Resonant motions in the presence of degeneracies for quasi-periodically perturbed systems

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Abstract

We consider one-dimensional systems in the presence of a quasi-periodic perturbation, in the analytical setting, and study the problem of existence of quasi-periodic solutions which are resonant with the frequency vector of the perturbation. We assume that the unperturbed system is locally integrable and anisochronous, and that the frequency vector of the perturbation satisfies the Bryuno condition. Existence of resonant solutions is related to the zeroes of a suitable function, called the Melnikov function – by analogy with the periodic case. We show that, if the Melnikov function has a zero of odd order and under some further condition on the sign of the perturbation parameter, then there exists at least one resonant solution which continues an unperturbed solution. If the Melnikov function is identically zero then one can push perturbation theory up to the order where a counterpart of Melnikov function appears and does not vanish identically: if such a function has a zero of odd order and a suitable positiveness condition is met, again the same persistence result is obtained. If the system is Hamiltonian, then the procedure can be indefinitely iterated and no positiveness condition must be required: as a byproduct, the result follows that at least one resonant quasi-periodic solution always exists with no assumption on the perturbation. Such a solution can be interpreted as a (parabolic) lower-dimensional torus.

1 Introduction

Melnikov theory studies the fate of homoclinic and periodic orbits of two-dimensional dynamical systems when they are periodically perturbed; see for instance [38] for an introduction to the subject. The problem can be stated as follows. Consider in $\mathbb{R}^2$ a dynamical system of the form

$$\begin{cases} \dot{x} = f_1(x,y) + \varepsilon g_1(x,y,t), \\ \dot{y} = f_2(x,y) + \varepsilon g_2(x,y,t), \end{cases} \tag{1.1}$$

with f_1, f_2, g_1, g_2 'sufficiently smooth', g_1 and g_2 T-periodic in t for some $T > 0$ and ε a small parameter, called the *perturbation parameter*. If $f_1 = \partial_y h$ and $f_2 = -\partial_x h$, for a suitable function h, the unperturbed system is Hamiltonian. Assume that for $\varepsilon = 0$ the system (1.1) admits a homoclinic orbit $u_1(t)$ to a hyperbolic saddle point p and that the bounded region of the phase space delimited by $\{u_1(t) : t \in \mathbb{R}\} \cup \{p\}$ is filled with a continuous family of periodic orbits $u_\delta(t)$, $\delta \in (0,1)$, whose periods tend monotonically to ∞ as $\delta \to 1$. Because of the assumptions, it is easy to see (as an application of the implicit function theorem) that for $\varepsilon \neq 0$ small enough, the system (1.1) admits a hyperbolic periodic orbit $\widetilde{u}(t,\varepsilon) = p + O(\varepsilon)$. Then one can ask whether the stable and unstable manifolds of $\widetilde{u}(t,\varepsilon)$

intersect transversely (in turn if this happens it can be used to prove that chaotic motions occur). Another natural question is what happens to the periodic orbits $u_\delta(t)$ when $\varepsilon \neq 0$. In particular one can investigate under which conditions 'periodic orbits persist', that is there are periodic orbits which are close to the unperturbed ones and reduce to them when the perturbation parameter is set equal to zero. If such orbits exist, they are called *subharmonic* or *resonant* orbits.

Both the existence of transverse intersection of the stable and unstable manifolds of $\widetilde{u}(t,\varepsilon)$ and the persistence of periodic orbits are related to the zeroes of suitable functions. More precisely if one define the *Melnikov function* as

$$M(t_0) := \int_{-\infty}^{\infty} \mathrm{d}t\, \big(f_2(u_1(t-t_0))g_2(u_1(t-t_0),t) - f_1(u_1(t-t_0))g_1(u_1(t-t_0),t)\big), \tag{1.2}$$

then if $M(t_0)$ has simple zeroes the stable and unstable manifolds of $\widetilde{u}(t,\varepsilon)$ intersect transversely, while if $M(t_0) \neq 0$ for all $t_0 \in \mathbb{R}$ no intersection occurs; essentially $M(t_0)$ measures the distance between the two manifolds along the normal to the homoclinic orbit at $u_1(t_0)$. Concerning the periodic orbits, if the period T_δ of $u_\delta(t)$ is not commensurable with the period T of the functions g_1, g_2, in general such an orbit will not persist under perturbations. Otherwise, set $T_\delta = mT/n$ and define the *(subharmonic) Melnikov function* as

$$M_{m/n}(t_0) := \int_{0}^{mT} \mathrm{d}t\, \big(f_2(u_\delta(t-t_0))g_2(u_\delta(t-t_0),t) - f_1(u_\delta(t-t_0))g_1(u_\delta(t-t_0),t)\big). \tag{1.3}$$

If $M_{m/n}(t_0)$ has a simple zero then (1.1) admits a subharmonic orbit $\overline{u}(t,\varepsilon)$ with period mT; in particular, if the functions f_1, f_2, g_1, g_2 are analytic, then $\overline{u}(t,\varepsilon)$ is analytic in both ε and t. If there are no zeroes at all, no periodic solution persists. The proof of the claims above is rather standard and it is essentially based on the application of the implicit function theorem. A possible approach for the case of subharmonic orbits consists in splitting the equations of motion into two separate sets of equations, the so-called *range equations* and *bifurcation equations*: one can solve the range equations in terms of the free parameter t_0 and then fix the latter by solving the bifurcation equations, which represent an implicit function problem.

The assumption that the zeroes of the Melnikov function are simple corresponds to a (generic) non-degeneracy condition on the perturbation. When the zeroes are not simple, the situation is slightly more complicated. In the case of subharmonic orbits, the same result of persistence extends to the more general case of zeroes of odd order [4], and interesting new analytical features of the solutions appear [1, 37, 18]; indeed the subharmonic solutions turn out to be analytic in a suitable fractional power of ε rather than ε itself. On the other hand if the zeroes are of even order one cannot predict a priori the persistence of periodic orbits. Finally, if the Melnikov function is identically zero, one has to consider higher order generalisations of it and study the existence and multiplicity of their zeroes to deal with the problem [18].

If one considers a quasi-periodic perturbation instead of a periodic one, that is $g_k(x,y,t) = G_k(x,y,\boldsymbol{\omega}t)$, with $G_k : \mathbb{R}^2 \times \mathbb{T}^d \to \mathbb{R}^2$ and $\boldsymbol{\omega} \in \mathbb{R}^d$, $d \geq 2$, one can still ask whether there exist hyperbolic sets run by quasi-periodic solutions with stable and unstable manifolds which intersect transversely and one can still study the existence of quasi-periodic solutions which are "resonant" with the *frequency vector* $\boldsymbol{\omega}$ of the perturbation; see below – after (1.4) – for a formal definition of resonant solution for quasi-periodic forcing.

Also in the quasi-periodic case, non-degeneracy assumptions are essential to prove transversality of homoclinic intersections. Existence of a quasi-periodic hyperbolic orbit close to the unperturbed saddle point and of its stable and unstable manifolds follows from general arguments, such as the invariant manifold theorem [25, 39], without even assuming any condition on the frequency vector $\boldsymbol{\omega}$. Palmer generalises the Melnikov method to the case of bounded perturbations [50] using the theory of exponential dichotomies [16]. A suitable generalisation of the Melnikov function for quasi-periodic forcing is also introduced by Wiggins [58]. He shows that if such a function has a simple zero then the stable and unstable manifolds intersect transversely. Then, generalising the Smale-Birkhoff homoclinic theorem to the case of orbits homoclinic to normally hyperbolic tori, he finds that there is an invariant set on which the dynamics of a suitable Poincaré map is conjugate to a subshift of finite type; in turn this yields the existence of chaos. Similar results hold also for more general almost periodic perturbations (which include the quasi-periodic ones as a special case): Meyer and Sell show that also in that case the dynamics near transverse homoclinic orbits behaves as a subshift of finite type [46] and Scheurle, relying on Palmer's results, finds particular solutions which have a random structure [55]; again, to obtain transversality the Melnikov function is assumed to have simple zeroes. Such assumption can be weakened to an assumption of "topological non-degeneracy" (i.e. the existence of an isolated minimum or maximum of the primitive of the Melnikov function) as in the case of subharmonic orbits, and one can deal with the problem by use of a variational approach; see for instance [17, 54, 3, 9].

A natural application for the study of homoclinic intersections, widely studied in the literature, is the quasi-periodically forced Duffing equation [59, 46, 61]. Often, especially in applications, the frequency vector is taken to be two-dimensional, with the two components which are nearly resonant with the proper frequency of the unperturbed system (see for instance [7, 61] and references therein). Then a different approach with respect to [58] is proposed by Yagasaki [61]: first, through a suitable change of coordinates, one arrives at a system with two frequencies, one fast and one slow, and then one uses averaging to reduce the analysis of the original system to that of a perturbation of a periodically forced system for which the standard Melnikov method applies: the persistence of hyperbolic periodic orbits and their stable and unstable manifolds for the original system is then obtained as a consequence of the invariant manifold theorem. Transversality of homoclic intersections plays also a crucial role in the phenomenon of Arnold diffusion [2, 44]: non-degeneracy assumptions on the perturbation are heavily used in the proofs existing in the literature (see e.g. [21, 31, 22]) in order to find lower bounds on the transversality, which in turn are fundamental to compute the diffusion times along the heteroclinic chains (see e.g. [11, 27, 8, 10, 57]). A physically relevant case, studied within the context of Arnold diffusion, is that with frequency vectors with two fast components [21, 30, 56] or with one component much faster and one component much slower than the proper frequency ('three scale system') [31, 32, 52]. In such cases the homoclinic splitting is exponentially small in the perturbation parameter and this makes the analysis rather delicate, as one has to check that the first order contribution to the splitting (the Melnikov function) really dominates; in particular non-degeneracy conditions on the perturbation are needed once more.

The problem of existence of quasi-periodic orbits close to the center of the unperturbed system is harder and does not follow from the invariant manifold theorem. Second-order approximations for the quasi-periodic solutions close to the centers of a forced oscillator are studied in [7], using the multiple scale technique for asymptotic expansions [43, 49]. But if one wants to really prove the existence of the solution, one must require additional assumption on $\boldsymbol{\omega}$ to deal with the presence of *small divisors*. In [47], Moser considers Duffing's equation with a quasi-periodic driving term and assumes that (i) the system is *reversible*, i.e. it can be written in the form $\dot{x} = f(x)$, with $f\colon \mathbb{R}^n \to \mathbb{R}^n$, and there exists an involution $I\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(Ix) = -If(x)$ (so that with $x(t)$ also $Ix(-t)$ is a solution), and

(ii) the frequency vector of the driving satisfies some Diophantine condition involving also the proper frequency of the unperturbed system linearised around its center. Then he shows that there exists a quasi-periodic solution, with the same frequency vector as the driving, to a slightly modified equation, in which the coefficient of the linear term is suitably corrected. If one tried to remove the correction then one should deal with an implicit function problem (see [6] for a similar situation), which, without assuming any non-degeneracy condition on the perturbation, would have the same kind of problems as in the present paper. Quasi-periodically forced Hamiltonian oscillators are also considered in [13], where the persistence of quasi-periodic solutions close to the centers of the unperturbed system is studied, including the case of resonance between the frequency vector of the forcing and the proper frequency. However, again, non-degeneracy conditions are assumed.

On the contrary the problem of persistence of quasi-periodic solutions far from the stationary points, corresponding to the subharmonic solutions of the periodic case, does not seem to have been studied a great deal. (We can mention a paper by Xu and Jing [60], who consider Duffing's equation with a two-frequency quasi-periodic perturbation and follow the approach in [61] to reduce the analysis to a one-dimensional backbone system; however the argument used to show the persistence of the two-dimensional tori is incomplete and requires further hypotheses.) Again the existence of resonant solutions is related to the zeroes of a suitable function, still called Melnikov function by analogy with the periodic case. If the zeroes are simple, assuming some Diophantine condition on $\boldsymbol{\omega}$, the analysis can be carried out so as to reach conclusions similar to the periodic case, that is the persistence of resonant solutions. In this paper we study the same problem in the case of zeroes of odd order and additionally investigate what can still be said when the Melnikov function is identically zero. As remarked before, considering non-simple zeroes means removing non-degeneracy – and hence genericity – conditions on the perturbation. This introduces nontrivial technical complications, because one is no longer allowed to separate the small divisor problem plaguing the range equations from the implicit function problem represented by the bifurcation equations. The method we use is based on the analysis and resummation of the perturbation series through renormalisation group techniques [29, 34, 35, 28]; for other renormalisation group approaches to small divisors problems in dynamical systems see for instance [12, 41, 45, 40, 23, 24, 42]. As in [19], the frequency vector of the perturbation will be assumed to satisfy the *Bryuno condition*; such a condition, originally introduced by Bryuno [14], has been studied recently in several small divisor problems arising in dynamical systems [34, 35, 36, 42, 45, 51]. With respect to [19], we consider here also non-Hamiltonian systems: what is required on the unperturbed system is a non-degeneracy condition on the frequency map of the periodic solutions (*anisochrony condition*). In the Hamiltonian case, such a condition becomes a convexity condition on the unperturbed Hamiltonian function, analogously to Cheng's paper [15], where the fate of resonant tori is studied. In the Hamiltonian case, the main difference with respect to [15] – and what prevents us from simply relying on that result – is that we consider isochronous perturbations (while in [15] the unperturbed Hamiltonian is convex in all action variables) and assume a weaker Diophantine condition on the frequency vector of the perturbation (the Bryuno condition instead of the standard one). Furthermore, as we said, our method covers also the non-Hamiltonian case, where Cheng's approach, based on a sequence of canonical transformations *à la* KAM, does not apply. In the Hamiltonian case, we do not require any further assumption on the perturbation (besides analyticity), as in [15]. In the non-Hamiltonian case we shall make some further assumptions. More precisely we shall require that some zeroes of odd order appear at some level of perturbation theory and a suitable positiveness condition holds; see Section 1 – in particular Hypotheses 3 and 4 – for a more formal statement.

Of course, one could also investigate what happens if the non-degeneracy condition on the unperturbed system is completely removed too. However, this would be a somewhat different problem and very likely a non-degeneracy condition could become necessary for the perturbation. Not even in the KAM theory for maximal tori, the fully degenerate case (no assumption on the unperturbed integrable system and no assumption on the perturbation, besides analyticity) has ever been treated in the literature – as far as we know.

The paper is organised as follows. We consider systems of the form (1.1) and assume that for $\varepsilon = 0$ there is a family of periodic solutions satisfying the same hypotheses as in the case of periodic forcing. In fact the analysis we will be interested in will be essentially local, so we can allow a more general setting and assume that, in suitable coordinates $(\beta, B) \in \mathbb{T} \times \mathfrak{B}$, with $\mathfrak{B}$ an open subset of $\mathbb{R}$, the unperturbed system reads

$$\begin{cases} \dot{\beta} = \omega_0(B), \\ \dot{B} = 0, \end{cases} \tag{1.4}$$

with ω_0 analytic and $\partial_B \omega_0(B) \neq 0$ (anisochrony condition). As a particular case we can consider that (B, β) are canonical coordinates (action-angle coordinates), but the formulation we are giving here is more general and applies also to non-Hamiltonian unperturbed systems; see also [5, 37]. Then we add to the vector field a small analytic quasi-periodic forcing term with frequency vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ which satisfy some weak Diophantine condition (Bryuno condition) and concentrate on a periodic solution of the unperturbed system which is resonant with $\boldsymbol{\omega}$, that is a solution with $B = \overline{B}_0$ such that $\omega(\overline{B}_0)\nu_0 + \omega_1\nu_1 + \ldots + \omega_d\nu_d = 0$ for suitable integers $\nu_0, \nu_1, \ldots, \nu_d$. In Section 2 we state formally our two main results on the persistence of such a solution: Theorem 2.2 takes into account the case in which the system is not assumed to be Hamiltonian and a zero of odd order appears at some order of perturbation theory, while Theorem 2.3 deals with the case in which the system is Hamiltonian and no further assumption is made on the perturbation. In Sections 3–5 we shall prove Theorem 2.2. As we shall see, the quasi-periodic solution will be only continuous in the perturbation parameter. In fact, in contrast to the case of periodic perturbations, in general the quasi-periodic solution is not expected to be analytic in ε nor in some fractional power of ε; already in the non-degenerate anisochronous Hamiltonian case the solution has been proved only to be C^∞ smooth in ε [28] and analyticity is very unlikely. In Section 6 we shall prove Theorem 2.3: we shall see that either (a) one is able to reduce the analysis to Theorem 2.2 or (b) suitable "cancellations" occur to all orders in the perturbation series formally defining the solution. In particular we shall see how the Hamiltonian structure of the equations of motion is fundamental in order to prove such cancellations. In turn this will imply, in case (b), the convergence of the perturbation series and hence the existence of a solution which is analytic in the perturbation parameter: we stress since now that this is a highly non-generic – and hence very unlikely – possibility. The cancellation mechanism turns out to be quite similar to the one performed in [20], where Moser's modifying terms theorem [48] is proved by using Cartesian coordinates instead of action-angle coordinates. It would be interesting to understand the deep reason of such a similarity.

2 Results

Let us consider the system

$$\begin{cases} \dot{\beta} = \omega_0(B) + \varepsilon F(\boldsymbol{\omega} t, \beta, B), \\ \dot{B} = \varepsilon G(\boldsymbol{\omega} t, \beta, B), \end{cases} \tag{2.1}$$

where $(\beta, B) \in \mathbb{T} \times \mathfrak{B}$, with $\mathfrak{B}$ an open subset of $\mathbb{R}$, $F, G \colon \mathbb{T}^{d+1} \times \mathfrak{B} \to \mathbb{R}$ and $\omega_0 \colon \mathfrak{B} \to \mathbb{R}$ are real analytic functions, $\boldsymbol{\omega} \in \mathbb{R}^d$ and ε is a real parameter called the *perturbation parameter*; hence the *perturbation* (F, G) is quasi-periodic in t with *frequency vector* $\boldsymbol{\omega}$. Without loss of generality we can assume that $\boldsymbol{\omega}$ has rationally independent components.

Denote by $\cdot$ the standard scalar product in $\mathbb{R}^d$, i.e. $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + \ldots + x_d y_d$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, and set $|\boldsymbol{x}| := \|\boldsymbol{x}\|_1 = |x_1| + \ldots + |x_d|$. If $f \colon \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, is a differentiable function, we shall denote (when no ambiguity arises) by $\partial_j f$ the derivative of f with respect to the j-th argument, i.e. $\partial_j f(x_1, \ldots, x_n) = \partial_{x_j} f(x_1, \ldots, x_n)$; if $n = 1$ we shall write also $f'(x_1) = \partial_1 f(x_1) = \partial f(x_1)$. Finally, for any finite set S we denote by $|S|$ its cardinality.

Take the solution for the unperturbed system given by $(\beta(t), B(t)) = (\beta_0 + \omega_0(\overline{B}_0)t, \overline{B}_0)$, with $\overline{B}_0$ such that $\omega_0(\overline{B}_0)$ is *resonant* with $\boldsymbol{\omega}$, i.e. such that there exists $(\overline{\nu}_0, \overline{\boldsymbol{\nu}}) \in \mathbb{Z}^{d+1}$ for which $\omega_0(\overline{B}_0)\,\overline{\nu}_0 + \boldsymbol{\omega} \cdot \overline{\boldsymbol{\nu}} = 0$. We want to study whether for some value of β_0, that is for a suitable choice of the *initial phase*, such a solution can be continued under perturbation.

The resonance condition between $\omega_0(\overline{B}_0)$ and $\boldsymbol{\omega}$ yields a "simple resonance" (or resonance of order 1) for the vector $(\omega_0(\overline{B}_0), \boldsymbol{\omega})$. The main assumptions on (2.1) are a Diophantine condition on the frequency vector of the perturbation and a non-degeneracy condition on the unperturbed system. More precisely we shall require that the vector $(\omega_0(\overline{B}_0), \boldsymbol{\omega})$ satisfies the condition

$$\sum_{n \geq 0} \frac{1}{2^n} \log \Big(\inf_{\substack{(\nu_0, \boldsymbol{\nu}) \in \mathbb{Z}^{d+1} \\ (\nu_0, \boldsymbol{\nu}) \nparallel (\overline{\nu}_0, \overline{\boldsymbol{\nu}}), 0 < |(\nu_0, \boldsymbol{\nu})| \leq 2^n}} |\omega_0(\overline{B}_0)\nu_0 + \boldsymbol{\omega} \cdot \boldsymbol{\nu}| \Big)^{-1} < \infty \tag{2.2}$$

and that $\omega_0'(\overline{B}_0) \neq 0$. Note that the condition (2.2) is weaker than requiring that the vector $(\omega_0(\overline{B}_0), \boldsymbol{\omega})$ satisfies the standard Diophantine condition $|\omega_0(\overline{B}_0)\nu_0 + \boldsymbol{\omega} \cdot \boldsymbol{\nu}| > \gamma\,(|\nu_0| + |\boldsymbol{\nu}|)^{-\tau}$ for suitable positive constants γ, τ and all $(\nu_0, \boldsymbol{\nu})$ non-parallel to $(\overline{\nu}_0, \overline{\boldsymbol{\nu}})$.

Up to a linear change of coordinates, we can (and shall) assume $\omega_0(\overline{B}_0) = 0$, so that the vector $\overline{\boldsymbol{\nu}}$, such that $\omega_0(\overline{B}_0)\,\overline{\nu}_0 + \boldsymbol{\omega} \cdot \overline{\boldsymbol{\nu}} = 0$, must be the null vector. Therefore it is not restrictive to formulate the assumptions on $\overline{B}_0$ and $\boldsymbol{\omega}$ as follows.

Hypothesis 1. $\omega(\overline{B}_0) = 0$ *and* $\boldsymbol{\omega}$ *satisfies the Bryuno condition* $\mathcal{B}(\boldsymbol{\omega}) < \infty$, *where*

$$\mathcal{B}(\boldsymbol{\omega}) = \sum_{n \geq 0} \frac{1}{2^n} \log \frac{1}{\alpha_n(\boldsymbol{\omega})}, \qquad \alpha_n(\boldsymbol{\omega}) = \inf_{\substack{\boldsymbol{\nu} \in \mathbb{Z}^d \\ 0 < |\boldsymbol{\nu}| \leq 2^n}} |\boldsymbol{\omega} \cdot \boldsymbol{\nu}|. \tag{2.3}$$

Hypothesis 2. $\omega_0'(\overline{B}_0) \neq 0$.

Let us write

$$F(\boldsymbol{\psi}, \beta, B) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu} \cdot \boldsymbol{\psi}} F_{\boldsymbol{\nu}}(\beta, B), \qquad G(\boldsymbol{\psi}, \beta, B) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu} \cdot \boldsymbol{\psi}} G_{\boldsymbol{\nu}}(\beta, B), \tag{2.4}$$

and note that, as F and G are real-valued functions, one has

$$F_{-\boldsymbol{\nu}}(\beta, B) = F_{\boldsymbol{\nu}}(\beta, B)^*, \qquad G_{-\boldsymbol{\nu}}(\beta, B) = G_{\boldsymbol{\nu}}(\beta, B)^*. \tag{2.5}$$

Here and henceforth $*$ denotes complex conjugation. By analogy with the periodic case, $G_{\boldsymbol{0}}(\beta, \overline{B}_0)$ will be called the (first order) *Melnikov function*.

