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KAM Theory in ConÞguration Space and Cancellations in the Lindstedt Series

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Abstract: The KAM theorem for analytic quasi-integrable anisochronous Hamiltonian systems yields that the perturbation expansion (Lindstedt series) for any quasi-periodic solution with Diophantine frequency vector converges. If one studies the Lindstedt series by following a perturbation theory approach, one Þnds that convergence is ultimately related to the presence of cancellations between contributions of the same perturbation order. In turn, this is due to symmetries in the problem. Such symmetries are easily visualised in action-angle coordinates, where the KAM theorem is usually formulated by exploiting the analogy between Lindstedt series and perturbation expansions in quantum Þeld theory and, in particular, the possibility of expressing the solutions in terms of tree graphs, which are the analogue of Feynman diagrams. If the unperturbed system is isochronous, MoserÕs modifying terms theorem ensures that an analytic quasi-periodic solution with the same Diophantine frequency vector as the unperturbed Hamiltonian exists for the system obtained by adding a suitable constant (counterterm) to the vector Þeld. Also in this case, one can follow the alternative approach of studying the perturbation expansion for both the solution and the counterterm, and again convergence of the two series is obtained as a consequence of deep cancellations between contributions of the same order. In this paper, we revisit MoserÕs theorem, by studying the perturbation expansion one obtains by working in Cartesian coordinates. We investigate the symmetries giving rise to the cancellations which makes possible the convergence of the series. We Þnd that the cancellation mechanism works in a completely different way in Cartesian coordinates, and the interpretation of the underlying symmetries in terms of tree graphs is much more subtle than in the case of action-angle coordinates.

1. Introduction

Consider an isochronous Hamiltonian system, described by the Hamiltonian $H(, A) = \cdot A + f(, A)$, with f real analytic in $\mathbb{T}^d \times \mathcal{A}$ and $\mathcal{A}$ an open subset of $\mathbb{R}^d$.

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The corresponding Hamilton equations are

$$\dot\alpha = \omega + \varepsilon\partial_A f(\alpha, A), \qquad \dot A = -\varepsilon\partial_\alpha f(\alpha, A). \tag{1.1}$$

Let $(\alpha_0(t), A_0(t)) = (\alpha_0 + \omega t, A_0)$ be a solution of (1.1) for $\varepsilon = 0$. For $\varepsilon \neq 0$, in general, there is no quasi-periodic solution to (1.1) with frequency vector ω which reduces to $(\alpha_0(t), A_0(t))$ as $\varepsilon \to 0$. However, one can prove that, if ε is small enough and ω satisfies some Diophantine condition, then there is a "correction" $\mu(\varepsilon, A_0)$, analytic in both ε and A_0, such that the modified equations

$$\dot\alpha = \omega + \varepsilon\partial_A f(\alpha, A) + \mu(\varepsilon, A_0), \qquad \dot A = -\varepsilon\partial_\alpha f(\alpha, A) \tag{1.2}$$

admit a quasi-periodic solution with frequency vector ω which reduces to $(\alpha_0(t), A_0(t))$ as $\varepsilon \to 0$. This is a well known result, called the *modifying terms theorem*, or *translated torus theorem*, first proved by Moser [20]. By writing the solution as a power series in ε (Lindstedt series), the existence of an analytic solution means that the series converges. This is ultimately related to some deep cancellations in the series; see [1] for a review.

Equations like (1.1) naturally arise when studying the stability of an elliptic equilibrium point. For instance, one can think of a mechanical system near a minimum point for the potential energy, where the Hamiltonian describing the system looks like

$$H(x_1, \dots, x_n, y_1, \dots, y_n) = \frac{1}{2}\sum_{j=1}^{d}\left(y_j^2 + \omega_j^2 x_j^2\right) + \varepsilon F(x_1, \dots, x_n, \varepsilon), \tag{1.3}$$

where F is a real analytic function at least of third order in its arguments, the vector $\omega = (\omega_1, \dots, \omega_d)$ satisfies some Diophantine condition, and the factor ε can be assumed to be obtained after a rescaling of the original coordinates – such rescaling makes sense if one wants to study the behaviour of the system near the origin. Indeed, the corresponding Hamilton equations, written in action-angle variables, are of the form (1.1).

Unfortunately, the action-angle variables are singular near the equilibrium, and hence there are problems in the region where one of the actions is much smaller than the others. Thus, it can be worthwhile to work directly in the original Cartesian coordinates. In fact, there has been a lot of interest for KAM theory in configuration space, that is, without action-angle variables; see for instance [6, 19, 22].

1.1. Set up of the problem. In this paper we consider the ordinary differential equations

$$\ddot x_j + \omega_j^2 x_j + \varepsilon f_j(x_1, \dots, x_d, \varepsilon) + \eta_j x_j = 0, \qquad j = 1, \dots, d, \tag{1.4}$$

where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, ε is real parameter (*perturbation parameter*), the function $f(x, \varepsilon) = (f_1(x, \varepsilon), \dots, f_d(x, \varepsilon))$ is real analytic in x and ε at $(x, \varepsilon) = (0, 0)$ and at least quadratic in x,

$$f_j(x, \varepsilon) = \sum_{p=1}^{\infty} \varepsilon^p \sum_{\substack{s_1, \dots, s_d \geq 0 \\ s_1 + \dots + s_d = p+1}} f_{j, s_1, \dots, s_d}\, x_1^{s_1} \dots x_d^{s_d}, \tag{1.5}$$

(by taking $f_j(x, \varepsilon) = -\partial_{x_j} F(x, \varepsilon)$ one recovers the Hamilton equations corresponding to the Hamiltonian (1.3)), $\eta = (\eta_1, \dots, \eta_d)$ is a vector of parameters, and the *frequency vector* (or *rotation vector*) $\omega = (\omega_1, \dots, \omega_d)$ satisfies the Diophantine condition

$$|\omega \cdot \nu| > \gamma_0 |\nu|^{-\tau} \qquad \forall \nu \in \mathbb{Z}^d_*, \tag{1.6}$$

with $\mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{0\}$, $> d-1$ and $_0 > 0$. Here and henceforth $\cdot$ denotes the standard scalar product in $\mathbb{R}^d$, and $| \; | = | \; _1| + \ldots + | \; _d|$.

In light of Moser's theorem of the modifying terms, one expects that, by taking the (arbitrary) unperturbed solution $x_{0,j}(t) = C_j \cos \; _j t + S_j \sin \; _j t = c_j e^{i \; _j t} + c_j^* e^{-i \; _j t}$, $j = 1, \ldots, d$, there exists a function $(\; , c)$, analytic both in and $c = (c_1, \ldots, c_d)$, such that, by fixing $_j = \; _j(\; , c)$, there exists a quasi-periodic solution to (1.4) with frequency vector , which reduces to the unperturbed one as $\to 0$. In fact, this is what happens: the result is just a rephrasing of Moser's modifying terms theorem, with the advantage that it extends to the regions of phase space where the action-angle variables cannot be defined, and hence is not surprising; see also [6]. What is less obvious is the cancellation mechanism which is behind the convergence of the perturbation series. The problem can be described as follows.

One can try to write again — as in action-angle variables — the solution as a power series in , and study directly the convergence of the series. In general, when considering the Lindstedt series of some KAM problem, first of all one identifies the terms of the series which are an obstruction to convergence: such terms are usually called resonances (or self-energy clusters, by analogy to what happens in quantum field theory). Crudely speaking, the series is given by the sum of infinitely many terms (finitely many for each perturbation order), and each term looks like a product of ‘small divisors’ times some harmless factors: a resonance is a particular structure in the product which allows a dangerous accumulation of small divisors. This phenomenon is very easily visualised when each term of the series is graphically represented as a tree graph (tree *tout court* in the following), that is, a set of points and lines connecting them in such a way that no loop arises; we refer to [10, 13, 15] for an introduction to the tree formalism. Shortly, in any tree, each line carries a label $j \in \{1, \ldots, d\}$ and a label $\in \mathbb{Z}^d$ (that one calls momentum, again inspired by the terminology of quantum field theory) and with each such line a small divisor $_j(\; \cdot \;)$ is associated; here $u \to \; _j(u)$ is a smooth function, which depends on both the model under study and the coordinates one is working with, for instance $_j(u) = u$ for (1.2), while $_j(u) = u^2 - \; _j^2$ for (1.4). Then a resonance becomes a subgraph which is between two lines $_1$ and $_2$ with the same small divisors, i.e. $_{j_1}(\; \cdot \; _1) = \; _{j_2}(\; \cdot \; _2)$. A tree with a chain of resonances represents a term of the series containing a factor $_j(\; \cdot \;)$ to a very large power, and this produces a factorial $k!$ to some positive power when bounding some terms contributing to the k^{th} order in of the Lindstedt series, so preventing a proof of convergence.

However, a careful analysis of the resonances shows that there are cancellations to all perturbation orders. This is what can be proved in the case of the standard anisochronous KAM theorem, as first pointed out by Eliasson [8]; see also [9, 10], for a proof which more deeply exploits the similarity with the techniques of quantum field theory.

More precisely the cancellation mechanism works in the following way. Given a tree and two lines $_1$ and $_2$ of with the same small divisor, consider all possible resonances which can be inserted between $_1$ and $_2$. For each possible resonance one obtains a different tree, which represents a term of the perturbation series, and each term can be written as the product of a numerical value corresponding to the resonance times a numerical value associated to the points and lines of which are outside the resonance: this second numerical value is the same for all such trees, and hence factorises out. When summing together the numerical values corresponding to all resonances, there are compensations and the sum is in fact much smaller than each summand (for more details we refer to [10, 13]).

For the isochronous case, already in action-angle variables [1], there are some kinds of resonances which do not cancel each other. Nevertheless there are other kinds of resonances for which the gain factor due to the cancellation is more than what is needed (that is, one has a second order instead of a Þrst order cancellation). Thus, the hope naturally arises that one can use the extra gain factors to compensate the lack of gain factors for the Þrst kind of resonances, and in fact this happens. Indeed, the resonances for which there is no cancellation cannot accumulate too much without entailing the presence of as many resonances with the extra gain factors, in such a way that the overall number of gain factors is, in average, one per resonance (this is essentially the meaning of Lemma 5.4 in [1]).

When working in Cartesian coordinates, one immediately meets a difÞculty. If one writes down the lowest order resonances, there is no cancellation at all. This is slightly surprising because a cancellation is expected somewhere: if the resonances do not cancel each other, in principle one can construct trees containing chains of arbitrarily many resonances, and these trees represent terms of the formal power series expansion for which a bound proportional to some factorial seems unavoidable. However, we shall show that there are cancellations, as soon as one has at least two resonances. So, one has the curious phenomenon that resonances which do not cancel each other are allowed, but they cannot accumulate too much. Moreover, the cancellation mechanism is more involved than in other cases (including the same problem in action-angle variables). First of all, the resonances are no longer diagonal in the momenta, that is, the lines $_1$ and $_2$ considered above can have different momenta $_1$ and $_2$. Second, the cancellation does not operate simply by collecting together all resonances to a given order and then summing the corresponding numerical values. As we mentioned, in this way no cancellation is produced: to obtain a cancellation one has to consider all possible ways to connect two resonances to each other. Thus, there is a cancellation only if there is a chain of at least two resonances.

What emerges eventually is that working in Cartesian coordinates rather complicates the analysis. On the other hand, as remarked above, it can be worthwhile to investigate the problem in Cartesian coordinates. Moreover, the cancellations are due to remarkable symmetries in the problem, which can be of interest on their own; in this regard we mention the problem of the reducibility of the skew-product ßows with Bryuno base [11], where the convergence of the corresponding Lindstedt series is also due to some cancellation mechanism and hence to some deep symmetry of the system.

In this paper we shall assume the standard Diophantine condition on the frequency vector ; see (1.6) below. Of course one could consider more general Diophantine conditions than the standard one (for instance a Bryuno condition [5]; see also [12] for a discussion using the Lindstedt series expansion). This would make the analysis slightly more complicated, without shedding further light on the problem. An important feature of the Lindstedt series method is that, from a conceptual point of view, the general strategy is exactly the same independent of the kind of coordinates one uses (and independent of the fact that the system is a discrete map or a continuous ßow, see [2,10,15]). What is really important for the analysis is the form of the unperturbed solution: the simpler such a solution is the easier the analysis. Of course, an essential issue is that the system one wants to study is a perturbation of one which is exactly soluble. This is certainly true in the case of quasi-integrable Hamiltonian systems, but of course the range of applicability is much wider, and includes also non-Hamiltonian systems; see for instance [14,16]. Moreover an assumption of this kind is more or less always implicit in whatever method one can envisage to deal with small divisor problems of this kind; see also [6].

In the anisochronous case, the cancellations are due to symmetry properties of the model — essentially the symplectic character of the problem, as first pointed out by Eliasson [8]. The cancellation mechanism for the resonances is deeply related to that assuring the formal solubility of the equations of motions, which in turn is due to a symmetry property as already shown by Poincaré [21]. We refer to [17] for a detailed comparison between Eliasson's method and the tree formalism that we are using here. Note that, despite what is sometimes claimed in the literature, Eliasson did not study how the resonances have to be regrouped in order to exhibit the cancellation; on the contrary, he proved that, because of aforementioned symmetry properties, the sum of (the leading parts of) all possible resonances must cancel out; a proof of the cancellation through a careful regrouping of the resonances was first given by Gallavotti [9]. Subsequently, stressing further the analogy with quantum field theory, Bricmont et al. showed that the cancellations can be interpreted as a consequence of suitable Ward identities of the corresponding field theory [4] (see also [7]): the symmetry property corresponds to the translation invariance of the field theoy. In the isochronous case, in terms of Cartesian coordinates the cancellation mechanism works in a completely different way with respect to action-angle coordinates. However, as we shall see, the cancellation is still related to underlying symmetry properties: it would be interesting to relate the symmetry properties that we find to invariance properties of the corresponding quantum field model, as done in [4] for the KAM theorem.

1.2. Statement of the results. Now, we give a formal statement of our results. As stressed above, the main point of the paper is not in the results themselves, but in the method used to prove them, in particular on the analysis of the perturbation series and of the cancellation mechanism which is at the base of the convergence of the series.

We look for quasi-periodic solutions $x(t)$ of (1.4) with frequency vector . Therefore we expand the function $x(t)$ by writing

$$x(t) = \sum_{\in\mathbb{Z}^d} e^{i \cdot t} x , \tag{1.7}$$

and we denote by $f(x,)$ the th Fourier coefficient of the function that we obtain by Taylor-expanding $f(x,)$ in powers of x and Fourier-expanding x according to (1.7).

Thus, in Fourier space (1.4) becomes

$$\left[(\cdot)^2 - {}_j^2\right] x_{j,} = f_{j,}(x,) + {}_j x_{j,} . \tag{1.8}$$

For $= 0$, $= 0$, the vector $x^{(0)}(t)$ with components

$$x_j^{(0)}(t) = c_j e^{i\, _j t} + c_j^* e^{-i\, _j t}, \qquad j = 1, \dots, d, \tag{1.9}$$

is a solution of (1.4) for any choice of the complex constants $c = (c_1, \dots, c_d)$. Here and henceforth $*$ denotes complex conjugation.

Define e_j as the vector with components $_j$ (Kronecker delta). Then we can split (1.8) into two sets of equations, called respectively the *bifurcation equation* and the *range equation*,

$$f_{j, e_j}(x,) + {}_j x_{j, e_j} = 0, \qquad j = 1, \dots, d, \qquad = \pm 1, \tag{1.10a}$$

$$\left[(\cdot)^2 - {}_j^2\right] x_{j,} = f_{j,}(x,) + {}_j x_{j,} , \qquad j = 1, \dots, d, \qquad \neq \pm e_j . \tag{1.10b}$$

We shall study both Eqs. (1.10) simultaneously, by showing that for all choices of the parameters c there exist suitable counterterms , depending analytically on and c, such that (1.10) admits a quasi-periodic solution with frequency vector , which is analytic in , c, and t. Moreover, with the choice $x_{j,e_j} = c_j$ for all $j = 1, \ldots, d$, the counterterms are uniquely determined.

We formulate the following result.

Theorem 1.1. Consider the system described by Eqs. (1.4) and let (1.9) be a solution at $= 0$, $= 0$. Set $(c) = \max\{|c_1|, \ldots, |c_d|, 1\}$. There exist a positive constant $_0$, small enough and independent of c, and a unique function $(, c)$, holomorphic in the domain $| \; | \; ^3(c) \leq \; _0$ and real for real , such that the system

$$\ddot{x}_j + \; ^2_j x_j + f_j(x_1, \ldots, x_d, \;) + \; _j(\; , c)\, x_j = 0, \qquad j = 1, \ldots, d,$$

admits a solution $x(t) = x(t, \; , c)$ of the form (1.7), holomorphic in the domain $| \; | \; ^3(c)e^{3| \; ||\mathrm{Im}\, t|} \leq \; _0$ and real for real , t, with Fourier coefficients $x_{j,e_j} = c_j$ and $x_{j,} = O(\;)$ if $\neq \pm e_j$ for $j = 1, \ldots, d$.

The proof is organised as follows. After introducing the small divisors and proving some simple preliminary properties in Sect. 2, we develop in Sect. 3 a graphical representation for the power series of the counterterms and the solution (tree expansion). In particular we perform a multiscale analysis which allows us to single out the contributions (self-energy clusters) which give problems when trying to bound the coefficients of the series. In Sect. 4 we show that, as far as such contributions are neglected, there is no difficulty in obtaining power-like estimates on the coefficients: these estimates, which are generalisations of the Siegel-Bryuno bounds holding for anisochronous systems [9, 10], would imply the convergence of the series and hence analyticity. In Sect. 5 we discuss how to deal with the self-energy clusters: in particular we single out the leading part of their contributions (localised values), which are proved in Sect. 6 to satisfy some deep symmetry properties. Finally, in Sect. 7 we show how the symmetry properties can be exploited in order to obtain cancellations involving the localised parts, in such a way that the remaining contributions can still be bounded in a summable way. This will yield the convergence of the full series and hence the analyticity of both the solution and the counterterms.

Note that the system dealt with in Theorem 1.1 can be non-Hamiltonian. On the other hand the most general case for a Hamiltonian system near a stable equilibrium allows for Hamiltonians of the form

$$H(x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{1}{2} \sum_{j=1}^{d} \left(y_j^2 + \; ^2_j x_j^2 \right) + \; F(x_1, \ldots, x_n, y_1, \ldots, y_n, \;), \tag{1.11}$$

which lead to the equations

$$\begin{cases} \dot{x}_j = y_j + \; _{y_i} F(x, y, \;), \\ \dot{y}_j = - \; ^2_j x_j - \; _{x_i} F(x, y, \;). \end{cases} \tag{1.12}$$

Also in this case one can consider the modified equations

$$\begin{cases} \dot{x}_j = y_j + \; _{y_i} F(x, y, \;), \\ \dot{y}_j = - \; ^2_j x_j - \; _{x_i} F(x, y, \;) + \; _j x_j, \end{cases} \tag{1.13}$$

which are not of the form considered in Theorem 1.1. However, a result in the same spirit as Theorem 1.1 still holds.

Theorem 1.2. Consider the system described by Eqs. (1.13) and let $(x^{(0)}(t), y^{(0)}(t))$ be a solution at $= 0$, $= 0$, with $x^{(0)}(t)$ given by (1.9) and $y^{(0)}(t) = \dot{x}^{(0)}(t)$. Set $(\ c) = \max\{|c_1|, \dots, |c_d|, 1\}$. Then there exist a positive constant $_0$, small enough and independent of , c, and a unique function $(\ ,\ c)$, holomorphic in the domain $|\ |\ ^3(c) \le\ _0$ and real for real , such that the system

$$\begin{cases} \dot{x}_j = y_j + \ _{y_j} F(x, y, \ ), \\ \dot{y}_j = -\ _j^2 x_j -\ _{x_j} F(x, y, \ ) +\ _j(\ ,\ c)\, x_j \end{cases}$$

admits a solution $(x(t,\ ,\ c), y(t,\ ,\ c))$, holomorphic in the domain $|\ |\ ^3(c) e^{3|\ ||\mathrm{Im}\, t|} \le\ _0$ and real for real , t, with Fourier coefÞcients $x_{j,e_j} = y_{j,e_j}/i\ _j = c_j$ and $x_{j,} = y_{j,} = O(\ )$ if $\ne \pm e_j$ for $j = 1, \dots, d$.

The proof follows the same lines as that of Theorem 1.1, and it is discussed in Appendices A and B. Finally in Appendix C we brießy sketch an alternative approach based on the resummation of the perturbation series.

