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# Melnikov theory to all orders and Puiseux series for subharmonic solutions

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# Contents

# Introduction

*Life is what happens while you are making other plans.*
*J. Lennon*

Subharmonic bifurcations have been extensively studied in the literature, and are by now a standard topic of many classical textbooks [6, 11].

An intuitive formulation of the problem, far from being formal, is the following. Imagine the cylinder $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ in the euclidean tridimensional space, and let us suppose a point moving on $\mathcal{C}$ along the curve $\gamma(t) = (\cos(\omega(A_0)(t + t_0)), \sin(\omega(A_0)(t + t_0)), A_0)$ for fixed height $A_0$ and initial time $t_0$. Suppose the frequency to change monotonically for varying height and let us call $\alpha_0(t) = \omega(A_0)t$ the angle described by $\gamma(t)$. Hence, the velocity field on $\mathcal{C}$ is constant at fixed height, *i.e.* is given by

$$\begin{cases} \dot{\alpha}_0 = \omega(A), \\ \dot{A} = 0, \end{cases} \tag{1}$$

with respect to the angle and the height.

Let us call $\widetilde{\gamma}(t) = (\cos(\omega(A_0)(t + t_0)), \sin(\omega(A_0)(t + t_0)))$, *i.e.* the projection of $\gamma$ on the $x, y$ plane, and let us consider the cylinder $\widetilde{\mathcal{C}} = \{\widetilde{\gamma}\} \times \mathbb{R}$ in the extended phase space.

Suppose to cut $\widetilde{\mathcal{C}}$ at zero and $T = T(A_0) = 2\pi/\omega(A_0)$ height, and to merge the two boundaries, so that we obtain a torus as depicted in Figure 2. Such an operation is known as *topological quotient* of the cylinder $\widetilde{\mathcal{C}}$ and we shall denote the obtained torus as $\{\widetilde{\gamma}\} \times \mathbb{R}/T(A_0)\mathbb{Z}$.

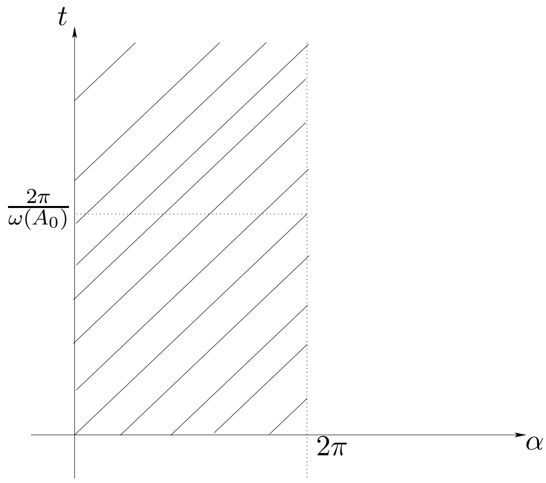


Figure 1: Trajectories on the plane $\alpha, t$ for some initial data.

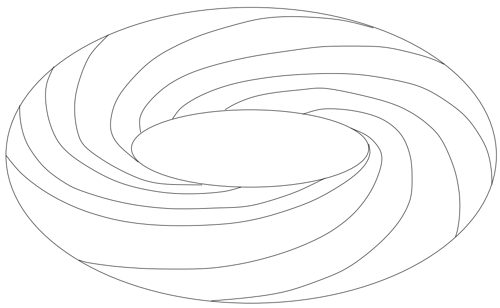

Figure 2: The torus $\{\widetilde{\gamma}\} \times \mathbb{R}/T(A_0)\mathbb{Z}$.

If we alternatively consider the torus $\{\widetilde{\gamma}\} \times \mathbb{R}/2\pi\mathbb{Z}$, we notice that, for varying $A_0$ some curves are closed while others are torus-filling curves. More precisely, the curves such that $\omega(A_0) = p/q \in \mathbb{Q}$, *i.e.* those having rational slope on the $\alpha, t$ plane, are still closed on the torus $\{\widetilde{\gamma}\} \times \mathbb{R}/2\pi\mathbb{Z}$.

Now, let us come back to the cylinder $\mathcal{C}$ and consider a small perturbation in the velocity field (1), both in the angular and in the vertical directions, which is $2\pi$-periodic with respect to the angle and to the time. We shall therefore write

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon F(\alpha, A, t), \\ \dot{A} = \varepsilon G(\alpha, A, t). \end{cases} \tag{2}$$

Of course not all curves will still be closed: in fact, if no further hypotheses are made on the perturbation, in general all the trajectories corresponding to the torus-filling curves in

$\{\widetilde{\gamma}\} \times \mathbb{R}/2\pi\mathbb{Z}$ will not survive under the perturbation, while some[1] of those with frequency $\omega(A_0) = p/q \in \mathbb{Q}$ are left. We are interested in studying whether some periodic solutions with period $T = 2\pi q/p$, persist under the action of the perturbation. Solutions with this property are called *subharmonic solutions of order $q/p$.*

We notice that our techniques applies also to systems of the form

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon F(\alpha, A, \varepsilon, t), \\ \dot{A} = \varepsilon G(\alpha, A, \varepsilon, t), \end{cases} \tag{3}$$

where the perturbation depends analytically on $\varepsilon$. Such a problem naturally arises, for instance, in the study of a periodically driven or forced system, in presence of dissipation, with one degree of freedom. In this case one has typically two parameters: the perturbation parameter $\varepsilon$ and the damping coefficient $\gamma$, *i.e.* one deals with an equation of the form

$$\ddot{x} + g(x) + \gamma\dot{x} = \varepsilon f(x,t), \quad x \in \mathbb{R}. \tag{4}$$

Hence, an interesting problem can be to study the region in the space of parameters where subharmonic solutions can occur and to determinate the bifurcation curves which divide the regions of existence and non-existence (see Figure 3) of these solutions; cf. for instance [2, 3, 13, 14], where Melnikov theory is applied to such situations.

As the computation for (3) are very similar to those performed for the system (2), we shall restrict our analysis to that case.

One can formulate the problem both in the $C^r$ Whitney topology and in the real-analytic setting. We shall choose the latter. From a technical point of view, this is mandatory since our techniques require for the system to be analytic. However, it is also very natural from a physical point of view, because in pratice, in any physical application, the functions appearing in the equations are analytic, often even polynomials, and when they are not analytic they are not even smooth. Moreover, we notice since now that, even though we restrict our analysis to the analytic setting, this does not mean at all that we can not deal with problems where non-analytic phenomena arise. The very case discussed here provides a counterexample.

The problem of subharmonic bifurcations was first considered by Melnikov [20], who showed that the existence of subharmonic solutions is related to the zeros of a suitable function, nowadays called the *subharmonic Melnikov function.* The standard Melnikov theory usually studies the case in which the Melnikov function has a simple (*i.e.* first order) zero. In such a case the problem can be reduced to an implicit function problem.

[1] The existence of closed orbits on $\mathcal{C}$ will depend on the initial datum $t_0$.

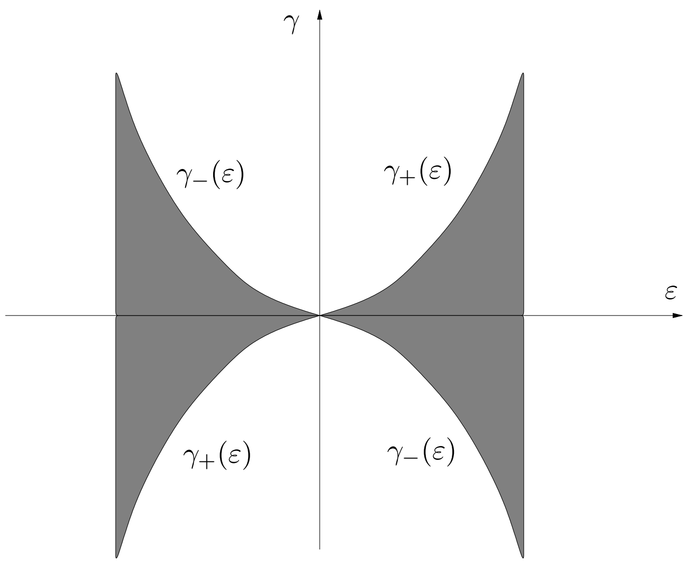


Figure 3: Set of existence (grey region) of subharmonic solutions in the plane $\varepsilon, \gamma$.

Nonetheless, it can happen that the subharmonic Melnikov function either vanishes identically or has a zero which is of order higher than one.

In the first case one can hopefully go to the higher orders, and if a suitable higher order generalisation of the subharmonic Melnikov function has a first order zero, then one can proceed very closely to the standard case, and existence of analytic subharmonic solutions is obtained. Most of the papers in the literature consider this kind of generalisations of Melnikov's theory; for instance Liu and Gu [19], gave an explicit expression for the second-order Melnikov function for the simple pendulum, and later Guo *et al.* [12], provided an explicit (and more general) expression for the second-order Melnikov function, and used it to obtain the existence of a solution for the equation of motion up to the first order in the perturbation parameter. However the explicit construction of such a solution was not performed there. Similar computation are performed in [31, 32], and also in [25, 30] a second order analysis is enough to settle the problem, under the hypothesis that the second-order Melnikov function has a simple zero. An explicit formula for the first two orders of the Melnikov function can be found in [18]. Another extension in this direction can be found in [5], where it is showed that, in a special case, there are at most three limit cycles bifurcating from the set of periodic orbits of the unperturbed system, and the first three orders of the Melnikov function are explicitly computed for that case. Also Iliev computed explicitly the first four orders of the Melnikov function in a special case [15]. Of course there are exceptions, such as [16, 28, 34, 35], dealing with analysis to arbitrarily high order. We remark that the papers cited above have to be considered just as examples with respect to the plenty of studies on Melnikov's theory.

The case in which the Melnikov function has a zero of finite order (higher than one) is more subtle. The problem can be still reduced to an implicit function problem, but the fact that the zeros are no longer simple prevents us from applying the implicit function theorem. Thus other arguments must be used, based on the Weierstrass preparation theorem [4, 6] and on the theory of the Puiseux series [1, 4, 6, 23]. However a systematic analysis is missing in the literature. Furthermore, in general, these arguments are not constructive: if on the one hand they allow us to prove (in certain cases) the existence of at least one subharmonic solution, on the other hand the problem of how many such solutions really exist and how they can be explicitly constructed has not been discussed in full generality.

The main difficulty for a constructive approach is that the solution of the implicit function equation has to be looked for by successive approximations. At each step of iteration, in order to find the correction to the approximate solution found at the previous one, one has to solve a new implicit function equation, which, in principle, still admits multiple roots. So, as far as the roots of the equations are not simple, one can not give an algorithm to produce systematically the corrections at the subsequent steps.

A careful discussion of a problem of the same kind can be found in [1], where the problem of bifurcations from multiple limit cycles is considered; cf. also [21, 22] where the problem is further investigated. There, under the hypothesis that a simple real zero is obtained at the first iteration step, it is proved that the bifurcating solutions can be expanded as fractional series (Puiseux series) of the perturbation parameter. The method to compute the coefficients of the series is based on the use of Newton's polygon [1, 4, 6], and allows one to go to arbitrarily high orders. However, the convergence of the series, and hence of the algorithm, relies on abstract arguments of algebraic and geometric theory.

To the best of our knowledge, the case of subharmonic bifurcations was not discussed in the literature. Of course, in principle one can think to adapt the same strategy as in [1] for the bifurcations of limit cycles. But still, there are issues which have not been discussed there: principally the explicit bound for the radius of convergence, and the case in which at the first iteration step one still has multiple roots. Moreover, we have a double aim. We are interested in results which are both general - not generic - and constructive. This means that we are interested in problems such as the following: which are the weaker conditions to impose on the perturbation, for a given integrable system and a given periodic solution, in order to prove the existence of subharmonic solutions? Of course the ideal result would be to have no restriction at all. At the same time, we are also interested in the explicit construction of such solutions, within any prefixed accuracy.

The problem of subharmonic solutions in the case of multiple zeros of the Melnikov functions has been considered in [33], where the following theorem is stated (without giving the proof) for $C^r$ smooth systems: if the subharmonic Melnikov function has a zero of odd order $n \leq r$, then there is at least one subharmonic solution. In any case the analyticity properties of the solutions are not discussed. In particular, the subharmonic solution is found as a function of two parameters - the perturbation parameter and the initial phase of the solution to be continued -, but the relation between the two parameters is not discussed. We point out that, in the analytic setting, it is exactly this relation which produces (as we shall see later) the lack of analyticity in the perturbation parameter. Furthermore, in [33] the case of zeros of even order is not considered: as we shall see, in that case the existence of subharmonic solutions can not be proved in general, but it can be obtained under extra assumptions.

Now we give a more detailed account of our results.

As said above, we shall consider systems which can be viewed as perturbations of integrable systems, with the perturbation which depends periodically in time. We shall use coordinates $(\alpha, A)$ such that, in the absence of the perturbation, $A$ is fixed to a constant value, while $\alpha$ rotates on the circle: hence all motions are periodic. As usual [11] we assume that, for $A$ varying in a finite interval, the periods change monotonically. More formal definitions will be given in Section 1.1.

Given an unperturbed periodic orbit $t \to (\alpha_0(t), A_0)$, we define the subharmonic Melnikov function $M(t_0)$ as the average over a period of the function $G(\alpha_0(t), A_0, t+t_0)$. By construction $M(t_0)$ is periodic in $t_0$. With the terminology introduced before, $\varepsilon$ is the perturbation parameter and $t_0$ is the initial phase. The following scenario arises.

- If $M(t_0)$ has no zero, then there is no subharmonic solution, that is no periodic solution which continues the unperturbed one at $\varepsilon \neq 0$.

- Otherwise, if $M(t_0)$ has zeros, the following two cases are possible: either $M(t_0)$ has a zero of finite order $\mathfrak{n}$ or $M(t_0)$ vanishes with all its derivatives. In the second case, because of analyticity, the function $M(t_0)$ is identically zero.

- If $M(t_0)$ has a simple zero (*i.e.* $\mathfrak{n} = 1$), then the usual Melnikov's theory applies. In particular there exists at least one subharmonic solution, and it is analytic in the perturbation parameter $\varepsilon$.

- If $M(t_0)$ has a zero of order $\mathfrak{n}$, then in general no result can be given about the existence of subharmonic solutions. However one can introduce an infinite sequence of polynomial equations, which are defined iteratively: if the first equation admits a real non-zero root and all the following equations admit a real root, then a subharmonic solution exists, and it is a function analytic in suitable fractional power of $\varepsilon$; more precisely it is analytic in $\eta = \varepsilon^{1/\mathfrak{p}}$, for some $\mathfrak{p} \leq \mathfrak{n}!$. If at some step the root is simple, an algorithm can be given in order to construct recursively all the coefficients of the series.

- If we further assume that the order $\mathfrak{n}$ of the zero is odd, then we have that all equations of the sequence satisfy the request made above on the roots, so that we can conclude that in such a case at least a subharmonic solution exists. Again, in order to really construct the solution, by providing an explicit recursive algorithm, we need that at a certain level of the iteration scheme a simple root appears.

- Moreover, we have at most $\mathfrak{n}$ periodic solutions bifurcating from the unperturbed one with initial phase $t_0$. Of course, to count all subharmonic solutions we have also to sum over all the zeros of the subharmonic Melnikov function.

- Finally, if $M(t_0)$ vanishes identically as a function of $t_0$, the solution $t \to (\alpha(t), A(t))$ is defined up to first order - as it is easy to check - and does not depend on the choice of $t_0$, so that one can expand the function $G(\alpha(t), A(t), t + t_0)$ up to first order: we call $M_1(t_0)$ its average over a period of the unperturbed solution. In particular if also $M_1(t_0)$ vanishes identically, then one can push the perturbation theory up to second order - as also the second order of the solution does not depend on $t_0$ - and, after expanding the function $G(\alpha(t), A(t), t + t_0)$ up to second order, one defines $M_2(t_0)$ as its average over a period, and so on. If at a finite step $k$ of such an iteration the $k$-th order Melnikov function has a zero of finite order $\mathfrak{n}_k$,one can repeat the analysis performed above for $M(t_0)$, and no further difficulties arise. Otherwise, if all the subharmonic Melnikov functions vanish identically in $t_0$, then one has a subharmonic solution for any $t_0$ small enough.

The methods we shall use to prove the results above will be of two different types. We shall rely on standard general techniques, based on the Weierstrass preparation theorem, in order to show that under suitable assumptions the solutions exist and to prove in this case the convergence of the series. Moreover, we shall use a combination of the (so called) Newton-Puiseux algorithm and the diagrammatic techniques based on the tree formalism [7, 8, 9, 10] in order to provide a recursive algorithm, when possible. Notice that in such a case the convergence of the Puiseux series follows by explicit construction of the coefficients, and an explicit bound of the radius of convergence is obtained through the estimates of the coefficients - on the contrary there is no way to provide quantitative bounds with the aforementioned abstract arguments. These results extend those in [10], where a special case was considered.

This thesis is organised in two parts

The first part (Chapter 1 and Chapter 2) is devoted to construct explicitly the (convergent) Puiseux series, and to provide an explicit bound for the radius of convergence, under some simplifying hypotheses. More precisely, in Section 1.1 we state the result (Theorem 1.4) about the existence of subharmonic solutions under the Hypothesis 2 and 3 namely, the Melnikov function has a $\mathfrak{n}$-th order zero and at the first iteration step one has a simple root for the polynomials above. The construction of a formal solution is performed in Section 1.2 while the convergence is proved in Section 2.3, using the diagrammatic techniques introduced in Section 2.1. Section 2.2 is devoted to prove that the formal solutions are analytic functions of two variables: the perturbation parameter and the initial phase.

The second part is devoted to generalise Theorem 1.4. In Section 3.1 we describe the Newton-Puiseux algorithm and we use it to prove Theorem 3.3 where Hypothesis 3 of Theorem 1.4 is substituted by the weaker Hypothesis 4, *i.e.* a simple zero for the polynomials equation is found at a finite step of iteration. In Section 3.2 we study the case in which there is no simple root at any iteration step. We shall see that in such a case the solution cannot be explicitly constructed. Finally, in Section 3.3 we consider the higher order Melnikov theory; in particular, we shall see how this situation can be led back to the previous one.

# Notations:

- Given a differentiable function of several arguments $F = F(x_1, \dots, x_n)$, we shall denote by $\partial^m_{x_k}$ the $m$-th derivative with respect to the $k$-th argument *i.e.* for $m \geq 1$
$$\partial^m_{x_k} F := \frac{\partial^m F}{\partial x_k^m},$$
while $m = 0$ has to be interpreted as $\partial^0_{x_k} F = F$.

- Given any $T$-periodic function $H$ we denote by $\langle H \rangle$ its mean over a period *i.e.*
$$\langle H \rangle := \frac{1}{T} \int_0^T \mathrm{d}\tau H(\tau).$$

- The ring of the formal power series in $n$ variables with coefficients in a field $K$ is denoted by $K[[x_1, \dots, x_n]]$.

- The ring of convergent power series in $n$ variables with coefficients in a field $K$ is denoted by $K\{x_1, \dots, x_n\}$.

- The ring of polynomials in $n$ variables with coefficients in an integral domain $D$ is $D[x_1, \dots, x_n]$.

- Given two polynomials $P_1, P_2 \in D[x_1, \dots, x_n]$, we denote the sum between $P_1, P_2$ where every common term is counted only once, by $P_1 \oplus P_2$. If $P_1, P_2$ have no common terms, we have simply $P_1 \oplus P_2 = P_1 + P_2$.

- Given a finite set $S$ we denote with $|S|$ the number of its elements.

- We set $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

# 1. Set up and Formal solution

In the first part of this chapter we state a result on the existence, under suitable conditions on the perturbation, of subharmonic solutions, while in the second part we prove the existence of such a solution as a formal fractional series.

## 1.1 Statement of the result

Let us consider the dynamical system

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon F(\alpha, A, t), \\ \dot{A} = \varepsilon G(\alpha, A, t), \end{cases} \tag{1.1}$$

where $(\alpha, A) \in \mathcal{M} := \mathbb{T} \times W$, with $W \subset \mathbb{R}$ an open set, the map $A \mapsto \omega(A)$ real analytic in $A$, and the functions $F$, $G$ depend analytically on their arguments and are $2\pi$-periodic in $\alpha$ and $t$. Finally $\varepsilon$ is a real parameter.

For $\varepsilon = 0$ one has the trivial solution $(\omega(A_0)t + \alpha_0, A_0)$ with $\alpha_0$, $A_0$ fixed as initial data. If we define $\alpha_0(t) = \omega(A_0)t$ and $A_0(t) = A_0$ in the *extended phase-space* $\mathcal{M} \times \mathbb{R}$ the solution $(\alpha_0(t), A_0(t), t + t_0)$ describes an invariant torus uniquely determined by $A_0$. The parameter $t_0$ is called *initial phase* and it fixes the initial datum $\alpha_0$ on the torus. Hence the motion in the extended phase space is quasiperiodic, and it is periodic if $\omega(A_0)$ is commensurate with 1. In the latter case, *i.e.* if $\omega(A_0)$ is rational, we say that the torus is *resonant*. In general, if no further hypotheses are made on the perturbation, all the non-resonant tori are destroyed. Most of resonant tori disappear too, but a finite number of periodic orbits lying on the unperturbed torus can survive under perturbation. The persisting trajectories are called *subharmonic solutions*.

Let us denote $T_0(A) = 2\pi/\omega(A)$ the period of the trajectory on an unperturbed torus and define $\omega'(A) := \mathrm{d}\omega(A)/\mathrm{d}A$. The value $A_0$ is fixed once and for all in such a way that

$$\omega(A_0) = \frac{p}{q} \in \mathbb{Q}, \tag{1.2}$$

where $p, q \in \mathbb{Z}$ are relatively prime integers, and we call $T = T(A_0) = 2\pi q$ the period of the trajectories in the extended phase space and say that the corresponding subharmonic solution has *order* $q/p$.

Furthermore we make the following assumption on the resonant torus with "energy" $A_0$.

**Hypothesis 1.** *One has* $\omega'(A_0) \neq 0$.

We define the *subharmonic Melnikov function* of order $q/p$ as

$$M(t_0) = \frac{1}{T} \int_0^T \mathrm{d}t\, G(\alpha_0(t), A_0, t + t_0), \tag{1.3}$$

and we point out that $M(t_0)$ is $2\pi$-periodic.

**Hypothesis 2.** *There exist* $t_0 \in [0, 2\pi)$ *and* $\mathfrak{n} \in \mathbb{N}$ *such that* $t_0$ *is a zero of order* $\mathfrak{n}$ *for the subharmonic Melnikov function, that is*

$$\frac{\mathrm{d}^k}{\mathrm{d}t_0^k} M(t_0) = 0 \quad \forall\, 0 \le k \le \mathfrak{n} - 1, \qquad D = D(t_0) := \frac{\mathrm{d}^{\mathfrak{n}}}{\mathrm{d}t_0^{\mathfrak{n}}} M(t_0) \neq 0. \tag{1.4}$$

In the following $t_0$ is fixed once and for all in such a way that it satisfies Hypothesis 2.

It is more convenient to write the system in the form

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon F(\alpha, A, t + t_0), \\ \dot{A} = \varepsilon G(\alpha, A, t + t_0), \end{cases} \tag{1.5}$$

so we can set equal to zero the initial angle of the unperturbed solution.