Hypothesis 3. *$\overline{\beta}_0$ is a zero of order $\mathfrak{n}$ for $G_{\mathbf{0}}(\beta_0,\overline{B}_0)$, with $\mathfrak{n}$ odd, and $\varepsilon\,\omega_0'(\overline{B}_0)\,\partial_{\beta_0}^{\mathfrak{n}}G_{\mathbf{0}}(\overline{\beta}_0,\overline{B}_0)>0$.*

We look for a quasi-periodic solution to (2.1) with frequency vector $\boldsymbol{\omega}$, that is a solution of the form $(\beta(t),B(t))=(\beta_0+b(t),B_0+\widetilde{B}(t))$, with

$$b(t)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}b_{\boldsymbol{\nu}},\qquad \widetilde{B}(t)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}B_{\boldsymbol{\nu}},\tag{2.6}$$

where $\mathbb{Z}^d_*=\mathbb{Z}^d\setminus\{\mathbf{0}\}$. Note that the existence of a quasi-periodic solution with frequency $\boldsymbol{\omega}$ in the variables in which $\omega_0(\overline{B}_0)=0$ implies the existence of a quasi-periodic solution with frequency resonant with $\boldsymbol{\omega}$ in terms of the original variables (that is, before performing the change of variables leading to $\omega_0(\overline{B}_0)=0$).

If we set $\Phi(t):=\omega_0(B(t))+\varepsilon F(\boldsymbol{\omega}t,\beta(t),B(t))$ and $\Gamma(t)=\varepsilon G(\boldsymbol{\omega}t,\beta(t),B(t))$ and write

$$\Phi(t)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}\Phi_{\boldsymbol{\nu}},\qquad \Gamma(t)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}\Gamma_{\boldsymbol{\nu}},\tag{2.7}$$

in Fourier space (2.1) becomes

$$(i\boldsymbol{\omega}\cdot\boldsymbol{\nu})b_{\boldsymbol{\nu}}=\Phi_{\boldsymbol{\nu}},\quad \boldsymbol{\nu}\neq\mathbf{0},\tag{2.8a}$$
$$(i\boldsymbol{\omega}\cdot\boldsymbol{\nu})B_{\boldsymbol{\nu}}=\Gamma_{\boldsymbol{\nu}},\quad \boldsymbol{\nu}\neq\mathbf{0},\tag{2.8b}$$
$$\Phi_{\mathbf{0}}=0,\tag{2.8c}$$
$$\Gamma_{\mathbf{0}}=0.\tag{2.8d}$$

According to the usual terminology, we shall call (2.8a) and (2.8b) the *range equations*, while (2.8c) and (2.8d) will be referred to as the *bifurcation equations*.

Our first result will be the following.

Theorem 2.1. *Consider the system (2.1) and assume Hypotheses 1, 2 and 3 to be satisfied. Then for ε small enough there exists at least one quasi-periodic solution $(\beta(t),B(t))$ with frequency vector $\boldsymbol{\omega}$ such that $(\beta(t),B(t))\to(\overline{\beta}_0,\overline{B}_0)$ for $\varepsilon\to 0$.*

Actually we shall prove a more general result which can be stated as follows. We look for a formal solution $(\beta(t),B(t))$, with

$$\begin{aligned}\beta(t)&=\beta(t;\varepsilon,\beta_0)=\beta_0+\sum_{k\geq1}\varepsilon^k b^{(k)}(t;\beta_0)=\beta_0+\sum_{k\geq1}\varepsilon^k\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}b^{(k)}_{\boldsymbol{\nu}}(\beta_0),\\ B(t)&=B(t;\varepsilon,\beta_0)=\overline{B}_0+\sum_{k\geq1}\varepsilon^k B^{(k)}(t;\beta_0)=\overline{B}_0+\sum_{k\geq1}\varepsilon^k\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}B^{(k)}_{\boldsymbol{\nu}}(\beta_0),\end{aligned}\tag{2.9}$$

and set $U(t):=\omega_0(B(t))-\omega_0'(\overline{B}_0)(B(t)-\overline{B}_0)$ and $\phi(t)=U(t)+\varepsilon F(\boldsymbol{\omega}t,\beta(t),B(t))$. Then define recursively for $k\geq1$

$$\begin{aligned}b^{(k)}_{\boldsymbol{\nu}}(\beta_0)&=\frac{1}{(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})}\phi^{(k)}_{\boldsymbol{\nu}}(\beta_0)+\frac{\omega_0'(\overline{B}_0)}{(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\Gamma^{(k)}_{\boldsymbol{\nu}}(\beta_0),\qquad \boldsymbol{\nu}\neq\mathbf{0}\\ B^{(k)}_{\boldsymbol{\nu}}(\beta_0)&=\frac{1}{(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})}\Gamma^{(k)}_{\boldsymbol{\nu}}(\beta_0),\qquad \boldsymbol{\nu}\neq\mathbf{0}\\ B^{(k)}_{\mathbf{0}}(\beta_0)&=-\frac{1}{\omega_0'(\overline{B}_0)}\phi^{(k)}_{\mathbf{0}}(\beta_0),\end{aligned}\tag{2.10}$$

where $\Gamma_{\boldsymbol{\nu}}^{(k)}(\beta_0) = [G(\boldsymbol{\omega}t,\beta(t),B(t))]_{\boldsymbol{\nu}}^{(k-1)}$ and $\phi_{\boldsymbol{\nu}}^{(k)}(\beta_0) = [U(t)]_{\boldsymbol{\nu}}^{(k)} + [F(\boldsymbol{\omega}t,\beta(t),B(t))]_{\boldsymbol{\nu}}^{(k-1)}$, with $U_{\boldsymbol{\nu}}^{(1)}(\beta_0)=0$, so that $\Gamma_{\boldsymbol{\nu}}^{(1)}(\beta_0)=G_{\boldsymbol{\nu}}(\beta_0,\overline{B}_0)$ and $\phi_{\boldsymbol{\nu}}^{(1)}(\beta_0)=F_{\boldsymbol{\nu}}(\beta_0,\overline{B}_0)$, while, for $k\geq 2$,

$$[U(t)]_{\boldsymbol{\nu}}^{(k)} = \sum_{s\geq 2}\frac{1}{s!}\partial_B^s\omega_0(\overline{B}_0) \sum_{\substack{\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu}\\ \boldsymbol{\nu}_i\in\mathbb{Z}^d,\, i=1,\ldots,s}} \sum_{\substack{k_1+\ldots+k_s=k,\\ k_i\geq 1}} \prod_{i=1}^s B_{\boldsymbol{\nu}_i}^{(k_i)}(\beta_0), \tag{2.11}$$

and

$$\begin{aligned}[P(\boldsymbol{\omega}t,\beta(t),B(t))]_{\boldsymbol{\nu}}^{(k-1)} = &\sum_{s\geq 1}\sum_{p+q=s} \sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu}\\ \boldsymbol{\nu}_0,\boldsymbol{\nu}_j\in\mathbb{Z}^d\; j=p+1,\ldots,s\\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*,\, i=1,\ldots,p}} \frac{1}{p!q!}\partial_\beta^p\partial_B^q P_{\boldsymbol{\nu}_0}(\beta_0,\overline{B}_0)\times \\ &\times \sum_{\substack{k_1+\ldots+k_s=k-1,\\ k_i\geq 1}} \prod_{i=1}^p b_{\boldsymbol{\nu}_i}^{(k_i)}(\beta_0) \prod_{i=p+1}^s B_{\boldsymbol{\nu}_i}^{(k_i)}(\beta_0), \qquad P=F,G.\end{aligned} \tag{2.12}$$

The series (2.9), with the coefficients defined as above and arbitrary β_0, turn out to be a formal solution of (2.8a)-(2.8c): the coefficients $b_{\boldsymbol{\nu}}^{(k)}(\beta_0)$, $B_{\mathbf{0}}^{(k)}(\beta_0)$ and $B_{\boldsymbol{\nu}}^{(k)}(\beta_0)$ are well defined for all $k\geq 1$ and all $\boldsymbol{\nu}\in\mathbb{Z}^d_*$, by Hypothesis 1, and solve (2.8a)-(2.8c) order by order – as it is straightforward to check (for instance by using the formalism introduced below in Section 3) –; moreover the functions $b^{(k)}(t;\beta_0)$ and $B^{(k)}(t;\beta_0)$ are analytic and quasi-periodic in t with frequency vector $\boldsymbol{\omega}$.

Assume that there exists $k_0\in\mathbb{N}$ such that all functions $\Gamma_{\mathbf{0}}^{(k)}(\beta_0)$ are identically zero for $0\leq k\leq k_0-1$; then we can solve the equation of motion up to order k_0-1 without fixing the parameter β_0 and moreover $\Gamma_{\mathbf{0}}^{(k_0)}$ is a well-defined function of β_0.

Hypothesis 4. *There exist $k_0\in\mathbb{N}$ and $\overline{\beta}_0$ such that $\Gamma_{\mathbf{0}}^{(k)}(\beta_0)$ vanish identically for $k<k_0$ and $\overline{\beta}_0$ is a zero of order $\bar{\mathfrak{n}}$ for $\Gamma_{\mathbf{0}}^{(k_0)}(\beta_0)$, with $\bar{\mathfrak{n}}$ odd. Moreover one has $\varepsilon^{k_0}\omega_0'(\overline{B}_0)\partial_{\beta_0}^{\bar{\mathfrak{n}}}\Gamma_{\mathbf{0}}^{(k_0)}(\overline{\beta}_0)>0$.*

We shall prove the following result.

Theorem 2.2. *Consider the system (2.1) and assume Hypotheses 1, 2 and 4 to be satisfied. Then for ε small enough there exists at least one quasi-periodic solution $(\beta(t),B(t))$ with frequency vector $\boldsymbol{\omega}$ such that $(\beta(t),B(t))\to(\overline{\beta}_0,\overline{B}_0)$ for $\varepsilon\to 0$.*

Note that Hypothesis 4 reduces to Hypothesis 3 if $k_0=1$. Therefore it will be enough to prove Theorem 2.2. The proof will be organised as follows. Besides the system (2.8) we shall consider first the system described by the range equations

$$(i\boldsymbol{\omega}\cdot\boldsymbol{\nu})b_{\boldsymbol{\nu}} = \Phi_{\boldsymbol{\nu}}, \quad \boldsymbol{\nu}\neq\mathbf{0}, \tag{2.13a}$$

$$(i\boldsymbol{\omega}\cdot\boldsymbol{\nu})B_{\boldsymbol{\nu}} = \Gamma_{\boldsymbol{\nu}}, \quad \boldsymbol{\nu}\neq\mathbf{0}, \tag{2.13b}$$

i.e. with no condition for $\boldsymbol{\nu}=\mathbf{0}$. In Sections 3 and 4 we shall prove that, if some further conditions (to be specified later on) are found to be satisfied, it is possible to find, for ε small enough and arbitrary β_0, B_0, a solution

$$(\beta_0+b(t), B_0+\widetilde{B}(t)), \tag{2.14}$$

to the system (2.13), with $b(t)$ and $\widetilde{B}(t)$ as in (2.6) depending on the free parameters ε,β_0,B_0; such a solution is obtained via a 'resummation procedure', starting from the formal solution of the range equations (2.13). The conditions mentioned above can be illustrated as follows. The resummation procedure turns out to be well-defined if the small divisors of the resummed series can be bounded proportionally to the square of the small divisors of the formal series. However, it is not obvious at all that this is possible, since the latter are of the form $(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})^{-1}$ with $\boldsymbol{\nu}\in\mathbb{Z}^d_*$, while the small divisors of the resummed series are of the form $(\det((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0,B_0)))^{-1}$, for suitable 2×2 matrices $\mathcal{M}^{[n]}$ (see Section 3). The bound on the small divisors of the resummed series is difficult to check without assuming any non-degeneracy condition on the perturbation. Therefore we replace $\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0)$ with $\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0)\xi_n(\det(\mathcal{M}^{[n]}(0;\varepsilon,\beta_0,B_0)))$, for suitable 'cut-off functions' ξ_n, in such a way that the bound automatically holds. The introduction of the cut-offs changes the series in such a way that if on the one hand the modified series are well-defined, on the other hand in principle they no longer solve the range equations: this turns out to be the case only if one can prove that the cut-offs can be removed. So, the last part of the proof consists in showing that, by suitably choosing the parameters β_0, B_0 as continuous functions of ε, this occurs and moreover, for the same choice of β_0, B_0, the bifurcation equations (2.8c) and (2.8d) hold; hence for such β_0, B_0, the function (2.14) is a solution of the whole system (2.1). Once Theorem 2.2 is proved, Theorem 2.1 will immediately follow taking $k_0 = 1$.

Next we shall see that if the system is Hamiltonian one can prove the same result as in Theorem 2.2 with the only assumptions in Hypotheses 1 and 2. More precisely, consider the Hamiltonian function

$$H(\boldsymbol{\alpha},\beta,\boldsymbol{A},B) := \boldsymbol{\omega}\cdot\boldsymbol{A} + h_0(B) + \varepsilon f(\boldsymbol{\alpha},\beta,B), \tag{2.15}$$

where $(\boldsymbol{\alpha},\beta)\in\mathbb{T}^{d+1}$ and $(\boldsymbol{A},B)\in\mathbb{R}^d\times\mathfrak{B}$, with $\mathfrak{B}$ an open subset of $\mathbb{R}$, are canonically conjugate (action-angle) variables and $f:\mathbb{T}^{d+1}\times\mathfrak{B}\to\mathbb{R}$ and $h_0:\mathfrak{B}\to\mathbb{R}$ are real analytic functions. Set $\omega_0(B)=\partial_B h_0(B)$. Then the corresponding Hamilton equations for the variables (β, B) are given by

$$\begin{cases} \dot\beta = \omega_0(B) + \varepsilon\partial_B f(\boldsymbol{\omega}t,\beta,B), \\ \dot B = -\varepsilon\partial_\beta f(\boldsymbol{\omega}t,\beta,B). \end{cases} \tag{2.16}$$

We shall prove in Section 6 the following result.

Theorem 2.3. *Consider the system (2.16) and assume Hypotheses 1 and 2 to be satisfied. Then for ε small enough there exists at least one quasi-periodic solution $(\beta(t), B(t))$ with frequency vector $\boldsymbol{\omega}$. Such a solution depends continuously on ε.*

Note that Hypothesis 2 is tantamount to requiring h_0 to be convex. Quasi-periodic solutions to (2.16) with frequency vector $\boldsymbol{\omega}$ describe lower-dimensional tori (d-dimensional tori for a system with $d+1$ degrees of freedom). Such tori are parabolic in the sense that the "normal frequency" vanishes for $\varepsilon = 0$. Theorem 2.3 can be seen as the counterpart of Cheng's result [15] in the case in which all "proper frequencies" are fixed (isochronous case) and the perturbation does not depend on the actions conjugated to the "fast angles" (otherwise one should add a correction like in [48]); moreover, with respect to [15], a weaker Diophantine condition is assumed on the proper frequencies.

3 Diagrammatic rules

Let us consider the range equations (2.13) and start by looking for a quasi-periodic solution which can be formally written as

$$\begin{aligned} \beta(t;\varepsilon,\beta_0,B_0) &= \beta_0 + \sum_{k\geq 1}\varepsilon^k b^{\{k\}}(t;\beta_0,B_0) = \beta_0 + \sum_{k\geq 1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}t} b^{\{k\}}(\beta_0,B_0), \\ B(t;\varepsilon,\beta_0,B_0) &= B_0 + \sum_{k\geq 1}\varepsilon^k B^{\{k\}}(t;\beta_0,B_0) = B_0 + \sum_{k\geq 1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}t} B^{\{k\}}(\beta_0,B_0), \end{aligned} \tag{3.1}$$

where a different notation for the Taylor coefficients has been used with respect to (2.9) to stress that now we are considering $B_0 = B_{\mathbf{0}}$ as a parameter. If we define recurively for $k \geq 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d_*$

$$b^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0) := \frac{\Phi^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0)}{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}}, \qquad B^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0) := \frac{\Gamma^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0)}{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}},$$

where we have set

$$\begin{aligned} \Phi^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0) :=& \sum_{s\geq 1}\frac{1}{s!}\partial_B^s\omega_0(B_0) \sum_{\substack{\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu} \\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*}} \sum_{\substack{k_1+\ldots+k_s=k \\ k_i\geq 1}} \prod_{i=1}^s B^{\{k_i\}}_{\boldsymbol{\nu}_i}(\beta_0,B_0) \\ &+ \sum_{\substack{s\geq 1 \\ p+q=s}} \sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu} \\ \boldsymbol{\nu}_0\in\mathbb{Z}^d,\boldsymbol{\nu}_i\in\mathbb{Z}^d_*}} \frac{1}{p!q!}\partial_\beta^p\partial_B^q F_{\boldsymbol{\nu}_0}(\beta_0,B_0) \sum_{\substack{k_1+\ldots+k_s=k-1 \\ k_i\geq 1}} \prod_{i=1}^p b^{\{k_i\}}_{\boldsymbol{\nu}_i}(\beta_0,B_0) \prod_{i=p+1}^s B^{\{k_i\}}_{\boldsymbol{\nu}_i}(\beta_0,B_0), \\ \Gamma^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0) :=& \sum_{\substack{s\geq 1 \\ p+q=s}} \sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu} \\ \boldsymbol{\nu}_0\in\mathbb{Z}^d,\boldsymbol{\nu}_i\in\mathbb{Z}^d_*}} \frac{1}{p!q!}\partial_\beta^p\partial_B^q G_{\boldsymbol{\nu}_0}(\beta_0,B_0) \sum_{\substack{k_1+\ldots+k_s=k-1 \\ k_i\geq 1}} \prod_{i=1}^p b^{\{k_i\}}_{\boldsymbol{\nu}_i}(\beta_0,B_0) \prod_{i=p+1}^s B^{\{k_i\}}_{\boldsymbol{\nu}_i}(\beta_0,B_0), \end{aligned}$$

for all $k \geq 1$ and all $\boldsymbol{\nu} \in \mathbb{Z}^d$, then (3.1) turns out to be a formal solution to the range equations (2.13). Note that we can see the formal expansion (2.9) as obtained from (3.1) by solving the bifurcation equation (2.8c) and further expanding $B_0 = B_{\mathbf{0}}(\varepsilon)$.

One could easily prove that (3.1) is formally well defined, that is that the coefficients $b^{\{k\}}(t;\beta_0,B_0)$ and $B^{\{k\}}(t;\beta_0,B_0)$ are well defined to all orders $k \geq 1$; for instance one could adapt the forthcoming diagrammatic formalism (which would rather simplify with respect to the discussion below). Unfortunately the power series may not be convergent – as far as we know –, so we have to look for a different approach: we shall see how to construct a series, convergent if β_0, B_0 are suitably chosen, whose formal expansion coincides with (2.9). To this aim we shall introduce a convenient graphical representation for the coefficients of such a series. We start by introducing some notations.

A graph is a set of points and lines connecting them. A *tree* θ is a connected graph with no cycle, such that all the lines are oriented toward a unique point (*root*) which has only one incident line ℓ_θ (*root line*). All the points in a tree except the root are called *nodes*. The orientation of the lines in a tree induces a partial ordering relation ($\preceq$) between the nodes and the lines: we can imagine that each line carries an arrow pointing toward the root. Given two nodes v and w, we shall write $w \prec v$ every time v is along the path (of lines) which connects w to the root. When drawing a tree, we shall put the root to the extreme left so that all the lines (and the corresponding arrows) will be directed from right to left; see Figure 3.1.

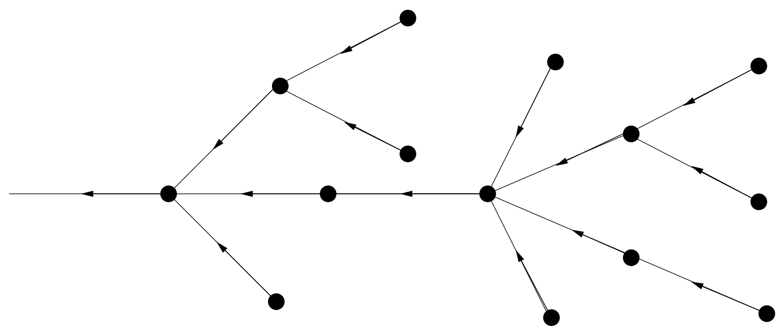

Figure 3.1: A tree with 14 nodes.

We denote by $N(\theta)$ and $L(\theta)$ the sets of nodes and lines in θ respectively. Since a line $\ell \in L(\theta)$ is uniquely identified by the node v which it leaves, we may write $\ell = \ell_v$. We write $\ell_w \prec \ell_v$ if $w \prec v$ and $w \prec \ell = \ell_v$ if $w \preceq v$; if ℓ and ℓ' are two distinct comparable lines, i.e. $\ell' \prec \ell$, we denote by $\mathcal{P}(\ell, \ell')$ the (unique) path of lines connecting ℓ' to ℓ, with ℓ and ℓ' not included (in particular $\mathcal{P}(\ell, \ell') = \emptyset$ if ℓ' enters the node ℓ exits).

Given a tree θ we associate labels with the nodes and the lines of θ, as follows.

With each node $v \in N(\theta)$ we associate a *mode label* $\boldsymbol{\nu}_v \in \mathbb{Z}^d$, a *component label* $h_v \in \{\beta, B\}$ and an *order label* $k_v \in \{0, 1\}$ with the constraint that $k_v = 1$ if $h_v = B$ or $\boldsymbol{\nu}_v \neq \mathbf{0}$. With each line $\ell = \ell_v$ we associate a pair of *component labels* $(e_\ell, u_\ell) \in \{\beta, B\}^2$, with the constraint that $u_\ell = h_v$, and a *momentum* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d_*$, except for the root line which can have either zero momentum or not, i.e. $\boldsymbol{\nu}_{\ell_\theta} \in \mathbb{Z}^d$. For any line ℓ, we call e_ℓ and u_ℓ the *e-component* and the *u-component* of ℓ, respectively

We denote by p_v and q_v the numbers of lines with e-component β and B, respectively, entering the node v and set $s_v = p_v + q_v$. If $k_v = 0$ for some $v \in N(\theta)$ we force also $p_v = 0$ and $q_v \geq 2$.

We impose the *conservation law*

$$\boldsymbol{\nu}_\ell = \sum_{v \prec \ell} \boldsymbol{\nu}_v \tag{3.2}$$

and we call *order* of θ the number

$$k(\theta) = \sum_{v \in N(\theta)} k_v. \tag{3.3}$$

Finally, we associate with each line ℓ also a *scale label* n_ℓ such that $n_\ell = -1$ if $\boldsymbol{\nu}_\ell = \mathbf{0}$, while $n_\ell \in \mathbb{Z}_+$ if $\boldsymbol{\nu}_\ell \neq \mathbf{0}$. Note that one can have $n_\ell = -1$ only if ℓ is the root line of θ.