2. Preliminary Results

We shall denote by $\mathbb{N}$ the set of (strictly) positive integers, and set $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For any $j = 1, \dots, d$ and $\in \mathbb{Z}^d$ deÞne the small divisors

$$_j(\ \cdot\ ) := \min\{\left|\ \cdot\ -\ _j\right|, \left|\ \cdot\ +\ _j\right|\} = |\ \cdot(\ -(\ , j)\, e_j)|, \qquad (2.1)$$

where $(\ , j)$ is the minimizer. Note that the Diophantine condition (1.6) implies that

$$_j(\ \cdot\ ) \ge\ |\ |^{-} \quad \forall j = 1, \dots, d, \quad \forall \ne 0,\ (\ , j)\, e_j, \qquad (2.2a)$$

$$_j(\ \cdot\ ) +\ _{j'}(\ \cdot\ ') \ge\ |\ -\ '|^{-} \quad \forall j, j' = 1, \dots, d,$$

$$\forall \ne\ ',\ -\ ' \ne (\ , j)\, e_j - (\ ', j')\, e_{j'}, \qquad (2.2b)$$

for a suitable positive > 0. We can (and shall) assume that is sufÞciently smaller than $_0$, and hence than $(0) = \min\{|\ _1|, \dots, |\ _d|\}$ and $:= \min\{||\ _i| - |\ _j|| : 1 \le i < j \le d\}$.

Lemma 2.1. Given $,\ ' \in \mathbb{Z}^d$, with $\ne\ '$, and $_j(\ \cdot\ ) =\ _{j'}(\ \cdot\ ')$ for some $j, j' \in \{1, \dots, d\}$, then either $|\ -\ '| \ge |\ | + |\ '| - 2$ or $|\ -\ '| = 2$.

Proof. One has $_j(\ \cdot\ ) = |\ \cdot\ -\ _j|$ and $_{j'}(\ \cdot\ ') = |\ \cdot\ ' -\ '\ _{j'}|$, with $= (\ , j)$ and $' = (\ ', j')$. Set $^- = \ -\ e_j$ and $^{-\prime} = \ ' -\ ' e_{j'}$. By the Diophantine condition (1.6) one can have $_j(\ \cdot\ ) =\ _{j'}(\ \cdot\ ')$, and hence $|\ \cdot\ ^-| = |\ \cdot\ ^{-\prime}|$, if and only if $^- = \pm\ ^{-\prime}$.

If $^- = -\ ^{-\prime}$ then for $= -\ '$ one has $|\ -\ '| = |\ | + |\ '|$, while for $=\ '$ one obtains $|\ -\ '| \ge |\ | + |\ '| - 2$. If $^- =\ ^{-\prime}$ and $j = j'$ one has $_i =\ _i'$ for all $i \ne j$ and $_j -\ =\ _j' -\ '$, and hence $|\ _j -\ _j'| = 2$. If $^- =\ ^{-\prime}$ and $j \ne j'$ then $_i =\ _i'$ for all $i \ne j, j'$, while $_j -\ =\ _j'$ and $_{j'} =\ _{j'}' -\ '$, and hence $|\ _j -\ _j'| = |\ _{j'} -\ _{j'}'| = 1$. □

Lemma 2.2. Let $,\ ' \in \mathbb{Z}^d$ be such that $\neq\ '$ and, for some $n \in \mathbb{Z}_+$, $j, j' \in \{1,\dots,d\}$, both $_j(\ \cdot\ ) \le 2^{-n}$ and $_{j'}(\ \cdot\ ') \le 2^{-n}$ hold. Then either $|\ -\ '| > 2^{(n-2)/}$ or $|\ -\ '| = 2$ and $_j(\ \cdot\ ) = \ _{j'}(\ \cdot\ ')$.

Proof. Write $_j(\ \cdot\ ) = |\ \cdot\ -\ _j|$ and $_{j'}(\ \cdot\ ') = |\ \cdot\ ' -\ '_{j'}|$, with $= (\ , j)$ and $' = (\ ', j')$, and set $\bar{} = \ - e_j$ and $\bar{}' = \ ' -\ ' e_{j'}$ as above.

If $\bar{} \neq \bar{}'$, by the Diophantine condition (2.2b), one has

$$|\bar{} - \bar{}'|^{-} < |\ \cdot(\bar{} - \bar{}')| \le |\ \cdot \bar{}| + |\ \cdot \bar{}'| < 2^{-(n-1)},$$

which implies $|\bar{} - \bar{}'| > 2^{(n-1)/}$, and hence we have $|\ -\ '| > 2^{(n-2)/}$ in such a case.

If $\bar{} = \bar{}'$ then, as in Lemma 2.1, one has $|\ -\ '| = 2$ and $_j(\ \cdot\ ) = \ _{j'}(\ \cdot\ ')$. □

Remark 2.3 Note that $|\ -\ '| \le 2$ and $_j(\ \cdot\ ) = \ _{j'}(\ \cdot\ ')$ if and only if $\ -\ ' = (\ , j)e_j - (\ ', j')e_{j'}$.

Lemma 2.4. Let $_1, \dots, \ _p \in \mathbb{Z}^d$ and $j_1, \dots, j_p \in \{1,\dots,d\}$, with $p \ge 2$, be such that $|\ _i -\ _{i-1}| \le 2$ and $_{j_i}(\ \cdot\ _i) = \ _{j_1}(\ \cdot\ _1) \le$ for $i = 2,\dots,p$. Then $|\ _1 -\ _p| \le 2$.

Proof. Set $_i = (\ _i, j_i)$ and $\bar{}_i = \ _i -\ _i e_{j_i}$ for $i = 1,\dots,p$. For all $i = 2,\dots,p$, the assumption $_{j_i}(\ \cdot\ _i) = \ _{j_{i-1}}(\ \cdot\ _{i-1})$ implies $\bar{}_i = \pm\bar{}_{i-1}$, which in turn yields $\bar{}_i = \bar{}_{i-1}$, since $|\ _i -\ _{i-1}| \le 2$. In particular $\bar{}_1 = \bar{}_p$, and hence $|\ _1 -\ _p| \le 2$. □

3. Multiscale Analysis and Diagrammatic Rules

As we are looking for $x(t, \ , c)$ and $(\ , c)$ analytic in , we formally write

$$x_{j,} = \sum_{k=0}^{\infty} {}^k x_{j,}^{(k)}, \qquad _j = \sum_{k=1}^{\infty} {}^k\ _j^{(k)}. \tag{3.1}$$

It is not difficult to see that using (3.1) in (1.10) one can recursively compute (at least formally) the coefficients $x_{j,}^{(k)}$, $_j^{(k)}$ to all orders. Here we introduce a graphical representation for each contribution to $x_{j,}^{(k)}$, $_j^{(k)}$, which will allow us to study the convergence of the series.

3.1. Trees.

A graph is a set of points and lines connecting them. A *tree* is a graph with no cycle, such that all the lines are oriented toward a unique point (*root*) which has only one incident line (*root line*). All the points in a tree except the root are called *nodes*. The orientation of the lines in a tree induces a partial ordering relation ($\preceq$) between the nodes and the lines: we can imagine that each line carries an arrow pointing toward the root; see Fig. 1. Given two nodes v and w, we shall write $w \prec v$ every time v is along the path (of lines) which connects w to the root.

We call $E(\ )$ the set of *end nodes* in , that is, the nodes which have no entering line, and $V(\ )$ the set of *internal nodes* in , that is, the set of nodes which have at least one entering line. Set $N(\ ) = E(\ ) \amalg V(\ )$. For all $v \in N(\ )$ denote by s_v the number of lines entering the node v.

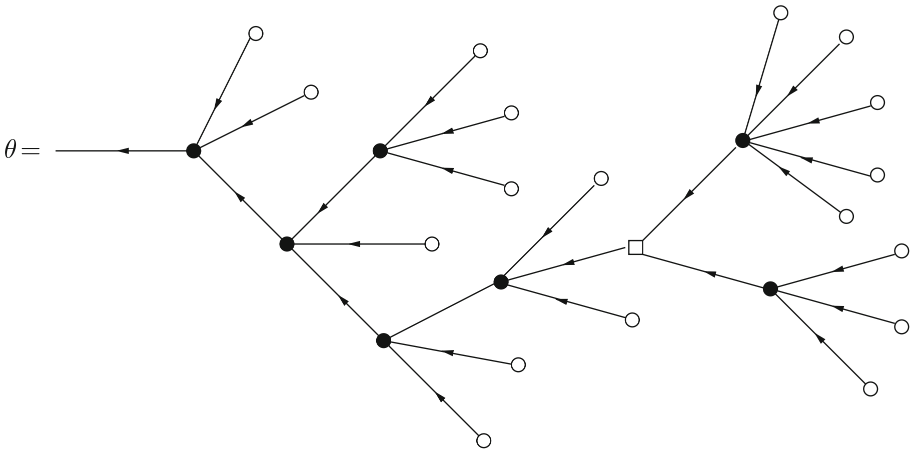

Fig. 1. An unlabelled tree: the arrows on the lines all point toward the root, according to the tree partial ordering

Remark 3.1. One has $\sum_{v\in V(\ )} s_v = |N(\ )| - 1$.

We denote by $L(\ )$ the set of lines in . We call an *internal line* a line exiting an internal node and an *end line* a line exiting an end node. Since a line $\in L(\ )$ is uniquely identiÞed with the node v which it leaves, we may write $= \ _v$. We write $\ _w \prec \ _v$ if $w \prec v$; we say that a node w precedes a line, and write $w \prec$, if $\ _w \preceq$.

Notation 3.2.

(1) If and $'$ are two comparable lines, i.e. $' \prec$, we denote by $\mathcal{P}(\ ,\ ')$ the (unique) path of lines connecting $'$ to , the lines and $'$ being excluded.
(2) Each internal line $\in L(\ )$ can be seen as the root line of the tree whose nodes and lines are those of which precede , that is, $N(\ ) = \{v' \in N(\ ) : v' \prec \ \}$ and $L(\ ) = \{\ ' \in L(\ ) : \ ' \preceq \ \}$.

3.2. Tree labels. With each end node $v \in E(\ )$ we associate a *mode* label $\ _v \in \mathbb{Z}^d$, a *component* label $j_v \in \{1,\ldots,d\}$, and a *sign* label $\ _v \in \{\pm\}$; see Fig. 2. We call $E_j(\ )$ the set of end nodes $v \in E(\ )$ such that $j_v = j$ and $\ _v =$.

With each internal node $v \in V(\ )$ we associate a *component* label $j_v \in \{1,\ldots,d\}$, and an *order* label $k_v \in \mathbb{Z}_+$. Set $V_0(\ ) = \{v \in V(\ ) : k_v = 0\}$ and $N_0(\ ) = E(\ ) \amalg V_0(\ )$. We also associate a sign label $\ _v \in \{\pm\}$ with each $v \in V_0(\ )$. The internal nodes v with $k_v \geq 1$ will be drawn as black bullets, while the end nodes and the internal nodes with $k_v = 0$ will be drawn as white bullets and white squares, respectively; see Fig. 2.

With each line we associate a *momentum* label $\ \in \mathbb{Z}^d$, a *component* label $j\ \in \{1,\ldots,d\}$, a *sign* label $\ \in \{\pm\}$, and *scale* label $n\ \in \mathbb{Z}_+ \cup \{-1\}$; see Fig. 3.

Denote by $s_{v,j}$ the number of lines with component label $j\ = j$ entering the node v, and with $r_{v,j,}$ the number of end lines with component label j and sign label which enter the node v. Of course $s_v = s_{v,1} + \cdots + s_{v,d}$ and $s_{v,j} \geq r_{v,j,+} + r_{v,j,-}$ for all $j = 1,\ldots,d$.

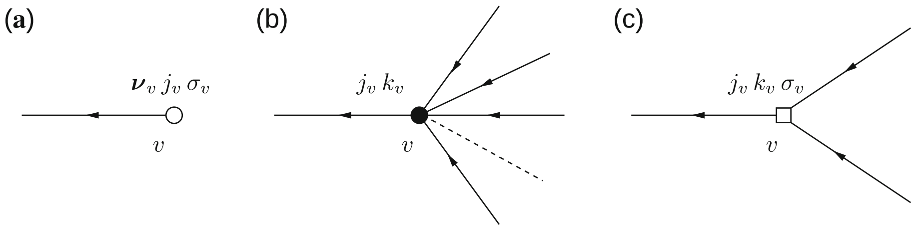

Fig. 2. Nodes and labels associated with the nodes: (a) end node v with $s_v = 0$, $j_v \in \{1,\ldots,d\}$, $_v \in \{\pm\}$, and $_v = \ _v e_{j_v}$ (cf. Sect. 3.3); (b) internal node v with $s_v \geq 2$, $j_v \in \{1,\ldots,d\}$, and $k_v = s_v - 1$ (cf. Sect. 3.3); (c) internal node v with $s_v = 2$, $j_v \in \{1,\ldots,d\}$ $k_v = 0$, $_v \in \{\pm\}$ (cf. Sect. 3.3)

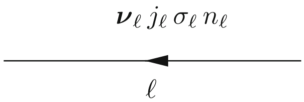

Fig. 3. Labels associated with a line. One has $= (\ ,j\ )$ (cf. Sect. 3.3) Moreover if $= \ _v$ then $j = j_v$; if $v \in V_0(\ )$ one has also $= \ _v$; if $= \ e_j$ then $n = -1$, otherwise $n \geq 0$ (cf. Sect. 3.3)

Finally call

$$k(\ ) := \sum_{v \in V(\ )} k_v$$

the order of the tree .

In the following we shall call trees tout court the trees with labels, and we shall use the term unlabelled trees for the trees without labels.

3.3. Constraints on the tree labels.

Constraint 3.3. We have the following constraints on the labels of the nodes (see Fig. 2).

(1) if $v \in V(\ )$ one has $s_v \geq 2$;
(2) if $v \in E(\ )$ one has $_v = \ _v e_{j_v}$;
(3) if $v \in V(\ )$ then $k_v = s_v - 1$, except for $s_v = 2$, where both $k_v = 1$ and $k_v = 0$ are allowed.

Constraint 3.4. The following constraints will be imposed on the labels of the lines:

(1) $j = j_v$, $= \ _v$, and $= \ _v$ if exits $v \in E(\ )$;
(2) $j = j_v$ if exits $v \in V(\ )$;
(3) if is an internal line then $= (\ ,j\ )$, i.e., $_j(\ \cdot\ ) = |\ \cdot\ - \ _j|$ (see (2.1) for notations);
(4) if $v \in V_0(\ )$ then (see Fig. 4)
   1. $s_v = 2$;
   2. both lines $_1$ and $_2$ entering v are internal and have $_1 = \ _2 = \ _v$ and $j_1 = j_2 = j_v$;
   3. either $_1 = \ _v e_{j_v}$ and $_2 \neq \ _v e_{j_v}$ or $_1 \neq \ _v e_{j_v}$ and $_2 = \ _v e_{j_v}$;
   4. $_v = \ _v$;
(5) if is an internal line and $= \ e_j$, then enters a node $v \in V_0(\ )$;
(6) $n \geq 0$ if $\neq \ e_j$ and $n = -1$ otherwise.

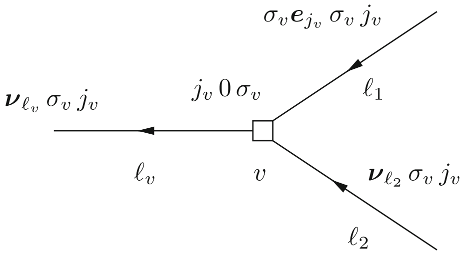

Fig. 4. If there is an internal node v with $k_v = 0$ then $s_v = 2$ and the following constraints are imposed on the other labels: $_v = {}_1 = {}_2 = {}_v$; $j_v = j_1 = j_2 = j_v$; either $_1 = {}_v e_{j_v}$ and $_2 \neq {}_v e_{j_v}$ (as in the Þgure) or $_2 = {}_v e_{j_v}$ and $_1 \neq {}_v e_{j_v}$. (The scale labels are not shown)

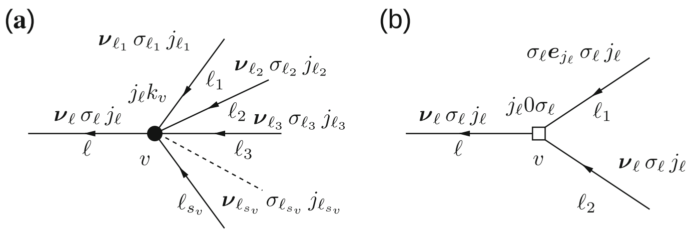

Fig. 5. Conservation law (a) v with $k_v = s_v - 1 \geq 1$, so that $= {}_1 + \ldots + {}_{s_v}$, (b) v with $s_v = 2$ and $k_v = 0$. (The scale labels are not shown)

Notation 3.5. Given a tree , call $_0$ its root line and consider the internal lines $_1, \ldots, {}_p \in L(\ )$ on scale -1 (if any) such that one has ≥ 0 for all $\in \mathcal{P}(\ _0, \ _i)$, $i = 1, \ldots, p$; we shall say that $_1, \ldots, {}_p$ are the lines on scale -1 which are closest to the root of . For each such line $_i$, call $_i = {}_i$. Then we call pruned tree $\breve{}$ the subgraph with set of nodes and set of lines

$$N(\breve{\ }) = N(\ ) \setminus \bigcup_{i=1}^{p} N(\ _i), \qquad L(\breve{\ }) = L(\ ) \setminus \bigcup_{i=1}^{p} L(\ _i),$$

respectively.

By construction, $\breve{}$ is a tree, except that, with respect to the constraints listed above, one has $s_v = 1$ whenever $k_v = 0$; moreover one has $\neq \ e_j$ (and hence $n \geq 0$) for all internal lines $\in L(\breve{\ })$ except possibly the root line.

Constraint 3.6. The modes of the end nodes and the momenta of the lines are related as follows: if $= {}_v$ one has the conservation law

$$= \sum_{\substack{w \in E(\ ) \\ w \preceq v}} {}_w - \sum_{\substack{w \in V_0(\ ) \\ w \preceq v}} {}_w e_{j_w} = \sum_{\substack{w \in E(\breve{\ }) \\ w \preceq v}} {}_w .$$

Note that by Constraint 3.6 one has $= {}_v$ if $v \in E(\ )$, and $= {}_1 + \cdots + {}_{s_v}$ if $v \in V(\ )$, $k_v \geq 1$, and $_1, \ldots, {}_{s_v}$ are the lines entering v; see Fig. 5. Moreover for any line $\in L(\ )$ one has $|\ | \leq |E(\breve{\ })|$.

Remark 3.7 In the following we shall repeatedly consider the operation of changing the sign label of the nodes. Of course this change produces the change of other labels, consistently with the constraints mentioned above: for instance, if we change the label $_v$ of an end node v into $-\ _v$, then also $_v$ is changed into $-\ _v$; if we change the sign labels of all the end nodes, then also the momenta of all the lines are changed, according to the conservation law (Constraint 3.6); and so on.

Two unlabelled trees are called **equivalent** if they can be transformed into each other by continuously deforming the lines in such a way that they do not cross each other. We shall call equivalent two trees if the same happens in such a way that all labels match.

Notation 3.8. We denote by $\mathfrak{T}^k_{j,}$ the set of inequivalent trees of order k with tree component j and tree momentum, that is, such that the component label and the momentum of the root line are j and , respectively. Finally for $n \geq -1$ deÞne $\mathfrak{T}^k_{j,}(n)$ the set of trees $\in \mathfrak{T}^k_{j,}$ such that $n \leq n$ for all $\in L(\ )$.

Remark 3.9 For $\in \mathfrak{T}^k_{j,}$, by writing $= (\ _1, \ldots,\ _d)$, one has $_i = |E^+_i(\breve{\ })| - |E^-_i(\breve{\ })|$ for $i = 1, \ldots, d$. In particular for $=\ e_j$, one has $|E_j(\breve{\ })| = |E^-_j(\breve{\ })| + 1 \geq 1$, and $|E_{j'}(\breve{\ })| = |E^-_{j'}(\breve{\ })|$ for all $j' \neq j$.

Lemma 3.10. The number of unlabelled trees with N nodes is bounded by 4^N. If $k(\ ) = k$ then $|E(\ )| \leq E_0 k$ and $|V(\ )| \leq V_0 k$, for suitable positive constants E_0 and V_0.

Proof. The bound $|V(\ )| \leq |E(\ )| - 1$ is easily proved by induction using that $s_v \geq 2$ for all $v \in V(\ )$. So it is enough to bound $|E(\ )|$. The definition of order and Remark 3.1 yield $|E(\ )| = 1 + k(\ ) + |V_0(\ )|$, and the bound $|V_0(\ )| \leq 2k(\ ) - 1$ immediately follows by induction on the order of the tree, simply using that $s_v \geq 2$ for $v \in V(\ )$. Thus, the assertions are proved with $E_0 = V_0 = 3$. □

3.4. Tree expansion. Now we shall see how to associate with each tree $\in \mathfrak{T}^k_{j,}$ a contribution to the coefÞcients $^{(k)}_{j,}$ and $^{(k)}_j$ of the power series in (3.1).