If we set $U(t) = \widetilde{U}(A(t)) = \omega(A(t)) - \omega(A_0) - \omega'(A_0)(A(t) - A_0)$, and

$$\begin{aligned} \Phi(t) &= \widetilde{\Phi}(\alpha(t), A(t), t + t_0) = \varepsilon F(\alpha(t), A(t), t + t_0) + U(t), \\ \Gamma(t) &= \widetilde{\Gamma}(\alpha(t), A(t), t + t_0) = \varepsilon G(\alpha(t), A(t), t + t_0), \end{aligned} \tag{1.6}$$

and denote by

$$W(t) = \begin{pmatrix} 1 & \omega'(A_0)t \\ 0 & 1 \end{pmatrix} \tag{1.7}$$

the Wronskian matrix for the unperturbed linearised system, then the solution of the system (1.5) can be written as

$$\begin{pmatrix} \alpha(t) \\ A(t) \end{pmatrix} = W(t) \begin{pmatrix} \alpha(0) \\ A(0) \end{pmatrix} + W(t) \int_0^t \mathrm{d}\tau\, W^{-1}(\tau) \begin{pmatrix} \Phi(\tau) \\ \Gamma(\tau) \end{pmatrix}, \tag{1.8}$$

or, more explicitly,

$$\begin{cases} \alpha(t) = \alpha(0) + t\omega'(A_0)A(0) + \int_0^t \mathrm{d}\tau\, \Phi(\tau) + \omega'(A_0) \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\, \Gamma(\tau') \\ A(t) = A(0) + \int_0^t \mathrm{d}\tau\, \Gamma(\tau). \end{cases} \tag{1.9}$$

In order to have a periodic solution of period $T$ we need $\langle \Gamma \rangle = 0$; in this case we fix also

$$\omega'(A_0)A(0) + \langle \Phi \rangle + \omega'(A_0)\langle \mathcal{G} \rangle = 0, \quad \mathcal{G}(t) = \int_0^t \mathrm{d}\tau\, (\Gamma(\tau) - \langle \Gamma \rangle), \tag{1.10}$$

so that the corresponding solution turns out to be periodic.

Hence we consider the system

$$\begin{cases} \alpha(t) = \alpha(0) + \int_0^t \mathrm{d}\tau\, (\Phi(\tau) - \langle \Phi \rangle) + \omega'(A_0) \int_0^t \mathrm{d}\tau\, \left(\mathcal{G}(t) - \langle \mathcal{G} \rangle\right) \\ A(t) = A(0) + \mathcal{G}(t) \\ \langle \Gamma \rangle = 0 \end{cases} \tag{1.11}$$

where $A(0)$ is determined according to (1.10) and it is well-defined as $\omega'(A_0) \neq 0$ by Hypothesis 1, while $\alpha(0)$ is considered as a free parameter. Our aim is to study whether it is possible to fix $\alpha(0)$ as a function of the perturbation parameter $\varepsilon$ in such a way that the latter equation in (1.11) is satisfied.

We start by removing the condition $\langle \Gamma \rangle = 0$ in (1.11), *i.e.* by considering the system

$$\begin{cases} \alpha(t) = \alpha(0) + \int_0^t \mathrm{d}\tau\, (\Phi(\tau) - \langle \Phi \rangle) + \omega'(A_0) \int_0^t \mathrm{d}\tau\, \left(\mathcal{G}(t) - \langle \mathcal{G} \rangle\right) \\ A(t) = A(0) + \mathcal{G}(t) \end{cases} \tag{1.12}$$

As we are looking for periodic solutions of period $T = 2\pi q$, *i.e.* of frequency $\omega = 1/q$, it is more convenient to work in Fourier space and write

$$\begin{aligned} \alpha(t) &= \alpha_0(t) + \beta(t), \qquad \beta(t) = \sum_{\nu \in \mathbb{Z}} e^{i\omega\nu t} \beta_\nu, \\ A(t) &= A_0(t) + B(t), \qquad B(t) = \sum_{\nu \in \mathbb{Z}} e^{i\omega\nu t} B_\nu. \end{aligned} \tag{1.13}$$

Hence we expand

$$G(\alpha, A, t + t_0) = \sum_{\sigma, \sigma' \in \mathbb{Z}} e^{i\sigma\alpha + i\sigma'(t+t_0)} G_{\sigma,\sigma'}(A), \qquad G_{\sigma,\sigma'}(A, t_0) := e^{i\sigma' t_0} G_{\sigma,\sigma'}(A) \tag{1.14}$$

with an analogous expression for $F(\alpha, A, t+t_0)$, and we obtain

$$\Phi(t) = \sum_{\nu\in\mathbb{Z}} e^{i\omega\nu t}\Phi_\nu, \qquad \Gamma(t) = \sum_{\nu\in\mathbb{Z}} e^{i\omega\nu t}\Gamma_\nu, \tag{1.15}$$

where we have defined

$$\begin{aligned}
\Gamma_\nu &= \varepsilon \sum_{m\geq 0} \sum_{\substack{r+s=m \\ r,s\in\mathbb{Z}_+}} \sum_{p\sigma_0+q\sigma_0'+\nu_1+\ldots+\nu_m=\nu} \frac{1}{r!s!}(i\sigma_0)^r \partial_A^s G_{\sigma_0,\sigma_0'}(A_0,t_0)\beta_{\nu_1}\ldots\beta_{\nu_r}B_{\nu_{r+1}}\ldots B_{\nu_m}, \\
\Phi_\nu &= \varepsilon \sum_{m\geq 0} \sum_{\substack{r+s=m \\ r,s\in\mathbb{Z}_+}} \sum_{p\sigma_0+q\sigma_0'+\nu_1+\ldots+\nu_m=\nu} \frac{1}{r!s!}(i\sigma_0)^r \partial_A^s F_{\sigma_0,\sigma_0'}(A_0,t_0)\beta_{\nu_1}\ldots\beta_{\nu_r}B_{\nu_{r+1}}\ldots B_{\nu_m} \\
&\quad + \sum_{s\geq 2} \sum_{\nu_1+\ldots+\nu_s=\nu} \frac{1}{s!}\partial_A^s\omega(A_0)B_{\nu_1}\ldots B_{\nu_s}.
\end{aligned} \tag{1.16}$$

Then (1.12) becomes

$$\begin{cases} (i\omega\nu)^2\beta_\nu = (i\omega\nu)\Phi_\nu + \omega'(A_0)\Gamma_\nu, \\ (i\omega\nu)B_\nu = \Gamma_\nu \end{cases} \tag{1.17}$$

for $\nu \neq 0$, provided one has

$$\begin{cases} \beta_0 = \alpha(0) - \displaystyle\sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} \frac{\Phi_\nu}{i\omega\nu} - \omega'(A_0) \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} \frac{\Gamma_\nu}{(i\omega\nu)^2} \\ \omega'(A_0)B_0 + \Phi_0 = 0, \end{cases} \tag{1.18}$$

for $\nu = 0$; similarly (1.11) can be written in the same form, with the constraint $\Gamma_0 = 0$. Remark that $\beta_0$ can be used as a free parameter instead of $\alpha(0)$. This means that, in Fourier space, (1.12) becomes

$$\begin{cases} \beta_\nu = \dfrac{\Phi_\nu}{i\omega\nu} + \omega'(A_0)\dfrac{\Gamma_\nu}{(i\omega\nu)^2}, \quad \nu \neq 0 \\ B_\nu = \dfrac{\Gamma_\nu}{i\omega\nu}, \quad \nu \neq 0 \\ B_0 = -\dfrac{\Phi_0}{\omega'(A_0)}, \end{cases} \tag{1.19}$$

with each Fourier coefficient depending on the free parameter $\beta_0$.

We can consider a solution $(\overline{\alpha}(t), \overline{A}(t))$ of (1.12) which can be formally expanded in Taylor series in $\varepsilon$ and $\beta_0$ as

$$\begin{aligned}
\overline{\alpha}(t) &= \overline{\alpha}(t;\varepsilon,\beta_0) = \alpha_0(t) + \beta_0 + \sum_{\substack{k\geq 1\\ j\geq 0}} \varepsilon^k\beta_0^j \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} e^{i\omega\nu t}\overline{\beta}_\nu^{(k,j)} \\
\overline{A}(t) &= \overline{A}(t;\varepsilon,\beta_0) = A_0 + \sum_{\substack{k\geq 1\\ j\geq 0}} \varepsilon^k\beta_0^j \sum_{\nu\in\mathbb{Z}} e^{i\omega\nu t}\overline{B}_\nu^{(k,j)}.
\end{aligned} \tag{1.20}$$

Also the functions $\overline{\Gamma}(t) = \varepsilon G(\overline{\alpha}(t), \overline{A}(t), t + t_0)$ and $\overline{\Phi}(t) = \widetilde{\Phi}(\overline{\alpha}(t), \overline{A}(t), t + t_0)$ can be formally expressed in power series of $\varepsilon$ and $\beta_0$, and one has

$$\overline{\Gamma}(t) = \overline{\Gamma}(t; \varepsilon, \beta_0) = \sum_{\substack{k \geq 1 \\ j \geq 0}} \varepsilon^k \beta_0^j \, \overline{\Gamma}^{(k,j)}(t) = \sum_{\substack{k \geq 1 \\ j \geq 0}} \varepsilon^k \beta_0^j \sum_{\nu \in \mathbb{Z}} e^{i\omega\nu t} \overline{\Gamma}_\nu^{(k,j)}, \tag{1.21}$$

where each term $\overline{\Gamma}_\nu^{(k,j)}$ depends on the Taylor coefficients of (1.20) of order strictly less than $k$, with an analogous expression holding for $\overline{\Phi}(t)$. For instance one has

$$\begin{aligned} \overline{\Gamma}_\nu^{(k,j)} = & \sum_{\substack{r' \geq 0 \\ s \geq 0}} \sum_{r + j_0 = r'} \sum_{p\sigma_0 + q\sigma_0' + \nu_1 + \ldots + \nu_{r+s} = \nu} \frac{(i\sigma_0)^{r'}}{r'!} \frac{\partial_A^s}{s!} G_{\sigma_0, \sigma_0'}(A_0, t_0) \\ & \times \sum_{\substack{k_1 + \ldots + k_{r+s} = k-1 \\ j_1 + \ldots + j_{r+s} = j - j_0 \\ k_i \geq 1, \, j_i \geq 0}} \overline{\beta}_{\nu_1}^{(k_1, j_1)} \cdots \overline{\beta}_{\nu_r}^{(k_r, j_r)} \overline{B}_{\nu_{r+1}}^{(k_{r+1}, j_{r+1})} \cdots \overline{B}_{\nu_{r+s}}^{(k_{r+s}, j_{r+s})}. \end{aligned} \tag{1.22}$$

Remark that by definition one has $\left\langle \overline{\Gamma}^{(k,j)} \right\rangle = \overline{\Gamma}_0^{(k,j)}$ and $\left\langle \overline{\Phi}^{(k,j)} \right\rangle = \overline{\Phi}_0^{(k,j)}$.

Hence one can formally write, for all $k \geq 1$ and $j \geq 0$

$$\begin{cases} \overline{\beta}_\nu^{(k,j)} = \dfrac{\overline{\Phi}_\nu^{(k,j)}}{i\omega\nu} + \omega'(A_0) \dfrac{\overline{\Gamma}_\nu^{(k,j)}}{(i\omega\nu)^2}, \quad \nu \neq 0 \\ \overline{B}_\nu^{(k,j)} = \dfrac{\overline{\Gamma}_\nu^{(k,j)}}{i\omega\nu}, \quad \nu \neq 0 \\ \overline{B}_0^{(k,j)} = -\dfrac{\overline{\Phi}_0^{(k,j)}}{\omega'(A_0)}. \end{cases} \tag{1.23}$$

For instance for $k = 1$ and $j = 0$ one has

$$\begin{aligned} \overline{\beta}_\nu^{(1,0)} = & \frac{1}{i\omega\nu} \sum_{p\sigma_0 + q\sigma_0' = \nu} F_{\sigma_0, \sigma_0'}(A_0, t_0) + \frac{\omega'(A_0)}{(i\omega\nu)^2} \sum_{p\sigma_0 + q\sigma_0' = \nu} G_{\sigma_0, \sigma_0'}(A_0, t_0), \\ \overline{B}_\nu^{(1,0)} = & \frac{1}{i\omega\nu} \sum_{p\sigma_0 + q\sigma_0' = \nu} G_{\sigma_0, \sigma_0'}(A_0, t_0), \end{aligned} \tag{1.24}$$

for $\nu \neq 0$, and

$$\overline{B}_0^{(1,0)} = -\frac{1}{\omega'(A_0)} \sum_{p\sigma_0 + q\sigma_0' = 0} F_{\sigma_0, \sigma_0'}(A_0, t_0), \tag{1.25}$$

for $\nu = 0$.

We also introduce the coefficients

$$\begin{aligned} \overline{\beta}_\nu^{(k)}(\beta_0) = \sum_{j \geq 0} \beta_0^j \, \overline{\beta}_\nu^{(k,j)}, \qquad \overline{B}_\nu^{(k)}(\beta_0) = \sum_{j \geq 0} \beta_0^j \, \overline{B}_\nu^{(k,j)}, \\ \overline{\Gamma}_\nu^{(k)}(\beta_0) = \sum_{j \geq 0} \beta_0^j \, \overline{\Gamma}_\nu^{(k,j)}, \qquad \overline{\Phi}_\nu^{(k)}(\beta_0) = \sum_{j \geq 0} \beta_0^j \, \overline{\Phi}_\nu^{(k,j)}, \end{aligned} \tag{1.26}$$

and remark that $\overline{\Gamma}_{\nu}^{(k)}(0) = \overline{\Gamma}_{\nu}^{(k,0)}$, and so on.

**Lemma 1.1.** *With the before introduced notations, if* $\overline{\Gamma}_0^{(k)}(0) = 0$ *for all* $k \in \mathbb{N}$*, then the formal solution*

$$\begin{aligned} \overline{\alpha}(t;\varepsilon,0) &= \alpha_0(t) + \sum_{k\geq 1} \varepsilon^k \overline{\beta}^{(k,0)}(t) \\ \overline{A}(t;\varepsilon,0) &= A_0 + \sum_{k\geq 1} \varepsilon^k \overline{B}^{(k,0)}(t) \end{aligned} \tag{1.27}$$

*that is the formal solution (1.20) of (1.12) for* $\beta_0 = 0$*, is also a formal solution of (1.11).*

*Proof.* The proof is a direct check. In fact if $\overline{\Gamma}_0^{(k)}(0) = 0$ for all $k \in \mathbb{N}$, then

$$\langle \overline{\Gamma} \rangle = \sum_{\substack{k\geq 1 \\ j\geq 1}} \varepsilon^k \beta_0^j \, \overline{\Gamma}_0^{(k,j)} \tag{1.28}$$

and it is equal to zero for $\beta_0 = 0$. Hence one obtain that (1.27) is also solution of (1.11). □

There is unfortunately no reason for the hypothesis $\overline{\Gamma}_0^{(k)}(0) = 0$ to hold true for all $k \in \mathbb{N}$; in general there will exist $k_0 \in \mathbb{N}$ such that $\overline{\Gamma}_0^{(k)}(0) = 0$ for $k = 0, \ldots, k_0$, while $\overline{\Gamma}_0^{(k_0+1)}(0) \neq 0$. Here and henceforth we deal with this case.

Note that by (1.21) one can define

$$\mathcal{F}(\varepsilon, \beta_0) := \sum_{k,j\geq 0} \varepsilon^k \beta_0^j \mathcal{F}_{k,j}, \qquad \mathcal{F}_{k,j} = \overline{\Gamma}_0^{(k+1,j)}, \tag{1.29}$$

so that

$$\varepsilon \mathcal{F}(\varepsilon, \beta_0) = \langle \overline{\Gamma}(\,\cdot\,; \varepsilon, \beta_0) \rangle. \tag{1.30}$$

Our aim is to find $\beta_0 = \beta_0(\varepsilon)$ such that $\mathcal{F}(\varepsilon, \beta_0(\varepsilon)) \equiv 0$. For such $\beta_0$ the formal solution (1.20) of (1.12) is also a formal solution of (1.11).

Note that $\mathcal{F}(\varepsilon, \beta_0)$ is $\beta_0$-general of order $\mathfrak{n}$, *i.e.* $\mathcal{F}_{0,j} = 0$ for $j = 0, \ldots, \mathfrak{n} - 1$ while $\mathcal{F}_{0,\mathfrak{n}} \neq 0$. This can be easily shown using the tree formalism introduced in Chapter 2. In fact for all $j$, $\mathcal{F}_{0,j} = \overline{\Gamma}_0^{(1,j)}$ is associated with a tree with 1 node and $j$ leaves (see the Remark 2.4). Hence one has

$$j!\overline{\Gamma}_0^{(1,j)} = \langle \partial_\alpha^j G(\alpha_0(\cdot), A_0, \cdot + t_0) \rangle = (-\omega(A_0))^{-j} \frac{\mathrm{d}^j M}{\mathrm{d}t_0^j}(t_0), \tag{1.31}$$

where the second equality is provided by Lemma 3.9 on [10]. Then $\mathcal{F}(\varepsilon, \beta_0)$ is $\beta_0$-general of order $\mathfrak{n}$ by Hypothesis 2.

Given a formal power series $\mathcal{F}(\varepsilon, \beta_0) \in \mathbb{R}[[\varepsilon, \beta_0]]$ as in (1.29), we shall call *carrier of* $\mathcal{F}$ the set

$$\Delta(\mathcal{F}) = \{(k, j) \in \mathbb{N} \times \mathbb{N} : \mathcal{F}_{k,j} \neq 0\}. \tag{1.32}$$

For all $v \in \Delta(\mathcal{F})$ let us consider the positive quadrant $\mathfrak{A}_v := \{v\} + (\mathbb{R}_+)^2$ moved up to $v$, and define

$$\mathfrak{A} := \bigcup_{v \in \Delta(\mathcal{F})} \mathfrak{A}_v. \tag{1.33}$$

Let $\mathcal{C}$ be the convex hull of $\mathfrak{A}$. The boundary $\partial\mathcal{C}$ consists of a compact polygonal path $\mathcal{P}$ and two half lines $\mathcal{R}_1$ and $\mathcal{R}_2$, as displayed in Figure 1.1.

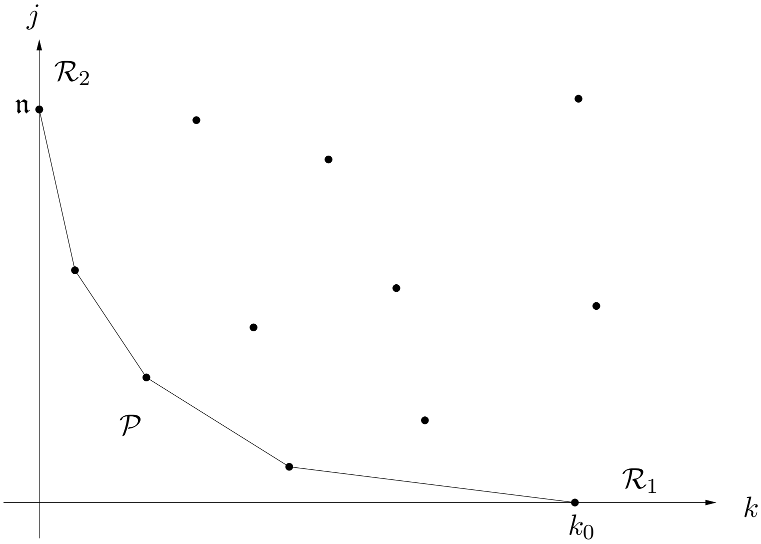


Figure 1.1: Newton polygon formed by four segments

The polygonal path $\mathcal{P}$ is called the *Newton polygon* of $\mathcal{F}$.

Notice that if the Newton polygon is a single point (see Figure 1.2) or, more generally, if $\mathcal{F}_{k,0} = 0$ for all $k \geq 0$ then there exists $\mathfrak{j} \geq 1$ such that $\mathcal{F}(\varepsilon, \beta_0) = \beta_0^{\mathfrak{j}} \cdot \overline{\mathcal{G}}(\varepsilon, \beta_0)$ with $\overline{\mathcal{G}}(\varepsilon, 0) \neq 0$ hence $\beta_0 \equiv 0$ is a solution of equation $\mathcal{F}(\varepsilon, 0) = 0$, that is the conclusion of Lemma 1.1.

Otherwise, *i.e.* in the case we are dealing with, there is at least a point of $\Delta(\mathcal{F})$ on each axis, then the Newton polygon $\mathcal{P}$ is formed by $N \geq 1$ segments $\mathcal{P}_1, \ldots, \mathcal{P}_N$ and we write $\mathcal{P} = \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_N$. For all $i = 1, \ldots, N$ let $-1/\mu_i \in \mathbb{Q}$ be the slope of the segment $\mathcal{P}_i$, so one can partition $\mathcal{F}$ according to the weights given by $\mu_i$:

$$\mathcal{F}(\varepsilon, \beta_0) = \widetilde{\mathcal{F}}_i(\varepsilon, \beta_0) + \mathcal{G}_i(\varepsilon, \beta_0) = \sum_{k + j\mu_i = \mathfrak{r}_i} \mathcal{F}_{k,j} \varepsilon^k \beta_0^j + \sum_{k + j\mu_i > \mathfrak{r}_i} \mathcal{F}_{k,j} \varepsilon^k \beta_0^j, \tag{1.34}$$

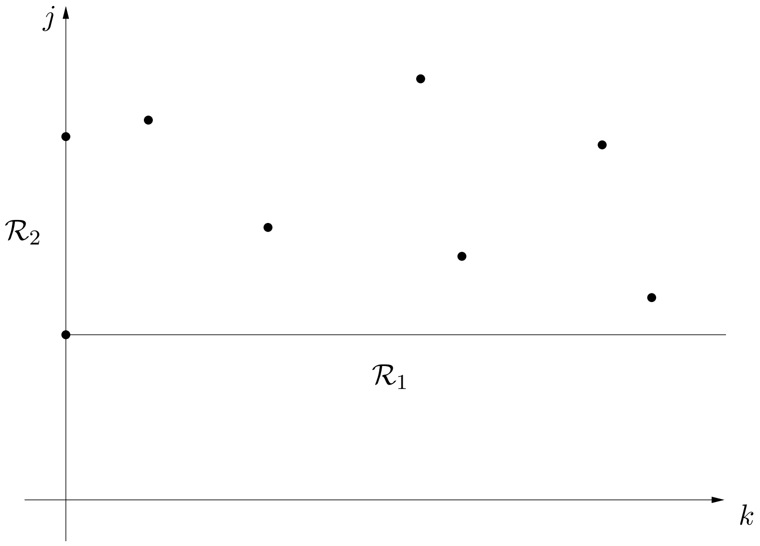


Figure 1.2: Newton polygon formed by a single point

where $\mathfrak{r}_i$ is the intercept on the $k$-axis of the continuation of $\mathcal{P}_i$ (see Figure 1.3).