In the following we shall call simply trees the trees with labels and we shall use the term *unlabelled tree* for the trees without labels.

We shall say that two trees are *equivalent* if they can be transformed into each other by continuously deforming the lines in such a way that these do not cross each other and also labels match. This provides an equivalence relation on the set of the trees. From now on we shall call trees tout court such equivalence classes.

A subset $T \subset \theta$ will be called a *subgraph* of θ if it is formed by a set of nodes $N(T) \subseteq N(\theta)$ and a set of of lines $L(T) \subseteq L(\theta)$ connecting them (possibly including the root line of θ) in such a way that

$N(T) \cup L(T)$ is connected. We call *order* of T the number

$$k(T) = \sum_{v \in N(T)} k_v. \tag{3.4}$$

We say that a line enters T if it connects a node $v \notin N(T)$ to a node $w \in N(T)$ and we say that a line exits T if it connects a node $v \in N(T)$ to a node $w \notin N(T)$. Of course, if a line ℓ enters or exits T, then $\ell \notin L(T)$. If θ is a labelled tree and T a subgraph of θ, then T inherits the labels of θ.

A *cluster* T on scale n is a maximal subgraph of a tree θ such that all the lines have scales $n' \leq n$ and there is at least one line with scale n. The lines entering the cluster T and the line coming out from it (unique if existing at all) are called the *external* lines of T. An example of clusters is in Figure 3.2.

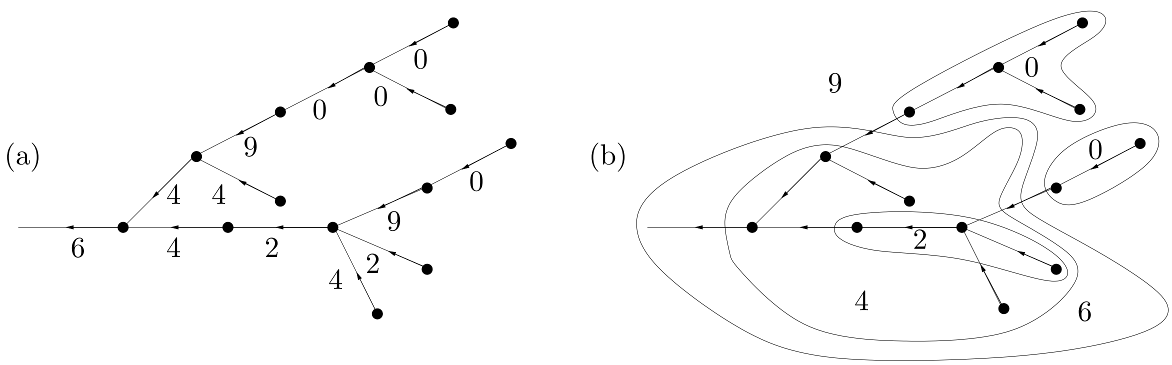

Figure 3.2: Example of clusters: (a) a tree θ is represented with (only) the scale labels associated with its lines; (b) the clusters in θ, corresponding to the same assignment of scale labels, are drawn.

A *self-energy cluster* is a cluster T such that (i) T has only one entering line ℓ'_T and one exiting line ℓ_T, (ii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$ and hence

$$\sum_{v \in N(T)} \boldsymbol{\nu}_v = \mathbf{0}. \tag{3.5}$$

Self-energy clusters will be represented graphically as in Figure 3.3. Examples of low order self-energy clusters are given in Figure 3.4.

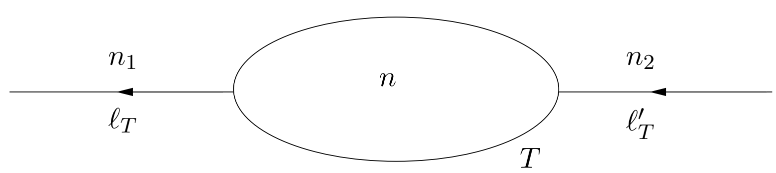

Figure 3.3: Graphical representation of the self-energy clusters T on scale n; by construction $n_1, n_2 \geq n+1$ and $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$. Note that neither ℓ'_T (the entering line) nor ℓ_T (the exiting line) belong to $L(T)$.

For any self-energy cluster T, set $\mathcal{P}_T = \mathcal{P}(\ell_T, \ell'_T)$. More generally, if T is a subgraph of θ with only one entering line ℓ' and one exiting line ℓ, we set $\mathcal{P}_T = \mathcal{P}(\ell, \ell')$. We shall say that a self-energy cluster T is on scale -1, if $N(T) = \{v\}$ (with of course $\boldsymbol{\nu}_v = \mathbf{0}$), so that $\mathcal{P}_T = \emptyset$. If a self-energy cluster is on a scale $n \geq 0$ then $|N(T)| \geq 2$ and $k(T) \geq 1$, as is easy to check.

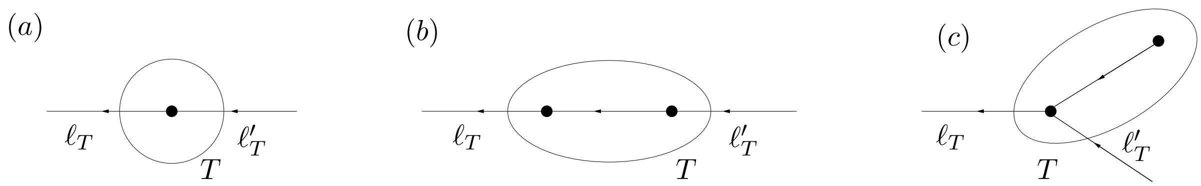

Figure 3.4: Self-energy clusters T with their external lines ℓ_T (exiting line) and ℓ'_T (entering line), for $|N(T)| = 1$ (case (a)) and $|N(T)| = 2$ (cases (b) and (c)), Note that in case (a) one has either $k(T) = 0$ or $k(T) = 1$, while in cases (b) and (c) one has either $k(T) = 1$ or $k(T) = 2$; in all cases, when $k(T) < |N(T)|$, one node $v \in N(T)$ has $k_v = 0$. The scale of T is $n = -1$ in case (a) and can be any $n \in \mathbb{Z}_+$ in cases (b) and (c).

Remark 3.1. Given a self-energy cluster T, the momenta of the lines in $\mathcal{P}_T$ depend on $\boldsymbol{\nu}_{\ell'_T}$ because of the conservation law (3.2). More precisely, for all $\ell \in \mathcal{P}_T$ one has $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}^0_\ell + \boldsymbol{\nu}_{\ell'_T}$ with

$$\boldsymbol{\nu}^0_\ell = \sum_{\substack{w \in N(T) \\ w \prec \ell}} \boldsymbol{\nu}_w,$$

while all the other momenta in T do not depend on $\boldsymbol{\nu}_{\ell'_T}$.

We shall say that two self-energy clusters T_1, T_2 have the same *structure* if setting $\boldsymbol{\nu}_{\ell'_{T_1}} = \boldsymbol{\nu}_{\ell'_{T_2}} = \mathbf{0}$ one has $T_1 = T_2$. This provides an equivalence relation on the set of all self-energy clusters. From now on we shall call self-energy clusters tout court such equivalence classes.

A *renormalised tree* is a tree in which no self-energy clusters appear; analogously a *renormalised subgraph* is a subgraph of a tree θ which does not contains any self-energy cluster. Note that if T is a renormalised self-energy cluster and $N(T) \geq 2$ then $k(T) \geq 2$.

Given a tree θ we call *total momentum* of θ the momentum associated with ℓ_θ and *total component* of θ the e-component of ℓ_θ. We denote by $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ the set of renormalised trees with order k, total momentum $\boldsymbol{\nu}$ and total component h; the set of renormalised self-energy clusters T on scale n such that $u_{\ell_T} = u$ and $e_{\ell'_T} = e$ will be denoted by $\mathfrak{R}_{n,u,e}$.

Lemma 3.2. *Let T be a subgraph of any tree θ. Then one has $|N(T)| \leq 3k(T) - 1$.*

Proof. We shall prove the result by induction on $k = k(T)$. For $k = 1$ the bound is trivially satisfied as a direct check shows. Assume then the bound to hold for all $k' < k$. Call v the node which ℓ_T (possibly ℓ_θ) exits, $\ell_1, \ldots, \ell_{s_v}$ the lines entering v and $T_1, \ldots, T_{s_v}$ the subgraphs of T with exiting lines $\ell_1, \ldots, \ell_{s_v}$. If $k_v = 1$ then by the inductive hypothesis one has

$$|N(T)| = 1 + \sum_{i=1}^{s_v} |N(T_i)| \leq 1 + 3(k-1) - s_v.$$

If $k_v = 0$ one has $s_v = q_v \geq 2$ and hence

$$|N(T)| = 1 + \sum_{i=1}^{q_v} |N(T_i)| \leq 1 + 3k - q_v.$$

so that in both cases the bound follows. ■

Remark 3.3. For any subgraph T, one has

$$\sum_{v \in N(T)} s_v \le |L(T)| + 1 \le |N(T)| + 1 \le 3k(T).$$

In particular $\sum_{v \in N(\theta)} s_v \le 3k(\theta)$.

For any $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ we associate with each node $v \in N(\theta)$ a *node factor*

$$\mathcal{F}_v = \mathcal{F}_v(\beta_0, B_0) := \begin{cases} \dfrac{1}{p_v! q_v!} \partial_\beta^{p_v} \partial_B^{q_v} F_{\boldsymbol{\nu}_v}(\beta_0, B_0), & h_v = \beta, k_v = 1, \\ \dfrac{1}{q_v!} \partial^{q_v} \omega_0(B_0), & h_v = \beta, k_v = 0, \\ \dfrac{1}{p_v! q_v!} \partial_\beta^{p_v} \partial_B^{q_v} G_{\boldsymbol{\nu}_v}(\beta_0, B_0), & h_v = B, k_v = 1. \end{cases} \tag{3.6}$$

With each line $\ell = \ell_v$ we associate a *propagator* $\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon, \beta_0, B_0)$ defined recursively as follows.

Let us introduce the sequences $\{m_n, p_n\}_{n \ge 0}$, with $m_0 = 0$ and, for all $n \ge 0$, $m_{n+1} = m_n + p_n + 1$, where $p_n := \max\{q \in \mathbb{Z}_+ : \alpha_{m_n}(\boldsymbol{\omega}) < 2\alpha_{m_n+q}(\boldsymbol{\omega})\}$. Then the subsequence $\{\alpha_{m_n}(\boldsymbol{\omega})\}_{n \ge 0}$ of $\{\alpha_m(\boldsymbol{\omega})\}_{m \ge 0}$ is decreasing. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function, non-increasing for $x \ge 0$ and non-decreasing for $x < 0$, such that

$$\chi(x) = \begin{cases} 1, & |x| \le 1/2, \\ 0, & |x| \ge 1. \end{cases} \tag{3.7}$$

Set $\chi_{-1}(x) = 1$ and $\chi_n(x) = \chi(8x/\alpha_{m_n}(\boldsymbol{\omega}))$ for $n \ge 0$. Set also $\psi(x) = 1 - \chi(x)$, $\psi_n(x) = \psi(8x/\alpha_{m_n}(\boldsymbol{\omega}))$, and $\Psi_n(x) = \chi_{n-1}(x)\psi_n(x)$, for $n \ge 0$; see Figure 3.5.

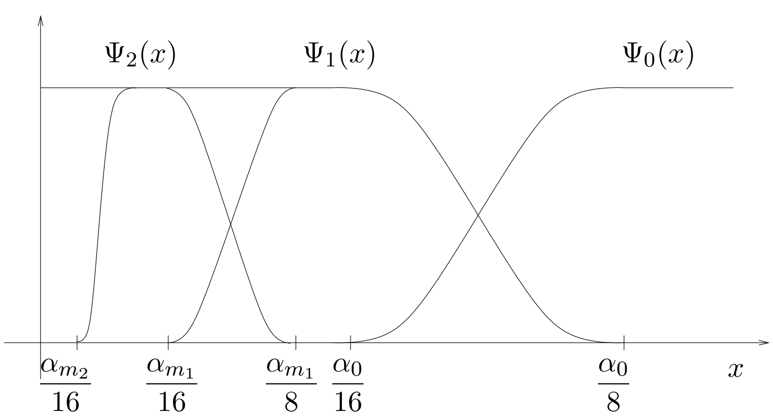

Figure 3.5: Graphs of some of the C^∞ functions $\Psi_n(x)$ partitioning the unity in $\mathbb{R} \setminus \{0\}$; here $\alpha_m = \alpha_m(\boldsymbol{\omega})$. The function $\chi_0(x) = \chi(8x/\alpha_0)$ is given by the sum of all functions $\Psi_n(x)$ for $n \ge 1$.

Lemma 3.4. *For all $x \neq 0$ and for all $p \ge 0$ one has*

$$\psi_p(x) + \sum_{n \ge p+1} \Psi_n(x) = 1.$$

Proof. For fixed $x \neq 0$ let $N = N(x) := \min\{n : \chi_n(x) = 0\}$ and note that $\max\{n : \psi_n(x) = 0\} \leq N-1$. Then if $p \leq N-1$

$$\psi_p(x) + \sum_{n \geq p+1} \Psi_n(x) = \psi_{N-1}(x) + \chi_{N-1}(x) = 1,$$

while if $p \geq N$ one has

$$\psi_p(x) + \sum_{n \geq p+1} \Psi_n(x) = \psi_p(x) = 1.$$

∎

Then for $n \geq 0$ we set formally

$$\begin{aligned} \mathcal{G}^{[n]}(x;\varepsilon,\beta_0,B_0) &= \begin{pmatrix} \mathcal{G}^{[n]}_{\beta,\beta}(x;\varepsilon,\beta_0,B_0) & \mathcal{G}^{[n]}_{\beta,B}(x;\varepsilon,\beta_0,B_0) \\ \\ \mathcal{G}^{[n]}_{B,\beta}(x;\varepsilon,\beta_0,B_0) & \mathcal{G}^{[n]}_{B,B}(x;\varepsilon,\beta_0,B_0) \end{pmatrix} \\ &:= \Psi_n(x) \left((\mathrm{i}x)\mathbb{1} - \mathcal{M}^{[n-1]}(x;\varepsilon,\beta_0,B_0) \right)^{-1}, \end{aligned} \tag{3.8}$$

where (here and henceforth) $\mathbb{1}$ is the 2×2 identity matrix and

$$\mathcal{M}^{[n-1]}(x;\varepsilon,\beta_0,B_0) := \sum_{q=-1}^{n-1} \chi_q(x) M^{[q]}(x;\varepsilon,\beta_0,B_0), \tag{3.9}$$

where, for $n \geq -1$, $M^{[n]}(x;\varepsilon,\beta_0,B_0)$ is the 2×2 matrix

$$M^{[n]}(x;\varepsilon,\beta_0,B_0) := \begin{pmatrix} M^{[n]}_{\beta,\beta}(x;\varepsilon,\beta_0,B_0) & M^{[n]}_{\beta,B}(x;\varepsilon,\beta_0,B_0) \\ \\ M^{[n]}_{B,\beta}(x;\varepsilon,\beta_0,B_0) & M^{[n]}_{BB}(x;\varepsilon,\beta_0,B_0) \end{pmatrix}, \tag{3.10}$$

with formally

$$M^{[n]}_{u,e}(x;\varepsilon,\beta_0,B_0) := \sum_{T \in \mathfrak{R}_{n,u,e}} \varepsilon^{k(T)} \mathscr{V}_T(x;\varepsilon,\beta_0,B_0), \tag{3.11}$$

and $\mathscr{V}_T(x;\varepsilon,\beta_0,B_0)$ is the *renormalised value* of T, defined as

$$\mathscr{V}_T(x;\varepsilon,\beta_0,B_0) := \left(\prod_{v \in N(T)} \mathcal{F}_v \right) \left(\prod_{\ell \in L(T)} \mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell;\varepsilon,\beta_0,B_0) \right). \tag{3.12}$$

Here and henceforth, the sums and the products over empty sets have to be considered as zero and 1, respectively. Note that $\mathscr{V}_T$ depends on ε – because the propagators do –; moreover it depends on $x = \boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'_T}$ only through the propagators associated with the lines $\ell \in \mathcal{P}_T$ (see Remark 3.1).

Set $\mathcal{M} := \{\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0)\}_{n \geq -1}$. We call *self-energies* the matrices $\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0)$.

Remark 3.5. One has

$$\partial_c \mathcal{G}^{[n]}_{e,u}(x;\varepsilon,\beta_0,B_0) = \left(\mathcal{G}^{[n]}(x;\varepsilon,\beta_0,B_0) \partial_c \mathcal{M}^{[n-1]}(x;\varepsilon,\beta_0,B_0) \left((\mathrm{i}x)\mathbb{1} - \mathcal{M}^{[n-1]}(x;\varepsilon,\beta_0,B_0) \right)^{-1} \right)_{e,u}$$

for both $c = \beta_0, B_0$.

Setting also $\mathcal{G}^{[-1]}(0;\varepsilon,\beta_0,B_0)=\mathbb{1}$, for any subgraph S of any $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ define the *renormalised value* of S as

$$\mathscr{V}(S;\varepsilon,\beta_0,B_0):=\left(\prod_{v\in N(S)}\mathcal{F}_v\right)\left(\prod_{\ell\in L(S)}\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0,B_0)\right). \tag{3.13}$$

We define

$$b^{[k]}_{\boldsymbol{\nu}}(\varepsilon,\beta_0,B_0):=\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\beta}}\mathscr{V}(\theta;\varepsilon,\beta_0,B_0),\qquad B^{[k]}_{\boldsymbol{\nu}}(\varepsilon,\beta_0,B_0):=\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},B}}\mathscr{V}(\theta;\varepsilon,\beta_0,B_0), \tag{3.14}$$

for any $\boldsymbol{\nu}\neq\mathbf{0}$, and

$$\Phi^{[k]}_{\mathbf{0}}(\varepsilon,\beta_0,B_0):=\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\mathbf{0},\beta}}\mathscr{V}(\theta;\varepsilon,\beta_0,B_0),\qquad \Gamma^{[k]}_{\mathbf{0}}(\varepsilon,\beta_0,B_0):=\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\mathbf{0},B}}\mathscr{V}(\theta;\varepsilon,\beta_0,B_0). \tag{3.15}$$

Set (again formally)

$$\begin{aligned}b^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)&:=\sum_{k\geq1}\varepsilon^k\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}b^{[k]}_{\boldsymbol{\nu}}(\varepsilon,\beta_0,B_0),\\ \widetilde{B}^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)&:=\sum_{k\geq1}\varepsilon^k\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}B^{[k]}_{\boldsymbol{\nu}}(\varepsilon,\beta_0,B_0),\end{aligned} \tag{3.16}$$

and

$$\Phi^{\mathcal{R}}_{\mathbf{0}}(\varepsilon;\beta_0,B_0):=\sum_{k\geq0}\varepsilon^k\Phi^{[k]}_{\mathbf{0}}(\varepsilon,\beta_0,B_0),\qquad \Gamma^{\mathcal{R}}_{\mathbf{0}}(\varepsilon;\beta_0,B_0):=\sum_{k\geq0}\varepsilon^k\Gamma^{[k]}_{\mathbf{0}}(\varepsilon,\beta_0,B_0), \tag{3.17}$$

and define $\beta^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)=\beta_0+b^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)$ and $B^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)=B_0+\widetilde{B}^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)$. Set also $\Theta^{\mathcal{R},n}_{k,\boldsymbol{\nu},h}=\{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}:n_\ell\leq n\text{ for all }\ell\in L(\theta)\}$ and define

$$\begin{aligned}\Phi^{\mathcal{R},n}_{\mathbf{0}}(\varepsilon;\beta_0,B_0)&:=\sum_{k\geq0}\varepsilon^k\sum_{\theta\in\Theta^{\mathcal{R},n}_{k,\mathbf{0},\beta}}\mathscr{V}(\theta;\varepsilon,\beta_0,B_0),\\ \Gamma^{\mathcal{R},n}_{\mathbf{0}}(\varepsilon;\beta_0,B_0)&:=\sum_{k\geq0}\varepsilon^k\sum_{\theta\in\Theta^{\mathcal{R},n}_{k,\mathbf{0},B}}\mathscr{V}(\theta;\varepsilon,\beta_0,B_0).\end{aligned} \tag{3.18}$$

Remark 3.6. One has

$$\mathcal{M}^{[-1]}(x;\varepsilon,\beta_0,B_0)=M^{[-1]}(x;\varepsilon,\beta_0,B_0)=\begin{pmatrix}\varepsilon\partial_{\beta_0}F_{\mathbf{0}}(\beta_0,B_0) & \omega_0'(B_0)+\varepsilon\partial_{B_0}F_{\mathbf{0}}(\beta_0,B_0)\\ \varepsilon\partial_{\beta_0}G_{\mathbf{0}}(\beta_0,B_0) & \varepsilon\partial_{B_0}G_{\mathbf{0}}(\beta_0,B_0)\end{pmatrix},$$

where $\omega_0'(B_0)\neq0$ for B_0 close enough to $\overline{B}_0$ by Hypothesis 2. In particular $\mathcal{M}^{[-1]}(x;\varepsilon,\beta_0,B_0)$ does not depend on x and is a real-valued matrix.

Remark 3.7. If T is a renormalised self-energy cluster, then $\mathscr{V}(T;\varepsilon,\beta_0,B_0)=\mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_T'};\varepsilon,\beta_0,B_0)$.

Remark 3.8. Given a renormalised tree θ such that $\mathscr{V}(\theta;\varepsilon,\beta_0,B_0)\neq0$, for any line $\ell\in L(\theta)$ (except possibly the root line) one has $\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\neq0$ and hence

$$\frac{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})}{16}<|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|<\frac{\alpha_{m_{n_\ell-1}}(\boldsymbol{\omega})}{8}, \tag{3.19}$$

where $\alpha_{m_{-1}}(\boldsymbol{\omega})$ has to be interpreted as $+\infty$. Note also that $\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\neq 0$ implies

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| < \frac{1}{8}\alpha_{m_{n_\ell-1}}(\boldsymbol{\omega}) < \frac{1}{4}\alpha_{m_{n_\ell-1}+p_{n_\ell-1}}(\boldsymbol{\omega}) = \frac{1}{4}\alpha_{m_{n_\ell}-1}(\boldsymbol{\omega}) < \alpha_{m_{n_\ell}-1}(\boldsymbol{\omega})$$

and hence, by definition of $\alpha_m(\boldsymbol{\omega})$, one has $|\boldsymbol{\nu}_\ell| > 2^{m_{n_\ell}-1}$. Moreover, by the definition of $\{\alpha_{m_n}(\boldsymbol{\omega})\}_{n\geq 0}$, the number of scales which can be associated with a line ℓ in such a way that the propagator does not vanishes is at most 2. The same considerations apply to any subgraph of θ and to any renormalised self-energy cluster.