For all $j = 1, \ldots, d$ set $c^+_j = c_j$ and $c^-_j = c^*_j$. We associate with each end node $v \in E(\ )$ a node factor

$$F_v := c^{\ _v}_{j_v}, \tag{3.2}$$

and with each internal node $v \in V(\ )$ a node factor

$$F_v := \begin{cases} \dfrac{s_{v,1}! \ldots s_{v,d}!}{s_v!}\, f_{j_v, s_{v,1},\ldots,s_{v,d}}, & k_v \geq 1, \\[2ex] -\dfrac{1}{2c^{\ _v}_{j_v}}, & k_v = 0, \end{cases} \tag{3.3}$$

where the coefÞcients $f_{j,s_1,\ldots,s_d}$ are deÞned in (1.5).

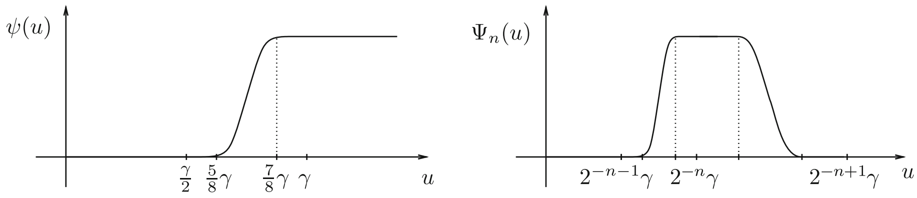

Fig. 6. The functions ψ and Ψ_n

Let ψ be a non-decreasing C^∞ function defined in $\mathbb{R}_+$, such that (see Fig. 6)

$$\psi(u) = \begin{cases} 1, & \text{for } u \geq 7\gamma/8, \\ 0, & \text{for } u \leq 5\gamma/8, \end{cases} \tag{3.4}$$

and set $\chi(u) := 1 - \psi(u)$. For all $n \in \mathbb{Z}_+$ define $\chi_n(u) := \chi(2^n u)$ and $\psi_n(u) := \psi(2^n u)$, and set (see Fig. 6)

$$\Psi_n(u) = \chi_{n-1}(u)\,\psi_n(u), \tag{3.5}$$

where $\chi_{-1}(u) = 1$. Note that $\chi_{n-1}(u) - \chi_n(u) = \Psi_n(u)$, and hence $\{\Psi_n(u)\}_{n\in\mathbb{Z}_+}$ is a partition of unity.

We associate with each line ℓ a propagator $g_\ell := g_j^{[n_\ell]}(\omega\cdot\nu_\ell)$, where

$$g_j^{[n]}(u) := \begin{cases} \dfrac{\Psi_n(\delta_j(u))}{u^2 - \lambda_j^2}, & n \geq 0, \\ 1, & n = -1. \end{cases} \tag{3.6}$$

Remark 3.11. The number of scale labels which can be associated with a line ℓ in such a way that $g_\ell \neq 0$ is at most 2. In particular, given a line ℓ with momentum $\nu_\ell = \nu$ and scale $n_\ell = n$, such that $\Psi_n(\delta_j(\omega\cdot\nu)) \neq 0$, then (see Fig. 6)

$$2^{-(n+1)}\gamma \leq \frac{5}{8}2^{-n}\gamma \leq \delta_j(\omega\cdot\nu) \leq \frac{7}{8}2^{-(n-1)}\gamma \leq 2^{-(n-1)}\gamma, \tag{3.7}$$

and if $\Psi_n(\delta_j(\omega\cdot\nu))\,\Psi_{n+1}(\delta_j(\omega\cdot\nu)) \neq 0$, then

$$\frac{5}{8}2^{-n}\gamma \leq \delta_j(\omega\cdot\nu) \leq \frac{7}{8}2^{-n}\gamma. \tag{3.8}$$

We define

$$\mathscr{V}(\theta) := \left(\prod_{\ell\in L(\theta)} g_\ell\right)\left(\prod_{v\in N(\theta)} F_v\right), \tag{3.9}$$

and call $\mathscr{V}(\theta)$ the *value of the tree* θ.

Remark 3.12. The number of trees $\theta \in \mathfrak{T}^k_{j,\nu}$ with $\mathscr{V}(\theta) \neq 0$ is bounded proportionally to C^k, for some positive constant C. This immediately follows from Lemma 3.10 and the observation that the number of trees obtained from a given unlabelled tree by assigning the labels to the nodes and the lines is also bounded by a constant to the power k (use Remark 3.11 to bound the number of allowed scale labels).

Remark 3.13. In any tree there is at least one end node with node factor c_j for each internal node v with $k_v = 0$, $_v =$ and $j_v = j$ (this is easily proved by induction on the order of the pruned tree): the node factors $-1/2c_j$ do not introduce any singularity at $c_j = 0$. Therefore for any tree the corresponding value $\mathscr{V}( )$ is well deÞned because both propagators and node factors are Þnite quantities. Remark 3.12 implies that also

$$\sum_{\in \mathfrak{T}^k_{j,}} \mathscr{V}( )$$

is well deÞned for all $k \in \mathbb{N}$, all $j \in \{1, \dots, d\}$, and all $\in \mathbb{Z}^d$.

Lemma 3.14. For all $k \in \mathbb{N}$, all $j = 1, \dots, d$, and any $\in \mathfrak{T}^k_{j, e_j}$, there exists $' \in \mathfrak{T}^k_{j,- e_j}$ such that $c_j^{-} \mathscr{V}( ) = c_j \mathscr{V}(')$. The tree $'$ is obtained from by changing the sign labels of all the nodes $v \in N_0( )$.

Proof. The proof is by induction on the order of the tree. For any tree $\in \mathfrak{T}^k_{j,e_j}$ consider the tree $' \in \mathfrak{T}^k_{j,-e_j}$ obtained from by replacing all the labels $_v$ of all nodes $v \in N_0( )$ with $-_v$, so that the mode labels $_v$ are replaced with $-_v$ and the momenta with $-$ (see Remark 3.7). Call $_1, \dots, _p$ the lines on scale -1 (if any) closest to the root of, and for $i = 1, \dots, p$ denote by v_i the node $_i$ enters and $_i =$ $_i$ (recall (2) in Notation 3.2). As an effect of the change of the sign labels, each tree $_i$ is replaced with a tree $_i'$ such that $c^{-}_{j_{v_i}} \mathscr{V}(_i) = c_{j_{v_i}} \mathscr{V}(_i')$, by the inductive hypothesis. Thus, for each node v_i the quantity $F_{v_i} \mathscr{V}(_i)$ is not changed. Moreover, neither the propagators of the lines $\in L(\breve{})$ nor the node factors corresponding to the internal nodes $v \in V(\breve{})$ with $k_v \neq 0$ change, while the node factors $c_{j_v}^{v}$ of the nodes $v \in E(\breve{})$ are changed into $c_{j_v}^{-v}$. On the other hand one has $|E_i^+(\breve{})| = |E_i^-(\breve{})|$ for all $i \neq j$, whereas $|E_j^+(\breve{})| = |E_j^-(\breve{})| + 1$ and $|E_j^+(\breve{}')| + 1 = |E_j^-(\breve{}')|$. Therefore one obtains $c_j^{-} \mathscr{V}( ) = c_j \mathscr{V}(')$, and the assertion follows. □

For $k \in \mathbb{N}$, $j \in \{1, \dots, d\}$, and $\in \{\pm\}$, deÞne

$$^{(k)}_{j,} = -\frac{1}{c_j} \sum_{\in \mathfrak{T}^k_{j, e_j}} \mathscr{V}( ).$$

Lemma 3.15. For all $k \in \mathbb{N}$ and all $j = 1, \dots, d$ one has $^{(k)}_{j,+} = ^{(k)}_{j,-}$.

Proof. Lemma 3.14 implies

$$c_j^{-} \sum_{\in \mathfrak{T}^k_{j,e_j}} \mathscr{V}( ) = c_j^{+} \sum_{\in \mathfrak{T}^k_{j,-e_j}} \mathscr{V}( )$$

for all $k \in \mathbb{N}$ and all $j = 1, \dots, d$, so that the assertion follows from the definition of $^{(k)}_{j,}$. □

Lemma 3.16. Equations (1.10) formally hold, i.e., they hold to all perturbation orders, provided that for all $k \in \mathbb{N}$ and $j = 1, \dots, d$ we set formally

$$x_{j,} = \sum_{k=1}^{\infty} {}^{k} x_{j,}^{(k)}, \qquad x_{j,}^{(k)} = \sum_{\in \mathfrak{T}_{j,}^{k}} \mathscr{V}(\ ) \quad \forall \ \in \mathbb{Z}^{d} \setminus \{\pm e_j\}, \qquad x_{j,\pm e_j}^{(k)} = 0, \tag{3.10}$$

$$_{j} = \sum_{k=1}^{\infty} {}^{k}\ {}_{j}^{(k)}, \qquad {}_{j}^{(k)} = -\frac{1}{c_j} \sum_{\in \mathfrak{T}_{j,e_j}^{k}} \mathscr{V}(\ ). \tag{3.11}$$

Proof. The proof is a direct check. □

Remark 3.17 In ${}_{j}^{(k)}$, deÞned as (3.11), there is no singularity in $c_j = 0$ because $\mathscr{V}(\breve{\ })$ contains at least one factor $c_j^{\dagger} = c_j$ by Remark 3.9.

In the light of Lemma 3.16 one can wonder why the definition of the propagators for $\neq e_j$ is so involved; as a matter of fact one could deÞne

$$G = \frac{1}{(\ \cdot\ )^2 - {}_{j}^{2}}.$$

However, since $\sum_{n\geq 0}\ {}_{n}(u) \equiv 1$, the two definitions are equivalent. We use the definition (3.6) so that we can immediately identify the factors $O(2^n)$ which could prevent the convergence of the power series (3.1). In what follows we shall make this idea more precise.

3.5. Clusters. A cluster T on scale n is a maximal set of nodes and lines connecting them such that all the lines have scales $n' \leq n$ and there is at least one line with scale n; see Fig. 7. The lines entering the cluster T and the line coming out from it (unique if existing at all) are called the *external* lines of the cluster T. We call $V(T)$, $E(T)$, and $L(T)$ the set of internal nodes, of end nodes, and of lines of T, respectively; note that the external lines of T do not belong to $L(T)$. DeÞne also $E_j(T)$ as the set of end nodes $v \in E(T)$ such that ${}_{v} = \ $ and $j_v = j$. By setting

$$k(T) := \sum_{v \in V(T)} k_v,$$

we say that the cluster T has *order* k if $k(T) = k$.

3.6. Self-energy clusters. We call *self-energy cluster* any cluster T such that (see Fig. 8)

(1) T has only one entering line and one exiting line,

(2) one has $n \leq \min\{n_{\ T}, n_{\ T'}\} - 2$ for any $\ \in L(T)$,

(3) one has $|\ _{T} - \ _{T}'| \leq 2$ and $j_{\ T}(\ \cdot\ _{T}) = j_{\ T'}(\ \cdot\ _{T}')$.

Notation 3.18. For any self-energy cluster T we denote by $_{T}$ and $'_{T}$ the exiting and the entering line of T respectively. We call $\mathcal{P}_T$ the path of lines $\in L(T)$ connecting $'_{T}$ to $_{T}$, i.e., $\mathcal{P}_T = \mathcal{P}(\ _{T},\ '_{T})$ (recall (1) in Notation 3.2), and set $n_T = \min\{n_{\ T}, n_{\ '_{T}}\}$.

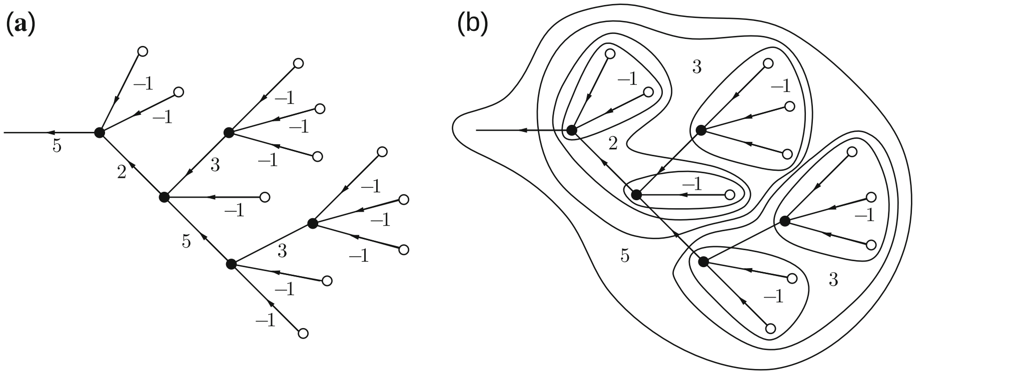

Fig. 7. Example of tree and the corresponding clusters: once the scale labels have been assigned to the lines of the tree as in (a), one obtains the cluster structure depicted in (b).

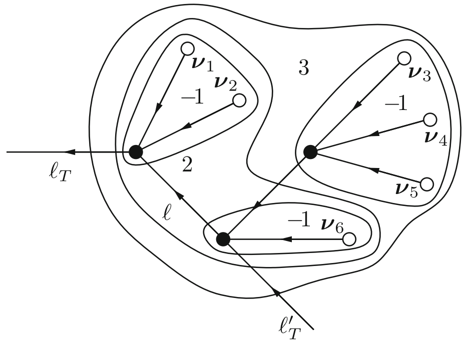

Fig. 8. Example of self-energy cluster: consider the cluster T on scale 3 in Fig. 7, and suppose that the mode labels of the end nodes are such that $|\ _1+\ _2+\ _3+\ _4+\ _5+\ _6| \le 2$ and $_{j_T}(\ \cdot\ _T) = \ _{j'_T}(\ \cdot\ '_T)$. Then T is a self-energy cluster with external lines $'_T$ (entering line) and $_T$ (exiting line). The path $\mathcal{P}_T$ is such that $\mathcal{P}_T = \{\ \}$.

Remark 3.19. Notice that, by Remark 2.3, for any self-energy cluster the label $_T$ is uniquely Þxed by the labels j_T, $_T$, j'_T, $'_T$, $'_T$. In particular, for Þxed and j such that $_j(\ \cdot\ ) \le$, there are only $2d-1$ momenta $' \neq$ such that $|\ ' - \ | \le 2$ and $_{j'}(\ \cdot\ ') = \ _j(\ \cdot\ )$ for some j' and $'$, depending on $'$. All the other $''$ with small divisor equal to $_j(\ \cdot\ )$ are far away from , according to Lemma 2.1.

We say that a line is a *resonant line* if it is both the exiting line of a self-energy cluster and the entering line of another self-energy cluster, that is, is resonant if there exist two self-energy clusters T_1 and T_2 such that $= \ '_{T_1} = \ _{T_2}$; see Fig. 9.

Remark 3.20. The notion of self-energy cluster was Þrst introduced by Eliasson, in the context of the KAM theorem, in [8], where it was called resonance. We prefer the term self-energy cluster to stress further the analogy with quantum Þeld theory.

The notion of equivalence given for trees can be extended in the obvious way to self-energy clusters.

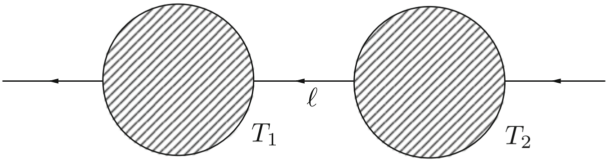

Fig. 9. Example of resonant line: is resonant if both T_1 and T_2 are self-energy clusters

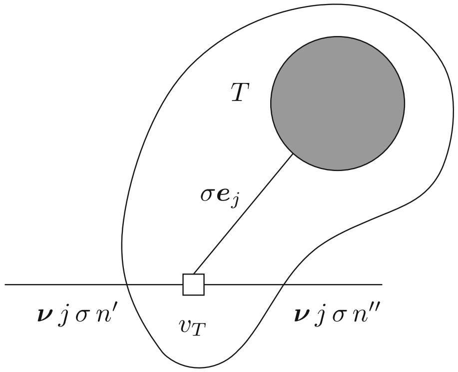

Fig. 10. A self-energy cluster in $\mathfrak{C}^k_{j, , j,}(\cdot , n)$; T contains at least one line on scale $\geq n$ and n such that $\min\{n', n''\} \geq n + 2$

Notation 3.21. We denote by $\mathfrak{R}^k_{j, , j', '}(\cdot ', n)$ the set of inequivalent self-energy clusters T on scale $\leq n$ of order k, such that $'_T = '$, $j_T = j$, $_T =$, $j'_T = j'$ and $'_T = '$. By definition of cluster for $T \in \mathfrak{R}^k_{j, , j', '}(\cdot ', n)$ one must have $n \leq n_T - 2$. For $j = j'$ and $= '$ deÞne also $\mathfrak{C}^k_{j, , j,}(\cdot ', n)$ the set of self-energy clusters $T \in \mathfrak{R}^k_{j, , j,}(\cdot ', n)$ such that (1) $'_T$ enters the same node v which $_T$ exits and (2) $k_v = 0$. We call v_T such a special node and set $\overline{\mathfrak{R}}^k_{j, , j,}(\cdot ', n) = \mathfrak{R}^k_{j, , j,}(\cdot ', n) \setminus \mathfrak{C}^k_{j, , j,}(\cdot ', n)$; see Fig. 10.

Notation 3.22. For any $T \in \mathfrak{C}^k_{j, , j,}(\cdot ', n)$ we call $_T$ the tree which has as root line the line $\in L(T)$ entering v_T (one can imagine to obtain $_T$ from T by ÒremovingÓ the node v_T); see Fig. 11. Note that $_T \in \mathfrak{T}^k_{j, e_j}(n)$.

Notation 3.23. Consider a self-energy cluster T such that $n \neq -1$ for all lines $\in \mathcal{P}_T$. If $T \in \mathfrak{C}^k_{j, , j,}(\cdot ', n)$ for some k, j, , $'$, n then we deÞne the pruned self-energy cluster $\breve{T}$ as the subgraph with $N(\breve{T}) = \{v_T\} \cup N(\breve{}_T)$ and $L(\breve{T}) = L(\breve{}_T)$. For all other self-energy clusters T, call $_1, \ldots, _p \in L(T)$ the internal lines on scale -1 (if any) which are closest to the exiting line of T, that is, such that $n \geq 0$ for all lines $\in \mathcal{P}(_T, _i)$, $i = 1, \ldots, p$. For each line $_i$ set $_i = _i$. Then the pruned self-energy cluster $\breve{T}$ is the subgraph with set of nodes and set of lines

$$N(\breve{T}) = N(T) \setminus \bigcup_{i=1}^{p} N(_i), \qquad L(\breve{T}) = L(T) \setminus \bigcup_{i=1}^{p} L(_i),$$

respectively.

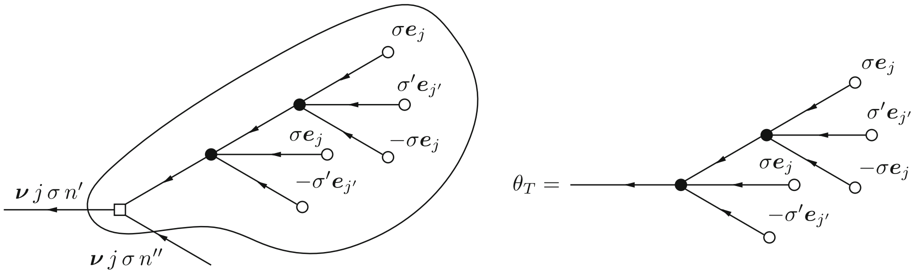

Fig. 11. An example of self-energy cluster $T \in \mathfrak{E}^k_{j,\ ,\ j,}$ (· , n) and the corresponding tree $_T$. (Only the mode labels of the end nodes are shown in T and $_T$.)

Remark 3.24. For $T \in \mathfrak{R}^k_{j,\ ,\ j',\ '}(\ \cdot\ ', n)$ such that $n \geq 0$ for all $\in \mathcal{P}_T$, one has $|E^+_i(\breve{T})| = |E^-_i(\breve{T})|$ for all $i \neq j, j'$. If $j \neq j'$ then $|E^-_{j'}\,'(\breve{T})| = |E_{j'}\,'(\breve{T})| + 1$ and $|E_j(\breve{T})| = |E^-_j(\breve{T})| + 1$; if $j = j'$, $= '$ and $T \in \overline{\mathfrak{R}}^k_{j,\ ,\ j,}(\ \cdot\ ', n)$ then $|E_j(\breve{T})| = |E^-_j(\breve{T})|$, while if $j = j'$ and $= -'$ then $|E_j(\breve{T})| = |E^-_j(\breve{T})| + 2$. Finally, for any $T \in \mathfrak{E}^k_{j,\ ,\ j,}(\ \cdot\ ', n)$ one has $|E_j(\breve{T})| = |E^-_j(\breve{T})| + 1 \geq 1$.