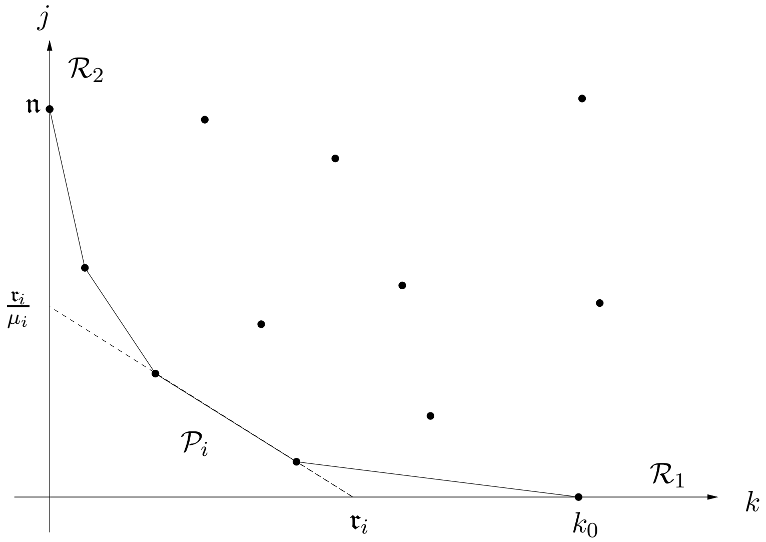


Figure 1.3: The intersections on the axes of the continuation of a segment $\mathcal{P}_i$.

We notice that we can associate with the whole Newton polygon a polynomial of degree $\mathfrak{n}$ in $\beta_0$, $\widetilde{\mathcal{F}}(\varepsilon, \beta_0) = \widetilde{\mathcal{F}}_1(\varepsilon, \beta_0) \oplus \ldots \oplus \widetilde{\mathcal{F}}_N(\varepsilon, \beta_0)$, which is the *lowest order of* $\mathcal{F}$. Notice that for each $j$ there is either one summand with $k = k_j = \mathfrak{r}_i - j\mu_i$ for some $i = 1, \ldots, N$, or no summands at all.

Hence the first approximate solutions of $\mathcal{F}(\varepsilon, \beta_0) = 0$ are the solutions of the quasi-homogeneous equations

$$\widetilde{\mathcal{F}}_i(\varepsilon, \beta_0) = \sum_{k\mathfrak{p}_i + j\mathfrak{h}_i = \mathfrak{s}_i} \mathcal{F}_{k,j}\varepsilon^k \beta_0^j = 0, \qquad i = 1, \dots, N, \tag{1.35}$$

where $\mathfrak{h}_i/\mathfrak{p}_i = \mu_i$, with $\mathfrak{h}_i, \mathfrak{p}_i$ relatively prime integers, and $\mathfrak{s}_i = \mathfrak{p}_i\mathfrak{r}_i$. Each formal solution $\beta_0' = \beta_0'(\varepsilon)$ of $\widetilde{\mathcal{F}}(\varepsilon, \beta_0) = 0$ can be expressed in *Puiseux series*, *i.e.* it is of the form

$$\beta_0' = \beta_0'(\varepsilon) = \sum_{h \geq \mathfrak{h}_i} c_h (\sigma\varepsilon)^{h/\mathfrak{p}_i}, \qquad \sigma := \text{sign}\,(\varepsilon), \tag{1.36}$$

(see for instance [4, 17, 23, 24]).

For all $i = 1, \dots, N$ we associate with $\widetilde{\mathcal{F}}_i$ a polynomial $P_i = P_i(c)$ in such a way that

$$\widetilde{\mathcal{F}}_i(\varepsilon, c(\sigma\varepsilon)^{\mu_i}) = (\sigma\varepsilon)^{\mathfrak{r}_i} \sum_{k\mathfrak{p}_i + j\mathfrak{h}_i = \mathfrak{s}_i} Q_{k,j} c^j = (\sigma\varepsilon)^{\mathfrak{r}_i} P_i(c), \qquad \sigma = \text{sign}\,(\varepsilon), \tag{1.37}$$

where $Q_{k,j} = \mathcal{F}_{k,j}\sigma^k$.

Remark that $P_i(c)$ is a polynomial of degree $d_i := \max\{j : k + j\mu_i = \mathfrak{r}_i\}$ then it has $d_i$ complex roots counting multiplicity. More precisely one has the following result.

**Lemma 1.2.** *With the notation introduced before, let $\Pi_i$ be the projection of the segment $\mathcal{P}_i$ on the $j$-axis and let $\ell_i = \ell(\Pi_i)$ be the length of $\Pi_i$. Then $P_i(c)$ has $\ell_i$ complex non-zero roots counting multiplicity.*

*Proof.* Let $m, n$ be respectively the maximum and the minimum among the exponents of the variable $\beta_0$ in $\widetilde{\mathcal{F}}_i$. Then $\ell_i = m - n$ as one can easily verify. Hence $P_i$ is a polynomial of degree $m$ and minimum power $n$. Therefore we can write $P_i(c) = c^n \widetilde{P}(c)$ where $\widetilde{P}$ has degree $\ell_i$ and $\widetilde{P}(0) \neq 0$. Fundamental theorem of algebra guarantees that $\widetilde{P}(c) = 0$ has $\ell_i$ complex solutions counting multiplicity, which are all the non-zero roots of $P_i$. $\square$

**Hypothesis 3.** *There exists a segment $\mathcal{P}_i$ of $\mathcal{P}$ associated with a quasi-homogeneous polynomial $\mathcal{F}_i(\varepsilon, \beta_0)$ such that the polynomial $P_i(c)$ has a simple root $c^* \in \mathbb{R}$.*

**Remark 1.3.** If $k_0 = 1$, then $\widetilde{\mathcal{F}}(\varepsilon, \beta_0) = D\beta_0^{\mathfrak{n}} + \mathcal{F}_{1,0}\varepsilon$: this situation, which is depicted in Figure 1.4, is explicitly considered in [10]. In this case we have $P(c) = \sigma D c^{\mathfrak{n}} + \mathcal{F}_{k_0,0}$, so that the root $c^* = (-\sigma\mathcal{F}_{k_0,0}/D)^{1/\mathfrak{n}}$ is simple. Moreover, if $\mathfrak{n}$ is odd, the polynomial admits the real root $\beta_0 = -(\varepsilon\mathcal{F}_{k_0,0}/D)^{1/\mathfrak{n}}$ for all values of $\varepsilon$, while for even $\mathfrak{n}$ a restriction on the sign of $\varepsilon$ must be imposed.

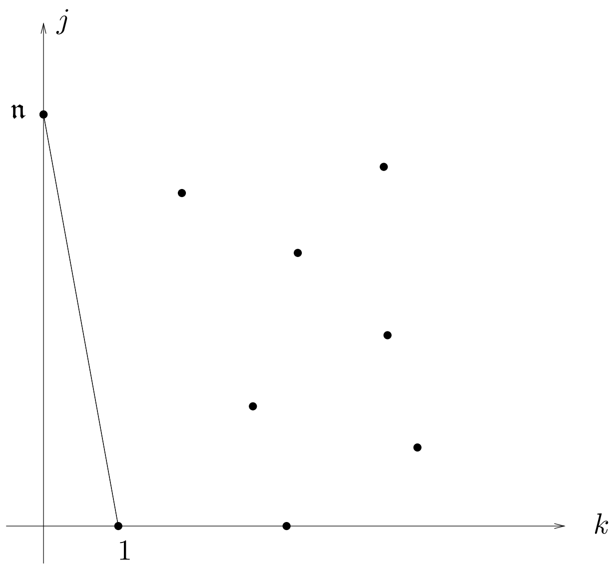


Figure 1.4: The Newton polygon for the case considered in [10].

Let $P_i = P_i(c)$ be the polynomial of Hypothesis 3. We introduce the auxiliary parameter $\eta := (\sigma\varepsilon)^{1/\mathfrak{p}_i}$, with $\sigma = \mathrm{sign}\,(\varepsilon)$, and set

$$\beta_0 = \beta_0(\eta) = \sum_{k \geq \mathfrak{h}_i} \eta^k \beta_0^{[k]}, \qquad \beta_0^{[\mathfrak{h}]} = c^*, \tag{1.38}$$

with $c^*$ the real solution of $P_i(c) = 0$ considered above. Then we shall look for a solution of (1.11) of the form

$$\begin{aligned} &\alpha(t) = \alpha_0(t) + \beta_0 + \widetilde{\beta}(t), \qquad \widetilde{\beta}(t) = \sum_{k \geq 1} \eta^k \widetilde{\beta}^{[k]}(t), \\ &A(t) = A + B(t), \qquad B(t) = \sum_{k \geq 1} \eta^k B^{[k]}(t), \end{aligned} \tag{1.39}$$

where $\widetilde{\beta}(t)$ and $A(t)$ are $T$-periodic functions (and $\widetilde{\beta}(t)$ has zero average). We shall see that it is possible to choose the coefficients $\beta_0^{[k]}$ in (1.38) in such a way that this can be achieved.

More precisely we shall prove the following result.

**Theorem 1.4.** *Consider a periodic solution with frequency $\omega = p/q$ for the system (1.1). Assume that Hypotheses 1, 2 and 3 are satisfied. Then there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the system (1.1) has at least one subharmonic solution of order $q/p$. Such a solution admits a convergent Puiseux series in $\varepsilon$.*

To prove Theorem 1.4 we shall give a recursive formula to compute the coefficients of the series (1.38) and (1.39) order by order, and this can be achieved if we assume Hypothesis 3. Remark that, as generically the roots of a polynomial are simple (see Appendix A for details), generically we can apply Theorem 1.4 to obtain as many subharmonic solutions as real non-zero roots for the polynomials $P_i$. Notice that generically, also the zeros of the Melnikov function are simple. The here considered case is thus non-generic too.

We shall see in Chapter 3 how to extend Theorem 1.4 to cases where Hypothesis 3 is not assumed.

## 1.2 Formal solubility of the equations of motion

Call $P(c)$ the polynomial $P_i(c)$ of Hypothesis 3 (*i.e.* from now on we drop the label $i$ to lighten the notation) and recall that it is of the form

$$P(c) = \sum_{k\mathfrak{p}+j\mathfrak{h}=\mathfrak{s}} Q_{k,j} c^j \tag{1.40}$$

with $k, j \geq 0$.

**Remark 1.5.** There are at least two pairs $(k_1, j_1)$ and $(k_2, j_2)$ with $k_1 \neq k_2$ and $j_1 \neq j_2$, satisfying the condition $k\mathfrak{p} + j\mathfrak{h} = \mathfrak{s}$ and if (for instance) $k_1 = 0$ then $j_1 = \mathfrak{n}$ while if $j_1 = 0$ then $k_1 = k_0 \geq 1$; in particular $\mathfrak{s} \geq \max\{\mathfrak{p}, \mathfrak{h}\}$.

Recall that we are looking for a solution $(\alpha(t), A(t))$, with $\alpha(t) = \alpha_0(t) + \beta_0 + \widetilde{\beta}(t)$ and $A(t) = A_0 + B(t)$, where

$$\beta_0 = \sum_{k\geq 1} \eta^k \beta_0^{[k]}, \qquad \widetilde{\beta}(t) = \sum_{k\geq 1} \eta^k \widetilde{\beta}^{[k]}(t), \qquad B(t) = \sum_{k\geq 1} \eta^k B^{[k]}(t). \tag{1.41}$$

In this section we shall show that it is possible to fix $\beta_0^{[k]}$ such that there exist two formal power series $\widetilde{\beta}(t)$ and $B(t)$ as in (1.41), whose coefficients are $T$-periodic functions, *i.e.*

$$\widetilde{\beta}^{[k]}(t) = \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} e^{i\nu\omega t} \widetilde{\beta}_\nu^{[k]}, \qquad B^{[k]}(t) = \sum_{\nu\in\mathbb{Z}} e^{i\nu\omega t} B_\nu^{[k]}. \tag{1.42}$$

with $\omega = 2\pi/T$ and solve (1.11) order by order. In other words we want to solve

$$\begin{cases} \widetilde{\beta}_\nu^{[k]} = \dfrac{\Phi_\nu^{[k]}}{i\omega\nu} + \omega'(A_0)\dfrac{\Gamma_\nu^{[k]}}{(i\omega\nu)^2}, & \nu \neq 0, \\ B_\nu^{[k]} = \dfrac{\Gamma_\nu^{[k]}}{i\omega\nu}, \quad \nu \neq 0, \\ B_0^{[k]} = -\dfrac{\Phi_0^{[k]}}{\omega'(A_0)}, \\ \Gamma_0^{[k]} = 0, \end{cases} \tag{1.43}$$

where $\Gamma_\nu^{[k]}$ and $\Phi_\nu^{[k]}$ are recursively defined as

$$\begin{aligned} \Gamma_\nu^{[k]} &= \sum_{m\geq 0}\sum_{r+s=m}\sum_{\substack{p\sigma_0+q\sigma_0'+\nu_1+\ldots+\nu_m=\nu \\ k_1+\ldots+k_m=k-\mathfrak{p}}} \frac{(i\sigma_0)^r}{r!}\frac{\partial_A^s}{s!}G_{\sigma_0,\sigma_0'}(A_0,t_0)\beta_{\nu_1}^{[k_1]}\cdots\beta_{\nu_r}^{[k_r]}B_{\nu_{r+1}}^{[k_{r+1}]}\cdots B_{\nu_m}^{[k_m]}, \\ \Phi_\nu^{[k]} &= \sum_{m\geq 0}\sum_{r+s=m}\sum_{\substack{p\sigma_0+q\sigma_0'+\nu_1+\ldots+\nu_m=\nu \\ k_1+\ldots+k_m=k-\mathfrak{p}}} \frac{(i\sigma_0)^r}{r!}\frac{\partial_A^s}{s!}F_{\sigma_0,\sigma_0'}(A_0,t_0)\beta_{\nu_1}^{[k_1]}\cdots\beta_{\nu_r}^{[k_r]}B_{\nu_{r+1}}^{[k_{r+1}]}\cdots B_{\nu_m}^{[k_m]}, \\ &\quad + \sum_{s\geq 2}\sum_{\substack{\nu_1+\ldots+\nu_s=\nu \\ k_1+\ldots+k_s=k}} \frac{\partial_A^s}{s!}\omega(A_0)B_{\nu_1}^{[k_1]}\ldots B_{\nu_s}^{[k_s]}, \end{aligned} \tag{1.44}$$

where $\beta_\nu^{[k]} = \widetilde{\beta}_\nu^{[k]}$ for $\nu \neq 0$ and we have used (1.14) and the analogous expression for $\Gamma$.

We say that the integral equations (1.11), and hence the equations (1.43), are satisfied up to order $\overline{k}$ if there exists a choice of the parameters $\beta_0^{[1]}, \ldots \beta_0^{[\overline{k}]}$ which make the relations (1.43) to be satisfied for all $k = 1, \ldots, \overline{k}$.

**Lemma 1.6.** *The equations (1.43) are satisfied up to order $k = \mathfrak{p} - 1$ with $\widetilde{\beta}_\nu^{[k]}$ and $B_\nu^{[k]}$ identically zero for all $k = 1, \ldots, \mathfrak{p} - 1$ and for any choice of the constants $\beta_0^{[1]}, \ldots \beta_0^{[\mathfrak{p}-1]}$.*

*Proof.* One has $\varepsilon = \sigma\eta^{\mathfrak{p}}$, so that $\Phi_\nu^{[k]} = \Gamma_\nu^{[k]} = 0$ for all $k < \mathfrak{p}$ and all $\nu \in \mathbb{Z}$, indipendently of the values of the constants $\beta_0^{[1]}, \ldots \beta_0^{[\mathfrak{p}-1]}$. Moreover, one has $\widetilde{\beta}_\nu^{[k]} = B_\nu^{[k]} = 0$ for all $k < \mathfrak{p}$. □

**Lemma 1.7.** *The equations (1.43) are satisfied up to order $k = \mathfrak{p}$, for any choice of the constants $\beta_0^{[1]}, \ldots \beta_0^{[\mathfrak{p}-1]}$.*

*Proof.* One has $\Gamma^{[\mathfrak{p}]} = G(\alpha_0(t), A_0, t + t_0)$ and $\Phi^{[\mathfrak{p}]} = F(\alpha_0(t), A_0, t + t_0)$, so that

$$\Gamma_\nu^{[\mathfrak{p}]} = \sum_{p\sigma_0+q\sigma_0'=\nu} G_{\sigma_0,\sigma_0'}(A_0), \qquad \Phi_\nu^{[\mathfrak{p}]} = \sum_{p\sigma_0+q\sigma_0'=\nu} F_{\sigma_0,\sigma_0'}(A_0). \tag{1.45}$$

Thus, $\widetilde{\beta}_\nu^{[\mathfrak{p}]}$ and $B_\nu^{[\mathfrak{p}]}$ can be obtained from (1.43). Finally $\Gamma_0^{[\mathfrak{p}]} = M(t_0)$ by definition, and one has $M(t_0) = 0$ by Hypothesis 2. Hence also the last equation of (1.43) is satisfied. ☐

**Lemma 1.8.** *The equations (1.43) are satisfied up to order* $k = \mathfrak{p} + \mathfrak{s} - 1$ *provided* $\beta_0^{[k']} = 0$ *for all* $k' \leq \mathfrak{h} - 1$.

*Proof.* If $\beta_0^{[k']} = 0$ for all $k' \leq \mathfrak{h} - 1$ one has $\Gamma_0^{[k']} = 0$ for $\mathfrak{p} < k' < \mathfrak{p} + \mathfrak{s}$. Moreover, $\Phi_\nu^{[k']}$ and $\Gamma_\nu^{[k']}$ are well-defined for such values of $k'$. Hence (1.43) can be solved up to order $k = \mathfrak{p} + \mathfrak{s} - 1$, indipendently of the values of the constants $\beta_0^{[k']}$ for $k' \geq \mathfrak{h}$. ☐

**Lemma 1.9.** *The equations (1.43) are satisfied up to order* $k = \mathfrak{p} + \mathfrak{s}$ *provided* $\beta_0^{[\mathfrak{h}]} = c^*$*, with* $c^*$ *the simple real root of* $P(c)$ *in (1.40).*

*Proof.* One has $\Gamma_0^{[\mathfrak{p}+\mathfrak{s}]} = \eta^{\mathfrak{p}} P(\beta_0^{[\mathfrak{h}]})$, so that $\Gamma_0^{[\mathfrak{p}+\mathfrak{s}]} = 0$ for $\beta_0^{[\mathfrak{h}]} = c^*$. ☐

**Lemma 1.10.** *The equations (1.43) are satisfied up to any order* $\overline{k} = \mathfrak{p} + \mathfrak{s} + \kappa$, $\kappa \geq 1$ *provided the constants* $\beta_0^{[\mathfrak{h}+\kappa']}$ *are suitably fixed up to order* $\kappa' = \kappa$.

*Proof.* By substituting (1.38) and $\varepsilon = \sigma\eta^{\mathfrak{p}}$ in $\Gamma_0(\varepsilon, \beta_0)$ we obtain

$$\Gamma_0(\sigma\eta^{\mathfrak{p}}, \beta_0(\eta)) = \sigma\eta^{\mathfrak{p}} \sum_{M \geq \mathfrak{s}} \sum_{\substack{s_1 \geq 0 \\ j \geq 0 \\ s_1\mathfrak{p} + j\mathfrak{h} = M}} \mathcal{F}_{s_1,j} \sigma^{s_1} \eta^M \sum_{n \geq 0} \eta^n \sum_{\substack{m_1 + \ldots + m_j = n \\ m_i \geq 0}} \beta_0^{[h+m_1]} \ldots \beta_0^{[h+m_j]}. \quad (1.46)$$

Call $Q_{s_1,j} = \mathcal{F}_{s_1,j}\sigma^{s_1}$. For any $\kappa \geq 1$ one has

$$\sigma\Gamma_0^{[\mathfrak{p}+\mathfrak{s}+\kappa]} = \sum_{n=0}^{\kappa} \sum_{\substack{s_1 \geq 0 \\ j \geq 0 \\ s_1\mathfrak{p} + j\mathfrak{h} = \mathfrak{s}+n}} Q_{s_1,j} \sum_{\substack{m_1 + \ldots + m_j = \kappa - n \\ m_i \geq 0}} \beta_0^{[\mathfrak{h}+m_1]} \ldots \beta_0^{[\mathfrak{h}+m_j]}, \quad (1.47)$$

so that one can write the last equation of (1.43) as

$$\begin{aligned} &\sum_{\substack{s_1 \geq 0 \\ j \geq 0 \\ s_1\mathfrak{p} + j\mathfrak{h} = \mathfrak{s}}} jQ_{s_1,j} \left(\beta_0^{[\mathfrak{h}]}\right)^{j-1} \beta_0^{[\mathfrak{h}+\kappa]} + \sum_{\substack{s_1 \geq 0 \\ j \geq 0 \\ s_1\mathfrak{p} + j\mathfrak{h} = \mathfrak{s}}} Q_{s_1,j} \sum_{\substack{m_1 + \ldots + m_j = \kappa \\ 0 \leq m_i \leq \kappa - 1}} \beta_0^{[\mathfrak{h}+m_1]} \ldots \beta_0^{[\mathfrak{h}+m_j]} \\ &\qquad + \sum_{n=1}^{\kappa} \sum_{\substack{s_1 \geq 0 \\ j \geq 0 \\ s_1\mathfrak{p} + j\mathfrak{h} = \mathfrak{s}+n}} Q_{s_1,j} \sum_{\substack{m_1 + \ldots + m_j = \kappa - n \\ m_i \geq 0}} \beta_0^{[\mathfrak{h}+m_1]} \ldots \beta_0^{[\mathfrak{h}+m_j]} = 0. \end{aligned} \quad (1.48)$$

Recall that by Hypothesis 3

$$\sum_{\substack{s_1 \geq 0 \\ j \geq 0 \\ s_1\mathfrak{p} + j\mathfrak{h} = \mathfrak{s}}} jQ_{s_1,j} \left(\beta_0^{[\mathfrak{h}]}\right)^{j-1} = \frac{\mathrm{d}P}{\mathrm{d}c}\left(\beta_0^{[\mathfrak{h}]}\right) =: C \neq 0, \quad (1.49)$$

so that we can use (1.48) to express $\beta_0^{[\mathfrak{h}+\kappa]}$ in terms of the coefficients $\beta_0^{[\mathfrak{h}+\kappa']}$ of lower orders $\kappa' < \kappa$. Thus we can conclude that the equations (1.43) are satisfied up to order $\overline{k}$ provided the coefficients $\beta_0^{[\mathfrak{h}+\kappa']}$ are fixed as

$$\beta_0^{[\mathfrak{h}+\kappa']} = -\frac{1}{C}\widetilde{G}^{[\kappa']}(\beta_0^{[\mathfrak{h}]}, \dots, \beta_0^{[\mathfrak{h}+\kappa'-1]}), \tag{1.50}$$

for all $1 \leq \kappa' \leq \kappa$ and

$$\begin{aligned}\widetilde{G}^{[\kappa']}(\beta_0^{[\mathfrak{h}]}, \dots, \beta_0^{[\mathfrak{h}+\kappa'-1]}) = & \sum_{\substack{s_1 \geq 0 \\ j \geq 0 \\ s_1\mathfrak{p}+j\mathfrak{h}=\mathfrak{s}}} Q_{s_1,j} \sum_{\substack{m_1+\dots+m_j=\kappa' \\ 0 \leq m_i \leq \kappa'-1}} \beta_0^{[\mathfrak{h}+m_1]} \dots \beta_0^{[\mathfrak{h}+m_j]} \\ & + \sum_{n=1}^{\kappa'} \sum_{\substack{s_1 \geq 0 \\ j \geq 0 \\ s_1\mathfrak{p}+j\mathfrak{h}=\mathfrak{s}+n}} Q_{s_1,j} \sum_{\substack{m_1+\dots+m_j=\kappa'-n \\ m_i \geq 0}} \beta_0^{[\mathfrak{h}+m_1]} \dots \beta_0^{[\mathfrak{h}+m_j]}.\end{aligned} \tag{1.51}$$

□

We can summarise the results above into the following statement.