For any renormalised subgraph S of any tree θ we denote by $\mathfrak{N}_n(S)$ the number of lines on scale $\geq n$ in S and set

$$K(S) := \sum_{v\in N(S)} |\boldsymbol{\nu}_v|.$$

Lemma 3.9. *For any $h\in\{\beta,B\}$, $\boldsymbol{\nu}\in\mathbb{Z}^d$, $k\geq 1$ and for any $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ such that $\mathscr{V}(\theta;\varepsilon,\beta_0,B_0)\neq 0$, one has $\mathfrak{N}_n(\theta)\leq 2^{-(m_n-2)}K(\theta)$ for all $n\geq 0$.*

Proof. First of all we note that if $\mathfrak{N}_n(\theta)\geq 1$, then there is at least one line ℓ with $n_\ell = n$ and hence $K(\theta)\geq|\boldsymbol{\nu}_\ell|\geq 2^{m_n-1}$ (see Remark 3.8). Now we prove the bound $\mathfrak{N}_n(\theta)\leq\max\{2^{-(m_n-2)}K(\theta)-1,0\}$ by induction on the order.

If the root line of θ has scale $n_{\ell_\theta} < n$ then the bound follows by the inductive hypothesis. If $n_{\ell_\theta}\geq n$, call $\ell_1,\ldots,\ell_r$ the lines with scale $\geq n$ closest to ℓ_θ (that is such that $n_{\ell'} < n$ for all lines $\ell'\in\mathcal{P}(\ell_\theta,\ell_i)$, $i=1,\ldots,r$). If $r=0$ then $\mathfrak{N}_n(\theta)=1$ and $|\boldsymbol{\nu}|\geq 2^{m_n-1}$, so that the bound follows. If $r\geq 2$ the bound follows once more by the inductive hypothesis. If $r=1$, then ℓ_1 is the only entering line of a cluster T which is not a renormalised self-energy cluster as $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ and hence $\boldsymbol{\nu}_{\ell_1}\neq\boldsymbol{\nu}$. But then

$$|\boldsymbol{\omega}\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}_{\ell_1})|\leq|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|+|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}|\leq\frac{1}{4}\alpha_{m_{n-1}}(\boldsymbol{\omega}) < \alpha_{m_{n-1}+p_{n-1}}(\boldsymbol{\omega}) = \alpha_{m_n-1}(\boldsymbol{\omega}),$$

as both ℓ_θ and ℓ_1 are on scale $\geq n$, so that one has $K(T)\geq|\boldsymbol{\nu}-\boldsymbol{\nu}_{\ell_1}|\geq 2^{m_n-1}$. Now, call θ_1 the subtree of θ with root line ℓ_1. Then one has $\mathfrak{N}_n(\theta)=1+\mathfrak{N}_n(\theta_1)\leq 1+\max\{2^{-(m_n-2)}K(\theta_1)-1,0\}$, so that $\mathfrak{N}_n(\theta)\leq 2^{-(m_n-2)}(K(\theta)-K(T))\leq 2^{-(m_n-2)}K(\theta)-1$, again by induction. ∎

Lemma 3.10. *For any $e,u\in\{\beta,B\}$, $n\geq 0$ and for any $T\in\mathfrak{R}_{n,u,e}$ such that $\mathscr{V}_T(x;\varepsilon,\beta_0,B_0)\neq 0$, one has $K(T) > 2^{m_n-1}$ and $\mathfrak{N}_p(T)\leq 2^{-(m_p-2)}K(T)$ for $0\leq p\leq n$.*

Proof. We first prove that for all $n\geq 0$ and all $T\in\mathfrak{R}_{n,u,e}$, one has $K(T)\geq 2^{m_n-1}$. In fact if $T\in\mathfrak{R}_{n,u,e}$ then T contains at least a line on scale n. If there is $\ell\in L(T)\setminus\mathcal{P}_T$ with $n_\ell = n$, then $K(T)\geq|\boldsymbol{\nu}_\ell| > 2^{m_n-1}$ (see Remark 3.8). Otherwise, let $\ell\in\mathcal{P}_T$ be the line on scale n which is closest to ℓ'_T. Call $\widetilde{T}$ the subgraph (actually the cluster) consisting of all lines and nodes of T preceding ℓ. Then $\boldsymbol{\nu}_\ell\neq\boldsymbol{\nu}_{\ell'_T}$, otherwise $\widetilde{T}$ would be a renormalised self-energy cluster. Therefore $K(T) > |\boldsymbol{\nu}_\ell-\boldsymbol{\nu}_{\ell'_T}| > 2^{m_n-1}$ as both ℓ,ℓ'_T are on scale $\geq n$.

Given a tree θ, call $\mathcal{C}(n,p)$ the set of renormalised subgraphs T of θ with only one entering line ℓ'_T and one exiting line ℓ_T both on scale $\geq p$, such that $L(T)\neq\emptyset$ and $n_\ell\leq n$ for any $\ell\in L(T)$. Note that $\mathfrak{R}_{n,u,e}\subset\mathcal{C}(n,p)$ for all $n,p\geq 0$ and $u,e\in\{\beta_0,B_0\}$. We prove that $\mathfrak{N}_p(T)\leq\max\{K(T)2^{-(m_p-2)}-1,0\}$ for $0\leq p\leq n$ and all $T\in\mathcal{C}(n,p)$. The proof is by induction on the order. Call $N(\mathcal{P}_T)$ the set of nodes in T connected by lines in $\mathcal{P}_T$. If all lines in $\mathcal{P}_T$ are on scale $< p$, then $\mathfrak{N}_p(T) = \mathfrak{N}_p(\theta_1) + \ldots + \mathfrak{N}_p(\theta_r)$ if $\theta_1, \ldots, \theta_r$ are the subtrees with root line entering a node in $N(\mathcal{P}_T)$ and hence the bound follows from (the proof of) Lemma 3.9. If there exists a line $\ell \in \mathcal{P}_T$ on scale $\geq p$, call T_1 and T_2 the subgraphs of T such that $L(T) = \{\ell\} \cup L(T_1) \cup L(T_2)$. Note that if $L(T_1), L(T_2) \neq \emptyset$, then $T_1, T_2 \in \mathcal{C}(n,p)$. Hence, by the inductive hypothesis one has

$$\mathfrak{N}_p(T) = 1 + \mathfrak{N}_p(T_1) + \mathfrak{N}_p(T_2) \leq 1 + \max\{2^{-(m_p-2)}K(T_1) - 1, 0\} + \max\{2^{-(m_p-2)}K(T_2) - 1, 0\}.$$

If both $\mathfrak{N}_p(T_1), \mathfrak{N}_p(T_2)$ are zero the bound follows as $K(T) \geq 2^{m_p-1}$, while if both are non-zero one has $\mathfrak{N}_p(T) \leq 2^{-(m_p-2)}(K(T_1) + K(T_2)) - 1 = 2^{-(m_p-2)}K(T) - 1$. Finally if only one is zero, say $\mathfrak{N}_p(T_1) \neq 0$ and $\mathfrak{N}_p(T_2) = 0$, then $\mathfrak{N}_p(T) \leq 2^{-(m_p-2)}K(T_1) = 2^{-(m_p-2)}K(T) - 2^{-(m_p-2)}K(T_2)$. On the other hand, in such a case T_2 is a cluster and hence $\boldsymbol{\nu}_\ell \neq \boldsymbol{\nu}_{\ell'_T}$, which implies $K(T_2) \geq 2^{m_p-1}$. The same argument can be used in the case $\mathfrak{N}_p(T_1) = 0$ and $\mathfrak{N}_p(T_2) \neq 0$. ■

Remark 3.11. Inequality (3.19) has been repeatedly used in the proof of Lemmas 3.9 and 3.10. In fact the proof works – as one can easily check – under the weaker condition that

$$\frac{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})}{32} < |\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell| < \frac{\alpha_{m_{n_\ell - 1}}(\boldsymbol{\omega})}{4} \tag{3.20}$$

as long as $\Psi_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) \neq 0$. This observation will be used later on (see Lemma 4.6 below).

4 Convergence of the resummed series: dimensional bounds

Now we shall prove that, under the assumption that the propagators $\mathcal{G}^{[n]}_{e,u}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon, \beta_0, B_0)$ are bounded proportionally to $1/(\boldsymbol{\omega} \cdot \boldsymbol{\nu})^2$, the series (3.16) converge and solve the range equations (2.13). Then, in the next section, we shall see that the assumption is justified at least along a curve $(\beta_0(\varepsilon), B_0(\varepsilon))$ satisfying also the bifurcation equations (2.8c) and (2.8d). We shall not write the dependence on $\varepsilon, \beta_0, B_0$ unless needed.

Definition 4.1. *We shall say that $\mathcal{M}$ satisfies property 1 if one has*

$$\Psi_{n+1}(x) \left|\det\left((\mathrm{i}x)\mathbb{1} - \mathcal{M}^{[n]}(x)\right)\right| \geq \Psi_{n+1}(x) x^2/2,$$

for all $n \geq -1$.

Definition 4.2. *We shall say that $\mathcal{M}$ satisfies property 1-p if one has*

$$\Psi_{n+1}(x) \left|\det\left((\mathrm{i}x)\mathbb{1} - \mathcal{M}^{[n]}(x)\right)\right| \geq \Psi_{n+1}(x) x^2/2.$$

for $-1 \leq n < p$.

Lemma 4.3. *Assume $\mathcal{M}$ to satisfy property 1-p. Then, for $0 \leq n \leq p$ and ε small enough, the self-energies are well defined and one has*

$$|M^{[n]}_{u,e}(x)| \leq |\varepsilon|\, K_1 \mathrm{e}^{-K_2 2^{m_n}}, \tag{4.1a}$$

$$|\partial^j_x M^{[n]}_{u,e}(x)| \leq |\varepsilon|\, C_j \mathrm{e}^{-\overline{C}_j 2^{m_n}}, \qquad j = 1, 2, \tag{4.1b}$$

for some constants $K_1, K_2, C_1, C_2, \overline{C}_1$ and $\overline{C}_2$.

Proof. We shall prove first (4.1a) by induction on n. Let $n \le p$ and $T \in \mathfrak{R}_{n,u,e}$. The analyticity of F, G and ω_0 implies that there exist positive constants F_1, F_2, ξ such that for all $v \in N(T)$ one has

$$|\mathcal{F}_v| \le F_1 F_2^{s_v} \mathrm{e}^{-\xi|\boldsymbol{\nu}_v|}.$$

Note that

$$\prod_{v\in N(T)} \mathrm{e}^{-\frac{1}{4}\xi|\boldsymbol{\nu}_v|} = \exp\Big(-\frac{1}{4}\xi\, K(T)\Big) < \exp\Big(-\frac{1}{8}\xi\, 2^{m_n}\Big),$$

by Lemma 3.10. Moreover by property 1-p and the inductive hypothesis, one has (for instance)

$$\begin{aligned}|\mathcal{G}^{[n']}_{\beta,\beta}(x)| &\le \frac{2}{x^2}\Big(|\mathrm{i}x| + \big|\mathcal{M}^{[n'-1]}_{B,B}(x)\big|\Big)\Psi_{n'}(x)\\ &\le \frac{2}{x^2}\Big(|x| + P_1 + |\varepsilon|^2 K_1 \sum_{q=0}^{n'-1} \mathrm{e}^{-K_2 2^{m_q}}\Big)\Psi_{n'}(x) \le \gamma_0\, \alpha_{m_{n'}}(\boldsymbol{\omega})^{-2}\end{aligned}$$

for all $0 \le n' \le n$ and for a suitable constant γ_0, where we used that any renormalised self-energy cluster T on scale ≥ 0 has $k(T) \ge 2$ and that there exists $P_1 \ge 0$ such that $|\mathcal{M}^{[-1]}_{u,e}| \le P_1$ (see Remark 3.6). Of course one can reason analogously for $\mathcal{G}^{[n']}_{\beta,B}(x)$, $\mathcal{G}^{[n']}_{B,\beta}(x)$ and $\mathcal{G}^{[n']}_{\beta,B}(x)$, possibly redefining γ_0. Hence by Lemmas 3.10 and 3.2 one can bound

$$\begin{aligned}\prod_{\ell\in L(T)} |\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)| &\le \prod_{q\ge 0}\left(\frac{\gamma_0}{\alpha_{m_q}(\boldsymbol{\omega})^2}\right)^{\mathfrak{N}_q(T)} \le \left(\frac{\gamma_0}{\alpha_{m_{n_0}}(\boldsymbol{\omega})^2}\right)^{3k(T)-1} \prod_{q\ge n_0+1}\left(\frac{\gamma_0}{\alpha_{m_q}(\boldsymbol{\omega})^2}\right)^{\mathfrak{N}_q(T)}\\ &\le \left(\frac{\gamma_0}{\alpha_{m_{n_0}}(\boldsymbol{\omega})^2}\right)^{3k(T)-1} \prod_{q\ge n_0+1}\left(\frac{\gamma_0^{1/2}}{\alpha_{m_q}(\boldsymbol{\omega})}\right)^{2^{-(m_q-3)}K(T)}\\ &\le D(n_0)^{3k(T)-1}\exp(\xi(n_0)K(T)),\end{aligned}$$

with

$$D(n_0) = \frac{\gamma_0}{\alpha_{m_{n_0}}(\boldsymbol{\omega})^2}, \qquad \xi(n_0) = 8\sum_{q\ge n_0+1}\frac{1}{2^{m_q}}\log\frac{\gamma_0^{1/2}}{\alpha_{m_q}(\boldsymbol{\omega})}.$$

Then, by Hypothesis 1, one can choose n_0 such that $\xi(n_0) \le \xi/2$. Furthermore, Lemma 3.2 ensures also that the sum over the other labels is bounded by a constant to the power $k(T)$ and hence one can bound, for some positive constants C and K_0,

$$|M^{[n]}_{u,e}(x)| \le \sum_{T\in\mathfrak{R}_{n,u,e}} |\varepsilon|^{k(T)} |\mathscr{V}_T(x)| \le \sum_{T\in\mathfrak{R}_{n,u,e}} |\varepsilon|^{k(T)} C^{k(T)} \mathrm{e}^{-K_0 K(T)} \le \sum_{k\ge 2} |\varepsilon|^k C^k \mathrm{e}^{-K_2 2^{m_n}}, \tag{4.2}$$

with $K_2 = K_0/2$, then (4.1a) is proved for ε small enough. Now we prove (4.1b), again by induction on n. For $n = 0$ the bound is obvious. Assume then (4.1b) to hold for all $n' < n$. For any $T \in \mathfrak{R}_{n,u,e}$ such that $\mathscr{V}_T(x) \ne 0$ one has

$$\partial_x \mathscr{V}_T(x) = \sum_{\ell\in\mathcal{P}_T}\Bigg(\prod_{v\in N(T)}\mathcal{F}_v\Bigg)\Bigg(\partial_x \mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(x_\ell) \prod_{\ell'\in L(T)\setminus\{\ell\}} \mathcal{G}^{[n_{\ell'}]}_{e_{\ell'},u_{\ell'}}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'})\Bigg), \tag{4.3}$$

where $x_\ell = \boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell = x + \boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell^0$ and

$$\partial_x \mathcal{G}^{[n_\ell]}(x_\ell) = \frac{d}{dx}\mathcal{G}^{[n_\ell]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell^0 + x) = \partial_x \Psi_{n_\ell}(x_\ell)\Big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n_\ell-1]}(x_\ell)\Big)^{-1} \\ - \Psi_{n_\ell}(x_\ell)\left((\mathrm{i}x)\mathbb{1} - \mathcal{M}^{[n_\ell-1]}(x_\ell)\right)^{-2}\left(\mathrm{i}\mathbb{1} - \partial_x\mathcal{M}^{[n_\ell-1]}(x_\ell)\right).$$

One has

$$|\partial_x \Psi_{n_\ell}(x_\ell)| \le |\partial_x \chi_{n_\ell-1}(x_\ell)| + |\partial_x \psi_{n_\ell}(x_\ell)| \le \frac{B_1}{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})},$$

for some constant B_1 and, by (4.1a), the inductive hypothesis and Hypothesis 1,

$$|\partial_x \mathcal{M}_{u,e}^{[n_\ell-1]}(x_\ell)| \le \sum_{q=0}^{n_\ell-1} |(\partial_x\chi_q(x_\ell))M_{u,e}^{[q]}(x_\ell)| + \sum_{q=0}^{n_\ell-1} |\partial_x M_{u,e}^{[q]}(x_\ell)| \\ \le |\varepsilon|\, B_1K_1 \sum_{q\ge0} \frac{1}{\alpha_{m_q}(\boldsymbol{\omega})}\mathrm{e}^{-K_2 2^{m_q}} + |\varepsilon|\, C_1 \sum_{q\ge 0} \mathrm{e}^{-\overline{C}_1 2^{m_q}} \le |\varepsilon|\, B_2,$$

for some constant B_2. Hence the differentiated propagator $\partial_x \mathcal{G}_{e_\ell,u_\ell}^{[n_\ell]}(x_\ell)$ can be bounded by $\gamma_1\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^{-4}$ for some constant γ_1. Possibly redefining the constant γ_1, also the propagators of the lines $\ell' \neq \ell$ in (4.3) can be bounded by $\gamma_1\, \alpha_{m_{n_{\ell'}}}(\boldsymbol{\omega})^{-4}$, and hence, at the cost of replacing the previous bound $\gamma_0\alpha_{m_n}(\boldsymbol{\omega})^{-2}$ for the propagators $\mathcal{G}^{[n]}(x)$ with $\gamma_1\alpha_{m_n}(\boldsymbol{\omega})^{-4}$, one can reason as in the proof of (4.1a) to obtain (4.1b) for $j=1$. For $j=2$ one can reason analogously. ■

Remark 4.4. From the proof of Lemma 4.3 it follows that if $\mathcal{M}$ satisfies property 1-p the matrices $\mathcal{M}^{[n]}(x)$ and $\mathcal{G}^{[n]}(x)$ are well defined for all $-1 \le n \le p$. In particular there exists $\gamma_0 > 0$ such that $|\mathcal{G}_{e,u}^{[n]}(x)| \le \gamma_0\, \alpha_{m_n}(\boldsymbol{\omega})^{-2}$ for all $0 \le n \le p$. Moreover if $\mathcal{M}$ satisfies property 1, the same considerations apply for all $n \ge 0$.

Lemma 4.5. *Assume $\mathcal{M}$ to satisfy property 1-p. Then one has*

$$\mathcal{M}^{[n]}(-x) = \mathcal{M}^{[n]}(x)^* \tag{4.4}$$

for all $-1 \le n \le p$.

Proof. We shall prove the result by induction on n. For $n=-1$ the result is obvious; see Remark 3.6. Assume (4.4) to hold up to scale $n-1$. Then, by definition, one has also $\mathcal{G}^{[q]}(-x) = \mathcal{G}^{[q]}(x)^*$ for all $0 \le q \le n$. For any renormalised self-energy cluster T contributing to $M^{[n]}(x)$, consider the renormalised self-energy cluster T' obtained from T by replacing the mode labels $\boldsymbol{\nu}_v$ with $-\boldsymbol{\nu}_v$ and changing the sign of the momentum of the entering line. Then the node factors are changed into their complex conjugated, and this holds also for the propagators because of the conservation law (3.2). Then $\mathscr{V}_{T'}(-x) = \mathscr{V}_T(x)^*$. This is enough to prove the assertion. ■

Lemma 4.6. *Assume $\mathcal{M}$ to satisfy property 1-p. Then, for $0 \le n \le p$ and ε small enough, one has*

$$\left| M_{u,e}^{[n]}(x) - M_{u,e}^{[n]}(0) - x\,\partial_x M_{u,e}^{[n]}(0)\right| \le |\varepsilon|\, K_3 \mathrm{e}^{-\overline{K}_4 2^{m_n}} x^2 \tag{4.5}$$

for some constants K_3 and K_4.

Proof. For $x^2 > |\varepsilon|$ the bound follows trivially from Lemma 4.3: thus, we may assume in the following $x^2 \le |\varepsilon|$. Consider a self-energy cluster T whose value $\mathscr{V}_T(x)$ contributes to $M^{[n]}_{u,e}(x)$ through (3.11) and set $\mathcal{A}_T(x) = \mathscr{V}_T(x) - \mathscr{V}_T(0) - x\,\partial_x \mathscr{V}_T(0)$. Define also

$$\overline{n} = \min\{n \in \mathbb{Z}_+ : K(T) \le 2^{m_n}\}.$$

Let us distinguish between the two cases: (a) $2^{m_{\overline{n}}-1} < K(T) \le 2^{m_{\overline{n}}}$ and (b) $2^{m_{\overline{n}-1}} < K(T) \le 2^{m_{\overline{n}}-1}$.

In case (a), if $\alpha_{m_{\overline{n}}}(\boldsymbol{\omega}) \le 4|x|$ then one can bound $|\mathcal{A}_T(x)| \le |\mathscr{V}_T(x)| + |\mathscr{V}_T(0)| + |x\,\partial_x \mathscr{V}_T(0)|$. As soon as $\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell) \ne 0$ for all $\ell \in L(T)$, by (the proof of) Lemma 4.3 – see in particular (4.2) – each contribution can be bounded as

$$\begin{aligned}|\varepsilon|^{k(T)} C^k \mathrm{e}^{-K_0 K(T)} &\le |\varepsilon|^{k(T)} C^k \mathrm{e}^{-(K_0/2)K(T)} \mathrm{e}^{-(K_0/2)2^{m_{\overline{n}}-1}} \\ &\le |\varepsilon|^{k(T)} C^k \mathrm{e}^{-(K_0/2)K(T)} \alpha_{m_{\overline{n}}}(\boldsymbol{\omega})^2 \le 16\,x^2 |\varepsilon|^{k(T)} C^k \mathrm{e}^{-(K_0/4)2^{m_n}}.\end{aligned}$$

If on the contrary $\alpha_{m_{\overline{n}}}(\boldsymbol{\omega}) > 4|x|$, one can reason as follows. For any line $\ell \in L(T)$ one has $|\boldsymbol{\nu}^0_\ell| \le K(T) \le 2^{m_{\overline{n}}}$ and hence $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| \ge \alpha_{m_{\overline{n}}}(\boldsymbol{\omega})$. Then for all $\tau \in [0,1]$

$$\frac{5}{4}\left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell\right| \ge \left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell\right| + |x| \ge \left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell + \tau x\right| \ge \left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell\right| - |x| \ge \frac{3}{4}\left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell\right|.$$

In particular $(5/4)|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| \ge |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| \ge (3/4)|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell|$ and therefore

$$2\left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell\right| \ge \left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell + \tau x\right| \ge \frac{1}{2}\left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell\right|. \tag{4.6}$$

This implies that the sizes of the propagators in $\mathscr{V}_T(tx)$ 'do not change too much' with respect to $\mathscr{V}_T(x)$: in particular (3.19) yields the bound (3.20) and hence, by Remark 3.11, Lemmas 3.9 and 3.10 still hold, so as to obtain $|\partial_1^2 \mathscr{V}_T(tx)| \le C'(C'')^{k(T)} \mathrm{e}^{-K_2 2^{m_n}}$, where ∂_1 denotes the derivative with respect to the (only) argument, for some constants C' and C''. Then

$$|\mathcal{A}_T(x)| \le \left| x^2 \int_0^1 \mathrm{d}\tau\,(1-\tau)\,\partial_1^2 \mathscr{V}_T(\tau x)\right| \le x^2 C'(C'')^k \mathrm{e}^{-K_2 2^{m_n}}, \tag{4.7}$$

By summing over all possible self-energy values contributing to $M^{[n]}_{u,e}(x)$ the bound (4.5) follows.