We shall deÞne

$$\mathscr{V}(T, \ \cdot\ '_T) := \Bigg(\prod_{\in L(T)} G \Bigg) \Bigg(\prod_{v \in N(T)} F_v \Bigg), \tag{3.12}$$

where $\mathscr{V}(T, \ \cdot\ '_T)$ will be called the value of the self-energy cluster. T

The value $\mathscr{V}(T, \ \cdot\ '_T)$ depends on $\cdot\ '_T$ through the propagators of the lines $\in \mathcal{P}_T$.

Remark 3.25. The value of a self-energy cluster $T \in \mathfrak{E}^k_{j,\ ,\ j,}(u, n)$ does not depend on u so that we shall write

$$\mathscr{V}(T, u) = \mathscr{V}(T) = -\frac{1}{2c_j}\mathscr{V}(\ _T).$$

We deÞne also for future convenience

$$M^{(k)}_{j,\ ,\ j',\ '}(\ \cdot\ ', n) := \sum_{T \in \mathfrak{R}^k_{j,\ ,\ j',\ '}(\ \cdot\ ', n)} \mathscr{V}(T, \ \cdot\ '). \tag{3.13}$$

Note that $M^{(k)}_{j,\ ,\ j,}(\ \cdot\ ', n) = \widetilde{M}^{(k)}_{j,\ ,\ j,}(n) + \overline{M}^{(k)}_{j,\ ,\ j,}(\ \cdot\ ', n)$, where $\widetilde{M}^{(k)}_{j,\ ,\ j,}(n)$ and $\overline{M}^{(k)}_{j,\ ,\ j,}(\ \cdot\ ', n)$ are deÞned as in (3.13) but for the sum restricted to the set $\mathfrak{E}^k_{j,\ ,\ j,}(\ \cdot\ ', n)$ and $\overline{\mathfrak{R}}^k_{j,\ ,\ j,}(\ \cdot\ ', n)$ respectively.

Remark 3.26. Both the quantities $M^{(k)}_{j,\ ,\ j',\ '}(\ \cdot\ ', n)$ and the coefÞcients $x^{(k)}_{j,}$ and $^{(k)}_j$ are well deÞned to all orders because the number of terms which one sums over is Þnite (by the same argument in Remark 3.12). At least formally, we can deÞne

$$M_{j,\ ,\ j',\ '}(\ \cdot\ ') = \sum_{k=1}^{\infty} {}^k \sum_{n \geq -1} M^{(k)}_{j,\ ,\ j',\ '}(\ \cdot\ ', n).$$

We deÞne the depth $D(T)$ of a self-energy cluster T recursively as follows: we set $D(T) = 1$ if there is no self-energy cluster containing T, and set $D(T) = D(T') + 1$ if T is contained inside a self-energy cluster T' and no other self-energy clusters inside T' (if any) contain T. We denote by $\mathfrak{S}_D( )$ the set of self-energy clusters of depth D in , and by $\mathfrak{S}_D(, T)$ the set of self-energy clusters of depth D in contained inside T.

Notation 3.27. Call $\hat{} = \setminus \mathfrak{S}_1( )$ the subgraph of formed by the set of nodes and lines of which are outside the set $\mathfrak{S}_1( )$ (the external lines of the self-energy clusters $T \in \mathfrak{S}_1( )$ being included in $\hat{}$), and, analogously, for $T \in \mathfrak{S}_D( )$ call $\hat{T} = T \setminus \mathfrak{S}_{D+1}(, T)$ the subgraph of T formed by the set of nodes and lines of T which are outside the set $\mathfrak{S}_{D+1}(, T)$. We denote by $V(\hat{T})$, $E(\hat{T})$, and $L(\hat{T})$ the set of internal nodes, of end nodes, and of lines of $\hat{T}$, and by $k(\hat{T})$ the order of $\hat{T}$, that is, the sum of the labels k_v of all the internal nodes $v \in V(\hat{T})$.

Lemma 3.28. Given a line $\in L( )$, if T is the self-energy cluster with largest depth containing (if any), $\in \mathcal{P}_T$ and there is no line $' \in \mathcal{P}_T$ preceding with $n_{'} = -1$, one can write $= {}^0 + {}_T^{\prime}$. Then one has $| {}^0| \le E_1 k(\hat{T})$, for a suitable positive constant E_1, if $k(\hat{T}) \ge 1$, and $| {}^0| \le 2$ if $k(\hat{T}) = 0$.

Proof. We Þrst prove that for any tree , if we denote by $_0$ its root line, one has

$$| _0| \le \begin{cases} E_1 k(\hat{}) - 2, & \text{if } _0 \text{ does not exit a self-energy cluster,} \\ E_1 k(\hat{}), & \text{if } _0 \text{ exits a self-energy cluster,} \end{cases} \tag{3.14}$$

for a suitable constant $E_1 \ge 4$. The proof is by induction on the order of the tree. If $k( ) = 1$ (and hence $\hat{} = $) then the only internal line of is $_0$ and $| _0| \le 2$, so that the assertion trivially holds provided $E_1 \ge 4$. If $k( ) > 1$ let v_0 be the node which $_0$ exits. If v_0 is not contained inside a self-energy cluster let $_1, \dots, _m$, $m \ge 0$, be the internal lines entering v_0 and $_i = $ $_i$ for all $i = 1, \dots, m$. Finally let $_{m+1}, \dots, _{m+m'}$ be the end-lines entering v_0. By definition we have $k(\hat{}) = k_{v_0} + k(\hat{}_1) + \dots + k(\hat{}_m)$. If $k_{v_0} > 0$, we have $_0 = _1 + \dots + _{m+m'}$. This implies in turn

$$\begin{aligned} | _0| &\le | _1| + \dots + | _m| + m' \le E_1 \Big(k(\hat{}_1) + \dots + k(\hat{}_m) \Big) + m' \\ &\le E_1 (k(\hat{}) - m - m' + 1) + m'. \end{aligned}$$

The assertion follows for $E_1 \ge 4$ by the inductive hypothesis (the worst possible case is $m = 0, m' = 2$).

If $k_{v_0} = 0$ then $s_v = 2$ and $m' = 0$. Moreover one of the lines, say $_1$, is on scale $n = -1$ while for the other line one has $_0 = _2$. Once more the bound follows from the inductive hypothesis since $| _2| \le E_1 k(\hat{}_2) \le E_1 (k(\hat{}) - 1)$.

Finally, if v_0 is contained inside a self-energy cluster, then $_0$ exits a self-energy cluster T_1. There will be p self-energy clusters $T_1, \dots, T_p$, $p \ge 1$, such that the exiting line of T_i is the entering line of T_{i-1}, for $i = 2, \dots, p$, while the entering line $'$ of T_p does not exit any self-energy cluster. By Lemma 2.4, one has $| _0 - _{'}| \le 2$ and $k(\hat{}) = k(\hat{}_{'})$. Then, by the inductive hypothesis, one Þnds $| _0| \le 2 + E_1 k(\hat{}_{'}) - 2 = E_1 k(\hat{})$.

Now for and T as in the statement we prove, by induction on the order of the self-energy cluster, the bound

$$| {}^0| \le \begin{cases} E_1 k(\hat{T}) - 2, & \text{if } k(\hat{T}) \ge 1, \\ 2 & \text{if } k(\hat{T}) = 0, \end{cases} \tag{3.15}$$

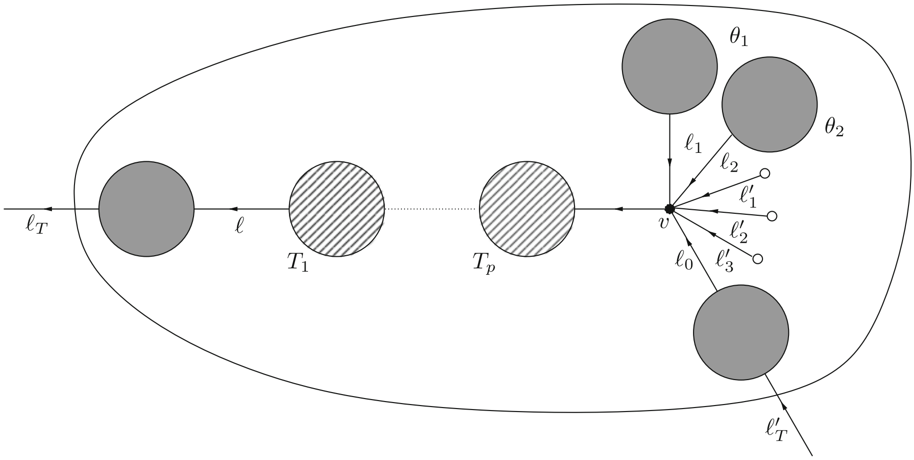

Fig. 12. The self-energy cluster T considered in the proof of Lemma 3.28, with $m = 2$, $m' = 3$, and a chain of p self-energy clusters between and $_v$ (one has $p \geq 0$, and $=$ $_v$ if $p = 0$)

where $T̂$ is the set of nodes and lines of $T̂$ which precede . The bound is trivially satisfied when $k(T̂) = 0$. Otherwise let v be the node in $V(T̂)$ between and $'_T$ which is closest to . If $k_v = 0$ the bound follows trivially by using the bound (3.14). If $k_v \geq 1$, call $_1, \ldots, _m$, $m \geq 0$, the internal lines entering v which are not along the path $\mathcal{P}_T$, and $_{m+1}, \ldots, _{m+m'}$ the end lines entering v; one has $m + m' \geq 1$. There is a further line $_0 \in \mathcal{P}_T$ entering v such that $_0 = {}^0_0 + {}'_T$; see Fig. 12. Using also Lemma 2.4 one has $|{}^0| \leq 2 + |{}^0_0| + |{}_1| + \cdots + |{}_m| + m'$. As $n_0 \leq n_{'_T} - 2$ one has $k(T̂_0) \geq 1$ and hence, by (3.14) and the inductive hypothesis, one has

$$| {}^0| \leq 2 + \Big(E_1 k(T̂_0) - 2\Big) + E_1 \Big(k(\hat{}_1) + \cdots + k(\hat{}_m)\Big) + m',$$

where $_i = {}_i$ for all $i = 1, \ldots, m$. Thus, since $k(T̂_0) + k(\hat{}_1) + \cdots + k(\hat{}_m) + (m + m') = k(T̂)$ and $m + m' \geq 1$, one finds

$$| {}^0| \leq E_1 \Big(k(T̂) - m - m'\Big) + m' \leq E_2 k(T̂) - 2,$$

provided $E_1 \geq 4$. Therefore, the assertion follows with, say, $E_1 = 4$. □

Notation 3.29. Given a tree and a line $\in L(\;)$, call $= (\;)$ the subgraph formed by the set of nodes and lines which do not precede ; see Fig. 13. Let us call $\hat{}$ the set of nodes and lines of which are outside any self-energy cluster contained inside .

Lemma 3.30. Given a tree let $_0$ and be the root line and an arbitrary internal line preceding $_0$. If $k(\hat{}) \geq 1$ one has $|{}_0 - {}| \leq E_2 k(\hat{})$, for a suitable positive constant E_2.

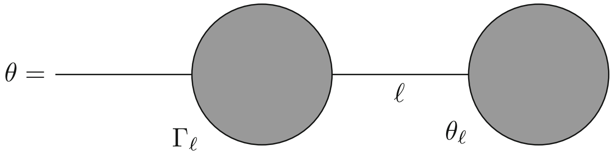

Fig. 13. The set $=$ ( ) and the subtree determined by the line $\in L($ $)$. If is the root line then $= \emptyset$

Proof. We prove by induction on the order of the bound

$$| \ _0 - \ | \leq \begin{cases} E_2 k(\hat{u}) - 2, & \text{if } \ _0 \text{ does not exit a self-energy cluster,} \\ E_2 k(\hat{u}), & \text{if } \ _0 \text{ exits a self-energy cluster.} \end{cases} \quad (3.16)$$

We mimic the proof of (3.14) in Lemma 3.28. The case $k(\hat{u}) = 1$ is trivial provided $E_2 \geq 3$, so let us consider $k(\hat{u}) > 1$ and call v_0 the node which $_0$ exits. If v_0 is not contained inside a self-energy cluster and $k_{v_0} \geq 1$ then $_0 = \ _1 + \cdots + \ _{m+m'}$, where $_1, \ldots, \ _m$ are the internal lines entering v_0, with (say) $_m \in \mathcal{P}(\ _0, \ ) \cup \{ \ \}$, and $_{m+1}, \ldots, \ _{m+m'}$ are the end lines entering v_0. Hence $k(\hat{u}) = k_{v_0} + k(\hat{u}_1) + \cdots + k(\hat{u}_{m-1}) + k(\hat{u}_m)$, where $_i = \ _i$ and $_m = \ (\ _m)(\ _m = \emptyset$ if $_m = \ )$. Thus, the assertion follows by (3.14) and the inductive hypothesis. If v_0 is not contained inside a self-energy cluster and $k_{v_0} = 0$ then two lines $_1$ and $_2$ enter v_0, and one of them, say $_1$, is such that $| \ _1| = 1$. If $= \ _2$ the result is trivial. If $_2 \in \mathcal{P}(\ _0, \ )$ the bound follows once more from the inductive hypothesis. If $= \ _1$ one has

$$| \ _0 - \ | \leq | \ _0| + 1 \leq E_1 k(\hat{u}_2) + 1 \leq E_2 k(\hat{u}) - 2,$$

where $_2 = \ _2$, provided $E_2 \geq E_1 + 3$, if E_1 is the constant deÞned in Lemma 3.28. If $_1 \in \mathcal{P}(\ _0, \ )$ denote by $'_1$ the line on scale -1 along the path $\{ \ _1\} \cup \mathcal{P}(\ _1, \ )$ which is closest to . Again call $_2 = \ _2$ and J_1 the subgraph formed by the set of nodes and lines preceding $'_1$ (with $'_1$ included) but not ; deÞne also $_1$ as the tree obtained from J_1 by (1) reverting the arrows of all lines along $\{ \ _1, \ \} \cup \mathcal{P}(\ '_1, \ )$, (2) replacing $'_1$ with an end line carrying the same sign and component labels as $_1$, and (3) replacing all the labels $_v$, $v \in N_0(J_1)$ with $- \ _v$. One has, by using also (3.14),

$$| \ _0 - \ | \leq | \ _0| + | \ | \leq E_1 k(\hat{u}_1) + E_1 k(\hat{u}_2) \leq E_2 k(\hat{u}) - 2,$$

provided $E_2 \geq E_1 + 2$ so that the bound follows once more. Finally, if v_0 is contained inside a self-energy cluster, then $_0$ exits a self-energy cluster T_1. There will be p self-energy clusters $T_1, \ldots, T_p$, $p \geq 1$, such that the exiting line of T_i is the entering line of T_{i-1}, for $i = 2, \ldots, p$, while the entering line $'$ of T_p does not exit any self-energy cluster. By Lemma 2.4, one has $| \ _0 - \ '| \leq 2$ and $k(\hat{u}) = k(\hat{u}')$, where $' = \ (\ ')$. Then, the inductive hypothesis yields $| \ _0 - \ | \leq 2 + | \ ' - \ | \leq 2 + E_2 k(\hat{u}') - 2 = E_2 k(\hat{u})$. Therefore the assertion follows with, say, $E_2 = E_1 + 3$ (and hence $E_2 = 7$ if $E_1 = 4$). □

Remark 3.31 Lemma 3.28 will be used in Sect. 5 to control the change of the momenta as an effect of the regularisation procedure (to be deÞned). Furthermore, both Lemmas 3.28 and 3.30 will be used in Sect. 7 to show that the resonant lines which are not regularised cannot accumulate too much.

4. Dimensional Bounds

In this section we discuss how to prove that the series (3.10) and (3.11) converge if the resonant lines are excluded. We shall see in the following sections how to take into account the presence of the resonant lines.

Call $\mathfrak{N}_n(\ )$ the number of non-resonant lines $\in L(\ )$ such that $n \geq n$, and $\mathfrak{N}_n(T)$ the number of non-resonant lines $\in L(T)$ such that $n \geq n$.

The analyticity assumption on yields that one has

$$|F_v| \leq \ ^{s_v+k_v} \qquad \forall v \in V(\ ) \setminus V_0(\ ), \tag{4.1}$$

for a suitable positive constant.

Lemma 4.1. Assume that $2^{-(n+2)} \leq \ _j(\ \cdot\ ) \leq 2^{-(n-2)}$ for all trees and all lines $\in L(\ )$. Then there exists a positive constant c such that for any tree one has $\mathfrak{N}_n(\ ) \leq c\,2^{-n/}\ k(\ )$.

Proof. We prove that $\mathfrak{N}_n(\ ) \leq \max\{0, c\,2^{-n/}\ k(\ ) - 2\}$ by induction on the order of.

1. First of all note that for a tree to have a line on scale $n \geq n$ one needs $k(\ ) \geq k_n = E_0^{-1} 2^{(n-2)/}$, as it follows from the Diophantine condition (2.2a) and Lemma 3.10. Hence the bound is trivially true for $k < k_n$.
2. For $k(\ ) \geq k_n$, let $_0$ be the root line of and set $= \ _0$ and $j = j_0$. If $n_0 < n$ the assertion follows from the inductive hypothesis. If $n_0 \geq n$, call $_1, \ldots, \ _m$ the lines with scale $\geq n-1$ which are closest to $_0$ (that is, such that $n \leq n-2$ for all $p = 1, \ldots, m$ and all lines $\in \mathcal{P}(\ _0, \ _p)$). The case $m = 0$ is trivial. If $m \geq 2$ the bound follows once more from the inductive hypothesis.
3. If $m = 1$, then $_1$ is the only entering line of a cluster T. Set $' = \ _1$, $j' = j_1$ and $n' = n_1$. By hypothesis one has $_j(\ \cdot\ ) \leq 2^{-(n-2)}$ and $_{j'}(\ \cdot\ ') \leq 2^{-(n-3)}$, so that, by Lemma 2.2, either $|\ -\ '| > 2^{(n-5)/}$ or $|\ -\ '| \leq 2$ and $_j(\ \cdot\ ) = \ _{j'}(\ \cdot\ ')$. In the first case, since
$$-\ ' = \sum_{w \in E(T)} \ _w - \sum_{\substack{w \in V(T)\\ k_w = 0}} \ _w e_{j_w} = \sum_{w \in E(\check{T})} \ _w,$$
the same argument used to prove Lemma 3.10 yields $|\ -\ '| \leq |E(T)| \leq E_0 k(T)$, and hence $k(T) \geq E_0^{-1} 2^{(n-5)/}$. Thus, if $_1 = \ _1$, one has $k(\ ) = k(T) + k(\ _1)$, so that
$$\begin{aligned}\mathfrak{N}_n(\ ) &= 1 + \mathfrak{N}_n(\ _1) \leq c\,2^{-n/}\ k(\ _1) - 1 \leq c\,2^{-n/}\ k(\ ) - c\,2^{-n/}\ k(T) - 1\\ &\leq c\,2^{-n/}\ k(\ ) - 2,\end{aligned}$$
provided $c \geq E_0 2^{5/}$.
4. If instead $|\ -\ '| \leq 2$ and $_j(\ \cdot\ ) = \ _{j'}(\ \cdot\ ')$, then the only way for T not to be a self-energy cluster is that $n_1 = n_0 - 1 = n - 1$ and there is at least a line $\in T$ with $n = n-2$. But then $_j(\ \cdot\ ) \neq \ _j(\ \cdot\ )$ so that $|\ -\ | > 2^{(n-6)/}$ and we can reason as in the previous case provided $c \geq E_0 2^{6/}$. Otherwise T is a self-energy cluster and $_1$ can be either resonant or not-resonant. Call $'_1, \ldots, \ '_{m'}$ the lines with scale $\geq n-1$ which are closest to $_1$. Once more the cases $m' = 0$ and $m' \geq 2$ are trivial.

5. If $m' = 1$, then $\ _1'$ is the only entering line of a cluster T'. If $\ _1' = \ _1'$, then $\mathfrak{N}_n(\ ) = 1 + \mathfrak{N}_n(\ _1')$ if $\ _1$ is resonant and $\mathfrak{N}_n(\ ) \le 2 + \mathfrak{N}_n(\ _1')$ if $\ _1$ is non-resonant. Consider Þrst the case of $\ _1$ being non-resonant. Set $\ '' = \ _1'$, $j'' = j_{\ _1'}$ and $n'' = n_{\ _1'}$. By reasoning as before we find that one has either $|\ ' - \ ''| > 2^{(n-5)/}$ or $|\ ' - \ ''| \le 2$ and $_{j'}(\ \cdot\ ') = \ _{j''}(\ \cdot\ '')$. If $|\ ' - \ ''| > 2^{(n-5)/}$ then $k(T') \ge E_0^{-1} 2^{(n-5)/}$; thus, by using that $k(\ ) = k(T) + k(T') + k(\ _1')$, we obtain

$$\mathfrak{N}_n(\ ) \le 2 + \mathfrak{N}_n(\ _1') \le c\,2^{-n/}\ k(\ ) - c\,2^{-n/}\ k(T) - c\,2^{-n/}\ k(T')$$
$$\le c\,2^{-n/}\ k(\ ) - c\,2^{-n/}\ k(T') \le c\,2^{-n/}\ k(\ ) - 2,$$

provided $c \ge 2E_0 2^{5/}$.