**Proposition 1.11.** *The equations (1.43) are satisfied to any order $k$ provided the constants $\beta_0^{[k]}$ are suitably fixed. In particular $\widetilde{\beta}_\nu^{[k]} = B_\nu^{[k]} = B_0^{[k]} = 0$ for $k < \mathfrak{p}$ and $\beta_0^{[k]} = 0$ for $k < \mathfrak{h}$.*

Proposition 1.11 shows that a formal power series which solves (1.11) can be constructed for each simple root of each polynomial $P_i$. The convergence of such series will be studied in Chapter 2 and this will imply the existence of subharmonic solutions for the system.

# 2. Trees and convergence of formal power series

In this chapter we introduce a graphical representation for the formal power series of the solution, and we use it prove the convergence of the series.

## 2.1 Labeled trees and diagrammatic rules

We shall start with some basic notation.

A *connected graph* $G$ is a set of points $\mathfrak{v}$ (*vertices*) and lines $\ell$ connecting all of them. A graph is said to be *planar* if it can be drawn in a plane without graph lines crossing. A *tree* $\theta$ is a planar graph such that for each pair of vertices there exists a unique path connecting them, *i.e.* a planar graph with no loops.

Given a tree $\theta$ we shall introduce a partial order in such a way that all the lines are consistently oriented towards a unique special vertex $\mathfrak{v}_0$. One can imagine that each line carries an arrow pointing towards the vertex $\mathfrak{v}_0$: the arrow will be thought of as superimposed on the line itself (see Figure 2.1).

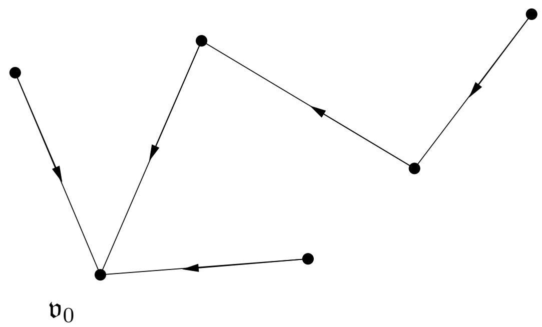


Figure 2.1: An oriented tree

If a line $\ell$ connects two vertices $\mathfrak{v}_1, \mathfrak{v}_2$ and is oriented from $\mathfrak{v}_2$ to $\mathfrak{v}_1$, we say that $\mathfrak{v}_2 \prec \mathfrak{v}_1$ and we shall write $\ell_{\mathfrak{v}_2} = \ell$. We shall say that $\ell$ exits from $\mathfrak{v}_2$ and enters $\mathfrak{v}_1$. We add an extra oriented line, exiting $\mathfrak{v}_0$, called *root line*. The end-point of the root line (which is not considered as a vertex) will be called *root*; such a tree will be called a *rooted tree*. More generally we write $\mathfrak{v}_2 \prec \mathfrak{v}_1$ when $\mathfrak{v}_1$ is on the path of lines connecting $\mathfrak{v}_2$ to the root: hence the orientation of the lines is opposite to the partial ordering relation $\prec$.

We denote with $V(\theta)$ and $L(\theta)$ the set of vertices and lines in $\theta$ respectively, and with $|V(\theta)|$ and $|L(\theta)|$ the number of vertices and lines respectively. Remark that one has $|V(\theta)| = |L(\theta)|$.

We shall say that two rooted trees are *equivalent* if they can be transformed into each other by continuously deforming the lines in the plane in such a way that these do not cross each other. This provides an equivalence relation on the set of trees as one can easily check. From now on, we will refer to equivalence class of trees simply by using the word *tree*.

**Lemma 2.1.** *The number of trees such that $|V(\theta)| = |L(\theta)| = k$ is bounded by $4^k$.*

*Proof.* Let $\theta$ be a tree with $k$ vertices. Starting from the root we "move" along the tree as follows:

- Arriving at a bifurcation, we choose the left way.
- If there is no entering line, we go back.
- When walking in the opposite way with respect to the orientation we place a label 1, and a label 0 otherwise.
- Coming back to the root, we interrupt the process.

Note that in such a way, we can associate with each tree a sequence of labels $1, 0$ (see Figure 2.2). Hence we obtain a sequence of $2k$ labels: $k$ of which are 1 and $k$ are 0. Then the number of the trees with $k$ nodes is bounded by the number of all the possible sequences of $2k$ values $\{1, 0\}$, which is $2^{2k} = 4^k$. □

We can consider two kinds of vertices: *nodes* and *leaves*. The leaves can only be end-points, *i.e.* points with no lines entering them, while the nodes can be either end-points or not (see Figure 2.3). We shall not consider the tree consisting of only one leaf and the line exiting from it, *i.e.* a tree must have at least the node from which the root line exits.

We shall denote with $N(\theta)$ and $E(\theta)$ the set of nodes and leaves respectively. Here and henceforth we shall denote with $\mathfrak{v}$ and $\mathfrak{e}$ the nodes and the leaves respectively.

Remark that $V(\theta) = N(\theta) \amalg E(\theta)$.

A *labeled tree* is a rooted tree together with a label function defined on $L(\theta)$ and $V(\theta)$.

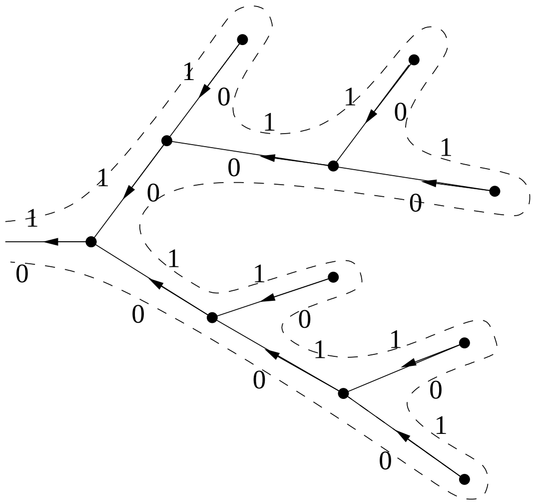

Figure 2.2: A tree with $k = 11$. The sequence of labels is $(1,1,1,0,1,1,0,1,0,0,0,1,1,0,1,1,0,1,0,0,0,0)$.

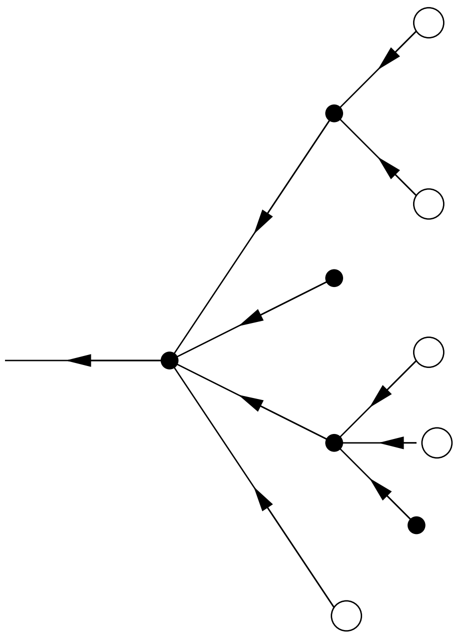

Figure 2.3: A tree with leaves. White bullets represent the leaves, while black bullets represent the nodes.

It is possible to extend the notion of equivalence also to labeled trees, simply by considering equivalent two labeled trees if they can be transformed into each other in such a way that also the labels match. As before, we shall refer to such equivalence classes simply as *labeled trees*.

With each line $\ell = \ell_{\mathfrak{v}}$, we associate three labels $(h_\ell, \delta_\ell, \nu_\ell)$, with $h_\ell \in \{\alpha, A\}$, $\delta_\ell \in \{1,2\}$ and $\nu_\ell \in \mathbb{Z}$, with the constraint that $\nu_\ell \neq 0$ for $h_\ell = \alpha$ and $\delta_\ell = 1$ for $h_\ell = A$. With each line $\ell = \ell_{\mathfrak{e}}$ we associate $h_\ell = \alpha$, $\delta_\ell = 1$ and $\nu_\ell = 0$. We shall say that $h_\ell$, $\delta_\ell$ and $\nu_\ell$ are the *component* label, the *degree* label and the *momentum* of the line $\ell$, respectively.

Given a node $\mathfrak{v}$, we call $r_{\mathfrak{v}}$ the number of the lines entering $\mathfrak{v}$ carrying a component label $h = \alpha$ and $s_{\mathfrak{v}}$ the number of the lines entering $\mathfrak{v}$ with component label $h = A$. We also introduce a *badge* label $b_{\mathfrak{v}} \in \{0,1\}$ with the constraint that $b_{\mathfrak{v}} = 1$ for $h_{\ell_{\mathfrak{v}}} = \alpha$ and $\delta_{\ell_{\mathfrak{v}}} = 2$, and for $h_{\ell_{\mathfrak{v}}} = A$ and $\nu_{\ell_{\mathfrak{v}}} \neq 0$, and two *mode* labels $\sigma_{\mathfrak{v}}, \sigma'_{\mathfrak{v}} \in \mathbb{Z}$. We call *global mode* label the sum

$$\nu_{\mathfrak{v}} = p\sigma_{\mathfrak{v}} + q\sigma'_{\mathfrak{v}}, \tag{2.1}$$

where $q, p$ are defined in (1.2), with the constraint that $\nu_{\mathfrak{v}} = 0$ when $b_{\mathfrak{v}} = 0$.

For all $\ell = \ell_{\mathfrak{v}}$, we set also the following *conservation law*

$$\nu_\ell = \nu_{\ell_{\mathfrak{v}}} = \sum_{\substack{\mathfrak{w} \in N(\theta) \\ \mathfrak{w} \preceq \mathfrak{v}}} \nu_{\mathfrak{w}}, \tag{2.2}$$

*i.e.* the momentum of the line exiting from $\mathfrak{v}$ is the sum of the momenta of the lines entering $\mathfrak{v}$ plus the global mode of the node $\mathfrak{v}$ itself.

Given a labeled tree $\theta$, where labels are defined as above, we associate with each line $\ell$ exiting from a node, a *propagator*

$$g_\ell = \begin{cases} \dfrac{\omega'(A_0)^{\delta_\ell - 1}}{(i\omega\nu_\ell)^{\delta_\ell}}, & h_\ell = \alpha, A, \quad \nu_\ell \neq 0, \\ -\dfrac{1}{\omega'(A_0)}, & h_\ell = A, \quad \nu_\ell = 0, \end{cases} \tag{2.3}$$

while for each line $\ell$ exiting from a leaf, we set $g_\ell = 1$.

Moreover, we associate with each node $\mathfrak{v}$ a *node factor*

$$\mathcal{N}_{\mathfrak{v}} = \begin{cases} \dfrac{(i\sigma_{\mathfrak{v}})^{r_{\mathfrak{v}}} \partial_A^{s_{\mathfrak{v}}}}{r_{\mathfrak{v}}! s_{\mathfrak{v}}!} F_{\sigma_{\mathfrak{v}}, \sigma'_{\mathfrak{v}}}(A_0, t_0), & h_{\ell_{\mathfrak{v}}} = \alpha, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 1, \quad \nu_{\ell_{\mathfrak{v}}} \neq 0, \\ \dfrac{\partial_A^{s_{\mathfrak{v}}}}{s_{\mathfrak{v}}!} \omega(A_0), & h_{\ell_{\mathfrak{v}}} = \alpha, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 0, \quad \nu_{\ell_{\mathfrak{v}}} \neq 0, \\ \dfrac{(i\sigma_{\mathfrak{v}})^{r_{\mathfrak{v}}} \partial_A^{s_{\mathfrak{v}}}}{r_{\mathfrak{v}}! s_{\mathfrak{v}}!} G_{\sigma_{\mathfrak{v}}, \sigma'_{\mathfrak{v}}}(A_0, t_0), & h_{\ell_{\mathfrak{v}}} = \alpha, \quad \delta_{\ell_{\mathfrak{v}}} = 2, \quad b_{\mathfrak{v}} = 1, \quad \nu_{\ell_{\mathfrak{v}}} \neq 0, \\ \dfrac{(i\sigma_{\mathfrak{v}})^{r_{\mathfrak{v}}} \partial_A^{s_{\mathfrak{v}}}}{r_{\mathfrak{v}}! s_{\mathfrak{v}}!} G_{\sigma_{\mathfrak{v}}, \sigma'_{\mathfrak{v}}}(A_0, t_0), & h_{\ell_{\mathfrak{v}}} = A, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 1, \quad \nu_{\ell_{\mathfrak{v}}} \neq 0, \\ \dfrac{(i\sigma_{\mathfrak{v}})^{r_{\mathfrak{v}}} \partial_A^{s_{\mathfrak{v}}}}{r_{\mathfrak{v}}! s_{\mathfrak{v}}!} F_{\sigma_{\mathfrak{v}}, \sigma'_{\mathfrak{v}}}(A_0, t_0), & h_{\ell_{\mathfrak{v}}} = A, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 1, \quad \nu_{\ell_{\mathfrak{v}}} = 0, \\ \dfrac{\partial_A^{s_{\mathfrak{v}}}}{s_{\mathfrak{v}}!} \omega(A_0), & h_{\ell_{\mathfrak{v}}} = A, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 0, \quad \nu_{\ell_{\mathfrak{v}}} = 0, \end{cases} \tag{2.4}$$

with the constraint that when $b_{\mathfrak{v}} = 0$ one has $r_{\mathfrak{v}} = 0$ and $s_{\mathfrak{v}} \geq 2$, and with each leaf $\mathfrak{e}$ a *leaf factor* $\mathcal{N}_{\mathfrak{e}} = \beta_0$.

Given a labeled tree $\theta$ with propagators and node and leaf factors associated as above, we define the *value* of $\theta$ the number

$$\mathrm{Val}(\theta) = \left( \prod_{\ell \in L(\theta)} g_\ell \right) \left( \prod_{\mathfrak{v} \in N(\theta)} \mathcal{N}_{\mathfrak{v}} \right). \tag{2.5}$$

Remark that $\mathrm{Val}(\theta)$ is a well-defined quantity because all the propagators and node factors are bounded quantities.

For each line $\ell$ exiting from a node $\mathfrak{v}$ we set $b_\ell = b_{\mathfrak{v}}$, while for each line $\ell$ exiting from a leaf we set $b_\ell = 0$. Given a labeled tree $\theta$, we call *order* of $\theta$ the number

$$k(\theta) = |\{\ell \in L(\theta) \ : \ b_\ell = 1\}|; \tag{2.6}$$

the momentum $\nu(\theta)$ of the root line will be the *total momentum*, and the component label $h(\theta)$ associated to the root line will be the *total component label*. Moreover, we set $j(\theta) = |E(\theta)|$.

Define $\mathcal{T}_{k,\nu,h,j}$ the set of all the trees $\theta$ with order $k(\theta) = k$, total momentum $\nu(\theta) = \nu$, total component label $h(\theta) = h$ and $j(\theta) = j$ leaves.

**Lemma 2.2.** *For any tree $\theta$ labeled as before, one has $|L(\theta)| = |V(\theta)| \leq 2k(\theta) + j(\theta) - 1$.*

*Proof.* We prove the bound $|N(\theta)| \leq 2k(\theta) - 1$ by induction on $k$.

For $k = 1$ the bound is trivially satisfied, as a direct check shows: in particular, a tree $\theta$ with $k(\theta) = 1$ has exactly one node and $j(\theta)$ leaves. In fact if $\theta$ has a line $\ell = \ell_{\mathfrak{v}}$ with $b_\ell = 0$, then $\mathfrak{v}$ has $s_{\mathfrak{v}} \geq 2$ lines with component label $h = A$ entering it. Hence there are at least two lines exiting from a node with $b_{\mathfrak{v}} = 1$.

Assume now that the bound holds for all $k' < k$, and let us show that then it holds also for $k$. Let $\ell_0$ be the root line of $\theta$ and $\mathfrak{v}_0$ the node from which the root line exits. Call $r$ and $s$ the number of lines entering $\mathfrak{v}_0$ with component labels $\alpha$ and $A$ respectively, and denote with $\theta_1, \ldots, \theta_{r+s}$ the subtrees which have those lines as root lines. Then

$$|N(\theta)| = 1 + \sum_{m=1}^{r+s} |N(\theta_m)|. \tag{2.7}$$

If $\ell_0$ has badge label $b_{\ell_0} = 1$ we have

$$|N(\theta)| \leq 1 + 2(k-1) - (r+s) \leq 2k - 1, \tag{2.8}$$

by the inductive hypothesis and by the fact that $k(\theta_1)+\ldots+k(\theta_{r+s})=k-1$. If $\ell_0$ has badge label $b_{\ell_0}=0$ we have

$$|N(\theta)| \leq 1+2k-(r+s) \leq 2k-1, \tag{2.9}$$

by the inductive hypothesis, by the fact that $k(\theta_1)+\ldots+k(\theta_{r+s})=k$, and the constraint that $s \geq 2$. Therefore the assertion is proved. □

## 2.2 Convergence of the formal power series

Our aim is to represent graphically the coefficients $\overline{\beta}_\nu^{(k,j)}$ and $\overline{B}_\nu^{(k,j)}$ in (1.20). By collecting together all the definitions given in Section 2.1, one obtains the following result.

**Lemma 2.3.** *The Fourier coefficients $\overline{\beta}_\nu^{(k,j)}$, $\nu \neq 0$, and $\overline{B}_\nu^{(k,j)}$ can be written in terms of trees as*

$$\begin{aligned} \overline{\beta}_\nu^{(k,j)} &= \sum_{\theta \in \mathcal{T}_{k,\nu,\alpha,j}} \mathrm{Val}(\theta), \quad \nu \neq 0, \\ \overline{B}_\nu^{(k,j)} &= \sum_{\theta \in \mathcal{T}_{k,\nu,A,j}} \mathrm{Val}(\theta), \quad \nu \in \mathbb{Z}, \end{aligned} \tag{2.10}$$

*for all $k \geq 1$, $j \geq 0$.*

*Proof.* First we consider trees without leaves, *i.e.* the coefficients $\overline{\beta}_\nu^{(k,0)}$, $\nu \neq 0$, and $\overline{B}_\nu^{(k,0)}$. For $k=1$ just compare (1.24) and (1.25) with the definition of trees in that case. Now let us suppose that the assertion holds for all $k<\overline{k}$. Let us write $f_\alpha=\overline{\beta}$, $f_A=\overline{B}$ and represent the coefficients $f_{\nu,h}^{(k,0)}$ with the graph element in Figure 2.4, as a line with label $\nu$ and $h=\alpha, A$ respectively, exiting from a ball with label $(k,0)$.

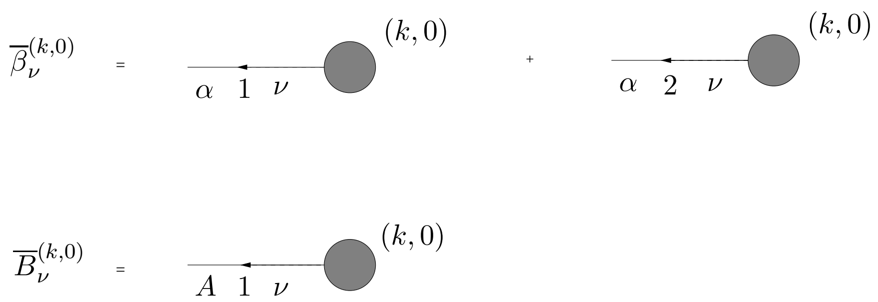


Figure 2.4: Graph element.

Then we can represent each equation of (1.23) graphically as in Figure 2.5. Simply represent each factor $f^{(k_i,0)}_{\nu_i,h_i}$ in the r.h.s. as a graph element according to Figure 2.4. The lines of all such graph elements enter the same node $\mathfrak{v}_0$.

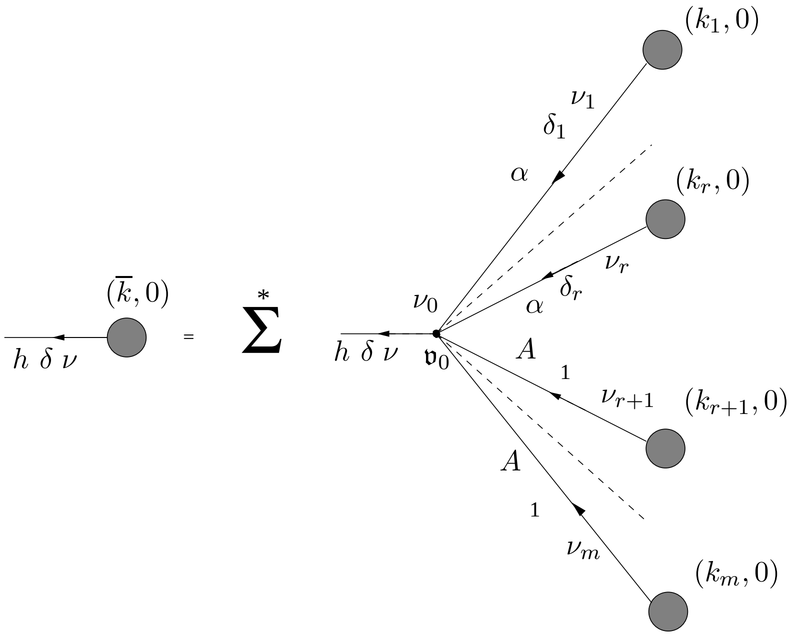


Figure 2.5: Graphical representation for the recursive equations.

The root line $\ell_0$ of such trees will carry a component label $h = \alpha, A$ for $f = \overline{\beta}, \overline{B}$ respectively, and a momentum label $\nu$.

Hence, by inductive hypothesis, one obtains

$$\begin{aligned} f^{(\overline{k},0)}_{\nu,h} &= \sum^* g_{\ell_0} \mathcal{N}_{\mathfrak{v}_0} f^{(k_1,0)}_{\nu_1,h_1} \dots f^{(k_m,0)}_{\nu_m,h_m} \\ &= \sum^* g_{\ell_0} \mathcal{N}_{\mathfrak{v}_0} \left( \sum_{\theta \in \mathcal{T}_{\kappa_1,\nu_1,h_1,0}} \mathrm{Val}(\theta) \right) \dots \left( \sum_{\theta \in \mathcal{T}_{\kappa_m,\nu_m,h_m,0}} \mathrm{Val}(\theta) \right) \\ &= \sum_{\theta \in \mathcal{T}_{\overline{k},\nu,h,0}} \mathrm{Val}(\theta). \end{aligned} \tag{2.11}$$

where $m = r_0 + s_0$, and we write $\sum^*$ for the sum over all the labels admitted by the constraints, so that the assertion is proved for all $k$ and for $j = 0$.