In case (b), if $\alpha_{m_{\overline{n}-1}}(\boldsymbol{\omega}) \le 8|x|$ then one can bound $|\mathcal{A}_T(x)| \le |\mathscr{V}_T(x)| + |\mathscr{V}_T(0)| + |x\,\partial_x \mathscr{V}_T(0)|$ and use that $K(T) > 2^{m_{\overline{n}-1}}$ to obtain

$$\mathrm{e}^{-(K_0/2)K(T)} \le \mathrm{e}^{-(K_0/2)2^{m_{\overline{n}-1}}} \le \alpha_{m_{\overline{n}-1}}(\boldsymbol{\omega})^2 \le 64x^2.$$

If $\alpha_{m_{\overline{n}-1}}(\boldsymbol{\omega}) > 8|x|$, for any line $\ell \in L(T)$ one has $|\boldsymbol{\nu}^0_\ell| \le K(T) \le 2^{m_{\overline{n}}-1}$ and hence

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| \ge \alpha_{m_{\overline{n}}-1}(\boldsymbol{\omega}) > \frac{1}{2}\alpha_{m_{\overline{n}-1}}(\boldsymbol{\omega}).$$

Then one can reason as done in case (a) and obtain (4.6) for all $t \in [0,1]$: in turn this yields the bound (4.7) and hence the bound (4.5) follows once more. ∎

Remark 4.7. From (4.1) and Lemmas 4.5 and 4.6 it follows that if property 1-p (respectively property 1) is satisfied then for all $n \le p$ (respectively for all $n \ge -1$) one has $\mathcal{M}^{[n]}(x) = \mathcal{M}^{[n]}(0) + \partial_x \mathcal{M}^{[n]}(0)x + O(\varepsilon x^2)$, where $\mathcal{M}^{[n]}(0)$ is a real-valued matrix, while $\partial_x \mathcal{M}^{[n]}(0)$ is a purely imaginary one. In particular this implies that if $\Psi_{n+1}(x)\,|x^2 - \det(\mathcal{M}^{[n]}(0))| \ge \Psi_{n+1}(x)x^2/2$ for all $-1 \le n < p$ (respectively for all $n \ge -1$) then property 1-p (respectively property 1) holds.

The following result will be crucial to check, in the forthcoming Section 5, that property 1 is satisfied by $\mathcal{M}$. The proof follows the lines of that for Lemma 4.8 in [19] and it is deferred to Appendix A (see (3.18) for the definition of $\Phi_{\mathbf{0}}^{\mathcal{R},p}$ and $\Gamma_{\mathbf{0}}^{\mathcal{R},p}$).

Lemma 4.8. *Assume $\mathcal{M}$ to satisfy property 1-p. Then*

$$\mathcal{M}^{[p]}(0) = \begin{pmatrix} \partial_{\beta_0}\Phi_{\mathbf{0}}^{\mathcal{R},p} + e_{p,\beta,\beta} & \partial_{B_0}\Phi_{\mathbf{0}}^{\mathcal{R},p} + e_{p,\beta,B} \\ \partial_{\beta_0}\Gamma_{\mathbf{0}}^{\mathcal{R},p} + e_{p,B,\beta} & \partial_{B_0}\Gamma_{\mathbf{0}}^{\mathcal{R},p} + e_{p,B,B} \end{pmatrix}, \tag{4.8}$$

with $|e_{p,u,e}| \le |\varepsilon|\, A_1 e^{-A_2 2^{m_{p+1}}}$, $u, e = \beta, B$, for suitable positive constants A_1 and A_2.

Lemma 4.9. *Assume $\mathcal{M}$ to satisfy property 1. Then the series (3.16) and (3.17) with the coefficients given by (3.14) and (3.15) respectively, converge for ε small enough.*

Proof. Let $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$. By Remark 4.4 one can bound $|\mathcal{G}^{[n]}_{e,u}(x)| \le \gamma_0\, \alpha_{m_n}(\boldsymbol{\omega})^{-2}$ for all $n \ge 0$ and hence by Lemma 3.9 one can reason as in the proof of the bound (4.1a) so as to obtain

$$\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}} |\mathscr{V}(\theta)| \le C_0 \overline{C}_0^k e^{-\xi|\boldsymbol{\nu}|/2},$$

for some constants C_0 and $\overline{C}_0$, which is enough to prove the assertion. ∎

Lemma 4.10. *Assume $\mathcal{M}$ to satisfy property 1. Then for ε small enough the function (3.16), with the coefficients given by (3.14), solve the equations (2.13).*

Proof. We shall prove that, the functions $b^{\mathcal{R}}, B^{\mathcal{R}}$ defined after (3.17) satisfy the range equations (2.13), i.e. we shall check that $f^{\mathcal{R}} := (b^{\mathcal{R}}, B^{\mathcal{R}}) = g\,\Xi(\boldsymbol{\omega}t, f^{\mathcal{R}})$, where g is the pseudo-differential operator with symbol $g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) = 1/(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}$ and $\Xi(\boldsymbol{\omega}t, f^{\mathcal{R}}) := \big(\omega(B^{\mathcal{R}}) + \varepsilon F(\boldsymbol{\omega}t, \beta^{\mathcal{R}}, B^{\mathcal{R}}), \varepsilon G(\boldsymbol{\omega}t, \beta^{\mathcal{R}}, B^{\mathcal{R}})\big)$. We can write the Fourier coefficients of $b^{\mathcal{R}}$ and $B^{\mathcal{R}}$ as

$$b^{\mathcal{R}}_{\boldsymbol{\nu}} = \sum_{n\ge0} b^{[n]}_{\boldsymbol{\nu}}, \qquad b^{[n]}_{\boldsymbol{\nu}} = \sum_{k\ge1}\varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\beta}(n)} \mathscr{V}(\theta),$$
$$B^{\mathcal{R}}_{\boldsymbol{\nu}} = \sum_{n\ge0} B^{[n]}_{\boldsymbol{\nu}}, \qquad B^{[n]}_{\boldsymbol{\nu}} = \sum_{k\ge1}\varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},B}(n)} \mathscr{V}(\theta),$$

where $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}(n)$ is the subset of $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ such that $n_{\ell_\theta} = n$. Set also $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n) := \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\beta}(n) \times \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},B}(n)$ and, for $\tau = (\theta,\theta') \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$, define $\mathscr{V}(\tau) := (\mathscr{V}(\theta), \mathscr{V}(\theta'))$.

Using Lemmas 3.4 and 4.9, in Fourier space one can write

$$\begin{aligned} g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\Xi(\boldsymbol{\omega}t, f^{\mathcal{R}})]_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n\ge0} \Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\Xi(\boldsymbol{\omega}t, f^{\mathcal{R}})]_{\boldsymbol{\nu}} \\ &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n\ge0} \Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(\mathcal{G}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}))^{-1}\mathcal{G}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\Xi(\boldsymbol{\omega}t, f^{\mathcal{R}})]_{\boldsymbol{\nu}} \\ &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n\ge0} \Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big) \sum_{k\ge1}\varepsilon^k \sum_{\tau\in\overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)} \mathscr{V}(\tau), \end{aligned}$$

where $\overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ differs from $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ as it also includes couples of trees where the root line of one or both of them is the exiting line of a renormalised self-energy cluster. If we separate such couples from the others, we obtain

$$
\begin{aligned}
g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\Xi(\boldsymbol{\omega}t, f^{\mathcal{R}})]_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg[\sum_{n\ge 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big) f^{[n]}_{\boldsymbol{\nu}} \\
&\qquad + \sum_{n\ge 0}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{p\ge n}\sum_{q=-1}^{n-1} M^{[q]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) f^{[p]}_{\boldsymbol{\nu}} + \sum_{n\ge 1}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{p=0}^{n-1}\sum_{q=-1}^{p-1} M^{[q]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) f^{[p]}_{\boldsymbol{\nu}}\Bigg] \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg[\sum_{n\ge 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big) f^{[n]}_{\boldsymbol{\nu}} + \sum_{p\ge 0}\Bigg(\sum_{q=-1}^{p-1} M^{[q]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge q+1}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg) f^{[p]}_{\boldsymbol{\nu}}\Bigg] \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg[\sum_{n\ge 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}\Big) f^{[n]}_{\boldsymbol{\nu}} + \sum_{n\ge 0}\Bigg(\sum_{q=-1}^{n-1} M^{[q]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\chi_q(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg) f^{[n]}_{\boldsymbol{\nu}}\Bigg] \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg[\sum_{n\ge 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big) f^{[n]}_{\boldsymbol{\nu}} + \sum_{n\ge 0}\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) f^{[n]}_{\boldsymbol{\nu}}\Bigg] = \sum_{n\ge 0} f^{[n]}_{\boldsymbol{\nu}} = f^{\mathcal{R}}_{\boldsymbol{\nu}},
\end{aligned}
$$

so that the proof is complete. ■

Remark 4.11. From Lemma 4.8 it follows that, if $\mathcal{M}$ satisfies property 1, one can define

$$
\mathcal{M}^{[\infty]}(x) := \lim_{n\to\infty}\mathcal{M}^{[n]}(x),
$$

and one has

$$
\mathcal{M}^{[\infty]}(0) = \begin{pmatrix} \partial_{\beta_0}\Phi^{\mathcal{R}}_{\mathbf{0}} & \partial_{B_0}\Phi^{\mathcal{R}}_{\mathbf{0}} \\ \partial_{\beta_0}\Gamma^{\mathcal{R}}_{\mathbf{0}} & \partial_{B_0}\Gamma^{\mathcal{R}}_{\mathbf{0}} \end{pmatrix}. \tag{4.9}
$$

Note that (4.9) is pretty much the same equality provided by Lemma 4.8 in [19], adapted to the present case.

Remark 4.12. If we take the formal expansion of the functions $\Phi^{\mathcal{R}}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)$, $\Gamma^{\mathcal{R}}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)$ and $\mathcal{M}^{[\infty]}_{u,e}(0;\varepsilon,\beta_0,B_0)$, $u,e\in\{\beta,B\}$, we obtain tree expansions where the self-energy clusters are allowed; see Section 6 where such a situation is discussed for the Hamiltonian case. Then it is easy to prove the identity (4.9) to any perturbation order; in particular, if one expands

$$
\det\left(\sum_{k=0}^{k_0-1}\varepsilon^k\left[\mathcal{M}^{[\infty]}\Big(0;\varepsilon,\beta_0,\overline{B}_0 + \sum_{h=1}^{k_0-1}\varepsilon^h B_0^{(h)} + O(\varepsilon^{k_0})\Big)\right]^{(k)}\right) = \sum_{k=0}^{k_0-1}\varepsilon^k\delta^{(k)} + O(\varepsilon^{k_0}),
$$

one has $\delta^{(k)} = \delta^{(k)}(\beta_0) \equiv 0$ for all $k = 0,\dots k_0-1$, if the coefficients $B_0^{(h)} = B_{\mathbf{0}}^{(h)}(\beta_0)$ are defined as in (2.10). Moreover, for such an expansion, if one writes

$$
\det\left(\sum_{k=0}^{k_0-1}\varepsilon^k\left[\mathcal{M}^{[n]}(0,\beta_0,\overline{B}_0 + \sum_{h=1}^{k_0-1}\varepsilon^h B_0^{(h)} + O(\varepsilon^{k_0})\Big)\right]^{(k)}\right) = \sum_{k=0}^{k_0-1}\varepsilon^k\delta^{(k)}_n + O(\varepsilon^{k_0})
$$

one has

$$\left|\sum_{k=0}^{k_0-1}\varepsilon^k\delta_n^{(k)}\right| \leq A_1 \mathrm{e}^{-A_2 2^{mn}}$$

for some positive constants A_1, A_2. However, under Hypotheses 1, 2 and 4, we are not able to prove the convergence of the series and we need to introduce some resummation procedure to give a meaning to the series.

Lemma 4.13. *Assume $\mathcal{M}$ to satisfy property 1. Then there exists $B_0 = B_0(\varepsilon, \beta_0)$, C^∞ in both ε, β_0, such that $B_0(\varepsilon, \beta_0) \to \overline{B}_0$ for $\varepsilon \to 0$ and $\Phi_{\mathbf{0}}^{\mathcal{R}}(\varepsilon, \beta_0, B_0(\varepsilon, \beta_0)) \equiv 0$ for any β_0 and ε small enough.*

Proof. One has $\Phi_{\mathbf{0}}^{\mathcal{R}}(\varepsilon; \beta_0, B_0) = \omega_0(B_0) + O(\varepsilon)$ and it is C^∞ in its arguments because of the assumption that $\mathcal{M}$ satisfies property 1. Then, by Hypothesis 2 one can apply the implicit function theorem to obtain the result. In particular one has

$$B_0(\varepsilon, \beta_0) = \overline{B}_0 + \sum_{h=1}^{k_0} \varepsilon^h B_{\mathbf{0}}^{(h)}(\beta_0) + O(\varepsilon^{k_0+1}),$$

where the coefficients $B_{\mathbf{0}}^{(h)}(\beta_0)$ coincide with those defined in (2.10). ■

Given $x_0 \in \mathbb{R}$ and an interval $(a, b) \subset \mathbb{R}$ such that $x_0 \in (a, b)$, we call half-neighbourhood of x_0 each of the two intervals (a, x_0) and (x_0, b).

Lemma 4.14. *Assume $\mathcal{M}$ to satisfy property 1 and set $g(\varepsilon, \beta_0) := \Gamma_{\mathbf{0}}^{\mathcal{R}}(\varepsilon; \beta_0, B_0(\varepsilon, \beta_0))$, where $B_0(\varepsilon, \beta_0)$ is the C^∞ function referred to in Lemma 4.13. Then there exists a continuous curve $\beta_0 = \beta_0(\varepsilon)$ such that $g(\varepsilon, \beta_0(\varepsilon)) = 0$ and moreover, at least in a suitable half-neighbourhood of $\varepsilon = 0$, one has* $\det\left(\mathcal{M}^{[\infty]}(0; \varepsilon, \beta_0(\varepsilon), B_0(\varepsilon, \beta_0(\varepsilon)))\right) \leq 0$.

Proof. Using the same argument in the proof of Lemma 4.11 in [19], as property 1 and Hypothesis 4 imply $g(\varepsilon, \beta_0) = \varepsilon^{k_0}\left(\Gamma_{\mathbf{0}}^{(k_0)}(\beta_0) + O(\varepsilon)\right)$ and $\omega_0'(B_0(\varepsilon, \beta_0))$ has the same sign of $\omega_0'(\overline{B}_0)$ for ε small enough, one can find a continuous curve $\beta_0 = \beta_0(\varepsilon)$ defined at least in a suitable half-neighbourhood of $\varepsilon = 0$ such that $g(\varepsilon, \beta_0(\varepsilon)) \equiv 0$ and $\partial_{\beta_0} g(\varepsilon, \beta_0(\varepsilon))\omega_0'(B_0(\varepsilon, \beta_0(\varepsilon))) \geq 0$. Moreover one has

$$\partial_{\beta_0} g(\varepsilon, \beta_0) = \partial_2 \Gamma_{\mathbf{0}}^{\mathcal{R}}(\varepsilon; \beta_0, B_0(\varepsilon, \beta_0)) + \partial_3 \Gamma_{\mathbf{0}}^{\mathcal{R}}(\varepsilon; \beta_0, B_0(\varepsilon, \beta_0))\partial_{\beta_0} B_0(\varepsilon, \beta_0),$$

(recall that ∂_j denotes the derivative with respect to the j-th argument), so that

$$\det\left(\mathcal{M}^{[\infty]}(0; \varepsilon, \beta_0(\varepsilon), B_0(\varepsilon, \beta_0(\varepsilon)))\right) = -\partial_{\beta_0} g(\varepsilon, \beta_0(\varepsilon))\left(\omega_0'(B_0(\varepsilon, \beta_0(\varepsilon))) + O(\varepsilon)\right)$$

and then the assertion follows. In particular if k_0 is even, the curve $\beta_0(\varepsilon)$ above can be defined in a whole neighbourood of $\varepsilon = 0$. ■

By the results above it follows that, if property 1 is satisfied, choosing $\beta_0 = \beta_0(\varepsilon)$ and $B_0 = B_0(\varepsilon, \beta_0(\varepsilon))$ as above, the series (3.16) solve the equation of motion (2.1).

In the forthcoming Section we shall prove that $\mathcal{M}$ satisfies property 1, at least along a continuous curve $C(\varepsilon) := (\beta_0(\varepsilon), B_0(\varepsilon, \beta_0(\varepsilon)))$ satisfying $\Phi_{\mathbf{0}}^{\mathcal{R}}(\varepsilon; C(\varepsilon)) \equiv \Gamma_{\mathbf{0}}^{\mathcal{R}}(\varepsilon; C(\varepsilon)) \equiv 0$, adapting the analogous proof in [19].

5 Convergence of the resummed series: fixing the initial phase

In this section, we shall complete the proof of Theorem 2.2 by showing that, under Hypotheses 1, 2 and 4, by suitably choosing β_0, B_0, then $\mathcal{M}$ turns out to satisfy property 1.

Define the C^∞ non-increasing function ξ such that

$$\xi(x) = \begin{cases} 1, & x \le 1/2, \\ 0, & x \ge 1, \end{cases} \tag{5.1}$$

and set $\xi_{-1}(x) = 1$ and $\xi_n(x) = \xi(2^8 x/\alpha^2_{m_{n+1}}(\boldsymbol{\omega}))$ for all $n \ge 0$. Set also

$$B_0(\varepsilon, \beta_0, B_0') := \overline{B}_0 + \sum_{h=1}^{k_0-1} \varepsilon^h B_{\mathbf{0}}^{(h)}(\beta_0) + \varepsilon^{k_0} B_0' \tag{5.2}$$

where the coefficients $B_{\mathbf{0}}^{(h)}(\beta_0)$ are defined as in (2.10) and k_0 is as in Hypothesis 4. For all $n \ge 0$ we define recursively the *regularised propagators* as

$$\overline{\mathcal{G}}^{[n]} = \overline{\mathcal{G}}^{[n]}(x; \varepsilon, \beta_0, B_0') := \Psi_n(x) \left((\mathrm{i}x)\mathbb{1} - \overline{\mathcal{M}}^{[n-1]}(x; \varepsilon, \beta_0, B_0') \xi_{n-1}(\Delta_{n-1}) \right)^{-1}, \tag{5.3}$$

where

$$\overline{\mathcal{M}}^{[n-1]}(x; \varepsilon, \beta_0, B_0') := \sum_{q=-1}^{n-1} \chi_q(x) \overline{M}^{[q]}(x; \varepsilon, \beta_0, B_0'), \tag{5.4}$$

with the 2×2 matrix $\overline{M}^{[q]}(x; \varepsilon, \beta_0, B_0')$ defined so as

$$\overline{M}^{[q]}_{u,e}(x; \varepsilon, \beta_0, B_0') := \sum_{T \in \mathfrak{R}_{q,u,e}} \varepsilon^{k(T)} \overline{\mathscr{V}}_T(x; \varepsilon, \beta_0, B_0'), \tag{5.5}$$

where

$$\overline{\mathscr{V}}_T(x; \varepsilon, \beta_0, B_0') := \left(\prod_{v \in N(T)} \widetilde{\mathcal{F}}_v \right) \left(\prod_{\ell \in L(T)} \overline{\mathcal{G}}^{[n_\ell]}_{e_\ell, u_\ell} \right), \tag{5.6}$$

with $\widetilde{\mathcal{F}}_v = \mathcal{F}_v(\beta_0, B_0(\varepsilon, \beta_0, B_0'))$ and

$$\Delta_{n-1} = \Delta_{n-1}(\varepsilon, \beta_0, B_0') := D_{n-1}(\varepsilon, \beta_0, B_0') - \sum_{k=0}^{k_0-1} \varepsilon^k \left[D_{n-1}(\varepsilon, \beta_0, B_0') \right]^{(k)},$$

with

$$D_{n-1}(\varepsilon, \beta_0, B_0') := \det \left(\overline{\mathcal{M}}^{[n-1]}(0; \varepsilon, \beta_0, B_0') \right).$$

For any $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$, define also, for all $k \ge 0$, $\boldsymbol{\nu} \in \mathbb{Z}^d$, $h = \beta, B$,

$$\overline{\mathscr{V}}(\theta; \varepsilon, \beta_0, B_0') := \left(\prod_{v \in N(T)} \widetilde{\mathcal{F}}_v \right) \left(\prod_{\ell \in L(T)} \overline{\mathcal{G}}^{[n_\ell]}_{e_\ell, u_\ell} \right).$$

Finally, set $\overline{\mathcal{M}} := \{ \overline{\mathcal{M}}^{[n]}(x; \varepsilon, \beta_0, B_0') \}_{n \ge -1}$ and $\overline{\mathcal{M}}^{\xi} := \{ \overline{\mathcal{M}}^{[n]}(x; \varepsilon, \beta_0, B_0') \xi_n(\Delta_n) \}_{n \ge -1}$.

Lemma 5.1. *For ε small enough, $\overline{\mathcal{M}}^{\xi}$ satisfies property 1.*

Proof. We shall prove that $\overline{\mathcal{M}}^{\xi}$ satisfies property 1-p for all $p \geq 0$, by induction on p. For $p = 0$ it is obvious if ε is small enough. Assume then that $\overline{\mathcal{M}}^{\xi}$ satisfies property 1-p. Then we can repeat almost word by word the proofs of Lemmas 4.3 and 4.5, so as to obtain

$$\overline{\mathcal{M}}^{[p]}(x;\varepsilon,\beta_0,B_0') = \overline{\mathcal{M}}^{[p]}(0;\varepsilon,\beta_0,B_0') + x\,\partial_x\overline{\mathcal{M}}^{[p]}(0;\varepsilon,\beta_0,B_0') + x^2\int_0^1 \mathrm{d}t\,(1-t)\,\partial_x^2\overline{\mathcal{M}}^{[n]}(tx;\varepsilon,\beta_0,B_0'),$$

with $\overline{\mathcal{M}}^{[p]}(0;\varepsilon,\beta_0,B_0')$ a real-valued matrix, $\partial_x\overline{\mathcal{M}}^{[p]}(0;\varepsilon,\beta_0,B_0')$ a purely imaginary one and

$$\left| x^2\int_0^1 \mathrm{d}t\,(1-t)\,\partial_x^2\overline{\mathcal{M}}^{[n]}(tx;\varepsilon,\beta_0,B_0')\right| \leq C\,|\varepsilon| x^2$$

for some constant C, by Lemma 4.6. Then we only have to prove that – see Remark 4.7 –

$$\Psi_{p+1}(x)\left|x^2 - D_p(\varepsilon,\beta_0,B_0')\xi_p(\Delta_p)^2\right| \geq \Psi_{p+1}(x)\frac{x^2}{2}.$$

Note that, by the definition of Δ_p, one has

$$\sum_{k=0}^{k_0-1}\varepsilon^k\left[D_p(\varepsilon,\beta_0,B_0')\right]^{(k)} = \sum_{k=0}^{k_0-1}\varepsilon^k\delta_p^{(k)}$$

with the coefficients $\delta_p^{(k)}$ as in Remark 4.12, and hence $\overline{\mathcal{M}}^{\xi}$ satisfies property 1-$(p+1)$ by the definition of the function ξ_p. ∎

Set

$$\overline{\mathcal{M}}^{[\infty]}(x;\varepsilon,\beta_0,B_0') := \lim_{n\to\infty}\overline{\mathcal{M}}^{[n]}(x;\varepsilon,\beta_0,B_0'), \tag{5.7}$$

and define

$$\overline{\Phi}(\varepsilon,\beta_0,B_0') := \sum_{k\geq 0}\varepsilon^k\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\mathbf{0},\beta}}\overline{\mathscr{V}}(\theta,\varepsilon,\beta_0,B_0'), \qquad \overline{\Gamma}(\varepsilon,\beta_0,B_0') := \sum_{k\geq 0}\varepsilon^k\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\mathbf{0},B}}\overline{\mathscr{V}}(\theta,\varepsilon,\beta_0,B_0'). \tag{5.8}$$

Lemma 5.2. *One has*

$$[\overline{P}(\varepsilon,\beta_0,B_0')]^{(k)} = [P^{\mathcal{R}}_{\mathbf{0}}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))]^{(k)}, \qquad P = \Phi,\Gamma,$$

for all $k = 0,\dots,k_0$.