6. Otherwise one has $|\ - \ '| \le 2$, $|\ ' - \ ''| \le 2$, and $_j(\ \cdot\ ) = \ _{j'}(\ \cdot\ ') = \ _{j''}(\ \cdot\ '')$. Since we are assuming $\ _1$ to be non-resonant then T' is not a self-energy cluster. But then there is at least a line $\ \in T$ with $n_{\ '} = n - 2$ and we can reason as in item 4.

7. So we are left with the case in which $\ _1$ is resonant and hence T' is a self-energy cluster. Let $\ _1'$ be the entering line of T'. Once more $\ _1'$ is either resonant or non-resonant. If it is non-resonant we repeat the same argument as done before for $\ _1$. If it is resonant, we iterate the construction, and so on. Therefore we proceed until either we Þnd a non-resonant line on scale $\ge n$, for which we can reason as before, or we reach a tree $\ '$ of order so small that it cannot contain any line on scale $\ge n$ (i.e., $k(\ ') < k_n$).

8. Therefore the assertion follows with, say, $c = 2E_0 2^{6/}$. □

Remark 4.2. One can wonder why in Lemma 4.1 did we assume $2^{-(n\ +2)} \le \ _j(\ \cdot\ ) \le 2^{-(n\ -2)}$ when Remark 3.11 assures the stronger condition $2^{-(n\ +1)} \le \ _j(\ \cdot\ ) \le 2^{-(n\ -1)}$. The reason is that later on we shall need to slightly change the momenta of the lines, in such a way that the scales in general no longer satisfy the condition (3.7) noted in Remark 3.11. However the condition assumed for proving Lemma 4.1 will still be satisÞed.

For any tree $\ $ we call $L_R(\ )$ and $L_{NR}(\ )$ the sets of resonant lines and of non-resonant lines respectively, in $L(\ )$. Then we can write

$$\mathscr{V}(\ ) = \left(\prod_{\ \in L_R(\ )} G\ \right)\mathscr{V}_{NR}(\ ),\qquad \mathscr{V}_{NR}(\ ) := \left(\prod_{\ \in L_{NR}(\ )} G\ \right)\left(\prod_{v\in N(\ )} F_v\right), \quad (4.2)$$

where each propagator $G\ $ can be bounded as $C_0 2^{n\ }$, for some constant C_0.

Lemma 4.3. For all trees $\ $ with $k(\ ) = k$ one has $|\mathscr{V}_{NR}(\ )| \le C^k\ ^{3k}(c)$, where $\ (c) := \max\{|c_1|, \ldots, |c_d|, 1\}$ and C is a suitable positive constant.

Proof. One has

$$|\mathscr{V}_{NR}(\ )| \le C_0^k\ ^{3k}(c)\ ^k \left(\prod_{\ \in L_{NR}(\ )} 2^{n\ }\right) \le C_0^k\ ^{3k}(c)\ ^k \prod_{n=0}^{\infty} 2^{n\mathfrak{N}_n(\ )}$$
$$\le C_0^k\ ^{3k}(c)\ ^k \exp\left(c\log 2\, k \sum_{n=1}^{\infty} 2^{-n/}\ n\right).$$

The last sum converges: this is enough to prove the lemma. □

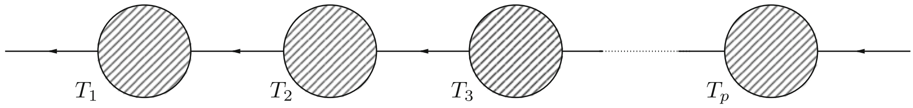

Fig. 14. A chain of self-energy clusters

So far the only bound that we have on the propagators of the resonant lines is $|G| \leq 1/|\ \ _j\ \ _j(\ \cdot\ )| \leq C_0 2^{n}$. What we need is to obtain a gain factor proportional to 2^{-n} for each resonant line with $n \geq 1$.

Lemma 4.4. Given such that $\mathscr{V}(\ ) \neq 0$, let $\in L(\ )$ be a resonant line and let T be the self-energy cluster of largest depth containing (if any). Then there is at least one non-resonant line in T on scale $\geq n - 1$.

Proof. Set $n = n$. There are in general $p \geq 2$ self-energy clusters $T_1, \ldots, T_p$, contained inside T, connected by resonant lines $_1, \ldots, _{p-1}$, and is one of such lines, while the entering line $_p$ of T_p and the exiting line $_0$ of T_1 are non-resonant. Moreover $(\ \cdot\ _i) = (\ \cdot\ )$ for all $i = 0, \ldots, p$, so that all the lines $_0, \ldots, _p$ have scales either $n, n-1$ or $n, n+1$, by Remark 3.11. In any case the lines $_0, _p$ must be in T by definition of the self-energy cluster. □

5. Renormalisation

Now we shall see how to deal with the resonant lines. In principle, one can have trees containing chains of arbitrarily many self-energy clusters (see Fig. 14), and this produces an accumulation of small divisors, and hence a bound proportional to $k!$ to some positive power for the corresponding values.

Let K_0 be such that $E_1 K_0 = 2^{-8/}$. For $T \in \mathfrak{R}^k_{j,\ ,\ j',\ '}(u,n)$, define the *localisation operator* $\mathscr{L}$ by setting

$$\mathscr{L}\mathscr{V}(T,u) := \begin{cases} \mathscr{V}(T,\ \ '\ \ _{j'}), & k(\hat{T}) \leq K_0 2^{n_T/},\ n \geq 0\ \forall \in \mathcal{P}_T, \\ 0, & \text{otherwise}, \end{cases} \tag{5.1}$$

which will be called the *localised value* of the self-energy cluster T. Define also $\mathscr{R} := \mathbb{1} - \mathscr{L}$, by setting, for $T \in \mathfrak{R}^k_{j,\ ,\ j',\ '}(u,n)$,

$$\mathscr{R}\mathscr{V}(T,u) = \begin{cases} \left(u - \ '\ _{j'}\right)\int_0^1 dt\ \partial_u \mathscr{V}(T,\ '\ _{j'} + t(u - \ '\ _{j'})), & k(\hat{T}) \leq K_0 2^{n_T/},\ n \geq 0\ \forall \in \mathcal{P}_T, \\ \mathscr{V}(T,u), & \text{otherwise}, \end{cases} \tag{5.2}$$

so that

$$\mathscr{L}M^{(k)}_{j,\ ,\ j',\ '}(u,n) = \sum_{T \in \mathfrak{R}^k_{j,\ ,\ j',\ '}(u,n)} \mathcal{L}\mathscr{V}(T,u), \tag{5.3a}$$

$$\mathscr{R}M^{(k)}_{j,\ ,\ j',\ '}(u,n) = \sum_{T \in \mathfrak{R}^k_{j,\ ,\ j',\ '}(u,n)} \mathscr{R}\mathscr{V}(T,u). \tag{5.3b}$$

We shall call $\mathscr{R}$ the *regularisation* operator and $\mathscr{R}\mathscr{V}(T,u)$ the *regularised value* of T.

Remark 5.1 If $T \in \mathfrak{C}^k_{j,\ ,\ j,}(u, n)$ the localisation operator acts as

$$\mathscr{L}\mathscr{V}(T) = \begin{cases} \mathscr{V}(T), & k(\hat{T}) \le K_0 2^{n_T/} \ , \\ 0, & k(\hat{T}) > K_0 2^{n_T/} \ . \end{cases}$$

Remark 5.2 If in a self-energy cluster T there is a line $\in \mathcal{P}_T$ such that $= e_j$ (and hence $n = -1$) then $\mathscr{L}\mathscr{V}(T', u) = 0$ for all self-energy clusters containing T such that $\in \mathcal{P}_{T'}$.

Recall the definition of the sets $\mathfrak{S}_D(\ )$ and $\mathfrak{S}_D(\ , T)$ after Remark 3.26. For any tree we can write its value as

$$\mathscr{V}(\ ) = \Bigg(\prod_{T \in \mathfrak{S}_1(\ )} \mathscr{V}(T, \ \cdot \ _T')\Bigg)\Bigg(\prod_{\in L(\ \setminus \mathfrak{S}_1(\ ))} G\Bigg)\Bigg(\prod_{v \in N(\ \setminus \mathfrak{S}_1(\ ))} F_v\Bigg), \quad (5.4)$$

and, recursively, for any self-energy cluster T of depth D we have

$$\mathscr{V}(T, \ \cdot \ _T') = \Bigg(\prod_{T' \in \mathfrak{S}_{D+1}(\ , T)} \mathscr{V}(T', \ \cdot \ _{T'}')\Bigg)\Bigg(\prod_{\in L(T \setminus \mathfrak{S}_{D+1}(\ , T))} G\Bigg) \times \Bigg(\prod_{v \in N(T \setminus \mathfrak{S}_{D+1}(\ , T))} F_v\Bigg). \quad (5.5)$$

Then we modify the diagrammatic rules given in Sect. 3 by assigning a further label $\mathscr{O}_T \in \{\mathscr{R}, \mathscr{L}\}$, which will be called the operator label, to each self-energy cluster T. Then, by writing $\mathscr{V}(\ )$ according to (5.4) and (5.5), one replaces $\mathscr{V}(T, \ \cdot \ _T')$ with $\mathscr{L}\mathscr{V}(T, \ \cdot \ _T')$ if $\mathscr{O}_T = \mathscr{L}$ and with $\mathscr{R}\mathscr{V}(T, \ \cdot \ _T')$ if $\mathscr{O}_T = \mathscr{R}$. When considering the regularised value of a self-energy cluster $T \in \mathfrak{R}^k_{j,\ ,\ j',\ ,}(u, n)$ with $k(\hat{T}) \le K_0 2^{n_T/}$ and $n \ge 0$ for all $\in \mathcal{P}_T$, then we have also an interpolation parameter t to consider: we shall denote it by t_T to keep trace of the self-energy cluster which it is associated with. We set $t_T = 1$ for a regularised self-energy cluster T with either $k(\hat{T}) > K_0 2^{n_T/}$ or $\mathcal{P}_T$ containing at least one line with $n = -1$.

We call renormalised trees the trees carrying the further labels $\mathscr{O}_T$, associated with the self-energy clusters T of . As an effect of the localisation and regularisation operators the arguments of the propagators of some lines are changed.

Remark 5.3 For any self-energy cluster T the localised value $\mathscr{L}\mathscr{V}(T, u)$ does not depend on the operator labels of the self-energy clusters containing T.

Given a self-energy cluster $T \in \mathfrak{R}^k_{j,\ ,\ j',\ ,}(u, n)$ such that no line along $\mathcal{P}_T$ is on scale -1, let be a line such that (1) $\in \mathcal{P}_T$, and (2) T is the self-energy cluster with largest depth containing . If one has $\mathscr{O}_T = \mathscr{R}$, then the quantity $\cdot$ is changed according to the operator labels of all the self-energy clusters T' such that (1) T' contains T, (2) no line along $\mathcal{P}_{T'}$ has scale -1, and (3) $\in \mathcal{P}_{T'}$. Call $T_p \subset T_{p-1} \subset \cdots \subset T_1$ such self-energy clusters, with $T_p = T$. If $\mathscr{O}_{T_i} = \mathscr{R}$ for all $i = 1,\dots,p$, then $\cdot$ is replaced with

$$\cdot \quad (\underline{t}\;) = \quad \cdot\; {}^0 + \quad {}_p \quad j_p + t_p \Big(\quad \cdot\; {}^0_p + \quad {}_{p-1} \quad j_{p-1} - \quad {}_p \quad j_p \Big)$$
$$+ \sum_{i=2}^{p-1} t_p \dots t_i \Big(\quad \cdot\; {}^0_i + \quad {}_{i-1} \quad j_{i-1} - \quad {}_i \quad j_i \Big)$$
$$+ t_p \dots t_1 \big(\quad \cdot \quad {}_1 - \quad {}_1 \quad j_1 \big)\,, \tag{5.6}$$

where we have set $\underline{t} = (t_1,\dots,t_p)$, ${}'_{T_i} = {}_i$ and $t_{T_i} = t_i$ for simplicity.

Otherwise let T_q be the self-energy cluster of highest depth, among $T_1,\dots,T_{p-1}$, with $\mathscr{O}_{T_q} = \mathscr{L}$ (so that $\mathscr{O}_{T_i} = \mathscr{R}$ for $i \ge q+1$). In that case, instead of (5.6), one has

$$\cdot \quad (\underline{t}\;) = \quad \cdot\; {}^0 + \quad {}_p \quad j_p + t_p \Big(\quad \cdot\; {}^0_p + \quad {}_{p-1} \quad j_{p-1} - \quad {}_p \quad j_p \Big)$$
$$+ \sum_{i=q+1}^{p-1} t_p \dots t_i \Big(\quad \cdot\; {}^0_i + \quad {}_{i-1} \quad j_{i-1} - \quad {}_i \quad j_i \Big)\,, \tag{5.7}$$

with the same notations used in (5.6).

If $\mathscr{O}_{T_p} = \mathscr{L}$, since $\cdot$ is replaced with $\cdot\; {}^0 + \quad {}'_T \quad j_{{}'_T}$ for $\in \mathcal{P}_T$, we can write $\cdot\; {}^0 + \quad {}'_T \quad j_{{}'_T}$ as in (5.6) by setting $t_p = 0$. More generally, if we set $t_T = 0$ whenever $\mathscr{O}_T = \mathscr{L}$, we see that we can always claim that, under the action of the localisation and regularisation operators, the momentum of any line $\in \mathcal{P}_T$ is changed to $(\underline{t}\;)$, in such a way that $\cdot \quad (\underline{t}\;)$ is given by (5.6).

Lemma 5.4. Given such that $\mathscr{V}(\;) \ne 0$, for all $\in L(\;)$ one has $4 \quad {}_j(\quad\cdot\quad) \le 5 \quad {}_j(\quad\cdot\quad(\underline{t}\;)) \le 6 \quad {}_j(\quad\cdot\quad)$.

Proof. The proof is by induction on the depth of the self-energy cluster.

1. Consider first the case that $\in \mathcal{P}_T$, with $\mathscr{O}_T = \mathscr{L}$. Set $n = n_{{}'_T}$, ${}' = {}_{{}'_T}$, ${}' = {}_{{}'_T}$, and $j' = j_{{}'_T}$. Then $\cdot\;{}'$ is replaced with ${}' \quad j'$, and, as a consequence, is replaced with $\cdot \quad (\underline{t}\;) = \quad \cdot\; {}^0 + \quad {}' \quad j'$. Define $\tilde{n}$ such that

$$2^{-(\tilde{n}+1)} \quad \le \quad {}_j\,(\quad\cdot\;{}^0 + \quad {}' \quad j') \le 2^{-(\tilde{n}-1)}\,, \tag{5.8}$$

where ${}_j(\quad\cdot\;{}^0 + \quad{}' \quad j') = |\quad\cdot\;{}^0 + \quad{}' \quad j' - \quad j\,| \ge \quad |\;{}^0|^{-}$ by the Diophantine condition (2.2b). Therefore $2^{\tilde{n}-1} \le |\;{}^0| \quad \le (E_1 k(\hat{T}))\quad \le (E_1 K_0)\;2^n = 2^{n-8}$, and hence $\tilde{n} \le n - 7$. Since $|\quad\cdot\;{}' - \quad{}' \quad j'| \le 2^{-n+2}$ by the inductive hypothesis, one has

$$\begin{aligned} {}_j\,(\quad\cdot\quad) &= \Big|\quad\cdot\;{}^0 + \quad\cdot\;{}' - \quad j\,\Big| \\ &\ge \Big|\quad\cdot\;{}^0 + \quad{}' \quad j' - \quad j\,\Big| - \big|\quad\cdot\;{}' - \quad{}' \quad j'\big| \ge \frac{15}{16}\quad {}_j\,(\quad\cdot\;{}^0 + \quad{}' \quad j'), \end{aligned}$$

because ${}_j(\quad\cdot\;{}^0 + \quad{}' \quad j') \ge 2^{-(\tilde{n}+1)} \quad \ge 2^{-n+6} \quad \ge 2^4\,|\quad\cdot\;{}' - \quad{}' \quad j'|$. In the same way one can bound ${}_j(\quad\cdot\quad) \le |\quad\cdot\;{}^0 + \quad{}' \quad j' - \quad j\,| + |\quad\cdot\;{}' - \quad{}' \quad j'|$, so that we conclude that

$$\frac{15}{16}\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'}) \le\ {}_j(\ \cdot\ ) \le \frac{17}{16}\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'}). \qquad (5.9)$$

This yields the assertion.

2. Consider now the case that $\mathscr{R}_{\mathrm{T}} = \mathscr{R}$. In that case $\cdot\ (\underline{t})$ is given by (5.6). DeÞne $\tilde{n}$ as in (5.8), with $' =\ _p$ and $j' = j\ _p$. We want to prove that

$$\frac{7}{8}\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'}) \le\ {}_j(\ \cdot\ (\underline{t})) \le \frac{9}{8}\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'}) \qquad (5.10)$$

for all $\underline{t} = (t_1, \ldots, t_p)$, with $t_i \in [0,1]$ for $i = 1, \ldots, p$. This immediately implies the assertion because, by using also (5.9), we obtain

$$\begin{aligned}\frac{14}{17}\ {}_j(\ \cdot\ ) &\le \frac{7}{8}\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'}) \le\ {}_j(\ \cdot\ (\underline{t}))\\ &\le \frac{9}{8}\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'}) \le \frac{18}{15}\ {}_j(\ \cdot\ ),\end{aligned}$$

and hence $4\ {}_j(\ \cdot\ ) \le 5\ {}_j(\ \cdot\ (\underline{t})) \le 6\ {}_j(\ \cdot\ )$.

By the inductive hypothesis and the discussion of the case (5.8), we have

$$\left|\ \cdot\ {}^0_i +\ {}_{i-1}\ {}_{j_{i-1}} -\ {}_i\ {}_{j_i}\right| \le 2^{-n_i+2}, \qquad i = 1, \ldots, p,$$

where $n_i = n\ _i$. Moreover one has $n_i \ge n_{i+1}$ for $i = 1, \ldots, p-1$, so that we obtain

$${}_j(\ \cdot\ (\underline{t})) \ge\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'}) - \sum_{i=1}^{p} 2^{-n_i+2} \ \ge\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'}) - 2^{-n+3}.$$

Since ${}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'}) \ge 2^{-(\tilde{n}+1)}$ and $\tilde{n} \le n-7$, one Þnds ${}_j(\ \cdot\ (\underline{t})) \ge (1-2^{-3})\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'})$. In the same way one has ${}_j(\ \cdot\ (\underline{t})) \le (1+2^{-3})\ {}_j(\ \cdot\ {}^0 +\ {}'\ {}_{j'})$, so that (5.10) follows. □

Remark 5.5. Given a renormalised tree, with $\mathscr{V}(\ ) \ne 0$, if a line $\in L(\ )$ has scale n then $_n(\ {}_j(\ \cdot\ )(\underline{t})) \ne 0$, and hence, by Lemma 5.4, one has $2^{(n+2)} \le\ {}_j(\ \cdot\ ) \le 2^{-(n-2)}$. Therefore, Lemma 4.1 still holds for the renormalised trees without any changes in the proof (see also Remark 4.2).

Remark 5.6. Another important consequence of Lemma 5.4 (and of Inequality (3.8) in Remark 3.11) is that the number of scale labels which can be associated with each line of a renormalised tree is still at most 2.

6. Symmetries and Identities

Now we shall prove some symmetry properties on the localized value of the self-energy clusters.

Lemma 6.1. If $T \in \mathfrak{E}^k_{j,\,,\,j,}(u,n)$ is such that $\breve{T}$ does not contain any end node v with $F_v = c^-_j$ then there exists $T' \in \overline{\mathfrak{R}}^k_{j,\,,\,j,}(u,n)$ such that $-2\mathscr{L}\mathscr{V}(T) = \mathscr{L}\mathscr{V}(T',u)$.