Now we consider $k$ as fixed and we prove the statement by induction on $j$. The case $j = 0$ has already been discussed. Finally we assume that the assertion holds for $j = j'$ and show that then it holds for $j'+1$. Notice that a tree $\theta \in \mathcal{T}_{k,\nu,h,j'+1}$, for both $h = \alpha, A$ can be obtained by considering a suitable tree $\theta_0 \in \mathcal{T}_{k,\nu,h,j'}$ attaching an extra leaf to a node of $\theta_0$ and applying an extra derivative $\partial_\alpha$ to the node factor associated to that node. If one considers all the trees that can be obtained in such a way from the same $\theta_0$ and sums together all those contributions, one finds a quantity proportional to $\partial_\alpha \mathrm{Val}(\theta_0)$. Then if we sum over all possible choices of $\theta_0$, we reconstruct $\overline{\beta}_\nu^{(k,j'+1)}$ for $h = \alpha$ and $\overline{B}_\nu^{(k,j'+1)}$ for $h = A$. Hence the assertion follows. □

**Remark 2.4.** The representation given in Lemma 2.3 is very helpful to prove the convergence of the formal power series in (1.20) as we shall see later. We now point out the fact that also $\overline{\Gamma}_0$ can be represented in terms of sum of trees with leaves. In fact if we consider the recursive equation (1.22) for $\nu = 0$ we immediately notice that we can repeat the construction above, simply by defining $\mathcal{T}_{k,0,\Gamma,j}$ as the set of the trees contributing to $\overline{\Gamma}_0^{(k,j)}$, and setting $g_{\ell_0} = 1$, $h_{\ell_0} = \Gamma$, $d_{\ell_0} = 1$ and

$$\mathcal{N}_{\mathfrak{v}_0} = \frac{(i\sigma_{\mathfrak{v}_0})^{r_{\mathfrak{v}_0}} \partial_A^{s_{\mathfrak{v}_0}}}{r_{\mathfrak{v}_0}! s_{\mathfrak{v}_0}!} G_{\sigma_{\mathfrak{v}_0},\sigma'_{\mathfrak{v}_0}}(A_0, t_0), \tag{2.12}$$

and no further difficulties arise.

Now we prove the convergence of the formal power series (1.20), for small $\varepsilon$ and $\beta_0$.

**Lemma 2.5.** *The formal solution (1.20) of the system (1.12), given by the recursive equations (1.23), converges for $\varepsilon$ and $\beta_0$ small enough.*

*Proof.* First of all we remark that by Lemma 2.1 and Lemma 2.2, the number of unlabeled trees of order $k$ and $j$ leaves is bounded by $4^{2k+j} \times 2^{2k+j} = 8^{2k+j}$. The sum over all labels except the mode labels and the momenta is bounded again by a constant to the power $k$ times a constant to the power $j$, simply because all such labels can assume only a finite number of values. Now by the analyticity assumption on the functions $F$ and $G$, we have the bound

$$\begin{aligned} \left| \frac{(i\sigma_0)^r}{r!} \frac{\partial_A^s}{s!} F_{\sigma_0,\sigma'_0}(A_0,t_0) \right| &\le \mathcal{Q} R^r S^s e^{-\kappa(|\sigma_0|+|\sigma'_0|)}, \\ \left| \frac{(i\sigma_0)^r}{r!} \frac{\partial_A^s}{s!} G_{\sigma_0,\sigma'_0}(A_0,t_0) \right| &\le \mathcal{Q} R^r S^s e^{-\kappa(|\sigma_0|+|\sigma'_0|)}, \end{aligned} \tag{2.13}$$

for suitable positive constants $\mathcal{Q}, R, S, \kappa$, and we can imagine, without loss of generality, that $\mathcal{Q}$ and $S$ are such that $|\partial_A^s \omega(A_0)/s!| \le \mathcal{Q} S^s$. This gives us a bound for the node factors. The propagators can be bounded by

$$|g_\ell| \le \max\left\{ \left| \frac{\omega'(A_0)}{\omega^2} \right|, \left| \frac{1}{\omega'(A_0)} \right|, \left| \frac{1}{\omega} \right|, 1 \right\}, \tag{2.14}$$

so that the product over all the lines can be bounded again by a constant to the power $k$ times a constant to the power $j$.

Thus the sum over the mode labels - which uniquely determine the momenta - can be performed by using for each node half the exponential decay factor provided by (2.13). Then we obtain

$$|\overline{\beta}_\nu^{(k,j)}| \leq C_1 C_2^k C_3^j e^{-\kappa|\nu|/2}, \quad |\overline{B}_\nu^{(k,j)}| \leq C_1 C_2^k C_3^j e^{-\kappa|\nu|/2}, \tag{2.15}$$

for suitable constants $C_1$, $C_2$ and $C_3$. This provides the convergence of the series (1.20) for $|\varepsilon| < C_2^{-1}$ and $|\beta_0| < C_2^{-1}$. □

**Remark 2.6.** The bound provided is far from being optimal. For instance the number of unlabeled trees with 3 nodes and 2 leaves is 30 while $8^{2\times3+2} = 8^8$. However it is enough to provide the convergence for the formal power series (1.20) and, as seen in Remark 2.4, also for $\Gamma_0^{(k,j)}$.

## 2.3 Diagrammatic rules for the formal power series in $\eta$

In order to give a graphical representation of the coefficients $\beta_0^{[k]}$, $\widetilde{\beta}_\nu^{[k]}$ and $B_\nu^{[k]}$ in (1.41) and (1.42), we shall consider a different tree expansion. We shall perform an iterative construction, similar to the one performed through the proof of Lemma 2.3, starting from equations (1.43) for the coefficients $\widetilde{\beta}_\nu^{[k]}$, $B_\nu^{[k]}$ for $k \geq \mathfrak{p}$, and from (1.50) for $\beta_0^{[k]}$, $k \geq \mathfrak{h}$.

First, for $k = \mathfrak{p}$ we represent the coefficients $\widetilde{\beta}_\nu^{[\mathfrak{p}]}$ and $B_\nu^{[\mathfrak{p}]}$ as a line exiting from a node, while for $k = \mathfrak{h}$ we represent $\beta_0^{[\mathfrak{h}]}$ as a line exiting from a leaf (see Figure 2.6).

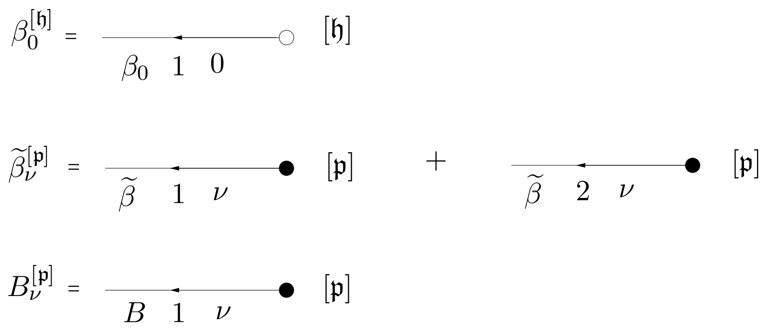


Figure 2.6: Graphical representation for $\beta_0^{[\mathfrak{h}]}$, $\widetilde{\beta}_\nu^{[\mathfrak{p}]}$ and $B_\nu^{[\mathfrak{p}]}$.

Now we represent each coefficient as a graph element according to Figure 2.7, as a line exiting from a gray bullet with *order label* $k$, with $k \geq \mathfrak{h}+1$ for the coefficients $\beta_0^{[k]}$, and

$k \geq \mathfrak{p}+1$ for the coefficients $\widetilde{\beta}_\nu^{[k]}$ and $B_\nu^{[k]}$; we associate with the line a component label $h \in \{\beta_0, \widetilde{\beta}, B\}$ and momentum $\nu_\ell \in \mathbb{Z}$, with the constraint that $\nu_\ell \neq 0$ for $h_\ell = \widetilde{\beta}$, while $\nu_\ell = 0$ for $h_\ell = \beta_0$.

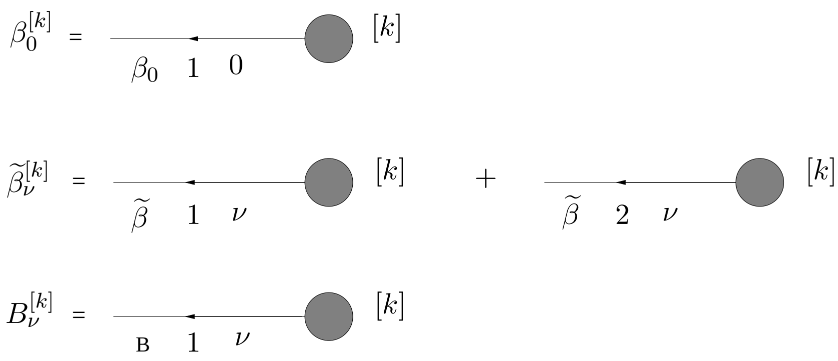


Figure 2.7: Graph elements.

Hence we can represent the first three equations in (1.43) graphically, representing each factor $\beta_{\nu_i}^{[k_i]}$ and $B_{\nu_i}^{[k_i]}$ in (1.44) as graph elements: again the lines of such graph elements enter the same node $\mathfrak{v}_0$, as depicted in Figure 2.8.

We associate with the root line $\ell_0 = \ell_{\mathfrak{v}_0}$ a degree label $\delta_{\ell_0} = 1, 2$, with the constraint that $\delta_{\ell_0} = 1$ for $h_{\ell_0} = B$, and we associate with $\mathfrak{v}_0$ a badge label $b_{\mathfrak{v}_0} \in \{0, 1\}$ by setting $b_{\mathfrak{v}_0} = 1$ for $h_{\ell_0} = \widetilde{\beta}$ and $\delta_{\ell_0} = 2$, and for $h_{\ell_0} = B$ and $\nu_{\ell_0} \neq 0$. We call also $r_{\mathfrak{v}_0}$ the number of the lines entering $\mathfrak{v}_0$ with component label $h = \beta = \beta_0, \widetilde{\beta}$, and $s_{\mathfrak{v}_0}$ the number of the lines entering $\mathfrak{v}_0$ with component label $h = B$, with the constraint that if $b_{\mathfrak{v}_0} = 0$ one has $r_{\mathfrak{v}_0} = 0$ and $s_{\mathfrak{v}_0} \geq 2$. Finally we associate with $\mathfrak{v}_0$ two mode labels $\sigma_{\mathfrak{v}_0}, \sigma_{\mathfrak{v}_0'} \in \mathbb{Z}$ and the global mode label $\nu_{\mathfrak{v}_0}$ as in (2.1), and we impose the conservation law

$$\nu_{\ell_{\mathfrak{v}_0}} = \nu_{\mathfrak{v}_0} + \sum_{i=1}^{r_{\mathfrak{v}_0}+s_{\mathfrak{v}_0}} \nu_{\ell_i}, \tag{2.16}$$

where $\ell_1, \ldots, \ell_{r_{\mathfrak{v}_0}+s_{\mathfrak{v}_0}}$ are the lines entering $\mathfrak{v}_0$.

We also force the following conditions on the order labels

$$\begin{aligned} \sum_{i=1}^{r_{\mathfrak{v}_0}+s_{\mathfrak{v}_0}} k_i = k - \mathfrak{p}, \qquad & b_{\mathfrak{v}_0} = 1, \\ \sum_{i=1}^{s_{\mathfrak{v}_0}} k_i = k, \qquad & b_{\mathfrak{v}_0} = 0, \end{aligned} \tag{2.17}$$

which reflect the condition on the sums in (1.44).

Finally we associate with $\mathfrak{v} = \mathfrak{v}_0$ a node factor

$$\mathcal{N}_{\mathfrak{v}}^* = \begin{cases} \sigma \dfrac{(i\sigma_{\mathfrak{v}})^{r_{\mathfrak{v}}} \partial_A^{s_{\mathfrak{v}}}}{r_{\mathfrak{v}}! s_{\mathfrak{v}}!} F_{\sigma_{\mathfrak{v}},\sigma'_{\mathfrak{v}}}(A_0,t_0), & h_{\ell_{\mathfrak{v}}} = \widetilde{\beta}, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 1, \quad \nu_{\ell_{\mathfrak{v}}} \neq 0, \\ \sigma \dfrac{\partial_A^{s_{\mathfrak{v}}}}{s_{\mathfrak{v}}!} \omega(A_0), & h_{\ell_{\mathfrak{v}}} = \widetilde{\beta}, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 0, \quad \nu_{\ell_{\mathfrak{v}}} \neq 0, \\ \sigma \dfrac{(i\sigma_{\mathfrak{v}})^{r_{\mathfrak{v}}} \partial_A^{s_{\mathfrak{v}}}}{r_{\mathfrak{v}}! s_{\mathfrak{v}}!} G_{\sigma_{\mathfrak{v}},\sigma'_{\mathfrak{v}}}(A_0,t_0), & h_{\ell_{\mathfrak{v}}} = \widetilde{\beta}, \quad \delta_{\ell_{\mathfrak{v}}} = 2, \quad b_{\mathfrak{v}} = 1, \quad \nu_{\ell_{\mathfrak{v}}} \neq 0, \\ \sigma \dfrac{(i\sigma_{\mathfrak{v}})^{r_{\mathfrak{v}}} \partial_A^{s_{\mathfrak{v}}}}{r_{\mathfrak{v}}! s_{\mathfrak{v}}!} G_{\sigma_{\mathfrak{v}},\sigma'_{\mathfrak{v}}}(A_0,t_0), & h_{\ell_{\mathfrak{v}}} = B, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 1, \quad \nu_{\ell_{\mathfrak{v}}} \neq 0, \\ \sigma \dfrac{(i\sigma_{\mathfrak{v}})^{r_{\mathfrak{v}}} \partial_A^{s_{\mathfrak{v}}}}{r_{\mathfrak{v}}! s_{\mathfrak{v}}!} F_{\sigma_{\mathfrak{v}},\sigma'_{\mathfrak{v}}}(A_0,t_0), & h_{\ell_{\mathfrak{v}}} = B, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 1, \quad \nu_{\ell_{\mathfrak{v}}} = 0, \\ \sigma \dfrac{\partial_A^{s_{\mathfrak{v}}}}{s_{\mathfrak{v}}!} \omega(A_0), & h_{\ell_{\mathfrak{v}}} = B, \quad \delta_{\ell_{\mathfrak{v}}} = 1, \quad b_{\mathfrak{v}} = 0, \quad \nu_{\ell_{\mathfrak{v}}} = 0, \end{cases} \tag{2.18}$$

where $\sigma = \mathrm{sign}\,(\varepsilon)$, and with the line $\ell = \ell_{\mathfrak{v}}$ a propagator

$$g_\ell^* = \begin{cases} \dfrac{\omega'(A_0)^{\delta_\ell - 1}}{(i\omega\nu_\ell)^{\delta_\ell}}, & h_\ell = \widetilde{\beta}, B, \quad \nu_\ell \neq 0, \\ -\dfrac{1}{\omega'(A_0)}, & h_\ell = B, \quad \nu_\ell = 0, \end{cases} \tag{2.19}$$

so that, if we sum over all labels admitted by the constraints, we obtain the graphical representation depicted in Figure 2.8.

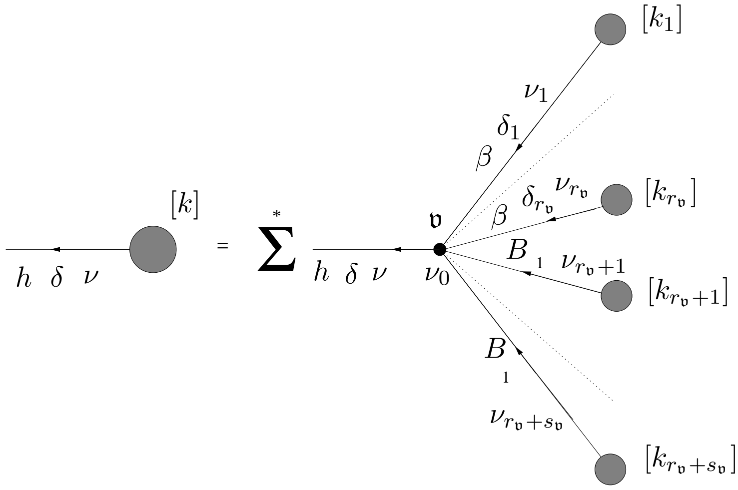


Figure 2.8: Graphical representation of (1.43). Again we write $\sum^*$ for the sum over all the labels admitted by the constraints.

The coefficients $\beta_0^{[k]}$, $k \geq \mathfrak{h} + 1$, have to be treated in a different way. Recall that the coefficients $Q_{s_1,j}$ in (1.51) are defined as $Q_{s_1,j} = \mathcal{F}_{s_1,j}\sigma^{s_1} = \overline{\Gamma}_0^{(s_1+1,j)}\sigma^{s_1}$ so that

$$Q_{s_1,j}\sigma^{s_1} = \sum_{\theta \in \mathcal{T}_{s_1+1,0,\Gamma,j}} \mathrm{Val}(\theta). \tag{2.20}$$

Hence the summands in (1.51) can be imagined as "some" of the trees in $\mathcal{T}_{s_1+1,0,\Gamma,j}$ where "some" leaves are substituted by graph elements with $h_\ell = \beta_0$. More precisely we shall consider only trees $\theta$ of the form depicted in Figure 2.9, with $s_1+1$ nodes, $s_0$ leaves and $s'_0$ graph elements with $h_\ell = \beta_0$, such that

$$\begin{aligned} & s_1\mathfrak{p} + (s_0 + s'_0)\mathfrak{h} = \mathfrak{s} + n, \\ & \sum_{i=1}^{s_0+s'_0} k_i = s_0\mathfrak{h} + \sum_{i=1}^{s'_0} k_i = (s_0 + s'_0)\mathfrak{h} + k - \mathfrak{h} - n, \end{aligned} \tag{2.21}$$

for a suitable $0 \le n \le k - \mathfrak{h}$, with the constraint that when $n = 0$ one has $s'_0 \ge 2$. We shall call $\ell_i$ the $s'_0$ lines with $h_{\ell_i} = \beta_0$. Again such conditions express the condition on the sums in (1.51).

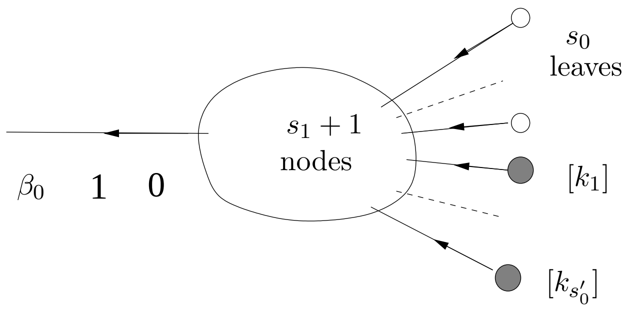


Figure 2.9: A tree contributing to $\beta_0^{[k]}$.

The propagators of the lines exiting from the $s_1+1$ nodes and the node factors of the nodes (except the root line and the node from which the root line exits) are $g^*_\ell = g_\ell$ and $\mathcal{N}^*_\mathfrak{v} = \sigma\mathcal{N}_\mathfrak{v}$ with the only difference that the component label can assume the values $\widetilde{\beta}, B$, which have the rôle of $\alpha, A$ respectively. We associate with the root line a propagator

$$g^*_{\ell_0} = -\frac{1}{C}, \tag{2.22}$$

where $C$ is defined in (1.49), while the node $\mathfrak{v}$ from which the root line exits will have a node factor

$$\mathcal{N}^*_\mathfrak{v} = \frac{(i\sigma_\mathfrak{v})^{r_\mathfrak{v}}\partial_A^{s_\mathfrak{v}}}{r_\mathfrak{v}!s_\mathfrak{v}!} G_{\sigma_\mathfrak{v},\sigma'_\mathfrak{v}}(A_0, t_0). \tag{2.23}$$

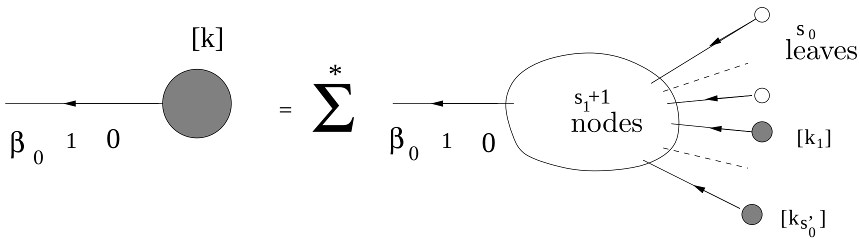


Figure 2.10: Graphical representation of (1.50). Again we write $\sum^*$ for the sum over all the labels admitted by the constraints.

Hence one obtain the graphical representation of Figure 2.10.

Finally we associate with each leaf $\mathfrak{e}$ a leaf factor $\mathcal{N}_{\mathfrak{e}}^* = c^*$, with $c^*$ the simple real root of the polynomial $P(c)$ in (1.40), while each line exiting from a leaf will have a propagator $g_\ell^* = 1$.

We now iterate the graphical representation of Figure 2.8 and Figure 2.10 until only simple nodes or leaves appear. More precisely, for all graph elements $\beta_{\nu_i}^{[k_i]}$, $B_{\nu_i}^{[k_i]}$ in both representation 2.8 and 2.10, we repeat the constructions above, considering a subtree with $\ell_i$ as root line. We shall call *allowed trees* all the trees obtained in such recursive way, and we shall denote with $\Theta_{k,\nu,h}$ the set of allowed trees with order $k$, total momentum $\nu$ and total component label $h$.

Given an allowed tree $\theta$ we denote with $L(\theta)$, $N(\theta)$ and $E(\theta)$ the set of lines, nodes and leaves respectively, and we shall define the *value* of $\theta$ as

$$\mathrm{Val}^*(\theta) = \left( \prod_{\ell \in L(\theta)} g_\ell^* \right) \left( \prod_{\mathfrak{v} \in N(\theta)} \mathcal{N}_{\mathfrak{v}}^* \right) \left( \prod_{\mathfrak{e} \in E(\theta)} \mathcal{N}_{\mathfrak{e}}^* \right). \tag{2.24}$$

Moreover, we denote with $\Lambda(\theta)$ the set of the lines (exiting from a node) in $\theta$ with component label $h = \beta_0$ and with $N^*(\theta)$ the nodes with $b_{\mathfrak{v}} = 1$; then we associate with each node in $N^*(\theta)$, with each leave and with each line in $\Lambda(\theta)$ a *weight* $\mathfrak{p}$, $\mathfrak{h}$ and $\mathfrak{h} - \mathfrak{p} - \mathfrak{s}$ respectively, and we call *order* of $\theta$ the number

$$k(\theta) = \mathfrak{p}|N^*(\theta)| + \mathfrak{h}|E(\theta)| + (\mathfrak{h} - \mathfrak{p} - \mathfrak{s})|\Lambda(\theta)|, \tag{2.25}$$

*i.e.* the weighted sum of nodes $N^*(\theta)$, leaves and lines $\Lambda(\theta)$. Notice that $\mathfrak{h} - \mathfrak{p} - \mathfrak{s} < 0$ as $\mathfrak{s} \geq \max\{\mathfrak{p}, \mathfrak{h}\}$ (see the Remark 1.5).

Hence we have the following result.

**Lemma 2.7.** *The Fourier coefficients $\beta_0^{[k]}$, $\widetilde{\beta}_\nu^{[k]}$ and $B_\nu^{[k]}$ can be written in terms of trees as*

$$\begin{aligned}
\beta_0^{[k]} &= \sum_{\theta\in\Theta_{k,0,\beta_0}} \mathrm{Val}^*(\theta), \qquad k \geq \mathfrak{h}, \\
\widetilde{\beta}_\nu^{[k]} &= \sum_{\theta\in\Theta_{k,\nu,\widetilde{\beta}}} \mathrm{Val}^*(\theta), \qquad k \geq \mathfrak{p}, \\
B_\nu^{[k]} &= \sum_{\theta\in\Theta_{k,\nu,B}} \mathrm{Val}^*(\theta), \qquad k \geq \mathfrak{p}.
\end{aligned} \tag{2.26}$$

*Proof.* We only have to prove that an allowed tree contributing to the Fourier coefficients $\beta_0^{[k]}$, $\widetilde{\beta}_\nu^{[k]}$ and $B_\nu^{[k]}$ has order $k$. We shall perform the proof by induction on $k \geq \mathfrak{h}$ for the coefficients $\beta_0^{[k]}$, and $k \geq \mathfrak{p}$ for $\widetilde{\beta}_\nu^{[k]}$ and $B_\nu^{[k]}$. Let us set $f_h = \widetilde{\beta}, B$. As depicted in Figure 2.6 an allowed tree $\theta$ contributing to $f_{\nu,h}^{[\mathfrak{p}]}$ has only one node so that $k(\theta) = \mathfrak{p}$ while an allowed tree $\overline{\theta}$ contributing to $\beta_0^{[\mathfrak{h}]}$ has only one leaf so that $k(\overline{\theta}) = \mathfrak{h}$.