Proof. Set $\Theta^{\mathcal{R}(n)}_{k,\boldsymbol{\nu},h} := \{\theta\in\Theta^{\mathcal{R},n}_{k,\boldsymbol{\nu},h} : \exists\ell\in L(\theta) \text{ such that } n_\ell = n\}$ and write

$$\overline{P}(\varepsilon,\beta_0,B_0') = \sum_{k\geq 0}\varepsilon^k\sum_{n\geq 0}\sum_{\theta\in\Theta^{\mathcal{R}(n)}_{k,\mathbf{0},h}}\overline{\mathscr{V}}(\theta,\varepsilon,\beta_0,B_0'),$$

with $h = \beta, B$ for $P = \Phi,\Gamma$, respectively, and note that if $\theta\in\Theta^{\mathcal{R}(n)}_{k,\boldsymbol{\nu},h}$ one has

$$\prod_{v\in N(\theta)}|\widetilde{\mathcal{F}}_v| \leq E_1^{|N(\theta)|}\mathrm{e}^{-E_2 2^{m_n}},$$

for some constants E_1, E_2. Moreover one can write formally

$$\overline{\mathcal{G}}_{n_\ell}(x) = \Psi_{n_\ell}(x) g_{n_\ell-1}(x) \Big(\mathbb{1} + \sum_{m\geq 1} \Big(g_{n_\ell-1}(x) \widetilde{\mathcal{M}}^{[n_\ell-1]}(x;\varepsilon,\beta_0,B_0') \xi_{n_\ell-1}(\Delta_{n_\ell-1}) \Big)^m \Big),$$

with

$$g_{n_\ell-1}(x) = \frac{1}{(\mathrm{i}x)^2} \begin{pmatrix} \mathrm{i}x & \omega_0'(\overline{B}_0)\xi_{n_\ell-1}(\Delta_{n_\ell-1}) \\ 0 & \mathrm{i}x \end{pmatrix}$$

and

$$\widetilde{\mathcal{M}}^{[n_\ell-1]}(x;\varepsilon,\beta_0,B_0') := \overline{\mathcal{M}}^{[n_\ell-1]}(x;\varepsilon,\beta_0,B_0') - \begin{pmatrix} 0 & \omega_0'(\overline{B}_0) \\ 0 & 0 \end{pmatrix} = O(\varepsilon),$$

and we can write $\xi_{n_\ell-1}(\Delta_{n_\ell-1}) = 1 + \xi'_{n_\ell-1}(\Delta^*)\Delta_{n_\ell-1}$ for some Δ^*, where $\Delta_{n_\ell-1} = O(\varepsilon^{k_0})$ and

$$|\xi'_{n_\ell-1}(\Delta^*)| \leq \frac{E_3}{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^2} \leq \frac{E_3}{\alpha_{m_n}(\boldsymbol{\omega})^2},$$

for some positive constant E_3 independent of n. Hence the assertion follows. ■

Introduce the C^∞ functions $\hat{\Phi}(\varepsilon,\beta_0,B_0)$, $\widetilde{\Phi}(\varepsilon,\beta_0,B_0)$, $\hat{\Gamma}(\varepsilon,\beta_0,B_0)$ and $\widetilde{\Gamma}(\varepsilon,\beta_0,B_0)$ such that
(1) the first k_0 coefficients of the Taylor expansion in ε of both $\hat{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))$ and $\widetilde{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))$ coincide with those of $\overline{\Phi}(\varepsilon,\beta_0,B_0')$,
(2) the first k_0 coefficients of the Taylor expansion in ε of both $\hat{\Gamma}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))$ and $\widetilde{\Gamma}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))$ coincide with those of $\overline{\Gamma}(\varepsilon,\beta_0,B_0')$,
(3) one has

$$\overline{\mathcal{M}}^{[\infty]}(0;\varepsilon,\beta_0,B_0') = \begin{pmatrix} \partial_2\hat{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) & \partial_3\widetilde{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) \\ \\ \partial_2\hat{\Gamma}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) & \partial_3\widetilde{\Gamma}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) \end{pmatrix}. \tag{5.9}$$

Define also, for all $n \geq -1$ the C^∞ functions $\hat{\Phi}_n(\varepsilon,\beta_0,B_0)$, $\widetilde{\Phi}_n(\varepsilon,\beta_0,B_0)$, $\hat{\Gamma}_n(\varepsilon,\beta_0,B_0)$ and $\widetilde{\Gamma}_n(\varepsilon,\beta_0,B_0)$ such that

$$\overline{\mathcal{M}}^{[n]}(0;\varepsilon,\beta_0,B_0') = \begin{pmatrix} \partial_2\hat{\Phi}_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) & \partial_3\widetilde{\Phi}_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) \\ \\ \partial_2\hat{\Gamma}_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) & \partial_3\widetilde{\Gamma}_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) \end{pmatrix}, \tag{5.10}$$

and

$$|P_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) - P(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))| \leq P_H |\varepsilon| \, \mathrm{e}^{-B_H 2^{m_n}}, \qquad P = \hat{\Phi}, \widetilde{\Phi}, \hat{\Gamma}, \widetilde{\Gamma}, \tag{5.11}$$

for some constants A_P, B_P. Then, by reasoning as in the proofs of Lemmas 4.13 and 4.14, we can find $\widetilde{B}_0 = \widetilde{B}_0(\varepsilon,\beta_0)$ and $\widetilde{\beta}_0 = \widetilde{\beta}_0(\varepsilon)$ such that
(i) $\hat{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,\widetilde{B}_0(\varepsilon,\beta_0))) \equiv 0$ for all β_0 and ε small enough,
(ii) $\hat{\Gamma}(\varepsilon,\widetilde{\beta}_0(\varepsilon),B_0(\varepsilon,\widetilde{\beta}_0(\varepsilon),\widetilde{B}_0(\varepsilon,\widetilde{\beta}_0(\varepsilon)))) \equiv 0$ for all ε small enough and
(iii) $\partial_3\widetilde{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,\widetilde{B}_0(\varepsilon,\beta_0)))\partial_{\beta_0}\hat{\Gamma}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,\widetilde{B}_0(\varepsilon,\beta_0)))\big|_{\beta_0=\widetilde{\beta}_0(\varepsilon)} \geq 0$, at least in a suitable half-neighbourhood of $\varepsilon = 0$.

Lemma 5.3. *Set $\widetilde{C}(\varepsilon) = (\widetilde{\beta}_0(\varepsilon), \widetilde{B}_0(\varepsilon, \widetilde{\beta}_0(\varepsilon)))$ with $\widetilde{B}_0(\varepsilon, \beta_0)$ and $\widetilde{\beta}_0(\varepsilon)$ as above. Then, along $\widetilde{C}(\varepsilon)$ one has $\xi_n(\Delta_n) \equiv 1$ for all $n \geq -1$.*

Proof. We shall prove the result by induction on n. For $n = -1$ it is obvious. Assume then $\xi_p(\Delta_p) \equiv 1$ for all $p = -1, \ldots, n-1$ along $\widetilde{C}(\varepsilon)$ and set $C(\varepsilon) = (\widetilde{\beta}_0(\varepsilon), B_0(\varepsilon, \widetilde{C}(\varepsilon)))$. Hence $\overline{\mathcal{G}}^{[p]}(x; \varepsilon, \widetilde{C}(\varepsilon)) \equiv \mathcal{G}^{[p]}(x; \varepsilon, C(\varepsilon))$ for all $p = 0, \ldots, n$ and thence $\overline{\mathcal{M}}^{[n]}(x; \varepsilon, \widetilde{C}(\varepsilon)) \equiv \mathcal{M}^{[n]}(x; \varepsilon, C(\varepsilon))$. In particular $\mathcal{M}$ satisfies property 1-n so that, using Lemma 4.8 one has

$$\overline{\mathcal{M}}^{[n]}(0; \varepsilon, \widetilde{C}(\varepsilon)) = \begin{pmatrix} \partial_2 \Phi_{\mathbf{0}}^{\mathcal{R},n}(\varepsilon, C(\varepsilon)) + e_{n,\beta,\beta} & \partial_3 \Phi_{\mathbf{0}}^{\mathcal{R},n}(\varepsilon, C(\varepsilon)) + e_{n,\beta,B} \\ \partial_2 \Gamma_{\mathbf{0}}^{\mathcal{R},n}(\varepsilon, C(\varepsilon)) + e_{n,B,\beta} & \partial_3 \Gamma_{\mathbf{0}}^{\mathcal{R},n}(\varepsilon, C(\varepsilon)) + e_{n,B,B} \end{pmatrix},$$

with $|e_{n,u,e}| \leq |\varepsilon| A_1 e^{-A_2 2^{m_{n+1}}}$, $u, e = \beta, B$. On the other hand one has

$$\begin{aligned} \partial_2 \hat{\Phi}(\varepsilon, C(\varepsilon)) &= -\partial_3 \hat{\Phi}(\varepsilon, C(\varepsilon))\, \partial_{\beta_0} B_0(\varepsilon, \beta_0, \widetilde{B}_0(\varepsilon, \beta_0))\Big|_{\beta_0 = \widetilde{\beta}_0(\varepsilon)}, \\ \partial_2 \hat{\Gamma}(\varepsilon, C(\varepsilon)) &= \partial_{\beta_0} \hat{\Gamma}(\varepsilon, \beta_0, B_0(\varepsilon, \beta_0, \widetilde{B}_0(\varepsilon, \beta_0)))\Big|_{\beta_0 = \widetilde{\beta}_0(\varepsilon)} - \partial_3 \hat{\Gamma}(\varepsilon, C(\varepsilon))\, \partial_{\beta_0} B_0(\varepsilon, \beta_0, \widetilde{B}_0(\varepsilon, \beta_0))\Big|_{\beta_0 = \widetilde{\beta}_0(\varepsilon)}, \end{aligned}$$

so that, without writing explicitly the dependence on $(\varepsilon, C(\varepsilon))$, one has

$$\overline{\mathcal{M}}^{[n]}(0; \varepsilon, \widetilde{C}(\varepsilon)) = \begin{pmatrix} -\partial_3 \Phi_{\mathbf{0}}^{\mathcal{R},n} \partial_{\beta_0} B_0 + \gamma_n & \partial_3 \Phi_{\mathbf{0}}^{\mathcal{R},n} + e_{n,\beta,B} \\ \partial_{\beta_0} \Gamma_{\mathbf{0}}^{\mathcal{R},n} - \partial_3 \Gamma_{\mathbf{0}}^{\mathcal{R},n} \partial_{\beta_0} B_0 + \gamma'_n & \partial_3 \Gamma_{\mathbf{0}}^{\mathcal{R},n} + e_{n,B,B} \end{pmatrix},$$

with $|\gamma_n|, |\gamma'_n| \leq |\varepsilon|\, C_1 e^{-C_2 2^{m_{n+1}}}$ for some C_1, C_2. Hence

$$\Delta_n = -\partial_{\beta_0} \Gamma_{\mathbf{0}}^{\mathcal{R},n} \partial_3 \Phi_{\mathbf{0}}^{\mathcal{R},n} + c_n = -\partial_{\beta_0} \hat{\Gamma}_n \partial_3 \widetilde{\Phi}_n + c'_n = -\partial_{\beta_0} \Gamma \partial_3 \widetilde{\Phi} + c''_n \leq c''_n,$$

with $|c_n|, |c'_n|, |c''_n| \leq |\varepsilon|\, D_1 e^{-D_2 2^{m_{n+1}}}$ for some constants D_1 and D_2, so that the assertion follows by the definition of ξ_n. ∎

Lemma 5.4. *Let $\widetilde{C}(\varepsilon)$ be as in Lemma 5.3 and set $C(\varepsilon) = (\widetilde{\beta}_0(\varepsilon), B_0(\varepsilon, \widetilde{C}(\varepsilon)))$. One can choose the functions $\hat{\Phi}, \widetilde{\Phi}, \hat{\Gamma}, \widetilde{\Gamma}$ such that $\hat{\Phi}(\varepsilon, C(\varepsilon)) = \widetilde{\Phi}(\varepsilon, C(\varepsilon)) = \Phi_{\mathbf{0}}^{\mathcal{R}}(\varepsilon, C(\varepsilon)) \equiv 0$ and $\hat{\Gamma}(\varepsilon, C(\varepsilon)) = \widetilde{\Gamma}(\varepsilon, C(\varepsilon)) = \Gamma_{\mathbf{0}}^{\mathcal{R}}(\varepsilon, C(\varepsilon)) \equiv 0$. In particular $(\beta(t, \varepsilon), B(t, \varepsilon)) = C(\varepsilon) + (b^{\mathcal{R}}(t; \varepsilon, C(\varepsilon)), B^{\mathcal{R}}(t; \varepsilon, C(\varepsilon)))$ defined in (3.16) solves the equation of motion (2.1)*

Proof. It follows from the results above. Indeed, for any $\hat{\Phi}$, $\hat{\Gamma}$ there is a curve $C(\varepsilon)$ along which $\mathcal{M} = \overline{\mathcal{M}} = \overline{\mathcal{M}}^{\xi}$ (hence $\mathcal{M}$ satisfies property 1) and $\hat{\Phi}(\varepsilon, C(\varepsilon)) = \hat{\Gamma}(\varepsilon, C(\varepsilon)) \equiv 0$. By Remark 4.11 also $\Phi_{\mathbf{0}}^{\mathcal{R}}$ and $\Gamma_{\mathbf{0}}^{\mathcal{R}}$ are among the primitives of $\mathcal{M}_{\beta,\beta}^{[\infty]}$ and $\mathcal{M}_{B,\beta}^{[\infty]}$ respectively. Then the assertion follows. ∎

Remark 5.5. Note that without Remark 4.11 we were able to prove only the existence of curves on which the solution of the range equations is well-defined. On the other hand Remark 4.11 (which follows from Lemma 4.8) guarantees that the solution of the bifurcation equations is one of such curves.

Lemma 5.4 completes the proof of Theorem 2.2: indeed the function $(\beta(t, \varepsilon), B(t, \varepsilon))$ is a quasi-periodic solution to (2.1) with frequency vector $\boldsymbol{\omega}$ and, by construction, it reduces to $(\overline{\beta}_0, \overline{B}_0)$ as ε tends to 0.

6 The Hamiltonian case

In this section we prove Theorem 2.3. Consider (2.1) of the form (2.16), i.e. with $F = \partial_B f$ and $G = -\partial_\beta f$, where f is the function appearing in (2.15). We look for a solution $(\beta(t), B(t))$ which can be formally written as in (2.9), with the coefficients given by (2.10). If there exists $k_0 \geq 1$ such that all the coefficients $\Gamma_{\mathbf{0}}^{(k)}(\beta_0)$ vanish identically for all $0 \leq k \leq k_0 - 1$, while $\Gamma_{\mathbf{0}}^{(k_0)}(\beta_0)$ is not identically zero, we can solve the equations of motion up to order k_0 without fixing the parameter β_0. Moreover one has $\Gamma^{(k_0)}(\beta_0) = \partial_{\beta_0} g^{(k_0)}(\beta_0)$, with

$$g^{(k_0)}(\beta_0) := [\overline{B}\,\dot{\overline{b}}]_{\mathbf{0}}^{(k_0)} - [h_0(\overline{B}_0 + \overline{B} + B^{(k_0)})]_{\mathbf{0}}^{(k_0)} - [f(\boldsymbol{\omega} t, \beta_0 + \overline{b}, \overline{B}_0 + \overline{B})]_{\mathbf{0}}^{(k_0-1)},$$

because, if we denote

$$\overline{b} = \sum_{k=1}^{k_0-1} b^{(k)}, \qquad \overline{B} = \sum_{k=1}^{k_0-1} B^{(k)},$$

one has

$$\begin{aligned}
\partial_{\beta_0}[f(\boldsymbol{\omega} t, \beta_0 + \overline{b}, \overline{B}_0 + \overline{B})]_{\mathbf{0}}^{(k_0-1)} &= [\partial_\beta f(\boldsymbol{\omega} t, \beta_0 + \overline{b}, \overline{B}_0 + \overline{B})(1 + \partial_{\beta_0}\overline{b})]_{\mathbf{0}}^{(k_0-1)} \\
&\quad + [\partial_B f(\boldsymbol{\omega} t, \beta_0 + \overline{b}, \overline{B}_0 + \overline{B})\partial_{\beta_0}\overline{B}]_{\mathbf{0}}^{(k_0-1)} \\
&= -\Gamma_{\mathbf{0}}^{(k_0)} - [\dot{\overline{B}}\partial_{\beta_0}\overline{b}]_{\mathbf{0}}^{(k_0)} + [\dot{\overline{b}}\partial_{\beta_0}\overline{B}]_{\mathbf{0}}^{(k_0)} \\
&\quad - [\omega_0(\overline{B}_0 + \overline{B} + B^{(k_0)})\partial_{\beta_0}(\overline{B} + B^{(k_0)})]_{\mathbf{0}}^{(k_0)} \\
&= -\Gamma_{\mathbf{0}}^{(k_0)} + \partial_{\beta_0}[\overline{B}\,\dot{\overline{b}}]_{\mathbf{0}}^{(k_0)} - \partial_{\beta_0}[h_0(\overline{B}_0 + \overline{B} + B^{(k_0)})]_{\mathbf{0}}^{(k_0)}.
\end{aligned}$$

Since $g^{(k_0)}$ is analytic and periodic, it has at least one maximum β_0' and one minimum β_0''. Then Hypothesis 4 automatically holds. Indeed, if $\varepsilon^{k_0}\omega_0'(\overline{B}_0) > 0$ one can choose $\overline{\beta}_0 = \beta_0''$, while if $\varepsilon^{k_0}\omega_0'(\overline{B}_0) < 0$ one can choose $\overline{\beta}_0 = \beta_0'$ and hence in both cases Hypothesis 4 is satisfied. Therefore the existence of a quasi-periodic solution with frequency vector $\boldsymbol{\omega}$ follows from Theorem 2.2.

Assume now that $\Gamma_{\mathbf{0}}^{(k)} \equiv 0$ for all $k \geq 0$. We shall prove the following result, which, together with the argument given above, implies Theorem 2.3.

Proposition 6.1. *Consider the system (2.16) and assume Hypotheses 1 and 2 to be satisfied. Assume also that $\Gamma_{\mathbf{0}}^{(k)} \equiv 0$ for all $k \geq 0$. Then for ε small enough there exists a resonant maximal torus run with frequency vector $\boldsymbol{\omega}$, which can be parameterised as $\beta = \beta_0 + \beta_\varepsilon(\boldsymbol{\psi}, \beta_0)$, $B = \overline{B}_0 + B_\varepsilon(\boldsymbol{\psi}, \beta_0)$, $\boldsymbol{\alpha} = \boldsymbol{\psi}$, $\boldsymbol{A} = \boldsymbol{A}_\varepsilon(\boldsymbol{\psi}, \beta_0)$, with $(\boldsymbol{\psi}, \beta_0) \in \mathbb{T}^{d+1}$, where the functions β_ε, B_ε and $\boldsymbol{A}_\varepsilon$ are analytic in ε, as well as in $\boldsymbol{\psi}$ and β_0, and are at least of order ε. The time evolution along the torus is given by $(\boldsymbol{\psi}, \beta_0) \to (\boldsymbol{\psi} + \boldsymbol{\omega} t, \beta_0)$.*

As we shall see, no resummation is needed in such a case and the formal expansion (2.9) turns out to converge, i.e. the solution is analytic in the perturbation parameter. However we still need a multiscale decomposition of the propagators to show that the small divisors ‘do not accumulate too much’. To this aim, we slightly change some of the definitions in Section 3 as follows.

First of all, when defining the labelled trees, the following changes are made. We associate with each node v a *mode label* $\boldsymbol{\nu}_v \in \mathbb{Z}^d$, a *component label* $h_v \in \{\beta, B\}$ and an *order label* $k_v \in \{0, 1\}$ with the constraint that $k_v = 1$ if $\boldsymbol{\nu}_v \neq \mathbf{0}$. With each line $\ell = \ell_v$, $\ell \neq \ell_\theta$, we associate a *component label* $h_\ell \in \{\beta, B\}$, with the constraint that $h_{\ell_v} = h_v$, and a *momentum label* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d$, with the constraint that $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ if $h_\ell = \beta$. We associate with the root line ℓ_θ a component label $h_{\ell_\theta} \in \{\beta, B, \Phi, \Gamma\}$ and a momentum label $\boldsymbol{\nu}_{\ell_\theta} \in \mathbb{Z}^d$ with the following constraints. Call v_0 the node which ℓ_θ exists: then (i) $h_{\ell_\theta} = B, \Gamma$ if $h_{v_0} = B$, while $h_{\ell_\theta} = \beta, \Phi$ if $h_{v_0} = \beta$ and (ii) $\boldsymbol{\nu}_{\ell_\theta} \neq \mathbf{0}$ for $h_{\ell_\theta} = \beta$, while $\boldsymbol{\nu}_{\ell_\theta} = \mathbf{0}$ for $h_{\ell_\theta} = \Phi, \Gamma$. We require $k_{v_0} = 1$ if $\ell = \ell_{v_0}$ is such that either $\ell = \ell_\theta$ and $h_\ell = \Gamma$ or $h_\ell = B$ and $\boldsymbol{\nu}_\ell \neq \mathbf{0}$. Denote by p_v and q_v the numbers of lines with component label β and B, respectively, entering the node v and set $s_v = p_v + q_v$: if $k_v = 0$ for some $v \in N(\theta)$, then we impose $p_v = 0$ and $q_v \geq 1$ if $h_v = \beta$, while we impose $p_v = 0$ and $q_v \geq 2$ if $h_v = B$. Finally with each line $\ell \in L(\theta)$ we associate a *scale label* $n_\ell \in \mathbb{Z}_+ \cup \{-1\}$, with the constraint that $n_\ell \geq 0$ if $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ and $n_\ell = -1$ if $\boldsymbol{\nu}_\ell = \mathbf{0}$.