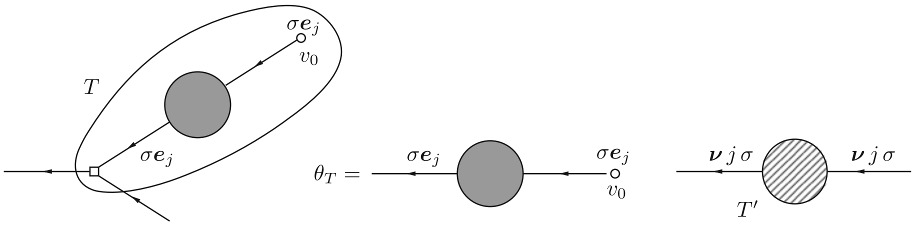

Fig. 15. The self-energy cluster T, the tree θ_T, and the self-energy cluster T' in the proof of Lemma 6.1

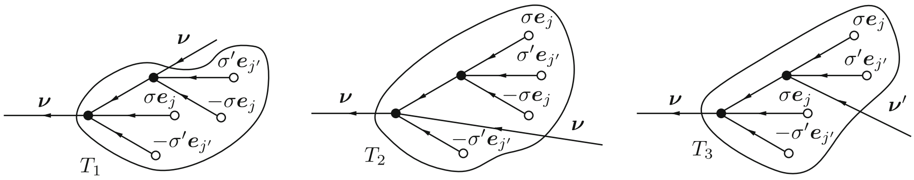

Fig. 16. The sets $\mathcal{F}_1(T) = \{T_1, T_2\}$ and $\mathcal{F}_2(T) = \{T_3\}$ corresponding to the self-energy cluster T in Fig. 11

Proof. If $T \in \mathfrak{C}^k_{j,\sigma,j,\sigma}(u, n)$ one has $|E_j(\breve{T})| = |E^-_j(\breve{T})| + 1$ (see Remark 3.24), so that if $|E^-_j(\breve{T})| = 0$, then also $|E_j(\breve{T})| = 1$. This means that $j_v \neq j$ for all $v \in E(\breve{T}) \setminus \{v_0\}$, if $E_j(\breve{T}) = \{v_0\}$. Consider the self-energy cluster $T' \in \overline{\mathfrak{R}}^k_{j,\sigma,j,\sigma}(u, n)$ obtained from θ_T by replacing the line exiting v_0 with an entering line carrying a momentum ν such that $\omega\cdot\nu = u$ and $n_{T'} = n_T$; see Fig. 15. With the exception of v_0, the nodes of θ_T have the same node factors as T'; in particular they have the same combinatorial factors. If we compute the propagators G_ℓ of $\ell \in L(T')$, by setting $u = \omega_j$, then they are the same as the corresponding propagators of T. Finally, as $n_{T'} = n_T$, one has $\mathscr{L}\mathscr{V}(T) = 0$ if and only if also $\mathscr{L}\mathscr{V}(T', u) = 0$. Thus, by recalling also Remark 3.25, one finds $-2\mathscr{L}\mathscr{V}(T) = \mathscr{L}\mathscr{V}(T', u)$. □

For $T \in \mathfrak{C}^k_{j,\sigma,j,\sigma}(u, n)$ let us call $\mathcal{F}_1(T)$ the set of all inequivalent self-energy clusters $T' \in \overline{\mathfrak{R}}^k_{j,\sigma,j,\sigma}(u, n)$ obtained from θ_T by replacing a line exiting an end node $v \in E_j(\breve{\theta}_T)$ with an entering line carrying a momentum ν such that $\omega\cdot\nu = u$ and with $n_{T'} = n_T$. Call also $\mathcal{F}_2(T)$ the set of all inequivalent self-energy clusters $T' \in \mathfrak{R}^k_{j,\sigma,j,-\sigma}(u', n)$, with $u' = u - 2\omega_j$, obtained from θ_T by replacing a line exiting an end node $v \in E^-_j(\breve{\theta}_T)$ (if any) with an entering line carrying a momentum ν' such that $\omega\cdot\nu' = u'$ and with $n_{T'} = n_T$; see Fig. 16.

Lemma 6.2. For all $T \in \mathfrak{C}^k_{j,\sigma,j,\sigma}(u, n)$ one has

$$\left(2c_j\,\mathscr{L}\mathscr{V}(T) + c_j \sum_{T'\in\mathcal{F}_1(T)} \mathscr{L}\mathscr{V}(T', u)\right) = c^-_j \sum_{T'\in\mathcal{F}_2(T)} \mathscr{L}\mathscr{V}(T', u'),$$

where $u' = u - 2\omega_j$ and the right hand side is meant as zero if $\mathcal{F}_2(T) = \emptyset$.

Proof. The case $k(T) > K_0 2^{n_T/\tau}$ is trivial so that we consider only the case $k(T) \leq K_0 2^{n_T/\tau}$. By construction any $T \in \mathfrak{C}^k_{j,\sigma,j,\sigma}(u, n)$ is such that $\breve{T}$ contains at least an end node v such that $F_v = c_j$, hence $|E_j(\check{T})| \geq 1$. By Lemma 6.1 either $|E_j^-(\check{T})| \geq 1$ or there exists $T' \in \overline{\mathfrak{R}}^k_{j,\,,\,j,}(u,n)$ such that $2\mathscr{L}\mathscr{V}(T) + \mathscr{L}\mathscr{V}(T',u) = 0$. Hence the assertion is proved if $E_j^-(\check{T}) = \emptyset$.

So, let us consider the case $|E_j^-(\check{T})| \geq 1$. First of all note that there is a 1-to-1 correspondence between the lines of and the lines and external lines, respectively, of both $T' \in \mathcal{F}_1(T)$ and $T' \in \mathcal{F}_2(T)$; the same holds for the internal nodes. Moreover the propagators both of any $T' \in \mathcal{F}_1(T)$ and of any $T' \in \mathcal{F}_2(T)$ are equal to the corresponding propagators of T when setting $u = \quad_j$ and $u' = - \quad_j$, respectively. Also the node factors of the internal nodes of all self-energy clusters $T' \in \mathcal{F}_1(T) \cup \mathcal{F}_2(T)$ are the same as those of T. For $T' \in \mathcal{F}_1(T)$ one has $|E_i^+(\check{T}')| = |E_i^-(\check{T}')|$ for all $i = 1, \ldots, d$, whereas for $T'' \in \mathcal{F}_2(T)$ one has $|E_i^+(\check{T}'')| = |E_i^-(\check{T}'')|$ for all $i \neq j$ and $|E_j(\check{T}'')| = |E_j^-(\check{T}'')| + 2$; thus, one has

$$\left(\prod_{v \in E(\check{T})} c_{j_v}^{\ v}\right) = c_j \left(\prod_{v \in E(\check{T}')} c_{j_v}^{\ v}\right) = c_j^- \left(\prod_{v \in E(\check{T}'')} c_{j_v}^{\ v}\right)$$

for all $T' \in \mathcal{F}_1(T)$ and all $T'' \in \mathcal{F}_2(T)$.

Therefore, if we write

$$-2c_j\, \mathscr{L}\mathscr{V}(T) = \mathscr{V}(\ _T) = \mathscr{A}(T) \left(\prod_{v \in E(\check{T})} c_{j_v}^{\ v}\right), \tag{6.1}$$

where $\mathscr{A}(T)$ depends only on T, then one Þnds

$$\sum_{T' \in \mathcal{F}_1(T)} \mathscr{L}\mathscr{V}(T',u) = \mathscr{A}(T) \frac{1}{c_j} \left(\prod_{v \in E(\check{T})} c_{j_v}^{\ v}\right) \sum_{v \in V(\check{T})} r_{v,j,}\ ,$$

with the same factor $\mathscr{A}(T)$ as in (6.1). Analogously one has

$$\sum_{T' \in \mathcal{F}_2(T)} \mathscr{L}\mathscr{V}(T',u') = \mathscr{A}(T) \frac{1}{c_j^-} \left(\prod_{v \in E(\check{T})} c_{j_v}^{\ v}\right) \sum_{v \in V(\check{T})} r_{v,j,-}\ ,$$

again with the same factor $\mathscr{A}(T)$ as in (6.1), so one can write

$$\begin{aligned} &\left(-2c_j\, \mathscr{V}(T) + c_j \sum_{T' \in \mathcal{F}_1(T)} \mathscr{L}\mathscr{V}(T',u)\right) - c_j^- \sum_{T' \in \mathcal{F}_2(T)} \mathscr{L}\mathscr{V}(T',u') \\ &= \mathscr{B}(T) \left(-1 + \sum_{v \in V(\check{T})} \left(r_{v,j,} - r_{v,j,-}\ \right)\right), \end{aligned} \tag{6.2}$$

where

$$\mathscr{B}(T) = \mathscr{A}(T) \left(\prod_{v \in E(\check{T})} c_{j_v}^{\ v}\right).$$

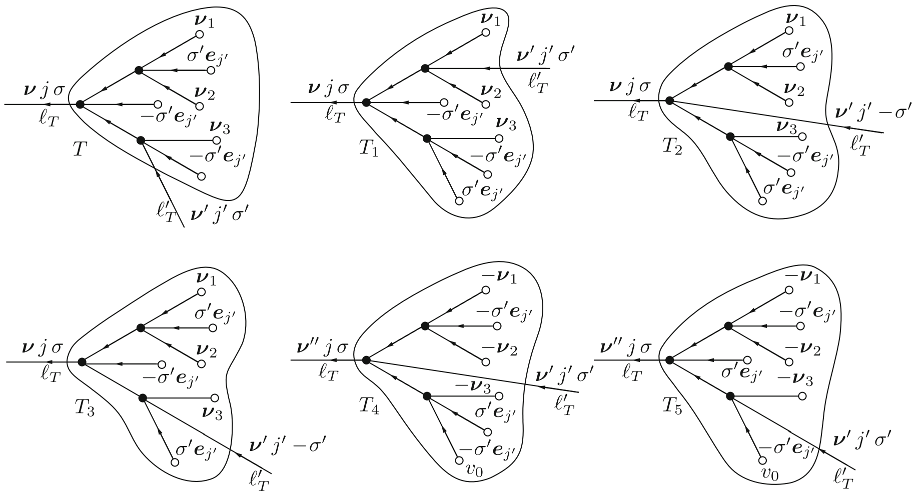

Fig. 17. A self-energy cluster T and the corresponding sets $\mathcal{G}_1(T) = \{T, T_1\}$, $\mathcal{G}_2(T) = \{T_2, T_3\}$, and $\mathcal{G}_3(T) = \{T_4, T_5\}$

On the other hand one has

$$\sum_{v \in V(\breve{T})} r_{v,j,} = |E_j(\breve{T})|,$$

so that the term in the last parentheses of (6.2) gives $-1 + |E_j(\breve{T})| - |E^-_j(\breve{T})| = 0$. Therefore the assertion is proved. □

For $T \in \mathfrak{R}^k_{j,,j',}(u,n)$ with $j \neq j'$ and $n \geq 0$ for all $\in \mathcal{P}_T$, call $\mathcal{G}_1(T)$ the set of self-energy clusters $T' \in \mathfrak{R}^k_{j,,j',}(u,n)$ obtained from T by exchanging the entering line $'_T$ with a line exiting an end node $v \in E_{j'}'(\breve{T})$ (if any). Call also $\mathcal{G}_2(T)$ the set of self-energy clusters $T' \in \mathfrak{R}^k_{j,,j',-}(u',n)$, with $u' = u - 2\ _j$, obtained from T by (1) replacing the momentum of $_T$ with a momentum $'$ such that $\cdot\ ' = u'$, (2) changing the sign label of an end node $v \in E^-_{j'}{}'(\breve{T})$ into $'$, and (3) exchanging the lines $_T$ and $_v$. Finally call $\mathcal{G}_3(T)$ the set of self-energy clusters $T' \in \mathfrak{R}^k_{j,-,j',}(u,n)$, obtained from T by (1) replacing the entering line $_T$ with a line exiting a new end node v_0 with $_{v_0} = \ '$ and $_{v_0} = \ 'e_{j'}$, (2) replacing all the labels $_v$ of the nodes $v \in N_0(T) \cup \{v_0\}$ with $-\ _v$ and (3) replacing a line exiting an end node $v \in E_{j'}{}'(\breve{T})$, with the entering line $'_T$; see Fig. 17. Again we force $n_{T'} = n_T$ for all $T' \in \mathcal{G}_1(T) \cup \mathcal{G}_2(T) \cup \mathcal{G}_3(T)$.

Lemma 6.3. For all $T \in \mathfrak{R}^k_{j,,j',}(u,n)$, with $j \neq j'$ and $n \geq 0$ for all $\in \mathcal{P}_T$, one has

$$c_{j'}{}' \sum_{T' \in \mathcal{G}_1(T)} \mathscr{L}\mathscr{V}(T',u) = c^-_{j'}{}' \sum_{T' \in \mathcal{G}_2(T)} \mathscr{L}\mathscr{V}(T',u'),$$

$$c^-_j\ c_{j'}{}' \sum_{T' \in \mathcal{G}_1(T)} \mathscr{L}\mathscr{V}(T',u) = c_j\, c^-_{j'}{}' \sum_{T' \in \mathcal{G}_3(T)} \mathscr{L}\mathscr{V}(T',u).$$

Proof. Again we consider only the case $k(\hat{T}) \le K_0 2^{n_T/}$. For Þxed $T \in \mathfrak{R}^k_{j,\,,\,j',\,'}(u,n)$, with $j \neq j'$, let $\in \mathfrak{T}^k_{j,\ e_j}(n)$ be the tree obtained from T by replacing the entering line $'_T$ with a line exiting a new end node v_0 with $_{v_0} = '$ and $_{v_0} = ' e_{j'}$. Note that in particular one has $|E_{j'}(\breve{\ })| = |E^-_{j'}(\breve{\ })|$. Any $T' \in \mathcal{G}_1(T)$ can be obtained from by replacing a line exiting an end node $v \in E_{j'}^{\,'}(\breve{\ })$ with an entering line $'_{T'}$, with the same labels as $'_T$, so that

$$c_{j'}^{\,'} \sum_{T' \in \mathcal{G}_1(T)} \mathscr{L}\mathscr{V}(T', u) = |E_{j'}^{\,'}(\breve{\ })|\,\mathscr{V}(\ ).$$

On the other hand, any $T' \in \mathcal{G}_2(T)$ can be obtained from by replacing a line exiting an end node $v \in E^{-\ '}_{j'}(\breve{\ })$ with an entering line $'_{T'}$, with labels $' - 2\ ' e_{j'}, j', -\ '$, hence

$$c^{-\ '}_{j'} \sum_{T' \in \mathcal{G}_2(T)} \mathscr{L}\mathscr{V}(T', u) = |E^{-\ '}_{j'}(\breve{\ })|\,\mathscr{V}(\ ),$$

so that the Þrst equality is proved.

Now, let $' \in \mathfrak{T}^k_{j,-\ e_j}(n)$ be the tree obtained from by replacing all the labels $_v$ of the nodes $v \in N_0(\ )$ with $-\ _v$. Any $T' \in \mathcal{G}_3(T)$ can be obtained from $'$ by replacing a line exiting an end node $v \in E_{j'}^{\,'}(\breve{\ }')$ with an entering line $'_{T'}$, carrying the same labels as $'_T$. Hence, by Lemma 3.14,

$$\begin{aligned} c_j^- \ c_{j'}^{\,'} \sum_{T' \in \mathcal{G}_1(T)} \mathscr{L}\mathscr{V}(T', u) &= c_j^- \ |E_{j'}^{\,'}(\breve{\ })|\,\mathscr{V}(\ ) = c_j\,|E^{-\ '}_{j'}(\breve{\ }')|\,\mathscr{V}(\ ') \\ &= c_j\, c^{-\ '}_{j'} \sum_{T' \in \mathcal{G}_3(T)} \mathscr{L}\mathscr{V}(T', u), \end{aligned}$$

which yields the second identity, and hence completes the proof. □

Lemma 6.4. For all $k \in \mathbb{Z}_+$, all $j, j' = 1, \ldots, d$, and all $,\ ' \in \{\pm\}$, one has

(i) $^{(k)} = \ ^{(k)}(|c_1|^2, \ldots, |c_d|^2)$, i.e., $^{(k)}$ depends on c only through the quantities $|c_1|^2, \ldots, |c_d|^2$;

(ii) $\mathscr{L}M^{(k)}_{j,\,,\,j',\,'}(u,n) = c_j^- \ c_{j'}^{\,'} M^{(k)}_{j,j'}(n)$, where $M^{(k)}_{j,j'}(n)$ does not depend on the indices $,\ '$.

Proof. One works on the single trees contributing to $\mathscr{L}M^{(k)}_{j,\,,\,j',\,'}(u,n)$. Then the proof follows from Lemma 3.14 and the results above. □

Remark 6.5 Note that Lemma 6.4 could be reformulated as

$$\mathscr{L}M^{(k)}_{j,\,,\,j',\,'}(u,n) = \ _{c_{j'}^{\,'}} c_j\, \mathscr{L}\widetilde{M}^{(k)}_{j,\,,\,j,\ }(n),$$

with $\widetilde{M}^{(k)}_{j,\,,\,j,\ }(n)$ deÞned after (3.13). We omit the proof of the identity, since it will not be used.

7. Cancellations and Bounds

We have seen in Sect. 4 that, as far as resonant lines are not considered, no problems arise in obtaining “good bounds”, i.e., bounds on the tree values of order k proportional to some constant to the power k (see Lemma 4.3). For the same bound to hold for all tree values we need a gain factor proportional to 2^{-n} for each resonant line on scale $n \geq 1$.

Let us consider a tree, and write its value as in (5.4). Let be a resonant line. Then exits a self-energy cluster T_2 and enters a self-energy cluster T_1, see Fig. 9. By construction $T_1 \in \mathfrak{R}^{k_1}_{j_1, \; 1, j_1', \; 1'}(\cdot \; '_{T_1}, n_1)$ and $T_2 \in \mathfrak{R}^{k_2}_{j_2, \; 2, j_2', \; 2'}(\cdot \; '_{T_2}, n_2)$, for suitable values of the labels, with the constraint $j_1 = j_2' = j$ and $1 = 2' =$.

If $\mathscr{O}_{T_1} = \mathscr{O}_{T_2} = \mathscr{L}$, we consider also all trees obtained from by replacing T_1 and T_2 with other clusters $T_1' \in \mathfrak{R}^{k_1}_{j_1, \; 1, j_1', \; 1'}(\cdot \; '_{T_1}, n_1)$ and $T_2' \in \mathfrak{R}^{k_2}_{j_2, \; 2, j_2', \; 2'}(\cdot \; '_{T_2}, n_2)$, respectively, with $\mathscr{O}_{T_1'} = \mathscr{O}_{T_2'} = \mathscr{L}$. In this way

$$\mathscr{L}\mathscr{V}(T_1, \cdot \; '_{T_1})\, G^{[n\;]}_j(\cdot \;)\, \mathscr{L}\mathscr{V}(T_2, \cdot \; _{T_2})$$

is replaced with

$$\mathscr{L}M^{(k_1)}_{j_1, \; 1, j \; ,}(\cdot \; '_{T_1}, n_1)\, G^{[n\;]}_j(\cdot \;)\, \mathscr{L}M^{(k_2)}_{j \; , \; , j_2', \; 2'}(\cdot \; '_{T_2}, n_2). \tag{7.1}$$

Then consider also all trees in which the factor (7.1) is replaced with

$$\mathscr{L}M^{(k_1)}_{j_1, \; 1, j \; ,-}(\cdot \; '_{T_1}, n_1)\, G^{[n\;]}_j(\cdot \; ')\, \mathscr{L}M^{(k_2)}_{j \; ,- \; , j_2', \; 2}(\cdot \; '_{T_2}, n_2), \tag{7.2}$$

with $'$ such that $\cdot \; - \; _j = \cdot \; ' + \; _j$; see Fig. 18. Because of Lemmas 6.2 and 6.3 the sum of the two contributions (7.1) and (7.2) gives

$$\mathscr{L}M^{(k_1)}_{j_1, \; 1, j \; ,}(\cdot \; '_{T_1}, n_1) \left(G^{[n\;]}_j(\cdot \;) + G^{[n\;]}_j(\cdot \; ') \right) \mathscr{L}M^{(k_2)}_{j \; , \; , j_2', \; 2'}(\cdot \; '_{T_2}, n_2),$$

where

$$G^{[n\;]}_j(\cdot \;) + G^{[n\;]}_j(\cdot \; ') = \frac{ \; _n(\; _j(\; \cdot \;))}{(\; \cdot \; - \; _j \;)} \left(\frac{1}{\cdot \; + \; _j} + \frac{1}{\cdot \; ' - \; _j} \right)$$

$$= \frac{2 \; _n(\; _j(\; \cdot \;))}{(\; \cdot \; + \; _j)(\; \cdot \; ' - \; _j \;)}, \tag{7.3}$$

and hence $|G^{[n\;]}_j(\cdot \;) + G^{[n\;]}_j(\cdot \; ')| \leq 2 \; _j^{-2}$. This provides the gain factor $O(2^{-n})$ we were looking for, with respect to the original bound $C_0 2^n$ on the propagator G .

If $\mathscr{O}_{T_1} = \mathscr{R}$ then if $k(\hat{T}_1) > K_0 2^{n_{T_1}/}$ we can extract a factor $C^{k(\hat{T}_1)}$ from $\mathscr{V}(T_1, \cdot \; '_{T_1})$ (C is the constant appearing in Lemma 4.3), and, after writing $C^{k(\hat{T}_1)} = C^{2k(\hat{T}_1)} C^{-k(\hat{T}_1)}$, use that $C^{-k(\hat{T}_1)} \leq C^{-K_0 2^{n_{T_1}/}} \leq \text{const} 2^{-n_{T_1}}$ in order to obtain a gain factor $O(2^{-n})$.