Let us suppose first that for all $k' < k$, an allowed tree $\theta'$ contributing to $\beta_0^{[k']}$ has order $k(\theta') = k'$. A tree $\theta$ contributing to $\beta_0^{[k]}$ is of the form depicted in Figure 2.9 with the conditions (2.21) holding. By the inductive hypothesis, the order of such a tree is

$$k(\theta) = (s_1+1)\mathfrak{p} + s_0\mathfrak{h} + \sum_{i=1}^{s_0'} k_i + \mathfrak{h} - \mathfrak{p} - \mathfrak{s}, \tag{2.27}$$

and *via* the conditions in (2.21) we obtain $k(\theta) = k$.

Let us suppose now that the inductive hypothesis holds for all trees $\theta'$ contributing to $f_{\nu,h}^{[k']}$, $k' < k$. An allowed tree $\theta$ contributing to $f_{\nu,h}^{[k]}$ is of the form depicted in Figure 2.11, where $s_0, s_1$ are the numbers of the lines exiting from a leaf and a simple node respectively and entering $\mathfrak{v}_0$, while $s_0', s_1'$ are the graph elements entering $\mathfrak{v}_0$ with component label $\beta_0$ and $f$ respectively.

If $b_{\mathfrak{v}_0} = 1$, by the inductive hypothesis the order of such a tree is given by

$$k(\theta) = (s_1+1)\mathfrak{p} + s_0\mathfrak{h} + \sum_{i=1}^{s_0'+s_1'} k_i, \tag{2.28}$$

and by the first condition in (2.17) we have $k(\theta) = k$. Otherwise if $b_{\mathfrak{v}_0} = 0$, we have $s_0 + s_0' = 0$ and, by the inductive hypothesis,

$$k(\theta) = s_1\mathfrak{p} + \sum_{i=1}^{s_1'} k_i = k, \tag{2.29}$$

*via* the second condition in (2.17). □

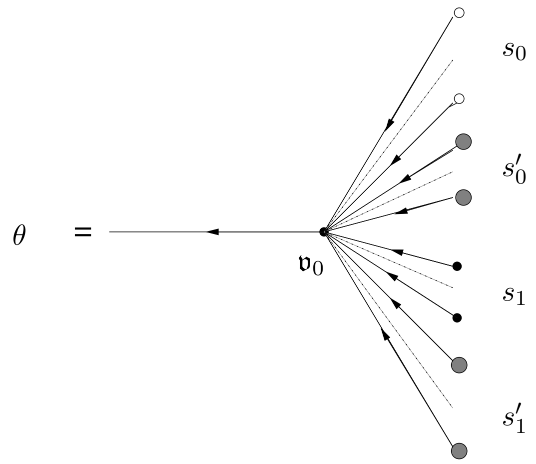


Figure 2.11: An allowed tree contributing to $f^{[\kappa]}_{\nu,h}$.

We now state the analogous of Lemma 2.2.

**Lemma 2.8.** *Let* $\mathfrak{q} := \min\{\mathfrak{h}, \mathfrak{p}\}$ *and let us define*

$$M = 2\frac{\mathfrak{s}}{\mathfrak{q}} + 3. \tag{2.30}$$

*Then for all* $\theta \in \Theta_{k,\nu,h}$ *one has*

$$|L(\theta)| \leq Mk. \tag{2.31}$$

As the proof is rather technical we shall perform it in Appendix B.

Now we prove the convergence of the series (1.41) for small $\eta$.

**Proposition 2.9.** *The formal solution (1.41) of the system (1.11), given by the recursive equations (1.43) and (1.50), converges for* $\eta$ *small enough.*

*Proof.* By Lemma 2.1 and Lemma 2.8, the number of unlabeled trees of order $k$ is bounded by $4^{Mk}$, so that the sum over all labels except the mode labels and the momenta is bounded by a constant to the power $k$ because all such labels can assume only a finite number of values. The bound for each node factor is the same as in Lemma 2.5, while the propagators can be bounded by

$$|g^*_\ell| \leq \max\left\{\left|\frac{\omega'(A_0)}{\omega^2}\right|, \left|\frac{1}{\omega'(A_0)}\right|, \left|\frac{1}{\omega}\right|, \left|\frac{1}{C}\right|, 1\right\}, \tag{2.32}$$

so that the product over all the lines can be bounded again by a constant to the power $k$. The product over the leaves factors is again bounded by a constant to the power $k$, while the sum over the mode labels - which uniquely determine the momenta - can be performed by using for each node half the exponential decay factor provided by (2.13). Thus we obtain

$$|\widetilde{\beta}_\nu^{[k]}| \le C_1 C_2^k e^{-\kappa|\nu|/2}, \quad |B_\nu^{[k]}| \le C_1 C_2^k e^{-\kappa|\nu|/2}, \tag{2.33}$$

for suitable constants $C_1$ and $C_2$. Hence we obtain the convergence for the series (1.41), for $|\eta| \le C_2^{-1}$ □

The discussion above ends the proof of Theorem 1.4.

# 3. Puiseux expansion for the degenerate case

In this Chapter we shall see how to extend Theorem 1.4 when Hypothesis 3 is released.

## 3.1 The Newton-Puiseux process

Let us come back to $\mathcal{F}(\varepsilon, \beta_0) \in \mathbb{R}\{\varepsilon, \beta_0\}$[1] defined in (1.29) and let us consider the Newton polygon $\mathcal{P}^{(0)} = \mathcal{P}_1^{(0)} \cup \ldots \cup \mathcal{P}_{N_0}^{(0)}$ of $\mathcal{F}^{(0)} := \mathcal{F}$. Recall that $\mathcal{F}$ is $\beta_0$-general of order $\mathfrak{n} = \mathfrak{n}_0$. We do not use the same notation introduced in Section 1.1: the reason of this choice will be clearer later. As done in Section 1.1 we consider the quasi-homogeneous polynomials $\widetilde{\mathcal{F}}_i^{(0)}$ associated with the segments $\mathcal{P}_i^{(0)}$ with slope $-1/\mu_i^{(0)}$, with $\mu_i^{(0)} = \mathfrak{h}_i^{(0)}/\mathfrak{p}_i^{(0)}$, where $\mathfrak{h}_i^{(0)}$ and $\mathfrak{p}_i^{(0)}$ are relatively prime integers for all $i = 1, \ldots, N_0$. Then we introduce the polynomials $P_i^{(0)} = P_i^{(0)}(c)$ in such a way that

$$\widetilde{\mathcal{F}}_i^{(0)}(\varepsilon, c(\sigma_0\varepsilon)^{\mu_i^{(0)}}) = (\sigma_0\varepsilon)^{\mathfrak{r}_i^{(0)}} \sum_{k\mathfrak{p}_i^{(0)} + j\mathfrak{h}_i^{(0)} = \mathfrak{s}_i^{(0)}} Q_{k,j}c^j = (\sigma_0\varepsilon)^{\mathfrak{r}_i^{(0)}} P_i^{(0)}(c), \quad \sigma_0 := \operatorname{sign}(\varepsilon), \tag{3.1}$$

where $Q_{k,j} = \mathcal{F}_{k,j}^{(0)}\sigma_0^k$. Recall that by Lemma 1.2, each polynomial $P_i^{(0)}$ has at least a non-zero complex root $c_i^{(0)}$.

Let $\Re_0$ be the set of all the non-zero real solutions of the polynomial equations $P_i^{(0)}(c) = 0$. If $\Re_0 = \emptyset$ the system (1.1) has no subharmonic solution.

Let us suppose then, that there exists $c^{(0)} \in \Re_0$, so that $c^{(0)}(\sigma_0\varepsilon)^{\mu_i^{(0)}}$ is a first approximate solution of the implicit equation $\mathcal{F}^{(0)}(\varepsilon, \beta_0) = 0$ for a suitable $i = 1, \ldots, N_0$. From now on we shall drop the label $i$ to lighten the notation. We now set $c_0 = c^{(0)}$ and $\varepsilon_1 = (\sigma_0\varepsilon)^{1/\mathfrak{p}^{(0)}}$, and, as $\varepsilon_1^{\mathfrak{s}^{(0)}}$ divides $\mathcal{F}^{(0)}(\sigma_0\varepsilon_1^{\mathfrak{p}^{(0)}}, c_0\varepsilon_1^{\mathfrak{h}^{(0)}} + y_1\varepsilon_1^{\mathfrak{h}^{(0)}})$, we obtain a new power series $\mathcal{F}^{(1)}(\varepsilon_1, y_1)$ given by

$$\mathcal{F}^{(0)}(\sigma_0\varepsilon_1^{\mathfrak{p}^{(0)}}, c_0\varepsilon_1^{\mathfrak{h}^{(0)}} + y_1\varepsilon_1^{\mathfrak{h}^{(0)}}) = \varepsilon_1^{\mathfrak{s}^{(0)}}\mathcal{F}^{(1)}(\varepsilon_1, y_1), \tag{3.2}$$

which is $y_1$-general of order $\mathfrak{n}_1$ for some $\mathfrak{n}_1 \geq 1$.

[1]See the Remark 2.6.

**Lemma 3.1.** *With the notations introduced before, let us write* $P^{(0)}(c) = g_0(c)(c - c_0)^{m_0}$ *with* $g_0(c_0) \neq 0$*. Then* $\mathfrak{n}_1 = m_0$*.*

*Proof.* This simply follows by the definitions of $\mathcal{F}^{(1)}$ and $P^{(0)}$. In fact we have

$$\varepsilon_1^{\mathfrak{s}^{(0)}} \mathcal{F}^{(1)}(\varepsilon_1, y_1) = \varepsilon_1^{\mathfrak{s}^{(0)}} \left( \sum_{k+\mu^{(0)} j = \mathfrak{r}^{(0)}} Q_{k,j}(c_0 + y_1)^j + \varepsilon_1(\dots) \right), \tag{3.3}$$

so that

$$\mathcal{F}^{(1)}(0, y_1) = P^{(0)}(c_0 + y_1) = g_0(c_0 + y_1) y_1^{m_0}, \tag{3.4}$$

and $g_0(c_0 + y_1) \neq 0$ for $y_1 = 0$. Hence $\mathcal{F}^{(1)}$ is $y_1$-general of order $\mathfrak{n}_1 = m_0$. □

**Remark 3.2.** If Hypothesis 3 is satisfied, by Lemma 3.1 we have $\mathfrak{n}_1 = 1$. Thus, we can apply the Implicit Function Theorem to solve the equation $\mathcal{F}^{(1)}(\varepsilon_1, y_1) = 0$ and we obtain $y_1 = y_1(\varepsilon_1)$ as a convergent power series, *i.e.* we shall obtain

$$y_1(\varepsilon_1) = \sum_{k \geq 0} y_1^{[k]} \varepsilon_1^k. \tag{3.5}$$

This is exactly the result obtained in Sections 1.2 and 2.3, only with a different notation. We used the tree formalism to give a bound for the convergence radius of the power series.

Now we restart the process just described: we construct the Newton polygon $\mathcal{P}^{(1)}$ of $\mathcal{F}^{(1)}$. If $\mathcal{F}_{k,0}^{(1)} = 0$ for all $k \geq 0$, then $\mathcal{F}^{(1)}(\varepsilon_1, 0) \equiv 0$ so that we have $\mathcal{F}(\varepsilon, c_0(\sigma_0\varepsilon)^{\mu^{(0)}}) \equiv 0$, *i.e.* $c_0(\sigma_0\varepsilon)^{\mu^{(0)}}$ is a solution of the implicit equation $\mathcal{F}(\varepsilon, \beta_0) = 0$. Otherwise we consider the segments $\mathcal{P}_1^{(1)}, \dots, \mathcal{P}_{N_1}^{(1)}$ with slopes $\mu_i^{(1)}$ for all $i = 1, \dots, N_1$, and we obtain the polynomials $P_i^{(1)}$; we call $\Re_1$ the set of the real roots of the polynomials $P_i^{(1)}$. If $\Re_1 = \emptyset$, we stop the process as there is no subharmonic solution. Otherwise we call $\mu^{(1)} = \mathfrak{h}^{(1)}/\mathfrak{p}^{(1)}$ the slope of the segment (again we omit the label $i$) associated with the polynomial $P^{(1)}$ which has a real root $c^{(1)}$, so that $c^{(1)}(\sigma_1\varepsilon_1)^{\mu^{(1)}}$, where $\sigma_1 := \operatorname{sign}(\varepsilon_1)$, is an approximate solution of the equation $\mathcal{F}^{(1)}(\varepsilon_1, y_1) = 0$. Again we call $c_1 = c^{(1)}$ and we substitute $\varepsilon_2 = (\sigma_1\varepsilon_1)^{1/\mathfrak{p}^{(1)}}$ and we obtain

$$\mathcal{F}^{(1)}(\sigma_1 \varepsilon_2^{\mathfrak{p}^{(1)}}, c_1 \varepsilon_2^{\mathfrak{h}^{(1)}} + y_2 \varepsilon_2^{\mathfrak{h}^{(1)}}) = \varepsilon_2^{\mathfrak{s}^{(1)}} \mathcal{F}^{(2)}(\varepsilon_2, y_2) \tag{3.6}$$

which is $y_2$-general of order $\mathfrak{n}_2 \leq \mathfrak{n}_1$, and so on. Iterating the process we eventually obtain a sequence of approximate solutions

$$\begin{aligned} \beta_0 &= \varepsilon^{\mu^{(0)}}(c_0 + y_1), \\ y_1 &= \varepsilon_1^{\mu^{(1)}}(c_1 + y_2), \\ y_2 &= \varepsilon_2^{\mu^{(2)}}(c_2 + y_3), \\ &\vdots \end{aligned} \tag{3.7}$$

so that the result is that

$$\beta_0 = c_0\varepsilon^{\mu^{(0)}} + c_1\varepsilon^{\mu^{(0)}+\mu^{(1)}/\mathfrak{p}^{(0)}} + c_2\varepsilon^{\mu^{(0)}+\mu^{(1)}/\mathfrak{p}^{(0)}+\mu^{(2)}/\mathfrak{p}^{(0)}\mathfrak{p}^{(1)}} + \dots \tag{3.8}$$

is a formal expansion of $\beta_0$ as a series in ascending fractional powers of $\varepsilon$. This iterating method is called *Newton-Puiseux process.* Of course this does not occurs if we have $\Re_n = \emptyset$ at a certain step $n$-th, with $n \geq 0$.

From now on we suppose that $\Re_n \neq \emptyset$ for all $n \geq 0$.

**Hypothesis 4.** *There exists $i_0 \geq 0$ such that at the $i_0$-th step of the iteration, there exists a polynomial $P^{(i_0)} = P^{(i_0)}(c)$ which has a simple root $c^* \in \mathbb{R}$.*

Notice that Hypothesis 4 is a "weakened version" of Hypothesis 3, so that we can easily obtain the following result.

**Theorem 3.3.** *Consider a periodic solution with frequency $\omega = p/q$ for the system (1.1). Assume that Hypotheses 1, 2 and 4 are satisfied. Then there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the system (1.1) has at least one subharmonic solution of order $q/p$. Such a solution admits a convergent Puiseux series in $\varepsilon$.*

*Proof.* If we assume Hypothesis 4, we can repeat the analysis of Sections 1.2 to prove the existence of a formal Puiseux series which solves $\mathcal{F}^{(i_0)}(\varepsilon_{i_0}, y_{i_0}) = 0$. More precisely, by setting $\eta := |\varepsilon|^{1/\mathfrak{p}}$, where $\mathfrak{p} = \mathfrak{p}^{(0)} \cdot \ldots \cdot \mathfrak{p}^{(i_0)}$, we obtain a formal solution $(\alpha(t), A(t))$ for (1.1), with $\alpha(t) = \alpha_0(t) + \beta_0 + \widetilde{\beta}(t)$ and $A(t) = A_0 + B(t)$, where

$$\beta_0 = \sum_{k \geq \mathfrak{h}^{(0)}} \eta^k \beta_0^{\{k\}}, \quad \widetilde{\beta}(t) = \sum_{\substack{\nu \in \mathbb{Z} \\ \nu \neq 0}} e^{i\nu\omega t} \sum_{k \geq \mathfrak{p}} \eta^k \widetilde{\beta}_\nu^{\{k\}}, \quad B(t) = \sum_{\nu \in \mathbb{Z}} e^{i\nu\omega t} \sum_{k \geq \mathfrak{p}} \eta^k B_\nu^{\{k\}}, \tag{3.9}$$

and the coefficients $\beta_0^{\{k\}}$, $\widetilde{\beta}_\nu^{\{k\}}$ and $B_\nu^{\{k\}}$ solve

$$\begin{cases} \widetilde{\beta}_\nu^{\{k\}} = \dfrac{\Phi_\nu^{\{k\}}}{i\omega\nu} + \omega'(A_0)\dfrac{\Gamma_\nu^{\{k\}}}{(i\omega\nu)^2}, \quad \nu \neq 0, \\ B_\nu^{\{k\}} = \dfrac{\Gamma_\nu^{\{k\}}}{i\omega\nu}, \quad \nu \neq 0, \\ B_0^{\{k\}} = -\dfrac{\Phi_0^{\{k\}}}{\omega'(A_0)}, \\ \Gamma_0^{\{k\}} = 0, \end{cases} \tag{3.10}$$

with the functions $\Gamma_\nu^{\{k\}}$ and $\Phi_\nu^{\{k\}}$ recursively defined as

$$\begin{aligned} \Gamma_\nu^{\{k\}} &= \sum_{m\geq 0}\sum_{r+s=m}\sum_{\substack{p\sigma_0+q\sigma_0'+\nu_1+\ldots+\nu_m=\nu \\ k_1+\ldots+k_m=k-\mathfrak{p}}} \frac{(i\sigma_0)^r}{r!}\frac{\partial_A^s}{s!}G_{\sigma_0,\sigma_0'}(A_0,t_0)\beta_{\nu_1}^{\{k_1\}}\cdots\beta_{\nu_r}^{\{k_r\}}B_{\nu_{r+1}}^{\{k_{r+1}\}}\cdots B_{\nu_m}^{\{k_m\}}, \\ \Phi_\nu^{\{k\}} &= \sum_{m\geq 0}\sum_{r+s=m}\sum_{\substack{p\sigma_0+q\sigma_0'+\nu_1+\ldots+\nu_m=\nu \\ k_1+\ldots+k_m=k-\mathfrak{p}}} \frac{(i\sigma_0)^r}{r!}\frac{\partial_A^s}{s!}F_{\sigma_0,\sigma_0'}(A_0,t_0)\beta_{\nu_1}^{\{k_1\}}\cdots\beta_{\nu_r}^{\{k_r\}}B_{\nu_{r+1}}^{\{k_{r+1}\}}\cdots B_{\nu_m}^{\{k_m\}}, \\ &\quad + \sum_{s\geq 2}\sum_{\substack{\nu_1+\ldots+\nu_s=\nu \\ k_1+\ldots+k_s=k}} \frac{\partial_A^s}{s!}\omega(A_0)B_{\nu_1}^{\{k_1\}}\ldots B_{\nu_s}^{\{k_s\}}, \end{aligned} \tag{3.11}$$

where $\beta_\nu^{\{k\}} = \widetilde{\beta}_\nu^{\{k\}}$ for $\nu \neq 0$ . We used a different notation for the Taylor coefficients to stress that we are expanding in a different fractional power of $\varepsilon$.

Notice that if we set

$$\begin{aligned} \mathfrak{h}_0 &= \mathfrak{h}^{(0)}\mathfrak{p}^{(1)}\cdot\ldots\cdot\mathfrak{p}^{(i_0)}, \\ \mathfrak{h}_1 &= \mathfrak{h}_0 + \mathfrak{h}^{(1)}\mathfrak{p}^{(2)}\cdot\ldots\cdot\mathfrak{p}^{(i_0)}, \\ \mathfrak{h}_2 &= \mathfrak{h}_1 + \mathfrak{h}^{(2)}\mathfrak{p}^{(3)}\cdot\ldots\cdot\mathfrak{p}^{(i_0)}, \\ &\vdots \\ \mathfrak{h}_{i_0} &= \mathfrak{h}_{i_0-1} + \mathfrak{h}^{(i_0)}, \end{aligned} \tag{3.12}$$

we obtain

$$\beta_0 = \beta_0^{\{\mathfrak{h}_0\}}\eta^{\mathfrak{h}_0} + \beta_0^{\{\mathfrak{h}_1\}}\eta^{\mathfrak{h}_1} + \ldots + \beta_0^{\{\mathfrak{h}_{i_0}\}}\eta^{\mathfrak{h}_{i_0}} + \sum_{k\geq 1}\beta_0^{\{\mathfrak{h}_{i_0}+k\}}\eta^{\mathfrak{h}_{i_0}+k}. \tag{3.13}$$

Thus, if we set $\beta_0^{\{\mathfrak{h}_j\}} = c_j$, for $j = 0,\ldots,i_0$, we can formally solve the equation of motion up to order $\mathfrak{p} + \mathfrak{s}^{(i_0)}$.

Finally, to solve the equation of motion up to any order $k = \mathfrak{p} + \mathfrak{s}^{(i_0)} + \kappa'$, $\kappa' \geq 1$, the higher orders have to be recursively defined by

$$\begin{aligned}\beta_0^{\{\mathfrak{h}_{i_0}+\kappa'\}} = &-\frac{1}{C_{i_0}} \sum_{\substack{s_1\ge 0\\ j\ge 0\\ s_1\mathfrak{p}+j\mathfrak{h}_{i_0}=\mathfrak{s}^{(i_0)}}} Q_{s_1,j}^{(i_0)} \sum_{\substack{m_1+\ldots+m_j=\kappa'\\ 0\le m_i\le\kappa'-1}} \beta_0^{\{\mathfrak{h}_{i_0}+m_1\}}\ldots\beta_0^{\{\mathfrak{h}_{i_0}+m_j\}}\\ &+\sum_{n=1}^{\kappa'} \sum_{\substack{s_1\ge 0\\ j\ge 0\\ s_1\mathfrak{p}+j\mathfrak{h}_{i_0}=\mathfrak{s}^{(i_0)}+n}} Q_{s_1,j}^{(i_0)} \sum_{\substack{m_1+\ldots+m_j=\kappa'-n\\ m_i\ge 0}} \beta_0^{\{\mathfrak{h}_{i_0}+m_1\}}\ldots\beta_0^{\{\mathfrak{h}_{i_0}+m_j\}},\end{aligned} \tag{3.14}$$

where $Q_{s_1,j}^{(i_0)} = \mathcal{F}_{s_1,j}^{(i_0)}\sigma_{i_0}^{s_1}$, and $C_{i_0} := [\mathrm{d}P^{(i_0)}/\mathrm{dc}](c_{i_0}) \neq 0$ by Hypothesis 4.