We do not change the definition of cluster, while it is more convenient to change slightly the definition of self-energy cluster: a cluster T on scale n is a *self-energy cluster* if (i) it has only one entering line ℓ'_T and one exiting line ℓ_T, (ii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$, (iii) either $n = -1$ and $\mathcal{P}_T = \emptyset$ or $n \geq 0$ and one has $n_\ell \geq 0$ and $\boldsymbol{\nu}^0_\ell \neq \mathbf{0}$ for all $\ell \in \mathcal{P}_T$. We shall say that a subgraph T constituted by only one node v with $\boldsymbol{\nu}_v = \mathbf{0}$ and $s_v = 1$, is also a self-energy cluster on scale -1.

We denote by $\Theta_{k,\boldsymbol{\nu},h}$ the set of trees with order k, total momentum $\boldsymbol{\nu}$ and total component h and by $\mathfrak{S}^k_{n,u,e}$ the set of self-energy clusters with order k, scale n and such that $h_{\ell'_T} = e$ and $h_{\ell_T} = u$, with $e, u \in \{\beta, B\}$. Note that self-energy clusters are allowed both in $\Theta_{k,\boldsymbol{\nu},h}$ and in $\mathfrak{S}^k_{n,u,e}$; in other words we are considering trees and subgraphs which are not renormalised.

For any tree θ or any subgraph S of θ we define their (non-renormalised) value as in (3.13), but with the node factors defined as

$$\mathcal{F}_v = \begin{cases} \dfrac{1}{q_v!}\partial_1^{q_v}\omega_0(\overline{B}_0), & k_v = 0, \\ \dfrac{1}{p_v!q_v!}\partial_\beta^{p_v}\partial_B^{q_v+1} f_{\boldsymbol{\nu}_v}(\beta_0, \overline{B}_0), & k_v = 1, \quad h_v = \beta, \\ -\dfrac{1}{p_v!q_v!}\partial_\beta^{p_v+1}\partial_B^{q_v} f_{\boldsymbol{\nu}_v}(\beta_0, \overline{B}_0), & k_v = 1, \quad h_v = B, \quad \boldsymbol{\nu}_{\ell_v} \neq \mathbf{0}, \\ \dfrac{1}{p_v!q_v!}\partial_\beta^{p_v}\partial_B^{q_v+1} f_{\boldsymbol{\nu}_v}(\beta_0, \overline{B}_0), & k_v = 1, \quad h_v = B, \quad \boldsymbol{\nu}_{\ell_v} = \mathbf{0}, \quad h_{\ell_v} = B, \\ -\dfrac{1}{p_v!q_v!}\partial_\beta^{p_v+1}\partial_B^{q_v} f_{\boldsymbol{\nu}_v}(\beta_0, \overline{B}_0), & k_v = 1, \quad h_v = B, \quad \boldsymbol{\nu}_{\ell_v} = \mathbf{0}, \quad h_{\ell_v} = \Gamma, \end{cases} \tag{6.1}$$

and the propagators defined as

$$\mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell) := \begin{cases} \dfrac{\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)}{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell}, & n_\ell \geq 0, \\ -\dfrac{1}{\omega_0'(\overline{B}_0)}, & n_\ell = -1, \quad h_\ell = B, \\ 1, & n_\ell = -1, \quad h_\ell = \Gamma, \Phi. \end{cases} \tag{6.2}$$

Lemma 6.2. *Let T be a subgraph of any tree θ. Then one has $|N(T)| \leq 4k(T) - 2$.*

Proof. We shall prove the result by induction on $k = k(T)$. For $k = 1$ the bound is trivially satisfied as a direct check shows. Assume then the bound to hold for all $k' < k$. Call v the node which ℓ_T (possibly ℓ_θ) exits, $\ell_1, \ldots, \ell_{s_v}$ the lines entering v and $T_1, \ldots, T_{s_v}$ the subgraphs of T with exiting lines $\ell_1, \ldots, \ell_{s_v}$. If $k_v = 1$ then by the inductive hypothesis one has

$$|N(T)| = 1 + \sum_{i=1}^{s_v} |N(T_i)| \leq 1 + 4(k-1) - 2s_v \leq 4k - 2.$$

If $k_v = 0$ and $h_v = B$ then one has $s_v = q_v \geq 2$ and hence

$$|N(T)| = 1 + \sum_{i=1}^{q_v} |N(T_i)| \leq 1 + 4k - 2q_v \leq 4k - 2.$$

If $k_v = 0$ and $h_v = \beta$, then if $q_v \geq 2$ one can reason as in the previous case. Otherwise, the line $\ell = \ell_w$ entering v is such that $h_\ell = B$, so that either $k_w = 1$ or $k_w = 0$ and $q_w \geq 2$. call $\ell'_1, \ldots, \ell'_{s_w}$ the lines entering w and $T'_1, \ldots, T'_{s_v}$ the subgraphs of T with exiting lines $\ell'_1, \ldots, \ell'_{s_w}$. In the first case one has

$$|N(T)| = 2 + \sum_{i=1}^{s_w} |N(T'_i)| \leq 2 + 4(k-1) - 2s_w \leq 4k - 2,$$

while in the second case one has

$$|N(T)| = 2 + \sum_{i=1}^{q_w} |N(T'_i)| \leq 2 + 4k - 2q_w \leq 4k - 2.$$

Therefore the bound follows. ■

From now on we shall not write explicitly the dependence on β_0 and $\overline{B}_0$ to lighten the notations. If T is a self-energy cluster we can (and shall) write $\mathscr{V}(T) = \mathscr{V}_T(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'_T})$ to stress the dependence on $\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'_T}$; see also Remark 3.7. For all $k \geq 1$, define

$$b_{\boldsymbol{\nu}}^{(k)} := \sum_{\theta \in \Theta_{k,\boldsymbol{\nu},\beta}} \mathscr{V}(\theta), \quad \boldsymbol{\nu} \in \mathbb{Z}^d_*, \qquad B_{\boldsymbol{\nu}}^{(k)} := \sum_{\theta \in \Theta_{k,\boldsymbol{\nu},B}} \mathscr{V}(\theta), \quad \boldsymbol{\nu} \in \mathbb{Z}^d, \tag{6.3a}$$

$$M_{u,e}^{(k)}(x,n) := \sum_{T \in \mathfrak{S}^k_{n,u,e}} \mathscr{V}_T(x), \quad \mathcal{M}_{u,e}^{(k)}(x,n) := \sum_{p=-1}^{n} M_{u,e}^{(k)}(x,p), \quad \mathcal{M}_{u,e}^{(k)}(x) := \lim_{n\to\infty} \mathcal{M}_{u,e}^{(k)}(x,n). \tag{6.3b}$$

Note that the coefficients (6.3a) coincide with those in (2.10). Set also

$$\Phi_{\mathbf{0}}^{(k)} := \sum_{\theta \in \Theta_{k,\mathbf{0},\Phi}} \mathscr{V}(\theta), \qquad \Gamma_{\mathbf{0}}^{(k)} := \sum_{\theta \in \Theta_{k,\mathbf{0},\Gamma}} \mathscr{V}(\theta), \tag{6.4}$$

and note that $\Phi_{\mathbf{0}}^{(k)} = [\omega_0(B(t)) + \varepsilon \partial_B f(\boldsymbol{\omega} t, \beta(t), B(t))]_{\mathbf{0}}^{(k)}$ and $\Gamma_{\mathbf{0}}^{(k)} = [-\varepsilon \partial_\beta f(\boldsymbol{\omega} t, \beta(t), B(t))]_{\mathbf{0}}^{(k)}$.

Remark 6.3. One has

$$\mathcal{M}_{\beta,\beta}^{(0)}(x,n) = \mathcal{M}_{B,B}^{(0)}(x,n) = \mathcal{M}_{B,\beta}^{(0)}(x,n) = 0, \qquad \mathcal{M}_{\beta,B}^{(0)}(x,n) = \omega'_0(\overline{B}_0),$$
$$\mathcal{M}_{\beta,\beta}^{(1)}(x,n) = \partial_\beta \partial_B f_{\mathbf{0}} = -\mathcal{M}_{B,B}^{(1)}(x,n),$$

for all $n \geq -1$ and all $x \in \mathbb{R}$.

We shall say that a line ℓ is *resonant* if there exist two self-energy clusters T, T', such that $\ell_{T'} = \ell = \ell'_T$, otherwise ℓ is *non-resonant*. Given any subgraph S of any tree θ, we denote by $\mathfrak{N}^*_n(S)$ the number of non-resonant lines on scale $\geq n$ in S. Define also, for any line $\ell \in \theta$, the *minimum scale* of ℓ as

$$\zeta_\ell := \min\{n \in \mathbb{Z}_+ : \Psi_n(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) \neq 0\}$$

and denote by $\mathfrak{N}^\bullet_n(S)$ the number of non-resonant lines $\ell \in L(S)$ such that $\zeta_\ell \geq n$. If $\mathscr{V}(S) \neq 0$, for each line $\ell \in L(S)$ either $n_\ell = \zeta_\ell$ or $n_\ell = \zeta_\ell + 1$. Then one can prove the following results.

Lemma 6.4. *For all $h \in \{\beta, B, \Phi, \Gamma\}$, $\boldsymbol{\nu} \in \mathbb{Z}^d$, $k \geq 1$ and for any $\theta \in \Theta_{k,\boldsymbol{\nu},h}$ with $\mathscr{V}(\theta) \neq 0$, one has $\mathfrak{N}_n^\bullet(\theta) \leq 2^{-(m_n-3)} K(\theta)$ for all $n \geq 0$.*

Lemma 6.5. *For all $e, u \in \{\beta, B\}$, $n \geq 0$, $k \geq 1$ and for any $T \in \mathfrak{S}^k_{n,u,e}$ with $\mathscr{V}_T(x) \neq 0$, one has $K(T) > 2^{m_n - 1}$ and $\mathfrak{N}_p^\bullet(T) \leq 2^{-(m_p-3)} K(T)$ for all $0 \leq p \leq n$.*

The proofs of the two results above follow the lines of those for Lemmas 3.9 and 3.10, and are given in Appendix B.

Lemma 6.6. *Let $k \geq 1$, $\boldsymbol{\nu} \in \mathbb{Z}^d$, $h \in \{\beta, B, \Phi, \Gamma\}$, $u, e \in \{\beta, B\}$ and $n \geq 0$ arbitrarily fixed. For any tree $\theta \in \Theta_{k,\boldsymbol{\nu},h}$ and any self-energy cluster $T \in \mathfrak{S}^k_{n,u,e}$ denote by $L_{NR}(\theta)$ and $L_{NR}(T)$ the sets of non-resonant lines in θ and T, respectively, and set*

$$\mathscr{V}_{NR}(\theta) := \Bigg(\prod_{v \in N(\theta)} \mathcal{F}_v \Bigg) \Bigg(\prod_{\ell \in L_{NR}(\theta)} \mathcal{G}_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) \Bigg), \quad \mathscr{V}_{T,NR}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'_T}) := \Bigg(\prod_{v \in N(T)} \mathcal{F}_v \Bigg) \Bigg(\prod_{\ell \in L_{NR}(T)} \mathcal{G}_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) \Bigg),$$

Then

$$|\mathscr{V}_{NR}(\theta)| \leq c_1^k e^{-\xi|\boldsymbol{\nu}|/2}, \tag{6.5a}$$

$$|\mathscr{V}_{T,NR}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'_T})| \leq c_2^k e^{-\xi K(T)/2}, \tag{6.5b}$$

for some positive constants c_1, c_2.

Proof. The proof follows the lines of the proof of the bound (4.1a) in Lemma 4.3. Indeed by Lemma 6.2 and the analyticity of f and ω_0 one has

$$\prod_{v \in N(\theta)} |\mathcal{F}_v| \leq A B^k e^{-\xi K(\theta)},$$

for some positive constants A, B, while

$$\begin{aligned}
\prod_{\ell \in L_{NR}(\theta)} |\mathcal{G}_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)| &\leq \prod_{n \geq 0} \left(\frac{16}{\alpha_{m_n}(\boldsymbol{\omega})} \right)^{\mathfrak{N}_n^\bullet(\theta)} \leq \left(\frac{16}{\alpha_{m_{n_0}}(\boldsymbol{\omega})} \right)^{4k-2} \prod_{n \geq n_0+1} \left(\frac{16}{\alpha_{m_n}(\boldsymbol{\omega})} \right)^{\mathfrak{N}_n^\bullet(\theta)} \\
&\leq \left(\frac{16}{\alpha_{m_{n_0}}(\boldsymbol{\omega})} \right)^{4k-2} \prod_{n \geq n_0+1} \left(\frac{16}{\alpha_{m_n}(\boldsymbol{\omega})} \right)^{2^{-(m_n-3)} K(\theta)} \\
&\leq D(n_0)^{4k-2} \exp(\xi(n_0) K(\theta)),
\end{aligned}$$

with

$$D(n_0) = \frac{16}{\alpha_{m_{n_0}}(\boldsymbol{\omega})}, \qquad \xi(n_0) = 8 \sum_{n \geq n_0+1} \frac{1}{2^{m_n}} \log \frac{16}{\alpha_{m_n}(\boldsymbol{\omega})}.$$

Then, by Hypothesis 2, one can choose n_0 such that $\xi(n_0) \leq \xi/2$, so that (6.5a) follows, recalling $K(\theta) \geq |\boldsymbol{\nu}|$. To obtain (6.5b) one can reason in the same way, simply with T playing the role of θ. ■

Remark 6.7. By using Remark 3.11 one can show that also $\partial_x^j \mathscr{V}_{T,NR}(\tau x)$ admits the same bound as $\mathscr{V}_{T,NR}(x)$ in (6.5b) for $j = 0, 1, 2$ and $\tau \in [0, 1]$, possibly with a different constant c_2. This will be used later on (in Appendix C).

In the light of the results above, although the propagators are bounded proportionally to $1/|x|$ (since no resummation is performed), in principle one can have accumulation of small divisors because of the presence of resonant lines. Therefore one needs a 'gain factor' proportional to $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$ for each resonant line ℓ, in order to prove the convergence of the power series (2.9). One could also envisage performing the resummation and exploiting cancellations between the self-energies, but in practice this would make the analysis more complicated.

Lemma 6.8. *Assume that* $\Gamma_{\mathbf{0}}^{(k)}\equiv 0$ *for all* $k\geq 0$*. Then for all* $k\geq 1$ *one has*

$$\mathcal{M}^{(k)}_{B,\beta}(0)+\sum_{k_1+k_2=k}\mathcal{M}^{(k_1)}_{B,B}(0)\partial_{\beta_0}B^{(k_2)}_{\mathbf{0}}=0, \tag{6.6a}$$

$$\mathcal{M}^{(k)}_{\beta,\beta}(0)+\sum_{k_1+k_2=k}\mathcal{M}^{(k_1)}_{\beta,B}(0)\partial_{\beta_0}B^{(k_2)}_{\mathbf{0}}=0, \tag{6.6b}$$

$$\mathcal{M}^{(k)}_{\beta,\beta}(0)=-\mathcal{M}^{(k)}_{B,B}(0). \tag{6.6c}$$

Proof. Both (6.6a) and (6.6b) follow from the fact that (see also Remark 4.12)

$$\begin{aligned}\partial_{\beta_0}\Gamma_{\mathbf{0}}^{(k)}&=\mathcal{M}^{(k)}_{B,\beta}(0)+\sum_{k_1+k_2=k}\mathcal{M}^{(k_1)}_{B,B}(0)\partial_{\beta_0}B^{(k_2)}_{\mathbf{0}},\\ \partial_{\beta_0}\Phi_{\mathbf{0}}^{(k)}&=\mathcal{M}^{(k)}_{\beta,\beta}(0)+\sum_{k_1+k_2=k}\mathcal{M}^{(k_1)}_{\beta,B}(0)\partial_{\beta_0}B^{(k_2)}_{\mathbf{0}},\end{aligned}$$

where we have used that, for any function $g=g(\beta_0,B_{\mathbf{0}}(\beta_0))$, one has $\partial_{\beta_0}g=\partial_1 g+\partial_{\beta_0}B_{\mathbf{0}}\,\partial_2 g$.

To obtain (6.6c) one can reason as follows. For any self-energy cluster T, denote by $\overline{N}(T)$ the set of nodes $v\in N(T)$ such that $\ell_v\in\mathcal{P}_T\cup\{\ell_T\}$ and the line $\ell'_v\in\mathcal{P}_T\cup\{\ell'_T\}$ entering v has component $h_{\ell'_v}=h_{\ell_v}$. Let $T\in\mathfrak{S}^k_{n,\beta,\beta}$ and consider the self-energy cluster $T'\in\mathfrak{S}^k_{n,B,B}$ obtained from T by changing all the component labels of the lines in $\mathcal{P}_T\cup\{\ell_T\}\cup\{\ell'_T\}$ and reversing their orientation; in particular the entering line ℓ'_T of T becomes the exiting line $\ell_{T'}$ of T' and, vice versa, the exiting line ℓ_T of T becomes the entering line $\ell'_{T'}$ of T'. Both $\mathcal{M}^{(k)}_{\beta,\beta}(0)$ and $\mathcal{M}^{(k)}_{B,B}(0)$ are given by (6.3b) with $x=0$, so that the external lines ℓ'_T and ℓ_T carry momenta $\boldsymbol{\nu}_{\ell'_T}=\boldsymbol{\nu}_{\ell_T}=\mathbf{0}$. Hence any line $\ell'\in\mathcal{P}_{T'}$ has momentum $\boldsymbol{\nu}'=-\boldsymbol{\nu}$ if $\boldsymbol{\nu}$ is the momentum of the corresponding line in $\mathcal{P}_T$ and the corresponding propagator changes sign (see (6.2) and recall that $n_\ell\geq 0$ for all $\ell\in\mathcal{P}_T$). Moreover, for any $v\in\overline{N}(T)$ the node factor $\mathcal{F}_v$ changes sign when regarded as a node in $\overline{N}(T')$, see (6.1). All the other factors remains the same i.e. we can write $\mathscr{V}_T(0)=\mathfrak{A}(T)\,\mathscr{V}(\mathcal{P}_T)$ and $\mathscr{V}_{T'}(0)=\mathfrak{A}(T')\,\mathscr{V}(\mathcal{P}_{T'})$, where

$$\mathscr{V}(\mathcal{P}_T):=\Bigg(\prod_{v\in\overline{N}(T)}\mathcal{F}_v\Bigg)\Bigg(\prod_{\ell\in\mathcal{P}_T}\mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\Bigg)$$

and $\mathscr{V}(\mathcal{P}_{T'})$ is analogously defined with T' instead of T, while $\mathfrak{A}(T)=\mathfrak{A}(T')$. Now, one has

$$\prod_{v\in\overline{N}(T)}\mathcal{F}_v=\sigma\prod_{v\in\overline{N}(T')}\mathcal{F}_v$$

with $\sigma=\pm1$. If $\sigma=1$, then $|\overline{N}(T)|=|\overline{N}(T')|$ is even and hence there is an odd number of lines in $\mathcal{P}_T$. If on the contrary $\sigma=-1$, then there is an even number of lines in $\mathcal{P}_T$. In both cases the assertion follows. ■

Lemma 6.9. *Assume that* $\Gamma_{\mathbf{0}}^{(k)} \equiv 0$ *for all* $k \geq 0$*. Then for all* $k \geq 1$ *one has*

$$\partial_x \mathcal{M}_{B,\beta}^{(k)}(0) = \partial_x \mathcal{M}_{\beta,B}^{(k)}(0) = 0, \tag{6.7a}$$

$$\partial_x \mathcal{M}_{\beta,\beta}^{(k)}(0) = \partial_x \mathcal{M}_{B,B}^{(k)}(0). \tag{6.7b}$$

Proof. One reason along the same lines as the proof of (6.6c) in Lemma 6.8. Let $T \in \mathfrak{S}_{n,\beta,B}^{k}$ and consider the self-energy cluster $T' \in \mathfrak{S}_{n,\beta,B}^{k}$ obtained from T by changing all the component labels of the lines in $\mathcal{P}_T \cup \{\ell_T\} \cup \{\ell'_T\}$ and reversing their orientation. The derivative ∂_x acts on the propagator of some line $\ell \in \mathcal{P}_T$. After differentiation, when computing the propagators at $x = 0$, any line $\ell' \in \mathcal{P}_{T'}$ turns out to have momentum $\boldsymbol{\nu}' = -\boldsymbol{\nu}$, if $\boldsymbol{\nu}$ is the momentum of the corresponding line in $\mathcal{P}_T$ and the corresponding propagator changes sign, except the differentiated propagator $\partial_x \mathcal{G}_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell^0 + x)\big|_{x=0}$, which is even in its argument. Moreover, for any $v \in \overline{N}(T)$ (we use the same notations as in the proof of Lemma 6.8) the node factor $\mathcal{F}_v$ changes sign when regarded as a node in $\overline{N}(T')$, while all the other factors remains the same. If $|\overline{N}(T)| = |\overline{N}(T')|$ is even (resp. odd) then there is an even (resp. odd) number of lines in $\mathcal{P}_T$, but, as we said, the differentiated propagator does not change sign. Therefore the two contributions have the same modulus but different sign, so that, once summed together, they gives zero. Therefore (6.7a) is proved.

To prove (6.7b) reason as in proving (6.6c): the only difference is that, as in the previous case, the differentiated propagator does not change sign, so that the two contributions are equal to (and not the opposite of) each other. ∎

Remark 6.10. Note that the Hamiltonian structure is fundamental in order to prove both the identity (6.6c) and the identities (6.7).

Given $p \geq 2$ self-energy clusters $T_1, \ldots, T_p$ of any tree θ, with $\ell'_{T_i} = \ell_{T_{i+1}}$ for $i = 1, \ldots, p-1$ and ℓ_{T_1}, ℓ'_{T_p} being non-resonant, we say that $C = \{T_1, \ldots, T_p\}$ is a *chain*. Define $\ell_0(C) := \ell_{T_1}$ and $\ell_i(C) := \ell'_{T_i}$ for $i = 1, \ldots, p$ and set $n_i(C) = n_{\ell_i(C)}$ for $i = 0, \ldots, p$; we also call $k(C) := k(T_1) + \ldots + k(T_p)$ the *total order* of the chain C and $p(C) = p$ the *length* of C. Given a chain $C = \{T_1, \ldots, T_p\}$ we define the *value* of C as

$$\mathscr{V}_C(x) = \prod_{i=1}^{p} \mathscr{V}_{T_i}(x). \tag{6.8}$$

We denote by $\mathfrak{C}(k; h, h'; n_0, \ldots, n_p)$ the set of all chains $C = \{T_1, \ldots, T_p\}$ with total order k and with fixed labels $h_{\ell_0(C)} = h$, $h_{\ell_p(C)} = h'$ and $n_i(C) = n_i$ for $i = 0, \ldots, p$.