If $k(\hat{T}_1) \leq K_0 2^{n_{T_1}/}$ and $n \geq 0$ for all $\in \mathcal{P}_T$, we obtain a gain factor proportional to 2^{-n} because of the first line of (5.2). Of course whenever one has such a case,

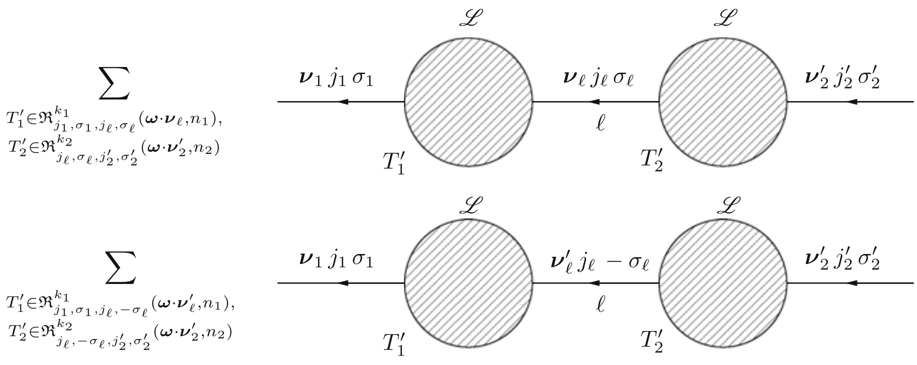

Fig. 18. Graphical representation of the cancellation mechanism discussed in the text: $' = -2\ e_j$. If we sum the two contributions we obtain a gain factor $O(2^{-n})$

then one has a derivative acting on $\mathscr{V}(T, u)$ Ð see (5.2). Therefore one needs to control derivatives like

$$\partial_u \mathscr{V}(T, u) = \sum_{\in \mathcal{P}_T} \partial_u G \left(\prod_{' \in L(T) \setminus \{\ \}} G_{'} \right) \left(\prod_{v \in N(T)} F_v \right), \tag{7.4}$$

where

$$\partial_u G = \frac{\partial_u\ n((\ \cdot\ ))}{(\ \cdot\ )^2 - \ _j^2} - 2\ \cdot\ \frac{n((\ \cdot\ ))}{((\ \cdot\ )^2 - \ _j^2)^2}. \tag{7.5}$$

The derived propagator (7.5) can be easily bounded by

$$|\partial_u G| \leq C_1 2^{2n}, \tag{7.6}$$

for some positive constant C_1.

In principle, given a line , one could have one derivative of G for each self-energy cluster containing . This should be a problem, because in a tree of order k, a propagator G could be derived up to $O(k)$ times, and no bound proportional to some constant to the power k can be expected to hold to order k. In fact, it happens that no propagator has to be derived more than once. This can be seen by reasoning as follows.

Let T be a self-energy cluster of depth $D(T) = 1$. If $\mathscr{O}_T = \mathscr{R}$ then a gain factor $O(2^{-n_T})$ is obtained. When writing $\partial_u \mathscr{V}(T, u)$ according to (7.4) one obtains $|\mathcal{P}_T|$ terms, one for each line $\in \mathcal{P}_T$. Then we can bound the derivative of G according to (7.6). By collecting together the gain factor and the bound (7.6) we obtain $2^{2n}\ 2^{-n_T'}$. We can interpret such a bound by saying that, at the cost of replacing the bound 2^n of the propagator G with its square 2^{2n}, we have a gain factor $2^{-n_T'}$ for the self-energy cluster T.

Suppose that is contained inside other self-energy clusters besides T, say $T_p \subset T_{p-1} \subset \cdots \subset T_1$ (hence T_p is that with largest depth, and $D(T_p) = p+1$). Then, when taking the contribution to (7.4) with the derivative ∂_u acting on the propagator G, we consider together the labels $\mathscr{O}_{T_i} = \mathscr{R}$ and $\mathscr{O}_{T_i} = \mathscr{L}$ for all $i = 1, \ldots, p$ (in other words we do not distinguish between localised and regularised values for such self-energy clusters), because we do not want to produce further derivatives on the propagator G. Of course we have obtained no gain factor corresponding to the entering lines of the self-energy clusters $T_1, \ldots, T_p$, and all these lines can be resonant lines. So, eventually we shall have to keep track of this.

Then we can iterate the procedure. If the self-energy cluster T does not contain any line whose propagator is derived, we split its value into the sum of the localised value plus the regularised value. On the contrary, if a line along the path $\mathcal{P}_T$ of T is derived we do not separate the localised value of T from its regularised value. Note that, if T is contained inside a regularised self-energy cluster, then both $\cdot$ and $\cdot\ '$ in (7.1) and (7.2) must be replaced with $\cdot\ (\underline{t})$ and $\cdot\ '(\underline{t})$, respectively, but still $\cdot\ (\underline{t}) - {}_j = \cdot\ '(\underline{t}) + {}_j$, so that the cancellation (7.3) still holds.

Let us call a *ghost line* a resonant line such that (1) is along the path $\mathcal{P}_T$ of a regularised self-energy cluster T and either (2a) enters or exits a self-energy cluster $T' \subset T$ containing a line whose propagator is derived or (2b) the propagator of is derived. Then, eventually one obtains a gain 2^{-n} for all resonant lines, except for the ghost lines. In other words we can say that there is an overall factor proportional to

$$\left(\prod_{\in L_R(\ )} 2^{-n} \right) \left(\prod_{\in L_G(\ )} 2^{n} \right), \tag{7.7}$$

where $L_G(\ )$ is the set of ghost lines. Indeed, in case (2a) there is no gain corresponding to the line , so that we can insert a “good” factor 2^{-n} provided we allow also a compensating “bad” factor 2^{n}. In case (2b) one can reason as follows. Call (with some abuse of notation) T_1 and T_2 the self-energy clusters which enters and exits, respectively. If $\mathscr{O}_{T_1} = \mathscr{O}_{T_2} = \mathscr{L}$, we consider

$$\mathscr{L}\mathscr{V}(T_1, \ \cdot \ {}'_{T_1})\ {}_uG_j^{[n\ ]}(\ \cdot\ (\underline{t}\ ))\,\mathscr{L}\mathscr{V}(T_2, \ \cdot \ {}_{T_2}),$$

and, by summing over all possible self-energy clusters as done in (7.1), we obtain

$$\mathscr{L}M^{(k_1)}_{j_1,\ 1, j\ ,}\ (\ \cdot\ {}'_{T_1}, n_1)\ {}_uG_j^{[n\ ]}(\ \cdot\ (\underline{t}\ ))\,\mathscr{L}M^{(k_2)}_{j\ ,\ \ ,j_2',\ {}_2'}(\ \cdot\ {}'_{T_2}, n_2);$$

then we sum this contribution with

$$\mathscr{L}M^{(k_1)}_{j_1,\ 1, j\ ,-}\ (\ \cdot\ {}'_{T_1}, n_1)\ {}_uG_j^{[n\ ]}(\ \cdot\ '(\underline{t}\ ))\,\mathscr{L}M^{(k_2)}_{j\ ,-\ \ ,j_2',\ 2}(\ \cdot\ {}'_{T_2}, n_2),$$

where $' = \ - 2\ e_j$; again we can use Lemmas 6.2 and 6.3 to obtain

$$\mathscr{L}M^{(k_1)}_{j_1,\ 1, j\ ,}\ (\ \cdot\ {}'_{T_1}, n_1) \left({}_uG_j^{[n\ ]}(\ \cdot\ (\underline{t}\ )) + {}_uG_j^{[n\ ]}(\ \cdot\ '(\underline{t}\ )) \right) \times \mathscr{L}M^{(k_2)}_{j\ ,\ \ ,j_2',\ {}_2'}(\ \cdot\ {}'_{T_2}, n_2),$$

where

$${}_uG_j^{[n\ ]}(\ \cdot\ (\underline{t}\ )) + {}_uG_j^{[n\ ]}(\ \cdot\ '(\underline{t}\ )) = \frac{2\ {}_u\ {}_n((\ \cdot\ (\underline{t}\ )))}{(\ \cdot\ (\underline{t}\ ) + {}_j)(\ \cdot\ '(\underline{t}\ ) - {}_j)} - \frac{4(\ \cdot\ (\underline{t}\ ) - {}_j)\ {}_n((\ \cdot\ (\underline{t}\ )))}{(\ \cdot\ (\underline{t}\ ) + {}_j)^2(\ \cdot\ '(\underline{t}\ ) - {}_j)^2},$$

so that we have not only the gain factor 2^{-n} due to the cancellation, but also a factor 2^{n} because of the term $u_n((\ \cdot\ ))$.

A trivial but important remark is that all the ghost lines contained inside the same self-energy cluster have different scales: in particular there is at most one ghost line on a given scale n. Therefore we can rely upon Lemma 4.4 and Lemma 5.4, to ensure that for each such line there is also at least one non-resonant line on scale $\geq n-3$ (inside the same self-energy cluster). Therefore we can bound the second product in (7.7) as

$$\left(\prod_{\in L_G(\ )} 2^{n} \right) \leq \prod_{n=1}^{\infty} 2^{n \mathfrak{N}_{n-3}(\ )},$$

which in turn is bounded as a constant to the power $k = k(\ )$, as argued in the proof of Lemma 4.3.

Finally if $k(\hat{u}_1) \leq K_0 2^{n_{T_1}/}$ and T_1 contains at least one line $\in \mathcal{P}_{T_1}$ with $n_{\ '} = -1$, in general there are $p \geq 1$ self-energy clusters $T'_p \subset T'_{p-1} \subset \cdots \subset T'_1 = T_1$ such that $\in \mathcal{P}_{T'_i}$ for $i = 1, \ldots, p$, and T'_p is the one with largest depth containing . For $i = 1, \ldots, p$ call $_i$ the exiting line of the self-energy cluster T'_i and $_i = \ _i$. Denote also, for $i = 1, \ldots, p-1$, by $_i = \ _{i+1}(\ _i)$ (recall Notation 3.29). By Lemma 3.30 one has $|\ _i - \ _{i+1}| \leq E_2 k(\hat{u}_i)$ for $i = 1, \ldots, p-1$. Moreover one has $|\ _1 - \ e_j\ | \leq E_2(k(\hat{u}_1) + \cdots + k(\hat{u}_{p-1}))$. On the other hand one has

$$\frac{}{|\ _i - \ _{i+1}|} \leq \ _{j_i}(\ \cdot\ _i) + \ _{j_{i+1}}(\ \cdot\ _{i+1}) \leq 2^{-n_{T'_{i+1}}+2},$$

$$\frac{}{|\ _1 - \ e_j\ |} \leq \ _{j_1}(\ \cdot\ _1) \leq 2^{-n_{T'_1}},$$

so that one can write

$$C^{k(\hat{u}_1)+\cdots+k(\hat{u}_{p-1}))} \leq C^{3k(\hat{u}_1)+\cdots+k(\hat{u}_{p-1}))} 2^{-n_{T'_1}} \prod_{i=2}^{p} 2^{-n_{T'_i}}, \tag{7.8}$$

which assures the gain factors for all self-energy clusters $T'_1, \ldots, T'_p$.

To conclude the analysis, if $\mathscr{O}_{T_1} = \mathscr{L}$ but $\mathscr{O}_{T_2} = \mathscr{R}$, one can reason in the same way by noting that $|n_{T'_2} - n\ | \leq 1$.

Lemma 7.1. Set $(\ c) = \max\{|c_1|, \ldots, |c_d|, 1\}$. There exists a positive constant C such that for $k \in \mathbb{N}$, $j \in \{1, \ldots, d\}$ and $\in \mathbb{Z}^d$ one has $|\sum_{\in \mathfrak{T}^k_{j,}} \mathscr{V}(\ )| \leq C^k\ ^{3k}(c)$.

Proof. Each time one has a resonant line , when summing together the values of all self-energy clusters, a gain $B_1 2^{-n}$ is obtained (either by the cancellation mechanism described at the beginning of this section or as an effect of the regularisation operator $\mathscr{R}$). The number of trees of order k is bounded by B_2^k for some constant B_2; see Remark 3.12. The derived propagators can be bounded by (7.6). By taking into account also the bound of Lemma 4.3, setting $B_3 = C_0$, and bounding by B_4^k, with

$$B_4 = \exp\left(3c \log 2 \sum_{n=0}^{\infty} 2^{-n/}\ n \right),$$

the product of the propagators (both derived and non-derived) of the non-resonant lines times the derived propagators of the resonant lines, we obtain the assertion with $C = B_1 B_2 B_3 B_4$. □

Lemma 7.2. The function (1.7), with $x_{j,}$ as in (3.10), and the counterterms $_j$ deÞned in (3.11) are analytic in and c, for $| \; | \;\; ^3(c) \leq \;_0$ with $_0$ small enough and $(\; c) = \max\{|c_1|, \ldots, |c_d|, 1\}$. Therefore the solution $x(t, \;, c)$ is analytic in $t, \;, c$ for $| \; | \;\; ^3(c) e^{3| \; ||\mathrm{Im}\, t|} \leq \;_0$, with $_0$ small enough.

Proof. Just collect together all the results above, in order to obtain the convergence of the series for $_0$ small enough and $| \; | \;\; (c) \leq \;_0$, for some constant. Moreover $x^{(k)}_{j,} = 0$ for $| \; | > \; k$, for the same constant Lemma 3.10 gives $= 3$. □

A. Momentum-Depending Perturbation

Here we discuss the Hamiltonian case in which the perturbation depends also on the coordinates $y_1, \ldots, y_d$, as in (1.13). As we shall see, differently from the y-independent case, here the Hamiltonian structure of the system is fundamental.

It is more convenient to work in complex variables z, $w = z^*$, with $z_j = (y_j + i \; _j x_j)/\sqrt{2 \; _j}$, where the Hamilton equations are of the form

$$\begin{cases} -i\dot{z}_j = \; _j z_j + \; _{w_j} F(z, w, \;) + \; _j z_j, \\ i\dot{w}_j = \; _j w_j + \; _{z_j} F(z, w, \;) + \; _j w_j, \end{cases} \tag{A.1}$$

with

$$F(z, w, \;) = \sum_{p=0}^{\infty} {}^p \sum_{\substack{s_1^+, \ldots, s_d^+, s_1^-, \ldots, s_d^- \geq 0 \\ s_1^+ + \cdots + s_d^+ + s_1^- + \cdots + s_d^- = p+3}} a_{s_1^+, \ldots, s_d^+, s_1^-, \ldots, s_d^-} z_1^{s_1^+} \ldots z_d^{s_d^+} w_1^{s_1^-} \ldots w_d^{s_d^-}. \tag{A.2}$$

Note that, since the Hamiltonian (1.11) is real, one has

$$a_{s^+, s^-} = a^*_{s^-, s^+}, \quad s^\pm = (s_1^\pm, \ldots, s_d^\pm) \in \mathbb{Z}_+^d. \tag{A.3}$$

Let us write

$$f_j^+(z, w, \;) = \; _{w_j} F(z, w, \;), \qquad f_j^-(z, w, \;) = \; _{z_j} F(z, w, \;)$$

so that

$$f_j (z, w, \;) = \sum_{p=1}^{\infty} {}^p \sum_{\substack{s^+, s^- \in \mathbb{Z}_+^d \\ s_1^+ + \cdots + s_d^+ + s_1^- + \cdots + s_d^- = p+1}} f_{j, s^+, s^-} z_1^{s_1^+} \ldots z_d^{s_d^+} w_1^{s_1^-} \ldots w_d^{s_d^-}, \qquad = \pm,$$

with $f^+_{j,s^+,s^-} = (s_j^- + 1) a_{s^+, s^- + e_j}$ and $f^-_{j,s^+,s^-} = (s_j^+ + 1) a_{s^+ + e_j, s^-}$, and hence

$$f^-_{j,s^+,s^-} = \left(f^+_{j,s^-,s^+} \right)^*, \qquad j = 1, \ldots, d, \quad s^+, s^- \in \mathbb{Z}^d, \tag{A.4a}$$

$$(s^{+}_{j_2}+1)\, f^{+}_{j_1,s^{+}+e_{j_2},s^{-}} = (s^{-}_{j_1}+1)\, f^{-}_{j_2,s^{+},s^{-}+e_{j_1}}, \qquad j_1, j_2 = 1,\ldots,d, \quad s^{+}, s^{-} \in \mathbb{Z}^d, \tag{A.4b}$$

$$(s^{-}_{j_2}+1)\, f^{+}_{j_1,s^{+},s^{-}+e_{j_2}} = (s^{-}_{j_1}+1)\, f^{+}_{j_2,s^{+},s^{-}+e_{j_1}}, \qquad j_1, j_2 = 1,\ldots,d, \quad s^{+}, s^{-} \in \mathbb{Z}^d, \tag{A.4c}$$

$$(s^{+}_{j_2}+1)\, f^{-}_{j_1,s^{+}+e_{j_2},s^{-}} = (s^{+}_{j_1}+1)\, f^{-}_{j_2,s^{+}+e_{j_1},s^{-}}, \qquad j_1, j_2 = 1,\ldots,d, \quad s^{+}, s^{-} \in \mathbb{Z}^d. \tag{A.4d}$$

Expanding the solution $(z(t), w(t))$ in Fourier series with frequency vector , (A.1) gives

$$\begin{cases} (\ \ \cdot\ \ -\ \ _j) z_{j,} = \ _j z_{j,} + f^{+}_{j,}(z, w, \ ), \\ (-\ \ \cdot\ \ -\ \ _j) w_{j,} = \ _j w_{j,} + f^{-}_{j,}(z, w, \ ). \end{cases} \tag{A.5}$$

We write the unperturbed solutions as

$$z^{(0)}_j(t) = c^{+}_j e^{i\ \ _j t}, \qquad w^{(0)}_j(t) = c^{-}_j e^{-i\ \ _j t}, \qquad j = 1,\ldots,d,$$

with $c_j = c^{+}_j \in \mathbb{C}$ and $c^{-}_j = c^{*}_j$. As in Sect. 1.2 we can split (A.5) into

$$f^{+}_{j,e_j}(z, w, \ ) + \ _j z_{j,e_j} = 0, \qquad j = 1,\ldots,d, \tag{A.6a}$$

$$f^{-}_{j,-e_j}(z, w, \ ) + \ _j w_{j,-e_j} = 0, \qquad j = 1,\ldots,d, \tag{A.6b}$$

$$\left[(\ \ \cdot\ \ ) - \ _j\right] z_{j,} = f^{+}_{j,}(z, w, \ ) + \ _j z_{j,}, \qquad j = 1,\ldots,d, \qquad \neq e_j, \tag{A.6c}$$

$$\left[-(\ \ \cdot\ \ ) - \ _j\right] w_{j,} = f^{-}_{j,}(z, w, \ ) + \ _j w_{j,}, \qquad j = 1,\ldots,d, \qquad \neq -e_j, \tag{A.6d}$$

so that Þrst of all one has to show that the same choice of $_j$ makes both (A.6a) and (A.6b) hold simultaneously, and that such $_j$ is real.

We consider a tree expansion very close to the one performed in Sect. 3. We simply drop (3) in Constraint 3.4. We denote by $\mathcal{T}^{k}_{j,\ ,\ }$ the set of inequivalent trees of order k, tree component j, tree momentum and tree sign that is, the sign label of the root line is .

We introduce ˘ and û as in Notation 3.5 and 3.27 respectively, and we deÞne the value of a tree as follows.

The node factors are deÞned as in (3.2) for the end nodes, while for the internal nodes $v \in V(\ )$ we deÞne

$$F_v = \begin{cases} \dfrac{s^{+}_{v,1}!\ldots s^{+}_{v,d}!\, s^{-}_{v,1}!\ldots s^{-}_{v,d}!}{s_v!}\, f^{\ \ v}_{j_v,s^{+}_v,s^{-}_v}, & k_v \geq 1, \\ -\dfrac{1}{2c^{\ \ v}_{j_v}}, & k_v = 0. \end{cases} \tag{A.7}$$

The propagators are deÞned as $G = 1$ if $\ \ = \ \ e_j$ and

$$G = G^{[n\ ]}_j(\ \ \cdot\ \ ), \qquad G^{[n]}_j(u) = \frac{_n(|u - \ _j|)}{u - \ _j}, \tag{A.8}$$

otherwise, and we deÞne $\mathcal{V}(\ )$ as in (3.9).

Finally we set $z_{j,e_j} = w^*_{j,-e_j} = c_j$, and formally deÞne

$$z_{j,} = \sum_{k=1}^{\infty} {}^k z^{(k)}_{j,}, \qquad z^{(k)}_{j,} = \sum_{\in \mathfrak{T}^k_{j,\,,+}} \mathscr{V}(\,), \qquad \neq e_j,$$

$$w_{j,} = \sum_{k=1}^{\infty} {}^k w^{(k)}_{j,}, \qquad w^{(k)}_{j,} = \sum_{\in \mathfrak{T}^k_{j,\,,-}} \mathscr{V}(\,), \qquad \neq -e_j, \tag{A.9}$$

and

$$_{j,} = \sum_{k=1}^{\infty} {}^k\, {}^{(k)}_{j,}, \qquad {}^{(k)}_{j,} = -\frac{1}{c_j} \sum_{\in \mathfrak{T}^k_{j,\,e_j,}} \mathscr{V}(\,). \tag{A.10}$$

Note that Remarks 3.9, 3.13 and 3.17 still hold.