Thus we can consider a tree expansion similar to the one performed in Section 2.3, prove the convergence (and bound the radius of convergence) of the series (3.9). More precisely, we consider a tree $\theta \in \Theta_{k,\nu,h}$ where now $\mathfrak{p} = \mathfrak{p}^{(0)}\cdot\ldots\cdot\mathfrak{p}^{(i_0)}$ and the constraints (2.21) have to be changed as

$$\begin{aligned} &s_1\mathfrak{p} + (s_0+s_0')\mathfrak{h}_{i_0} = \mathfrak{s}^{(i_0)} + n,\\ &\sum_{i=1}^{s_0+s_0'} k_i = s_0\mathfrak{h}_{i_0} + \sum_{i=1}^{s_0'} k_i = (s_0+s_0')\mathfrak{h}_{i_0} + k - \mathfrak{h}_{i_0} - n,\end{aligned} \tag{3.15}$$

for a suitable $0 \le n \le k - \mathfrak{h}_{i_0}$, with the constraint that when $n = 0$ one has $s_0' \ge 2$.

Now we associate with each leaf $\mathfrak{e}$ a *leaf label* $\mathfrak{a} = \mathfrak{a}_\mathfrak{e} = 0,\ldots,i_0$, with the constraint that if $\theta$ contributes to $\beta_0^{\{k\}}$ for some $k \ge \mathfrak{h}_{i_0}$, than each leaf has leaf label $\mathfrak{a} = i_0$. Again we denote with $N(\theta)$, $L(\theta)$ and $E(\theta)$ the set of nodes, lines and leaves of $\theta$ respectively, and we denote with $E_\mathfrak{a}(\theta)$ the set of leaves in $\theta$ with leaf label $\mathfrak{a}$. We point out that $E(\theta) = E_0(\theta) \amalg \ldots \amalg E_{i_0}(\theta)$.

Moreover, we associate with each node $\mathfrak{v}$ a node factors $\widetilde{\mathcal{N}}_\mathfrak{v} = \mathcal{N}_\mathfrak{v}^*$ and with each line $\ell$ a propagator $\widetilde{g}_\ell = g_\ell^*$, while the leaf factor will be $\widetilde{\mathcal{N}}_\mathfrak{e} = c_{\mathfrak{a}_\mathfrak{e}}$. We shall define the *value* of $\theta$ as

$$\widetilde{\mathrm{Val}}(\theta) = \left(\prod_{\ell\in L(\theta)} \widetilde{g}_\ell\right)\left(\prod_{\mathfrak{v}\in N(\theta)} \widetilde{\mathcal{N}}_\mathfrak{v}\right)\left(\prod_{\mathfrak{e}\in E(\theta)} \widetilde{\mathcal{N}}_\mathfrak{e}\right). \tag{3.16}$$

Finally, we denote with $\Lambda(\theta)$ the set of the lines (exiting from a node) in $\theta$ with component label $h = \beta_0$ and with $N^*(\theta)$ the nodes with $b_\mathfrak{v} = 1$; then we associate with each node in $N^*(\theta)$, with each leave in $E_\mathfrak{a}(\theta)$ and with each line in $\Lambda(\theta)$ a *weight* $\mathfrak{p}$, $\mathfrak{h}_\mathfrak{a}$ and $\mathfrak{h}_{i_0} - \mathfrak{p} - \mathfrak{s}^{(i_0)}$ respectively, and we call *order* of $\theta$ the number

$$k(\theta) = \mathfrak{p}|N^*(\theta)| + (\mathfrak{h}_{i_0} - \mathfrak{p} - \mathfrak{s}^{(i_0)})|\Lambda(\theta)| + \sum_{\mathfrak{a}=0}^{i_0} \mathfrak{h}_\mathfrak{a}|E_\mathfrak{a}(\theta)|, \tag{3.17}$$

*i.e.* the weighted sum of nodes $N^*(\theta)$, leaves and lines $\Lambda(\theta)$.

Hence we obtain

$$\begin{aligned} \beta_0^{\{k\}} &= \sum_{\theta\in\Theta_{k,0,\beta_0}} \widetilde{\mathrm{Val}}(\theta), \qquad k \geq \mathfrak{h}_{i_0}, \\ \widetilde{\beta}_\nu^{\{k\}} &= \sum_{\theta\in\Theta_{k,\nu,\widetilde{\beta}}} \widetilde{\mathrm{Val}}(\theta), \qquad k \geq \mathfrak{p}, \\ B_\nu^{\{k\}} &= \sum_{\theta\in\Theta_{k,\nu,B}} \widetilde{\mathrm{Val}}(\theta), \qquad k \geq \mathfrak{p}, \end{aligned} \tag{3.18}$$

as in Lemma 2.7, so that we can perform the bound of the radius of convergence, as in Proposition 2.9, provided the bound

$$|L(\theta)| \leq Mk, \quad M = 2\frac{\mathfrak{s}^{(i_0)}}{\mathfrak{q}} + 3, \quad \mathfrak{q} = \min\{\mathfrak{h}_{i_0}, \mathfrak{p}\} \tag{3.19}$$

which can be proved similarly to Lemma 2.8, and no further difficulties arise. □

We notice that Theorem 1.4 is a special case of Theorem 3.3, with $i_0 = 0$.

## 3.2 The degenerate case

Here we want to show that in the general case, *i.e.* when not even Hypothesis 4 is assumed, it is possible to prove the convergence of (3.8). Unfortunately we shall not obtain a bound for the convergence radius, as the proof of the convergence of (3.8) in the general case, is non-constructive.

First we show that the denominators in the exponents in (3.8) do not increase indefinitely.

**Lemma 3.4.** *With the notation introduced before if* $\mathfrak{n}_{i+1} = d_i := \deg(P^{(i)})$ *for some* $i$*, then* $\mu^{(i)}$ *is integer.*

*Proof.* Without loss of generality we shall prove the result for the case $i = 0$. Recall that

$$\begin{aligned} \mathcal{F}^{(1)}(0, y_1) &= \sum_{k+\mu^{(0)}j=\mathfrak{r}^{(0)}} Q_{k,j}(c_0 + y_1)^j \\ &= P^{(0)}(c_0 + y_1), \end{aligned} \tag{3.20}$$

with $\mathfrak{r}^{(0)} = \mu^{(0)}d_0$. If $d_0 = \mathfrak{n}_1$, then $P^{(0)}$ is of the form

$$P^{(0)}(c) = R_0(c - c_0)^{\mathfrak{n}_1}, \qquad R_0 \neq 0. \tag{3.21}$$

In particular this means that $Q_{k,\mathfrak{n}_1-1} \neq 0$ for some integer $k \geq 0$ with the constraint $k + \mu^{(0)}(\mathfrak{n}_1 - 1) = \mu^{(0)}\mathfrak{n}_1$. Hence $\mu^{(0)} = k$ is integer. □

**Lemma 3.5.** *With the notations introduced before, there exists* $i_0 \geq 0$ *such that* $\mu^{(i)}$ *is integer for all* $i \geq i_0$*.*

*Proof.* The series $\mathcal{F}^{(i)}$ are $y_i$-general of order $\mathfrak{n}_i$, and the $\mathfrak{n}_i$ and the $d_i$ form a descending sequence of natural numbers

$$\mathfrak{n} = \mathfrak{n}_0 \geq d_0 \geq \mathfrak{n}_1 \geq d_1 \geq \dots \tag{3.22}$$

By Lemma 3.4, $\mu^{(i)}$ fails to be integers only if $d_i > \mathfrak{n}_{i+1}$, and this may happen only finitely often. Hence from a certain $i_0$ onwards all the $\mu^{(i)}$ are integers. □

By the results above, we can define $\mathfrak{p} := \mathfrak{p}^{(0)} \cdot \ldots \cdot \mathfrak{p}^{(i_0)}$ such that we can write (3.8) as

$$\beta_0 = \beta_0(\varepsilon) = \sum_{h \geq h_0} \beta_0^{[h]} \varepsilon^{h/\mathfrak{p}}, \tag{3.23}$$

where $h_0 = \mathfrak{h}^{(0)}\mathfrak{p}^{(1)} \cdot \ldots \cdot \mathfrak{p}^{(i_0)}$. By construction $\mathcal{F}(\varepsilon, \beta_0(\varepsilon))$ vanishes to all orders, so that (3.23) is a formal solution of the implicit equation $\mathcal{F}(\varepsilon, \beta_0) = 0$.

We shall say that (3.23) is a *Puiseux series* for the plane curve defined by $\mathcal{F}(\varepsilon, \beta_0) = 0$.

**Lemma 3.6.** *For all* $i \geq 0$ *we can bound* $\mathfrak{p}^{(i)} \leq \mathfrak{n}_i$*.*

*Proof.* Without loss of generality we prove the result for $i = 0$. By definition, there exist $k', j'$ integers, with $j' \leq \mathfrak{n}_0$, such that

$$\frac{\mathfrak{h}^{(0)}}{\mathfrak{p}^{(0)}} = \mu^{(0)} = \frac{\mathfrak{r}^{(0)} - k'}{j'}, \tag{3.24}$$

and $\mathfrak{h}^{(0)}, \mathfrak{p}^{(0)}$ are relatively prime integers, so that $\mathfrak{p}^{(0)} \leq j' \leq \mathfrak{n}_0$. □

Remark that by Lemma 3.6 we can bound $\mathfrak{p} \leq \mathfrak{n}_0 \cdot \ldots \cdot \mathfrak{n}_{i_0} \leq \mathfrak{n}_0! = \mathfrak{n}!$.

Now we want to prove the convergence of the formal Puiseux series (3.23). Differently from the analogous statement when Hypothesis 4 is satisfied, we shall not bound the convergence radius for the series, but we shall use some well-known results to obtain the convergence.

Let us consider a polynomial $P(x; y) \in \mathbb{C}\{x\}[y]$ of the form

$$P(x; y) = y^n + c_1(x)y^{n-1} + \ldots + c_n(x), \tag{3.25}$$

such that $c_i(0) = 0$ for all $i = 1, \ldots, n$; then $P(x; y)$ is called a *Weierstrass polynomial.*

Recall that a power series

$$u(x, y) = \sum_{k,j \geq 0} u_{k,j} x^k y^j, \tag{3.26}$$

is a *unit* in $\mathbb{C}\{x,y\}$ if and only if the constant coefficient $u(0,0)=u_{0,0}$ is different to zero.

We now state some results well-known in the literature, which will be helpful later.

**Theorem 3.7 (Weierstrass Preparation Theorem).** *Let $f(x,y)\in\mathbb{C}\{x,y\}$ be $y$-general of order $n$ in a neighbourhood of the origin. Then there exist a Weierstrass polynomial of degree $n$ $P_f(x;y)\in\mathbb{C}\{x\}[y]$, and a unit $u(x,y)\in\mathbb{C}\{x,y\}$ such that*

$$f(x,y)=u(x,y)P_f(x;y). \tag{3.27}$$

*Moreover $P_f(x;y)$ and $u(x,y)$ are uniquely determined.*

Theorem 3.7 follows from Theorem 3.8 which is a kind of division with a remainder for convergent power series.

**Theorem 3.8 (Special division theorem).** *Let $P_n(c_1,\ldots,c_n;y)\in\mathbb{C}\{c_1,\ldots,c_n\}[y]$ be the general polynomial of degree $n$,* i.e.

$$P_n(c_1,\ldots,c_n;y)=y^n+c_1y^{n-1}+\ldots+c_n. \tag{3.28}$$

*Then, for all $f(x,y,c_1,\ldots,c_n)\in\mathbb{C}\{x,y,c_1,\ldots,c_n\}$ there exists $q\in\mathbb{C}\{x,y,c_1,\ldots,c_n\}$ and a polynomial $r\in\mathbb{C}\{x,c_1,\ldots,c_n\}[y]$ of degree $d\leq n-1$ such that*

$$f=qP_n+r \tag{3.29}$$

A complete proof of this statement is available for example in [4, 6]. Here we shall prove how Theorem 3.7 follows from Theorem 3.8.

*Proof. (Theorem 3.8 implies Theorem 3.7.)* Let $f\in\mathbb{C}\{x,y\}$ be $y$-general of order $n$, and let us write $f$ in the form

$$f(x,y)=\sum_{k,j\geq 0}f_{k,j}x^ky^j, \tag{3.30}$$

with $f_{0,0}=\ldots=f_{0,n-1}=0$ while $f_{0,n}\neq 0$; notice that we can regard $f(x,y)$ as an element of $\mathbb{C}\{x,y,c_1,\ldots,c_n\}$.

By Theorem 3.8, for all $c_1,\ldots,c_n$, we have

$$f(x,y)=q(x,y,c_1,\ldots,c_n)(y^n+c_1y^{n-1}+\ldots+c_n)+r(x,c_1,\ldots,c_n;y), \tag{3.31}$$

where $r\in\mathbb{C}\{x,c_1,\ldots,c_n\}[y]$ and $q\in\mathbb{C}\{x,y,c_1\ldots,c_n\}$. Our aim is to replace the general coefficients $c_i$ with suitable holomorphic functions $c_i(x)$ so that $r$ vanishes identically. Let us write

$$r(x,c;y)=a_1(c,x)y^{n-1}+\ldots+a_n(c,x),\quad c:=(c_1,\ldots,c_n),\quad a_i(c,x)\in\mathbb{C}\{c,x\}. \tag{3.32}$$

First of all we notice that

$$\partial_{c_j} a_i(0,0) = \begin{cases} 0, & i > j \\ -f_{0,n}, & i = j. \end{cases} \tag{3.33}$$

In fact if one sets $c_i = x = 0$ in (3.31) for all $i = 1, \ldots, n$, and compares the coefficients order by order in $y$, then one obtains

$$a_i(0,0) = 0, \quad \text{and} \quad q(0,0,0) = f_{0,n}. \tag{3.34}$$

Differentiating both sides of (3.31) with respect to the variable $c_j$ and comparing the coefficients order by order in $y$, then (3.33) follows. Hence the matrix $A_{i,j} = \partial_{c_j} a_i(0,0)$ is an upper triangular matrix with determinant $(-f_{0,n})^n \neq 0$. Hence the equations $a_i(c,x) = 0$ satisfy the hypotheses of the Implicit Function Theorem; then there exists $c(x) \in (\mathbb{C}\{x\})^n$, with $c(0) = 0$, such that $a_i(c(x), x) \equiv 0$ for all $i = 1, \ldots, n$.

Now, substituting $c = c(x)$ in (3.31) and setting $u(x,y) = q(x,y,c(x))$, we obtain

$$f(x,y) = u(x,y)(y^n + c_1(x)y^{n-1} + \ldots + c_n(x)), \tag{3.35}$$

and $u(0,0) = f_{0,n} \neq 0$. Hence Theorem 3.7 follows. □

**Remark 3.9.** Theorem 3.7 states that a convergent power series is equal, up to units, to a Weierstrass polynomial; in other words, since $u(x,y)$ is nowhere vanishing in a neighbourhood of the origin, then $f(x,y) = 0$ has the same solutions of the polynomial equation

$$y^n + c_1(x)y^{n-1} + \ldots + c_n(x) = 0. \tag{3.36}$$

Let us denote with $B_{\delta_1,\delta_2}$ the polydisc $B_{\delta_1,\delta_2} := \{(x,y) \in \mathbb{C}^{2m} : |y| < \delta_1, |x| < \delta_2\}$, where $|\cdot|$ denotes the euclidean norm in $\mathbb{C}^m$.

**Theorem 3.10.** *Let $\mathcal{F}(x,y) \in \mathbb{C}\{x,y\}$ be irreducible and $y$-general of order $n \geq 1$. Then there exists $\delta_0 > 0$ such that for all $0 < \delta_1 < \delta_0$ there exists $\delta_2 > 0$ such that if we define*

$$X := \{(x,y) \in B_{\delta_1,\delta_2} : \mathcal{F}(x,y) = 0\}, \tag{3.37}$$

*then there exists a convergent power series $y(z) \in \mathbb{C}\{z\}$ for which the map*

$$\pi : D \longrightarrow \mathbb{C}^2, \tag{3.38}$$

*from the disc $D := \{z \in \mathbb{C} : |z| < \delta_2^{1/n}\}$ to $\mathbb{C}^2$ with $\pi(z) = (z^n, y(z))$ is biholomorphic on $X$,* i.e. *$\pi$ is holomorphic, bijective on $X$ and $\pi^{-1} : X \to D$ is holomorphic. Moreover the restriction*

$$\pi : D \setminus \{0\} \longrightarrow X \setminus \{0\}, \tag{3.39}$$

*is biholomorphic and $\pi^{-1}(0) = 0$.*

This result is proved for instance in [4]. Here we shall see how the convergence of the Puiseux series (3.23) follows form Theorem 3.10.

**Lemma 3.11.** *Let $\mathcal{F}(\varepsilon, \beta_0) \in \mathbb{R}\{\varepsilon, \beta_0\}$ be $\beta_0$-general of order $\mathfrak{n}$ and let us suppose that $\Re_n \neq \emptyset$ for all $n \geq 0$. Then the series (3.23) which formally solves $\mathcal{F}(\varepsilon, \beta_0(\varepsilon)) \equiv 0$, is convergent for $\varepsilon$ small enough.*

*Proof.* Let $P_{\mathcal{F}}(\varepsilon; \beta_0)$ be the Weierstrass polynomial of $\mathcal{F}$ in $\mathbb{C}\{\varepsilon\}[\beta_0]$. If $\mathcal{F}$ is irreducible in $\mathbb{C}\{\varepsilon, \beta_0\}$, then by Theorem 3.10 we have a convergent series $\beta_0(\varepsilon^{1/\mathfrak{n}})$ which solves the equation $\mathcal{F}(\varepsilon, \beta_0) = 0$. Then all the following

$$\beta_0\left(\varepsilon^{1/\mathfrak{n}}\right),\ \beta_0\left((e^{2\pi i}\varepsilon)^{1/\mathfrak{n}}\right), \ldots, \beta_0\left((e^{2\pi(\mathfrak{n}-1)i}\varepsilon)^{1/\mathfrak{n}}\right), \tag{3.40}$$

are solutions of the equation $\mathcal{F}(\varepsilon, \beta_0) = 0$. Thus we have $\mathfrak{n}$ distinct roots of the Weierstrass polynomial $P_{\mathcal{F}}$ and they are all convergent series in $\mathbb{C}\{\varepsilon^{1/\mathfrak{n}}\}$. But also the series (3.23) is a solution of the equation $\mathcal{F}(\varepsilon, \beta_0) = 0$. Then, as a polynomial of degree $\mathfrak{n}$ has exactly $\mathfrak{n}$ (complex) roots counting multiplicity, (3.23) is one of the (3.40). In particular this means that (3.23) is convergent for $\varepsilon$ small enough.

In general, we can write

$$\mathcal{F}(\varepsilon, \beta_0) = \prod_{i=1}^{N} (\mathcal{F}_i(\varepsilon, \beta_0))^{m_i}, \tag{3.41}$$

for some $N \geq 1$, where the $\mathcal{F}_i$ are the irreducible factors of $\mathcal{F}$. Then the Puiseux series (3.23) satisfies one of the equations $\mathcal{F}_i(\varepsilon, \beta_0) = 0$, and hence, by what said above, it is convergent for $\varepsilon$ small enough. □

**Remark 3.12.** Notice that the proof of Lemma 3.11 is non-constructive: in fact it deals with the problem of decomposing a series in its irreducible factors. Moreover, if $\mathfrak{n}$ is even we can not say *a priori* if $\Re_n \neq \emptyset$ for all $n \geq 0$, *i.e.* we can not say if a formal solution exists at all.

As a consequence of Lemma 3.11 we obtain the following corollary.

**Theorem 3.13.** *Consider a periodic solution with frequency $\omega = p/q$ for the system (1.1). Assume that Hypotheses 1 and 2 are satisfied with $\mathfrak{n}$ odd. Then there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the system (1.1) has at least one subharmonic solution of order $q/p$. Such a solution admits a convergent Puiseux series in $\varepsilon$.*

*Proof.* If $\mathfrak{n}$ is odd, then $\Re_n \neq \emptyset$ for all $n \geq 0$. This trivially follows from the fact that if $\mathfrak{n}$ is odd, then there exists at least a polynomial $P_i^{(0)}$ of degree $d_i$, with $d_i$ odd. Thus such a polynomial admits a real root with odd multiplicity $\mathfrak{n}_1$, so that $\mathcal{F}^{(1)}(\varepsilon_1, y_1)$ is $y_1$-general of odd order $\mathfrak{n}_1$ and so on.

Hence we can apply the Newton-Puiseux process to obtain a subharmonic solution as a Puiseux series in $\varepsilon$ which is convergent for $\varepsilon$ sufficiently small by Lemma 3.11. □

Remark that Theorem 3.13 extends the results of [33]. First it gives the explicit dependence of the parameter $\beta_0$ on $\varepsilon$. Second, it shows that it is possible to express the subharmonic solution as a convergent fractional power series in $\varepsilon$, and this allow us to push perturbation theory to arbitrarily high order. Finally a subharmonic solution can be constructed for ant non-zero real root of each odd-degree polynomial $P_i^{(n)}$ associated with each segment of $\mathcal{P}^{(n)}$ to all step of iteration.

## 3.3 Higher order Melnikov functions

Now we shall see how to extend the results above when the Melnikov function vanishes identically.

We are searching for a solution of the form $(\alpha(t), A(t))$ with $\alpha(t) = \alpha_0(t) + \beta_0 + \widetilde{\beta}(t)$ and $A(t) = A_0 + B(t)$, where

$$\begin{aligned} \widetilde{\beta}(t) &= \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} e^{i\nu\omega t}\beta_\nu(\varepsilon,\beta_0), \\ B(t) &= \sum_{\nu\in\mathbb{Z}} e^{i\nu\omega t} B_\nu(\varepsilon,\beta_0). \end{aligned} \tag{3.42}$$

First of all, we notice that we can formally write the equations of motion as

$$\begin{cases} \overline{\beta}_\nu^{(k)}(\beta_0) = \dfrac{\overline{\Phi}_\nu^{(k)}(\beta_0)}{i\omega\nu} + \omega'(A_0)\dfrac{\overline{\Gamma}_\nu^{(k)}(\beta_0)}{(i\omega\nu)^2}, \quad \nu\neq 0 \\ \overline{B}_\nu^{(k)}(\beta_0) = \dfrac{\overline{\Gamma}_\nu^{(k)}(\beta_0)}{i\omega\nu}, \quad \nu\neq 0 \\ \overline{B}_0^{(k)}(\beta_0) = -\dfrac{\overline{\Phi}_0^{(k)}(\beta_0)}{\omega'(A_0)}. \end{cases} \tag{3.43}$$

where the notations in (1.26) have been used,up to any order $k$, provided

$$\overline{\Gamma}_0(\varepsilon,\beta_0) = 0. \tag{3.44}$$

If $M(t_0)$ vanishes identically, by (1.31) we have $\overline{\Gamma}_0^{(1,j)} = 0$ for all $j \geq 0$, that is

$$\overline{\Gamma}_0^{(1)}(\beta_0) = 0, \tag{3.45}$$

for all $\beta_0$, and hence $\overline{\Gamma}_0(\varepsilon, \beta_0) = \varepsilon^2 \mathcal{F}^{(2)}(\varepsilon, \beta_0)$, with $\mathcal{F}^{(2)}$ a suitable function analytic in $\varepsilon$, $\beta_0$.

