\mathcal{N}_{Y/X} \cong \mathcal{O}_Y(2h)$.

We have $h^0(\mathcal{O}_Y(2e_1)) = 1$. Moreover, $h^2(\mathcal{O}_Y(2e_1)) = h^0(\mathcal{O}_Y(-2e_1 - h)) = 0$ by the Serre duality theorem, hence the Riemann–Roch theorem on Y yields $h^1(\mathcal{O}_Y(2e_1)) = 0$. The cohomology of the sequence (2.2) for the pair $Z \subset Y$ tensored by $\mathcal{O}_Y(2e_1)$ then yields

$$h^i(\mathcal{N}_{Z/Y}) = 0 \tag{5.3}$$

for each i. Now the Serre duality theorem and the cohomology of the sequence (2.2) for the pair $Z \subset \mathbb{P}^4$ imply

$$\begin{aligned} h^1(\mathcal{O}_Z \otimes \mathcal{N}_{Y/X}) &= h^1(\mathcal{O}_Z \otimes \mathcal{O}_X(2h)) = \\ &= h^0(\mathcal{O}_Z \otimes \mathcal{O}_X(-5h)) = h^1(\mathcal{I}_{Z/X}(-5h)) = 0 \end{aligned} \tag{5.4}$$

thanks to Proposition 5.2. The cohomology of the sequence (2.2) for the pair $Z \subset Y$ tensored by $\mathcal{O}_X(2h)$ returns

$$h^0(\mathcal{O}_Z \otimes \mathcal{N}_{Y/X}) = \chi(\mathcal{O}_Z \otimes \mathcal{O}_X(2h)) = 7 \tag{5.5}$$

thanks to the Riemann–Roch theorem on Y. The assertions on $h^i(\mathcal{N}_{Z/X})$ then follow from the cohomology of (5.2) and the equalities (5.3), (5.4), (5.5). □

As an immediate by–product of the above corollary, we deduce that the scheme $\mathcal{R}$ is smooth of dimension 7. Moreover, as pointed out in the proof of Proposition 5.2, the scheme $\mathcal{R}$ is dominated by an open subset of the Hilbert scheme of lines $L \subset X$ (which is isomorphic to $\mathbb{P}^3$) times the set of 3×2 matrices with entries in $\mathbb{C}[x_3, x_4]$. Thus $\mathcal{R}$ is unirational.

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