Lemma A.1. With the notations introduced above, one has $^*_{j,+} = {}_{j,-}$ and $z^*_{j,} = w_{j,-}$.

Proof. By definition we only have to prove that for any $\in \mathfrak{T}^k_{j,\,,+}$ there exists $' \in \mathfrak{T}^k_{j,-\,,-}$ such that $\mathscr{V}(\,)^* = \mathscr{V}(\,')$. The proof is by induction on the order of the tree. Given $\in \mathfrak{T}^k_{j,\,,+}$, let us consider the tree $'$ obtained from by replacing the labels $_v$ of all the nodes $v \in N_0(\,)$ with $-\,_v$ and the labels of all the lines $\in L(\,)$ with $-$. Call $_1, \dots, {}_p$ the lines on scale -1 (if any) closest to the root of, and denote by $\mathbf{v}_i$ the node $_i$ enters and by $_i$ the tree with root line $_i$. Each tree $_i$ is then replaced with a tree $'_i$ such that $\mathscr{V}(\,_i)^* = \mathscr{V}(\,'_i)$ by the inductive hypothesis. Moreover, as for any internal line in the momentum becomes $-$, the propagators do not change. Finally, for any $v \in V(\breve{\ })$ the node factor is changed into

$$F'_v = \begin{cases} \dfrac{s^-_{v,1}! \cdots s^-_{v,d}!\, s^+_{v,1}! \cdots s^+_{v,d}!}{s_v!}\, f^{-\,v}_{j_v, s^-_v, s^+_v}, & k_v \geq 1, \\[2ex] -\dfrac{1}{2c^{-\,v}_{j_v}}, & k_v = 0. \end{cases} \tag{A.11}$$

Hence by (A.4a) one has $\mathscr{V}(\,)^* = \mathscr{V}(\,')$. □

Lemma A.2. With the notations introduced above, one has $_{j,+} \in \mathbb{R}$.

Proof. We only have to prove that for any $\in \mathfrak{T}^k_{j,e_j,+}$ there exists $'' \in \mathfrak{T}^k_{j,e_j,+}$ such that

$$c^+_j \mathscr{V}(\,)^* = c^-_j \mathscr{V}(\,'').$$

Let $v_0 \in E^+_j(\breve{\ })$ (existing by Remark 3.9) and let us consider the tree $''$ obtained from by (1) exchanging the root line $_0$ with $_{v_0}$, (2) replacing all the labels $_v$ of all the nodes $v \in N_0(\,) \setminus \{v_0\}$ with $-\,_v$, and (3) replacing all the labels of all the internal lines with $-$, except for those in $\mathcal{P}(\,_{v_0},\,_0)$ which remain the same. The propagators do not change; this is trivial for the lines outside $\mathcal{P}(\,_{v_0},\,_0)$, while for $\in \mathcal{P}(\,_{v_0},\,_0)$ one can reason as follows. The line divides $E(\breve{\ })\setminus\{v_0\}$ into two disjoint sets of end nodes $E(\breve{\ }, p)$ and $E(\breve{\ }, s)$ such that if $= {}_w$ one has $E(\breve{\ }, p) = \{v \in E(\breve{\ })\setminus\{v_0\} : v \prec w\}$ and $E(\breve{\ }, s) = (E(\breve{\ })\setminus\{v_0\})\setminus E(\breve{\ }, p)$. If

$$^{(p)} = \sum_{v\in E(\breve{\ }, p)} {}_v, \qquad ^{(s)} = \sum_{v\in E(\breve{\ }, s)} {}_v,$$

one has $^{(p)} + {}^{(s)} = 0$. When considering as a line in one has $= {}^{(p)} + e_j$ while in $''$ one has $= -{}^{(s)} + e_j$. Hence, as we have not changed the sign label, also G does not change. The node factors of the internal nodes are changed into their complex conjugates; this can be obtained as in Lemma A.1 for the internal nodes w such that ${}_w \notin \mathcal{P}({}_{v_0}, {}_0)$ while for the other nodes one can reason as follows.

First of all if v is such that ${}_v \in \mathcal{P}({}_{v_0}, {}_0)\cup\{{}_{v_0}\}$, there is a line ${}'_v \in \mathcal{P}({}_{v_0}, {}_0)\cup\{{}_0\}$ entering v. We shall denote $j_{\,v} = j_1$, ${}_v =$, $j_{\,'_v} = j_2$, and ${}_{'_v} = {}'$. Moreover we call s_i'' the number of lines outside $\mathcal{P}({}_{v_0}, {}_0)\cup\{{}_0\}$ with component label i and sign label $''$ entering v. Let us consider Þrst the case $= {}' = +$. When considering v as node of one has

$$F_v^* = \left(\frac{s_1^+!\cdots s_d^+!s_1^-!\cdots s_d^-!(s_{j_2}^+ + 1)}{s_v!} f^+_{j_1, s^+ + e_{j_2}, s^-}\right)^* = \frac{s_1^+!\cdots s_d^+!s_1^-!\cdots s_d^-!(s_{j_2}^+ + 1)}{s_v!} f^-_{j_1, s^-, s^+ + e_{j_2}}.$$

When considering v as node of $''$ one has $s_v^+ = s^- + e_{j_1}$ and $s_v^- = s^+$, so that

$$F_v'' = \frac{s_1^+!\cdots s_d^+!s_1^-!\cdots s_d^-!(s_{j_1}^- + 1)}{s_v!} f^+_{j_2, s^- + e_{j_1}, s^+},$$

and hence by (A.4b) $F_v^* = F_v''$. Reasoning analogously one obtains $F_v^* = F_v''$ also in the cases $= {}' = -$ and $\neq {}'$, using again (A.4b) when $= {}' = -$, and (A.4c) and (A.4d) for $= -$, ${}' = +$ and $= +$, ${}' = -$ respectively. Hence the assertion is proved. □

We deÞne the self-energy clusters as in Sect. 3.6, but replacing the constraint (3) with (3′) one has $|{}_T - {}'_T| \le 2$ and $|{}_T \cdot {}_T - {}_{j_T}| = |{}'_T \cdot {}'_T - {}_{j'_T}|$. We introduce $\breve{T}$ and $\hat{T}$ as in Notation 3.23 and 3.27 respectively, and we can deÞne $\mathscr{V}(T)$ as in (3.12) and the localisation and the regularisation operators as in Sect. 5.

Note that the main difference with the y-independent case is in the role of the sign label . In fact, here the sign label of a line does not depend on its momentum and component labels, and the small divisor is given by $(\cdot) = | \cdot - {}_j|$.

Hence the dimensional bounds of Sect. 4 and the symmetries discussed in Sect. 6 and summarised in Lemma 6.1 can be proved word by word as in the y-independent case, except for the second equality in Lemma 6.3 where one has to take into account a change of signs. More precisely for $T \in \mathfrak{R}^k_{j, \,, j', \,'}(u, n)$, with $j \neq j'$ and $n \geq 0$ for all $\in \mathcal{P}_T$, we deÞne $\mathcal{G}_1(T)$ as in Sect. 6 and $\mathcal{G}_3(T)$ as in Sect. 6 but replacing also the sign labels of the lines $\in L(T)$ with $-$.

Lemma A.3. For all $T \in \mathfrak{R}^k_{j,\,,\,j',\,'}(u,n)$, with $j \neq j'$ and $n \geq 0$ for all $\in \mathcal{P}_T$, one has

$$c_j^{-}\ c_{j'}^{\ '} \sum_{T'\in\mathcal{G}_1(T)} \mathscr{L}\mathscr{V}(T',u) = c_j\, c_{j'}^{-\ '} \sum_{T'\in\mathcal{G}_3(T)} \mathscr{L}\mathscr{V}(T',u). \qquad \text{(A.12)}$$

Proof. We consider only the case $\kappa(\hat{T}) \leq K_0 2^{n_T/}$. For fixed $T \in \mathfrak{R}^k_{j,\,,\,j',\,'}(u,n)$, with $j \neq j'$, let $\in \mathfrak{T}^k_{j,\ e_j,}(n)$ be the tree obtained from T by replacing the entering line $'_T$ with a line exiting a new end node v_0 with $_{v_0} = \ '$ and $_{v_0} = \ ' e_{j'}$. As in the proof of Lemma 6.3 one has

$$c_{j'}^{\ '} \sum_{T'\in\mathcal{G}_1(T)} \mathscr{L}\mathscr{V}(T',u) = |E_{j'}^{\ '}(\breve{\ })|\,\mathscr{V}(\ ).$$

Now, let $' \in \mathfrak{T}^k_{j,-\ e_j,-}(n)$ be the tree obtained from by replacing all the labels $_v$ of the nodes $v \in N_0(\ )$ with $-\ _v$, and the labels of all the lines $\in L(\ )$ with $-$. Any $T' \in \mathcal{G}_3(T)$ can be obtained from $'$ by replacing a line exiting an end node $v \in E_{j'}^{\ '}(\breve{\ }')$ with entering line $'_{T'}$, carrying the same labels as $_v$. Hence, by Lemma A.1,

$$c_j^{-}\ c_{j'}^{\ '} \sum_{T'\in\mathcal{G}_1(T)} \mathscr{L}\mathscr{V}(T',u) = c_j^{-}\ |E_{j'}^{\ '}(\breve{\ })|\,\mathscr{V}(\ ) = c_j^{-}\ |E_{j'}^{\ '}(\breve{\ }')|\,\mathscr{V}(\ ')^* $$
$$= c_j^{-}\ c_{j'}^{-\ '} \sum_{T'\in\mathcal{G}_3(T)} (\mathscr{L}\mathscr{V}(T',u))^*.$$

On the other hand, exactly as in Lemma A.2 one can prove that for any $T' \in \mathcal{G}_3(T)$ there exists $T'' \in \mathcal{G}_3(T)$ such that

$$c_j^{-}\ (\mathscr{L}\mathscr{V}(T',u))^* = c_j\, \mathscr{L}\mathscr{V}(T'',u),$$

and hence the assertion follows. □

The cancellation mechanism and the bounds proved in Sect. 7 follow by the same reasoning (in fact it is even simpler); see the next appendix for details.

B. Matrix Representation of the Cancellations

As we have discussed in Sect. 5 the only obstacle to convergence of the formal power series of the solution is given by the accumulation of resonant lines; see Fig. 14.

The cancellation mechanism described in Sect. 7 can be expressed in matrix notation. This is particularly helpful in the y-dependent case. For this reason, and for the fact that the formalism introduced in Appendix A includes the y-independent case, we prefer to work here with the variables (z, w).

We first develop a convenient notation. Given such that $(\ ,1) = +$ and $_{1,+}(\ \cdot\ ) <$ let us group together, in an ordered set $S(\ )$, all the $'$ such that $' = \ '(j,\ ) := -\ e_1 + \ e_j$, $= \pm 1$ and $j = 1,\ldots,d$, see Remark 3.19. By definition one has $_{1,+}(\ \cdot\ ) = \ _{j,}(\ \cdot\ '(j,\ ))$ for all $j = 1,\ldots,d$ and $= \pm$. Then we construct a $2d \times 2d$ localised self-energy matrix $\mathscr{L}M^{(k)}(\ \cdot\ ,n)$ with entries $\mathscr{L}M^{(k)}_{j,\,,\,j',\,'}(\ \cdot\ '(j',\ '),n)$. We also define the $2d \times 2d$ diagonal propagator matrix $\mathscr{G}^{[n]}(\ \cdot\ )$ with entries $\mathscr{G}^{[n]}_{j,\,,\,j',\,,}(\;\cdot\;) = \;_{j,j'}\;\;\,_{,}\,G^{[n]}_{j'}(\;\cdot\;{}'(j,\;))$, with $G^{[n]}_{j'}(u)$ deÞned according to (A.8), and $_{a,b}$ is the Kronecker delta.

As in Sect. 7 let us consider a chain of two self-energy clusters; see Fig. 9. By definition its value is

$$\mathscr{L}\mathscr{V}(T_1,\;\cdot\;_1)\,G^{[n\;]}_{j}(\;\cdot\;)\,\mathscr{L}\mathscr{V}(T_2,\;\cdot\;_2),$$

with $_1 = \;{}'_{T_1}$ and $_2 = \;_{T_2}$.

Notice that, if one sets also for the sake of simplicity $_1 = \;{}'_{T_1}$, $j_1 = j_{T_1'}$, $_2 = \;_{T_2}$, and $j_2 = j_{T_2}$, by the constraint (3) in the definition of self-energy clusters given in Appendix A, one has $_1 - \; = \;_1 e_{j_1} - \;e_j$ and $- \;_2 = \;e_j - \;_2 e_{j_2}$; moreover $_1,\;,\;_2$ all belong to a single set $\mathcal{S}(\;)$ for some .

As done in Sect. 7 let us sum together the values of all the possible self-energy clusters T_1 and T_2 with Þxed labels associated with the external lines, and of Þxed orders k_1 and k_2, respectively. We obtain

$$\mathscr{L}M^{(k_1)}_{j_1,\;_1,j\;,}\;(\;\cdot\;{}'(j\;,\;\;),n_{T_1})\,\mathscr{G}^{[n\;]}_{j\;,\;\;,j\;,}\;(\;\cdot\;)\,\mathscr{L}M^{(k_2)}_{j\;,\;\;,j_2,\;_2}(\;\cdot\;{}'(j_2,\;_2),n_{T_2}).$$

If we also sum over all possible values of the labels $j,$ we get

$$\sum_{=\pm}\sum_{j=1}^{d}\mathscr{L}M^{(k_1)}_{j_1,\;_1,j\;,}\;(\;\cdot\;{}'(j\;,\;\;),n_{T_1})\mathscr{G}^{[n\;]}_{j\;,\;\;,j\;,}\;(\;\cdot\;)\mathscr{L}M^{(k_2)}_{j\;,\;\;,j_2,\;_2}(\;\cdot\;{}'(j_2,\;_2),n_{T_2})$$
$$= \left[\mathscr{L}M^{(k_1)}(\;\cdot\;,n_{T_1})\,\mathscr{G}^{[n\;]}(\;\cdot\;)\,\mathscr{L}M^{(k_2)}(\;\cdot\;,n_{T_2})\right]_{j_1,\;_1,j_2,\;_2},$$

(i.e. the entry $j_1,\;_1,\;j_2,\;_2$ of the matrix in square brackets).

By the definition (A.8) of the propagators and by the symmetries of Lemma 6.1, $\mathscr{G}^{[n]}(\;\cdot\;)$ and $\mathscr{L}M^{(k)}(\;\cdot\;,n)$ have the form

$$\mathscr{G}^{[n]}(\;\cdot\;) = \frac{_n(|\;\cdot\;-\;_1|)}{\;\cdot\;-\;_1}\begin{pmatrix} \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{pmatrix}, \tag{B.1}$$

and

$$\mathscr{L}M^{(k)}(\;\cdot\;,n) = \begin{pmatrix} M^{(k)}_{1,1}(n)\begin{pmatrix}c_1^*c_1 & c_1^*c_1^*\\ c_1c_1 & c_1c_1^*\end{pmatrix} & \cdots & M^{(k)}_{1,d}(n)\begin{pmatrix}c_1^*c_d & c_1^*c_d^*\\ c_1c_d & c_1c_d^*\end{pmatrix}\\ \vdots & M^{(k)}_{j,j'}(n)\begin{pmatrix}c_j^*c_{j'} & c_j^*c_{j'}^*\\ c_jc_{j'} & c_jc_{j'}^*\end{pmatrix} & \vdots\\ M^{(k)}_{d,1}(n)\begin{pmatrix}c_d^*c_1 & c_d^*c_1^*\\ c_dc_1 & c_dc_1^*\end{pmatrix} & \cdots & M^{(k)}_{d,d}(n)\begin{pmatrix}c_d^*c_d & c_d^*c_d^*\\ c_dc_d & c_dc_d^*\end{pmatrix}\end{pmatrix},$$

respectively. A direct computation gives

$$\left[\mathscr{L}M^{(k_1)}(\ \cdot\ , n_{T_1})\,\mathscr{G}^{[n\ ]}(\ \cdot\ )\,\mathscr{L}M^{(k_2)}(\ \cdot\ , n_{T_2})\right]_{j_1,\ _1, j_2,\ _2}$$

$$= \frac{\ _n\ (|\ \cdot\ -\ _1|)}{\ \cdot\ -\ _1} c_{j_1}^{-\ 1} c_{j_2}^{\ 2} \sum_{j=1}^{d} M_{j_1, j}(n_{T_1})\, M_{j, j_2}(n_{T_2})\, |c_j|^2 \sum_{=\pm} (-1)^{1+\ 1} = 0, \qquad \text{(B.2)}$$

for all choices of the scales n , n_{T_1}, n_{T_2} and of the orders k_1, k_2.

This proves the necessary cancellation. Note that this is an exact cancellation in terms of the variables (z, w): all chains of localised self-energy clusters of length $p \geq 2$ can be ignored as their values sum up to zero. In the t-independent case, and in terms of the variables x, the cancellation is only partial, and one only finds $\mathscr{L}M^{(k_1)} G^{[n]} \mathscr{L}M^{(k_2)} = O(2^{-n})$, as discussed in Sect. 7.

C. Resummation of the Perturbation Series

The fact that the series obtained by systematically eliminating the self-energy clusters converges, as seen in Sect. 4, suggests that one may follow another approach, alternative to what we have described so far, and leading to the same result. Indeed, one can consider a *resummed expansion*, where one really gets rid of the self-energy clusters at the price of changing the propagators into new *dressed propagators* — again terminology is borrowed from quantum field theory. This is a standard procedure, already exploited in the case of KAM tori [10], lower-dimensional tori [10,12], skew-product systems [11], etc. The convergence of the perturbation series reflects the fact that the dressed propagators can be bounded proportionally to (a power of) the original ones for all values of the perturbation parameter. In our case, the latter property can be seen as a consequence of the cancellation mechanism just described. In a few words — and oversimplifying the strategy — the dressed propagators are obtained starting from a tree expansion where no self-energy clusters are allowed, and then "inserting arbitrary chains of self-energy clusters": this means that each propagator $\mathscr{G}^{[n]} = \mathscr{G}^{[n]}(\ \cdot\ )$ is replaced by a dressed propagator

$$^{[n]} = \mathscr{G}^{[n]} + \mathscr{G}^{[n]} M \mathscr{G}^{[n]} + \mathscr{G}^{[n]} M \mathscr{G}^{[n]} M \mathscr{G}^{[n]} + \cdots, \qquad \text{(C.1)}$$

where $M = M(\ \cdot\ )$ denotes the insertion of all possible self-energy clusters compatible with the labels of the propagators of the external lines (M is the matrix with entries $M_{j,\ ,\ j',\ '}(\ \cdot\ '(j',\ '))$ formally defined in Remark 3.26). Then, formally, one can sum together all possible contributions in (C.1), so as to obtain

$$^{[n]} = \mathscr{G}^{[n]}\left(\mathbb{1} - M\mathscr{G}^{[n]}\right)^{-1} = \left(A^{-1} - B\right)^{-1}, \qquad A := \mathscr{G}^{[n]}, \quad B := M. \quad \text{(C.2)}$$

For sake of simplicity, let us also identify the self-energy values with their localised parts, so as to replace in (C.1), and hence in (C.2), M with $\mathscr{L}M$, if $\mathscr{L}$ is the localisation operator. Then, in the notations we are using, the cancellation (B.2) reads $BAB = 0$, which implies

$$^{[n]} = A + ABA.$$

Therefore one finds $\|{}^{[n]}\| \le \|A\| + \|A\|^2\|B\| = O(2^{2n})$. So the values of the trees appearing in the resummed expansion can be bounded as done in Sect. 4, with the only difference that now, instead of the propagators G bounded proportionally to 2^n, one has the dressed propagators ${}^{[n]}$ bounded proportionally to 2^{2n}.

Of course, the argument above should be made more precise. First of all one should have to take into account also the regularised values of the self-energy clusters. Moreover, the dressed propagators should be defined recursively, by starting from the lower scales: indeed, the dressed propagator of a line on scale n is defined in terms of the values of the self-energy clusters on scales $< n$, as in (C.2), and the latter in turn are defined in terms of (dressed) propagators on scales $< n$, according to (3.13). As a consequence, the cancellation mechanism becomes more involved because the propagators are no longer of the form (B.1); in particular the symmetry properties of the self-energy values should be proved inductively on the scale label. In conclusion, really proceeding by following the strategy outlined above requires some work (essentially the same amount as performed in this paper). We do not push forward the analysis, which in principle could be worked out by reasoning as done in the papers quoted above.

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