Thus, we can solve the equations of motion up to the first order in $\varepsilon$, and the parameter $\beta_0$ is left undetermined. More precisely we obtain

$$\begin{aligned} \beta_\nu &= \varepsilon \overline{\beta}_\nu^{(1)} + \varepsilon \widetilde{\beta}_\nu^{(1)}(\varepsilon, \beta_0), \\ B_\nu &= \varepsilon \overline{B}_\nu^{(1)} + \varepsilon \widetilde{B}_\nu^{(1)}(\varepsilon, \beta_0), \end{aligned} \tag{3.46}$$

where $\overline{\beta}_\nu^{(1)}$, $\overline{B}_\nu^{(1)}$ solve the equation of motion up to the first order in $\varepsilon$, while $\widetilde{\beta}_\nu^{(1)}$, $\widetilde{B}_\nu^{(1)}$ are the corrections to be determined.

Now, let us set

$$M_0(t_0) = M(t_0), \qquad M_1(t_0) = \overline{\Gamma}_0^{(2)}(0, t_0), \tag{3.47}$$

where $\overline{\Gamma}_\nu^{(k)}(\beta_0, t_0) = \overline{\Gamma}_\nu^{(k)}(\beta_0)$ *i.e.* we are stressing the dependence of $\overline{\Gamma}_\nu^{(k,j)}$ on $t_0$. We refer to $M_1(t_0)$ as the *second order subharmonic Melnikov function.*

Notice that $M_0(t_0) = \overline{\Gamma}_0^{(1)}(0, t_0)$.

If there exist $t_0 \in [0, 2\pi)$ and $\mathfrak{n}_1 \in \mathbb{N}$ such that $t_0$ is a zero of order $\mathfrak{n}_1$ for the second order subharmonic Melnikov function, that is

$$\frac{\mathrm{d}^k}{\mathrm{d}t_0^k} M_1(t_0) = 0 \quad \forall\, 0 \le k \le \mathfrak{n}_1 - 1, \qquad D = D(t_0) := \frac{\mathrm{d}^{\mathfrak{n}_1}}{\mathrm{d}t_0^{\mathfrak{n}_1}} M_1(t_0) \neq 0, \tag{3.48}$$

then we can repeat the analysis of the previous Sections to obtain the existence of a subharmonic solution. In fact, we have

$$\mathcal{F}^{(2)}(\varepsilon, \beta_0) := \sum_{k,j \ge 0} \varepsilon^k \beta_0^j \mathcal{F}_{k,j}^{(2)}, \qquad \mathcal{F}_{k,j}^{(2)} = \overline{\Gamma}_0^{(k+2,j)}(t_0), \tag{3.49}$$

where $t_0$ has to be fixed as the zero of $M_1(t_0)$, so that, as

$$(-\omega(A_0))^{-j} \frac{\mathrm{d}^j}{\mathrm{d}t_0^j} M_1(t_0) = j! \overline{\Gamma}_0^{(2,j)}(t_0), \tag{3.50}$$

for all $j$, as proved in [10] with a different notation, we can construct the Newton polygon of $\mathcal{F}^{(2)}$, which is $\beta_0$-general of order $\mathfrak{n}_1$ by (3.48), to obtain $\widetilde{\beta}^{(1)}$, $\widetilde{B}^{(1)}$ and $\beta_0$ as Puiseux series in $\varepsilon$, provided at each step of the iteration of the Newton-Puiseux algorithm one has a real root.

Otherwise, if $M_1(t_0)$ vanishes identically, we have $\overline{\Gamma}_0^{(2)}(\beta_0) = 0$ for all $\beta_0$, so that we can solve the equations of motion up to the second order in $\varepsilon$ and the parameter $\beta_0$ is still undetermined. Hence we set $M_2(t_0) = \overline{\Gamma}_0^{(3)}(0, t_0)$ and so on.

In general if $M_{k'}(t_0) \equiv 0$, for all $k' = 0, \ldots, \kappa - 1$, we have $\overline{\Gamma}_0(\varepsilon, \beta_0) = \varepsilon^{k'} \mathcal{F}^{(k')}(\varepsilon, \beta_0)$, so that we can solve the equations of motion up to the $\kappa$-th order in $\varepsilon$, and obtain

$$\begin{aligned} \beta_\nu &= \varepsilon \overline{\beta}_\nu^{(1)} + \ldots + \varepsilon^\kappa \overline{\beta}_\nu^{(\kappa)} + \varepsilon^\kappa \widetilde{\beta}_\nu^{(k)}(\varepsilon, \beta_0), \\ B_\nu &= \varepsilon \overline{B}_\nu^{(1)} + \ldots + \varepsilon^\kappa \overline{B}_\nu^{(\kappa)} + \varepsilon^\kappa \widetilde{B}_\nu^{(k)}(\varepsilon, \beta_0), \end{aligned} \tag{3.51}$$

where $\overline{\beta}_\nu^{(k')}$, $\overline{B}_\nu^{(k')}$, $k' = 0, \ldots, \kappa - 1$ solve the equation of motion up to the $\kappa$-th order in $\varepsilon$, while $\widetilde{\beta}_\nu^{(\kappa)}$, $\widetilde{B}_\nu^{(\kappa)}$ are the correction to be determined.

Hence we can weaken Hypotheses 2 and 4 as follows.

**Hypothesis 5.** *There exists $\kappa \geq 0$ such that for all $k' = 0, \ldots, \kappa - 1$, $M_{k'}(t_0)$ vanishes identically, and there exist $t_0 \in [0, 2\pi)$ and $\mathfrak{n} \in \mathbb{N}$ such that $t_0$ is a zero of order $\mathfrak{n}$ for the $\kappa$-th order subharmonic Melnikov function, that is*

$$\frac{\mathrm{d}^j}{\mathrm{d}t_0^j} M_\kappa(t_0) = 0 \quad \forall\, 0 \leq j \leq \mathfrak{n} - 1, \qquad D = D(t_0) := \frac{\mathrm{d}^{\mathfrak{n}}}{\mathrm{d}t_0^{\mathfrak{n}}} M_\kappa(t_0) \neq 0. \tag{3.52}$$

**Hypothesis 6.** *There exists $i_0 \geq 0$ such that at the $i_0$-th step of the iteration of the Newton-Puiseux algorithm for $\mathcal{F}^{(\kappa)}$, there exists a polynomial $P^{(i_0)} = P^{(i_0)}(c)$ which has a simple root $c^* \in \mathbb{R}$.*

Thus we have the following result.

**Theorem 3.14.** *Consider a periodic solution with frequency $\omega = p/q$ for the system (1.1), and assume that Hypotheses 1, 5 and 6 are satisfied. Then there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the system (1.1) has at least one subharmonic solution of order $q/p$. Such a solution admits a convergent Puiseux series in $\varepsilon$.*

The proof can be easily obtained suitably modifying the proof of Theorem 3.3.

Now, call $\mathfrak{R}_n^{(\kappa)}$ the set of real roots of the polynomials obtained at the $n$-th step of iteration of the Newton-Puiseux process for $\mathcal{F}^{(\kappa)}$. Again if $\mathfrak{n}$ is even we can not say *a priori* whether a formal solution exists at all. However, if $\mathfrak{R}_n^{(\kappa)} \neq \emptyset$ for all $n \geq 0$, then we obtain a convergent Puiseux series as in Section 3.2.

Finally, as a corollary, we have the following result.

**Theorem 3.15.** *Consider a periodic solution with frequency $\omega = p/q$ for the system (1.1). Assume that Hypotheses 1 and 5 are satisfied with $\mathfrak{n}$ odd. Then there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the system (1.1) has at least one subharmonic solution of order $q/p$. Such a solution admits a convergent Puiseux series in $\varepsilon$.*

Again the proof is a suitable modification of the proof of Theorem 3.13.

# A. On Hypothesis 3

Here we want to show that Hypothesis 3 holds for "almost all" polynomials. First we make this assertion precise.

A subset $\mathcal{A}$ of a topological space $X$ is called *residual* if it is the intersection of a countable number of subsets of $X$, each of which is open and dense. If all the residual subsets of $X$ are themselves dense in $X$, then $X$ is called a *Baire space.* Given a Baire space $X$, a property is said to be *generic on* $X$ if it holds on a subset of $X$ containing a residual set.

Notice that, as $\mathbb{R}^n$ is a complete metric space, by the Baire Category Theorem (see for instance [26]), it is a Baire space.

Hence we want to show that Hypothesis 3 is generic on the space of the coefficients of the polynomials.

More precisely, we shall show that given a polynomial of the form

$$P(a, c) = \sum_{i=0}^{n} a_{n-i} c^i, \qquad n \geq 1, \qquad a := (a_0, \dots, a_n), \tag{A.1}$$

the set of parameters $(a_0, \dots, a_n) \in \mathbb{R}^{n+1}$ for which $P(a, c)$ has multiples roots, is a proper Zariski-closed[1] subset of $\mathbb{R}^{n+1}$.

Notice that a polynomial $P = P(a, c)$ has a multiple root $c^*$ if and only if also the derivative $\partial P / \partial c$ vanishes at $c^*$.

Recall that, given two polynomials

$$\begin{aligned} P_1(c) &= \sum_{i=0}^{n} a_{n-i} c^i, \\ P_2(c) &= \sum_{i=0}^{m} b_{m-i} c^i, \end{aligned} \tag{A.2}$$

with $n, m \geq 1$, the *Sylvester matrix of* $P_1, P_2$ is an $n + m$ square matrix where the columns 1 to $m$ are formed by "shifted sequences" of the coefficients of $P_1$, while the columns $m + 1$ to

[1]See for instance [27].

$m+n$ are formed by "shifted sequences" of the coefficients of $P_2$, *i.e.*

$$\mathrm{Syl}(P_1,P_2) := \begin{pmatrix} a_0 & 0 & \dots & 0 & b_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 & b_1 & b_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n-1} & 0 & 0 & \dots & b_{m-1} \\ 0 & 0 & \dots & a_n & 0 & 0 & \dots & b_m \end{pmatrix}, \tag{A.3}$$

and the *resultant* $R(P_1,P_2)$ *of* $P_1,P_2$ is defined as the determinant of the Sylvester matrix.

**Lemma A.1.** *Let* $c_{1,1},\dots,c_{1,n}$ *and* $c_{2,1},\dots,c_{2,m}$ *be the complex roots of* $P_1,P_2$ *respectively. Then*

$$R(P_1,P_2) = a_0^m b_0^n \prod_{i=1}^{n}\prod_{j=1}^{m}(c_{1,i}-c_{2,j}). \tag{A.4}$$

A complete proof is performed for instance in [29].

In particular, Lemma A.1 implies that two polynomials have a common root if and only if $R(P_1,P_2)=0$.

Recall also that given a polynomial $P=P(c)$, the *discriminant* $D(P)$ *of* $P$ is the resultant of $P$ and its first derivative with respect to $c$, *i.e.* $D(P) := R(P,P')$, where $P' := \mathrm{d}P/\mathrm{d}c$.

Thus, a polynomial $P = P(a,c)$ of the form (A.1) has a multiple root if and only if its discriminant is equal to zero.

Now let us consider the set

$$V := \{a=(a_0,\dots,a_n)\in\mathbb{R}^{n+1} \,:\, P(a,c) \text{ has a multiple root }\}. \tag{A.5}$$

Notice that the discriminant of $P(a,c)$ is a polynomial in the parameters $a=(a_0,\dots,a_n)$ *i.e.* $D_P(a) = D(P)\in\mathbb{R}[a_0,\dots,a_n]$, hence we can write

$$V = \{a=(a_0,\dots,a_n)\in\mathbb{R}^{n+1} \,:\, D_P(a)=0\}. \tag{A.6}$$

Such a set is, by definition, a proper Zariski-closed subset of $\mathbb{R}^{n+1}$.

As the complement of a proper Zariski-closed subset of $\mathbb{R}^{n+1}$ is open and dense also in the Euclidean topology, then Hypothesis 3 is generic.

# B. Proof of Lemma 2.8

First we shall prove by induction on $k$ that for all $\theta \in \Theta_{k,0,\beta_0}$ one has

$$|L(\theta)| \leq M(k-\mathfrak{h}) - \left(1 + \frac{\mathfrak{s}}{\mathfrak{q}}\right), \tag{B.1}$$

for all $k \geq \mathfrak{h}+1$. In fact for $k = \mathfrak{h}$ the bound (2.31) is trivially satisfied as $L(\theta) = 1$. Recall that, by Lemma 2.7, for all $k \geq \mathfrak{h}$, each tree in $\Theta_{k,0,\beta_0}$ contributes to $\beta_0^{[k]}$.

For $k = \mathfrak{h}+1$ one has

$$\beta_0^{[\mathfrak{h}+1]} = -\frac{1}{C} \sum_{s_1 \mathfrak{p} + j\mathfrak{h} = \mathfrak{s}+1} Q_{s_1,j} (c^*)^j, \tag{B.2}$$

so that any tree $\theta$ contributing to $\beta_0^{[\mathfrak{h}+1]}$ has $s_1+1$ nodes and $j$ leaves, hence $|L(\theta)| = s_1+1+j$. Notice that the set $\Sigma_1 = \{(x,y) \in \mathbb{R}^2_+ : x\mathfrak{p} + y\mathfrak{h} = \mathfrak{s}+1\}$ is a segment (see Figure B.1) with the same slope of the segment $\overline{\mathcal{P}}$ of the Newton polygon, associated with the polynomial $P(c)$ in (1.40).

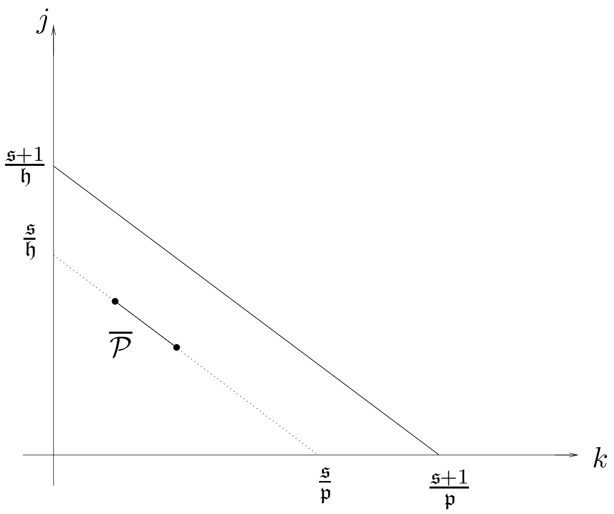


Figure B.1: The segment $\Sigma_1$ parallel to $\overline{\mathcal{P}}$.

Hence one has

$$|L(\theta)| = 1 + s_1 + j \le 1 + \frac{\mathfrak{s}+1}{\mathfrak{q}}. \tag{B.3}$$

Moreover for $k = \mathfrak{h}+1$ the r.h.s. in (B.1) is equal to $2+\mathfrak{s}/\mathfrak{q}$, so that the bound (B.1) holds, because one has $\mathfrak{q} \ge 1$. Assume now that the bound (B.1) holds for all $k' < k$ and let us show that then it holds also for $k$.

We call $M_0 = M\mathfrak{h} + 1 + \mathfrak{s}/\mathfrak{q}$, so that the inductive hypothesis can be written as

$$|L(\theta')| \le Mk(\theta') - M_0, \tag{B.4}$$

for all $\theta' \in \Theta_{k',0,\beta_0}$, $k' < k$.

Recall that a tree contributing to $\beta_0^{[k]}$ is of the form depicted in Figure 2.9 with the constraints (2.21) holding. Hence, by the inductive hypothesis, we have

$$|L(\theta)| \le 1 + s_1 + s_0 - s_0' M_0 + M \sum_{i=1}^{s_0'} k(\theta_i). \tag{B.5}$$

Let us set $m := k - \mathfrak{h} \ge 1$. Hence, *via* the conditions (2.21) we can write (B.5) as

$$|L(\theta)| \le 1 + s_1 + s_0 - s_0' M_0 + M(\mathfrak{s} + m - s_1\mathfrak{p} - s_0\mathfrak{h}). \tag{B.6}$$

Hence we shall prove that

$$1 + s_1 + s_0 - s_0' M_0 + M(\mathfrak{s} + m - s_1\mathfrak{p} - s_0\mathfrak{h}) \le mM - 1 - \frac{\mathfrak{s}}{\mathfrak{q}}, \tag{B.7}$$

or, in other words

$$(s_1\mathfrak{p} + (s_0 + s_0')\mathfrak{h})M + s_0'\left(1 + \frac{\mathfrak{s}}{\mathfrak{q}}\right) \ge \mathfrak{s}M + s_0 + s_1 + \frac{\mathfrak{s}}{\mathfrak{q}} + 2, \tag{B.8}$$

for all $s_0, s_0', s_1 \ge 0$ admitted by conditions (2.21).

First of all for $s_0' = 0$ by the first condition in (2.21) we have $s_1\mathfrak{p} + s_0\mathfrak{h} = \mathfrak{s} + m \ge \mathfrak{s} + 1$. Moreover $(s_1 + s_0)\mathfrak{q} \le s_1\mathfrak{p} + s_0\mathfrak{h} = \mathfrak{s} + m$, hence

$$s_1 + s_0 \le \frac{\mathfrak{s}+m}{\mathfrak{q}}, \tag{B.9}$$

so that one can bound (B.8) as

$$mM \ge 2\frac{\mathfrak{s}}{\mathfrak{q}} + 2 + \frac{m}{\mathfrak{q}}, \tag{B.10}$$

and, by substituting (2.30) one has

$$m\left(2\frac{\mathfrak{s}}{\mathfrak{q}} + 3\right) \ge 2\frac{\mathfrak{s}}{\mathfrak{q}} + 2 + \frac{m}{\mathfrak{q}}, \tag{B.11}$$

that is satisfied for all $m \geq 1$.

For $s_0' = 1$ the first conditions (2.21) can be written as $s_1\mathfrak{p} + (s_0+1)\mathfrak{h} = \mathfrak{s} + m \geq \mathfrak{s}+1$, so that

$$s_1 + s_0 \leq \frac{\mathfrak{s} - \mathfrak{h} + m}{\mathfrak{q}}. \tag{B.12}$$

Hence we can bound (B.8) as

$$mM \geq \frac{\mathfrak{s} + m - \mathfrak{h}}{\mathfrak{q}}, \tag{B.13}$$

and again (B.13) is satisfied for all $m \geq 1$.

Finally for $s_0' \geq 2$ the first condition in (2.21) can be written $s_1\mathfrak{p} + (s_0+s_0')\mathfrak{h} \geq \mathfrak{s}$, so that $s_1 + s_0 < s_1 + s_0 + s_0' \leq \frac{\mathfrak{s}}{\mathfrak{q}}$, and we can bound (B.8) as

$$\mathfrak{s}M + s_0'\left(\frac{\mathfrak{s}}{\mathfrak{q}} + 1\right) \geq \mathfrak{s}M + 2\frac{\mathfrak{s}}{\mathfrak{q}} + 2, \tag{B.14}$$

that is satisfied as we are assuming $s_0' \geq 2$.

Notice that by (2.21) this exhausts the discussion over all the choices of $s_0, s_0', s_1$.

Let us show now that

$$|L(\theta)| \leq Mk - 1, \tag{B.15}$$

for all $\theta \in \Theta_{k,\nu,f}$, $f = \widetilde{\beta}, B$, $k \geq \mathfrak{p}$.

Again recall that a tree $\theta \in \Theta_{k,\nu,f}$ contributes to $f_\nu^{[k]}$ with $f = \widetilde{\beta}, B$, so that the bound (B.15) is trivially satisfied for $k = \mathfrak{p}$ because one has $|L(\theta)| = 1$.

Let us suppose now that the bound holds for all $k' < k$; again we shall prove that then it holds also for $k$.

Recall that a tree contributing to $f_\nu^{[k]}$ is of the form depicted in Figure 2.11, where $s_0, s_1$ are the numbers of the lines exiting from a leaf and a simple node respectively and entering $\mathfrak{v}_0$, while $s_0', s_1'$ are the graph elements entering $\mathfrak{v}_0$ with component label $\beta_0$ and $f$ respectively. Hence, by the inductive hypothesis and by the bound (B.1), we have

$$|L(\theta)| \leq 1 + s_0 + s_1 - s_0'M_0 - s_1' + M\sum_{i=1}^{s_0'+s_1'} k(\theta_i). \tag{B.16}$$

Let us supposte first $b_{\mathfrak{v}_0} = 1$, and set $m = k - \mathfrak{p} \geq 1$; thus, *via* the first condition in (2.17), we shall prove the bound

$$1 + s_0 + s_1 + M(m - s_0\mathfrak{h} - s_1\mathfrak{p}) - s_0'M_0 - s_1' \leq Mk - 1, \tag{B.17}$$

or, in other words,

$$s_0(M\mathfrak{h} - 1) + s_1(M\mathfrak{p} - 1) + M\mathfrak{p} + s_0'M_0 + s_1' \geq 2, \tag{B.18}$$

and this is obviously satisfied as $M\mathfrak{h}, M\mathfrak{p} \geq 3$.

Finally if $b_{\mathfrak{v}_0} = 0$ we have

$$\begin{aligned} &\sum_{i=1}^{s_1'} k(\theta_i) = k - s_1\mathfrak{p}, \\ &s_0 + s_0' = 0, \qquad s_1 + s_1' \geq 2, \end{aligned} \tag{B.19}$$

so that we shall prove the bound

$$1 + s_1 + M(k - s_1\mathfrak{p}) - s_1' \leq Mk - 1, \tag{B.20}$$

or, in other words

$$s_1(M\mathfrak{p} - 1) + s_1' \geq 2, \tag{B.21}$$

and again this is obviously satisfied as $s_1 + s_1' \geq 2$ and $M\mathfrak{p} > 1$. □

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*La matematica è come una piramide rovesciata:*
*si regge sulla punta*
*che deve essere ben salda,*
*e per poter costruire un piano*
*è necessario farlo in tutte le direzioni,*
*altrimenti la piramide perde l'equilibrio.*
*E. Sernesi*

# Ringraziamenti

Eccomi qua, alla fine di un percorso. Fa un po' strano a dirlo, a pensarci... Mi guardo indietro pensando alle parole giuste da scrivere per ringraziare tutti in modo adeguato, e mi accorgo che tutte le parole del mondo, probabilmente, non renderebbero comunque l'idea: la mia solita "ansia da pagina bianca"... però le voglio trovare - le parole - anche se non so bene che cosa ne uscirà.

Pronti... via!

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L'altro è il prof. Pontecorvo: professo', alla fine ce l'ho fatta ad andarmene, ha visto? è contento? (rido molto). No, vabbeh, siamo seri. Professore, non dimenticherò le nostre lunghe chiacchierate, il sostegno che mi ha dato ogni volta che stavo per crollare, la disponibilità che ha avuto nei miei confronti, le risate nei momenti bui... Grazie.

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Misi, perché sei "l'altro cattivo del quartetto", per tutte le lunghe discussioni su Locke e Hume (ci si capisce, no?)...

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Romano... lo sai. Devo anche scriverlo? Ok. Grazie per essere stato un padre, un amico, un fratello, un figlio, un "amante", un nonno, un maestro... e potrei andare avanti per delle ore...

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*Ed io vi spio e mi commuovo,*
*Ed io vi spio e mi commuovo.*

*E grazie...*
*per tutto questo: grazie.*
*Per tutto quanto: grazie.*
*Per tutto: grazie.*

*E grazie.*
*Semplicemente grazie.*
*Perdutamente grazie.*
*Solennemente grazie.*