Remark 6.11. Let ℓ be a resonant line. Then there exists a chain C such that $\ell = \ell_i(C)$ for some $i = 1, \ldots, p(C) - 1$. If there exists a minimal self-energy cluster T containing ℓ, then T contains the whole chain C and all lines $\ell_0(C), \ldots, \ell_p(C)$ (this follows from the fact that, by definition of self-energy cluster, $\boldsymbol{\nu}_{\ell'}^0 \neq \mathbf{0}$ for all $\ell' \in \mathcal{P}_T$). In particular $L(T)$ contains the two non-resonant lines $\ell_0(C)$ and $\ell_p(C)$, with $\zeta_{\ell_0(C)} = \zeta_{\ell_p(C)} = \zeta_\ell$.

Lemma 6.12. *Assume that* $\Gamma_{\mathbf{0}}^{(k')} \equiv 0$ *for all* $k' \geq 0$*. Then for all* $p \geq 2$*, all* $k \geq 1$*, all* $h, h' \in \{\beta, B\}$ *and all* $\overline{n}_0, \ldots, \overline{n}_p \in \mathbb{Z}_+$ *such that* $\Psi_{\overline{n}_i}(x) \neq 0$*,* $i = 0, \ldots, p$*, one has*

$$\left| \sum_{C \in \mathfrak{C}(k; h, h'; \overline{n}_0, \ldots, \overline{n}_p)} \mathscr{V}_C(x) \right| \leq B^k |x|^{p-1}, \tag{6.9}$$

for some constant $B > 0$.

The proof is deferred to Appendix C.

The bound (6.9) provides exactly the gain factor which is needed in order to prove the convergence of the power series. Indeed given a tree θ, sum together the values of the trees obtained from θ by replacing each maximal chain C (i.e. each chain which is not contained inside any other chain) with any other chain which has the same total order, the same length, the same scale labels associated with the lines $\ell_0(C), \ldots, \ell_p(C)$ and the same component labels associated with the lines $\ell_0(C)$ and $\ell_p(C)$; in other words, if $C \in \mathfrak{C}(k; h, h'; \overline{n}_0, \ldots, \overline{n}_p)$ for some values of the labels, sum over all possible chains belonging to the set $\mathfrak{C}(k; h, h'; \overline{n}_0, \ldots, \overline{n}_p)$. Then we can bound the product of the propagators of the non-resonant lines outside the maximal chains thanks to Lemma 6.4, while the product of the propagators of the lines $\ell_1(C), \ldots, \ell_{p-1}(C)$ of any chain C times the sum of the corresponding chain values is bounded through Lemma 6.12. Then we can reason as in the proof of Lemma 4.9, by using that the propagator of any line ℓ is bounded proportionally to $\alpha_{m_{\zeta_\ell}}(\boldsymbol{\omega})^{-1}$.

Remark 6.13. We obtained the convergence of the power series (2.9) for any β_0 and any ε small enough. Thus the solution turns out to be analytic in both ε and β_0. Moreover, since the solution is parameterised by $\beta_0 \in \mathbb{T}$, in that case the full resonant torus survives. Of course, such a situation is highly non-generic and hence very unlikely.

A Proof of Lemma 4.8

A *left-fake cluster* T on scale n is a connected subgraph of a tree θ with only one entering line ℓ'_T and one exiting line ℓ_T such that (i) all the lines in T have scale $\leq n$ and there is in T at least one line on scale n, (ii) ℓ'_T is on scale $n+1$ and ℓ_T is on scale n and (iii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$. Analogously a *right-fake cluster* T on scale n is a connected subgraph of a tree θ with only one entering line ℓ'_T and one exiting line ℓ_T such that (i) all the lines in T have scale $\leq n$ and there is in T at least one line on scale n, (ii) ℓ'_T is on scale n and ℓ_T is on scale $n+1$ and (iii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$. Roughly speaking, a left-fake (respectively right-fake) cluster T fails to be a self-energy cluster (or even a cluster) only because the exiting (respectively the entering) line is on scale equal to the scale of T. Left-fake and right-fake clusters will be represented graphically as in Figure A.1.

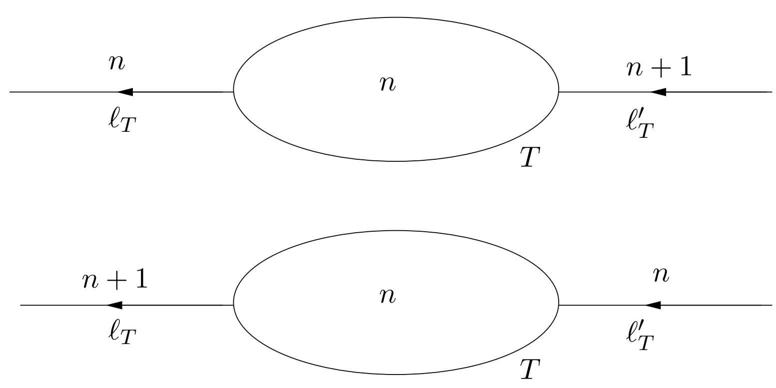

Figure A.1: Graphical representation of the left-fake (top) and right-fake (bottom) clusters T on scale n; by construction $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$. As for the self-energy clusters in Figure 3.4 one has $\ell_T, \ell'_T \notin L(T)$.

The sets of renormalised left-fake clusters and renormalised right-fake clusters T on scale n such that $u_{\ell_T} = u$ and $e_{\ell'_T} = e$ will be denoted by $\mathfrak{LF}_{n,u,e}$ and $\mathfrak{RF}_{n,u,e}$, respectively.

Remark A.1. If T is a renormalised left-fake (respectively right-fake) cluster, we can (and shall) write $\mathscr{V}(T;\varepsilon,\beta_0,B_0) = \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T};\varepsilon,\beta_0,B_0)$ to stress that the propagators of the lines in $\mathcal{P}_T$ depend on $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T}$. In particular one has

$$\sum_{T\in\mathfrak{LF}_{n,u,e}} \varepsilon^{k(T)}\,\mathscr{V}_T(x;\varepsilon,\beta_0,B_0) = \sum_{T\in\mathfrak{RF}_{n,u,e}} \varepsilon^{k(T)}\,\mathscr{V}_T(x;\varepsilon,\beta_0,B_0) = M^{[n]}_{u,e}(x;\varepsilon,\beta_0,B_0).$$

Throughout this appendix, for sake of simplicity, we shall omit the adjective "renormalised" referred to trees, self-energy clusters, left-fake clusters and right-fake clusters.

We shall prove explicitly only the bound

$$|\mathcal{M}^{[p]}_{\beta,\beta}(0;\varepsilon,\beta_0,B_0) - \partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)| \le |\varepsilon|\,A_1 e^{-A_2 2^{m_{p+1}}}, \tag{A.1}$$

as the others relations in (4.8) can be proved exactly in the same way.

We want to compute $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)$, with $\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)$ given by the first line of (3.16). We start by considering trees $\theta\in\Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ such that

$$\max_{\ell\in\Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}} \{n\in\mathbb{Z}_+ : \Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\neq 0\} \le p, \tag{A.2}$$

and shall see later how to deal with trees in $\Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ for which the condition (A.2) is not satisfied (see case **7** at the end).

First of all, for any tree θ set

$$\partial_v\mathscr{V}(\theta;\varepsilon,\beta_0,B_0) := \partial_{\beta_0}\mathcal{F}_v\Bigg(\prod_{w\in N(\theta)\setminus\{v\}}\mathcal{F}_w\Bigg)\Bigg(\prod_{\ell\in L(\theta)}\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0,B_0)\Bigg), \tag{A.3}$$

and

$$\begin{aligned}\partial_\ell\mathscr{V}(\theta;\varepsilon,\beta_0,B_0) &:= \partial_{\beta_0}\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(x_\ell;\varepsilon,\beta_0,B_0)\Bigg(\prod_{v\in N(\theta)}\mathcal{F}_v\Bigg)\Bigg(\prod_{\lambda\in L(\theta)\setminus\{\ell\}}\mathcal{G}^{[n_\lambda]}_{e_\lambda,u_\lambda}(x_\lambda;\varepsilon,\beta_0,B_0)\Bigg)\\ &= \mathcal{A}_\ell(\theta,x_\ell;\varepsilon,\beta_0,B_0)\,\partial_{\beta_0}\mathcal{G}^{[n_\ell]}_{,e_\ell,u_\ell}(x_\ell;\varepsilon,\beta_0,B_0)\,\mathcal{B}_\ell(\theta;\varepsilon,\beta_0,B_0),\end{aligned} \tag{A.4}$$

where $x_\ell := \boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$, $\partial_{\beta_0}\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(x_\ell;\varepsilon,\beta_0,B_0)$ is written according to Remark 3.5 and

$$\mathcal{A}_\ell(\theta,x_\ell;\varepsilon,\beta_0,B_0) := \Bigg(\prod_{\substack{v\in N(\theta)\\ v\not\prec\ell}}\mathcal{F}_v\Bigg)\Bigg(\prod_{\substack{\ell'\in L(\theta)\\ \ell'\not\preceq\ell}}\mathcal{G}^{[n_{\ell'}]}_{e_{\ell'},u_{\ell'}}(x_{\ell'};\varepsilon,\beta_0,B_0)\Bigg), \tag{A.5a}$$

$$\mathcal{B}_\ell(\theta;\varepsilon,\beta_0,B_0) := \Bigg(\prod_{\substack{v\in N(\theta)\\ v\prec\ell}}\mathcal{F}_v\Bigg)\Bigg(\prod_{\substack{\ell'\in L(\theta)\\ \ell'\prec\ell}}\mathcal{G}^{[n_{\ell'}]}_{e_{\ell'},u_{\ell'}}(x_{\ell'};\varepsilon,\beta_0,B_0)\Bigg). \tag{A.5b}$$

Let us define in the analogous way $\partial_v \mathscr{V}_T(x;\varepsilon,\beta_0,B_0)$ and $\partial_\ell \mathscr{V}_T(x;\varepsilon,\beta_0 B_0)$ for any self-energy cluster T and let us write

$$\partial_{\beta_0} \mathscr{V}(\theta;\varepsilon,\beta_0,B_0) = \partial_N \mathscr{V}(\theta;\varepsilon,\beta_0,B_0) + \partial_L \mathscr{V}(\theta;\varepsilon,\beta_0,B_0), \tag{A.6}$$

where

$$\partial_N \mathscr{V}(\theta;\varepsilon,\beta_0,B_0) := \sum_{v\in N(\theta)} \partial_v \mathscr{V}(\theta;\varepsilon,\beta_0,B_0), \tag{A.7}$$

and

$$\partial_L \mathscr{V}(\theta;\varepsilon,\beta_0,B_0) := \sum_{\ell\in L(\theta)} \partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0,B_0). \tag{A.8}$$

Let us also write

$$\partial_{\beta_0} \mathscr{V}_T(x;\varepsilon,\beta_0,B_0) = \partial_N \mathscr{V}_T(x;\varepsilon,\beta_0,B_0) + \partial_L \mathscr{V}_T(x;\varepsilon,\beta_0,B_0), \tag{A.9}$$

for any $T \in \mathfrak{R}_{n,u,e}$, $n \geq 0$ and $u,e \in \{\beta, B\}$, where the derivatives ∂_N and ∂_L are defined analogously with the previous cases (A.7) and (A.8), with $N(T)$ and $L(T)$ replacing $N(\theta)$ and $L(\theta)$, respectively, so that we can split

$$\begin{aligned}
\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(x;\varepsilon,\beta_0,B_0) &= \partial_N\Phi^{\mathcal{R},p}_{\mathbf{0}}(x;\varepsilon,\beta_0,B_0) + \partial_L\Phi^{\mathcal{R},p}_{\mathbf{0}}(x;\varepsilon,\beta_0,B_0),\\
\partial_{\beta_0}M^{[n]}(x;\varepsilon,\beta_0,B_0) &= \partial_N M^{[n]}(x;\varepsilon,\beta_0,B_0) + \partial_L M^{[n]}(x;\varepsilon,\beta_0,B_0),\\
\partial_{\beta_0}\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0) &= \partial_N \mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0) + \partial_L \mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0),
\end{aligned} \tag{A.10}$$

again with obvious meaning of the symbols.

Remark A.2. We can interpret the derivative ∂_v as all the possible ways to attach an extra line ℓ (with $\boldsymbol{\nu}_\ell = \mathbf{0}$ and $u_\ell = \beta$) to the node v, so that

$$\sum_{k\geq 0} \varepsilon^{k+1} \sum_{\theta\in\Theta^{\mathcal{R},p}_{k+1,\mathbf{0}}} \partial_N \mathscr{V}(\theta;\varepsilon,\beta_0,B_0),$$

produces contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0;\varepsilon,\beta_0,B_0)$.

In order to compute $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)$, we have to study the derivative (A.6) for any $\theta \in \Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$. The terms (A.7) produce immediately contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0;\varepsilon,\beta_0,B_0)$ by Remark A.2. Thus, we have to study the derivatives $\partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0,B_0)$ appearing in the sum (A.8). From now on, we shall not write any longer explicitly the dependence on ε,β_0 and B_0, in order not to overwhelm the notation.

For any $\theta \in \Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ satisfying the condition (A.2) and for any line $\ell \in L(\theta)$, either there is only one scale n such that $\Psi_n(x_\ell) \neq 0$ (and in that case $\Psi_n(x_\ell) = 1$ and $\Psi_{n'}(x_\ell) = 0$ for all $n' \neq n$) or there exists only one $0 \leq n \leq p-1$ such that $\Psi_n(x_\ell)\Psi_{n+1}(x_\ell) \neq 0$. To help following the argument below, we divide the discussion into several steps (cases **1** to **7**), marking the end of each step with a white box (□).

1. If $\Psi_n(x_\ell) = 1$ one has

$$\begin{aligned}
\partial_\ell \mathscr{V}(\theta) &= \mathcal{A}_\ell(\theta,x_\ell)\left(\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\left((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell)\right)^{-1}\right)_{e_\ell,u_\ell}\mathcal{B}_\ell(\theta)\\
&= \mathcal{A}_\ell(\theta,x_\ell)\left(\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n]}(x_\ell)\right)_{e_\ell,u_\ell}\mathcal{B}_\ell(\theta),
\end{aligned} \tag{A.11}$$

with $\mathcal{A}_\ell(\theta,x_\ell)$ and $\mathcal{B}_\ell(\theta)$ defined in (A.5). □

Remark A.3. Note that if we split $\partial_{\beta_0} = \partial_N + \partial_L$ in (A.11), the term with $\partial_N \mathcal{M}^{[n-1]}(x_\ell)$ is a contribution to $\mathcal{M}^{[p-1]}_{\beta,\beta}(0)$ and hence to $\mathcal{M}^{[p]}_{\beta,\beta}(0)$.

If there is only one $0 \le n \le p-1$ such that $\Psi_n(x_\ell)\Psi_{n+1}(x_\ell) \neq 0$, then $\Psi_n(x_\ell) + \Psi_{n+1}(x_\ell) = 1$ and $\chi_q(x_\ell) = 1$ for all $q = -1, \dots, n-1$, so that $\psi_{n+1}(x_\ell) = 1$ and hence $\Psi_{n+1}(x_\ell) = \chi_n(x_\ell)$. Moreover it can happen only (see Remark 3.8) $n_\ell = n$ or $n_\ell = n+1$.

2. Consider first the case $n_\ell = n+1$. One has

$$\partial_\ell \mathscr{V}(\theta) = \mathcal{A}_\ell(\theta, x_\ell) \, \Big(\mathcal{G}^{[n+1]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n]}(x_\ell) ((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell))^{-1} \Big)_{e_\ell, u_\ell} \mathcal{B}_\ell(\theta), \tag{A.12}$$

with

$$\begin{aligned}
&\mathcal{G}^{[n+1]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n]}(x_\ell)\big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1} \\
&\quad = \mathcal{G}^{[n+1]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\big(\Psi_n(x_\ell) + \Psi_{n+1}(x_\ell)\big)\big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1} \\
&\qquad + \mathcal{G}^{[n+1]}(x_\ell)\partial_{\beta_0}M^{[n]}(x_\ell)\,\chi_n(x_\ell)\big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1} \\
&\quad = \mathcal{G}^{[n+1]}(x_\ell)\left(\sum_{q=-1}^{n} \partial_{\beta_0} M^{[q]}(x_\ell)\right)\mathcal{G}^{[n+1]}(x_\ell) + \mathcal{G}^{[n+1]}(x_\ell)\left(\sum_{q=-1}^{n-1} \partial_{\beta_0} M^{[q]}(x_\ell)\right)\mathcal{G}^{[n]}(x_\ell) \\
&\qquad + \mathcal{G}^{[n+1]}(x_\ell)\left(\sum_{q=-1}^{n-1} \partial_{\beta_0} M^{[q]}(x_\ell)\right)\mathcal{G}^{[n]}(x_\ell)M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell),
\end{aligned} \tag{A.13}$$

where we have used that $\chi_n(x_\ell) = \Psi_{n+1}(x_\ell)$ and

$$\big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1}\big(\mathbb{1} + M^{[n]}(x_\ell)\Psi_{n+1}(x_\ell)\big((ix_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1}\big)^{-1} = \big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell)\big)^{-1}.$$

We represent graphically the three contributions in (A.13) as in Figure A.2: we represent the derivative ∂_{β_0} as an arrow pointing toward the graphical representation of the differentiated quantity. □

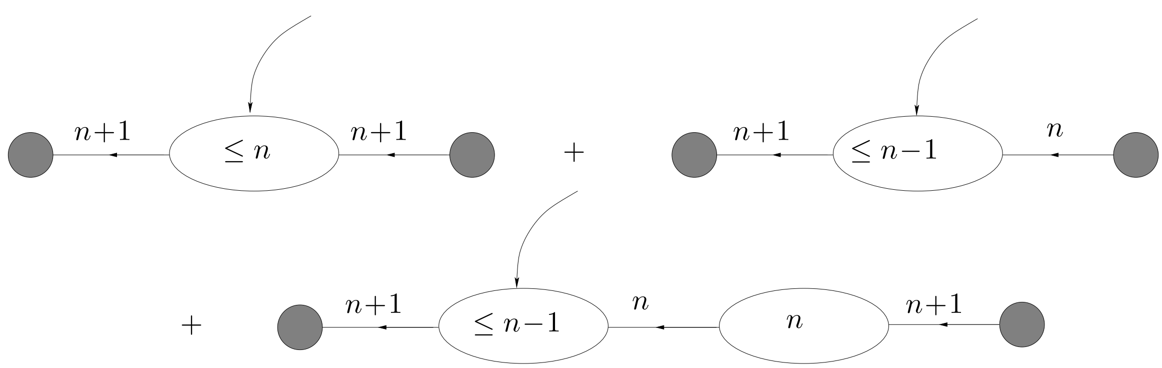

Figure A.2: Graphical representation of the derivative $\partial_\ell \mathscr{V}(\theta)$ according to (A.13).

Remark A.4. Note that the $M^{[n]}(x_\ell)$ appearing in the latter line of (A.13) has to be interpreted (see Remark 3.7) as the matrix with components

$$\sum_{T \in \mathfrak{L}\mathfrak{F}_{n,u,e}} \varepsilon^{k(T)} \, \mathscr{V}_T(x_\ell).$$

Note also that, again, if we split $\partial_{\beta_0} = \partial_N + \partial_L$ in (A.13), all the terms with $\partial_N M^{[q]}(x_\ell)$ are contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0)$.

Now consider the case $n_\ell = n$. We distinguish among several cases (see Remark A.5 below for the meaning of "removal" and "insertion" of left-fake clusters):
(a) ℓ *does not* exit any left-fake cluster and one *can* insert a left-fake cluster, together its entering line, between ℓ and the node ℓ exists without creating any self-energy cluster (case **3** below);
(b) ℓ *does not* exit any left-fake cluster and one *cannot* insert any left-fake cluster between ℓ and the node ℓ exists because this way a self-energy cluster would appear (case **4** below);
(c) ℓ *does* exit a left-fake cluster and one *can* remove the left-fake cluster, together its entering line, without creating a self-energy cluster (case **3** below);
(d) ℓ *does* exit a left-fake cluster and one *cannot* remove the left-fake cluster because a self-energy cluster would be produced (case **5** below).

Remark A.5. Here and henceforth, if S is a subgraph with only one entering line $\ell'_S = \ell_v$ and one exiting line ℓ_S, by saying that we "remove" S together with ℓ'_S, we mean that we change u_{ℓ_S} into h_v and we also reattach the line ℓ_S to the node v (so that ℓ_S becomes the line exiting v). Analogously, whenever we "insert" a subgraph S with only one entering line ℓ' between a line ℓ and the node v which ℓ exits, we mean that we set $u_{\ell'} = h_v$ and change u_ℓ into h_w if $w \in N(S)$ is the node to which we reattach ℓ (and ℓ becomes the line ℓ_w exiting S).

3. If ℓ is not the exiting line of a left-fake cluster, set $\bar\theta = \theta$; otherwise, if ℓ is the exiting line of a left-fake cluster T, define – if possible – $\bar\theta$ as the tree obtained from θ by removing T and ℓ'_T In both cases, define – if possible – $\tau_1(\bar\theta, \ell)$ as the set constituted by all the renormalised trees θ' obtained from $\bar\theta$ by inserting a left-fake cluster, together with its entering line, between ℓ and the node v which ℓ exits; see Figure A.3.

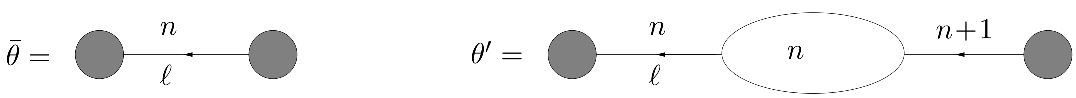

Figure A.3: The renormalised tree $\bar\theta$ and the renormalised trees θ' of the set $\tau_1(\bar\theta, \ell)$ associated with $\bar\theta$.

Remark A.6. The construction of the set $\tau_1(\bar\theta, \ell)$ could be impossible if the removal or the insertion of a left-fake cluster T, together with its entering line ℓ'_T, would produce a self-energy cluster. We shall see later (see cases **4** and **5** below) how to deal with these cases.

Then one has

$$\partial_\ell \mathscr{V}(\bar\theta) + \partial_\ell \sum_{\theta' \in \tau_1(\bar\theta,\ell)} \mathscr{V}(\theta') = \mathcal{A}_\ell(\bar\theta, x_\ell) \left(\partial_{\beta_0} \mathcal{G}^{[n]}(x_\ell) \left(\mathbb{1} + M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \right) \right)_{e_\ell, u_\ell} \mathcal{B}_\ell(\bar\theta), \qquad \text{(A.14)}$$

with

$$\begin{aligned}
&\partial_{\beta_0}\mathcal{G}^{[n]}(x_\ell)\left(\mathbb{1}+M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)\right)\\
&\quad=\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n]}(x_\ell)\\
&\qquad+\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\Psi_{n+1}(x_\ell)\left((\mathrm{i}x_\ell)\mathbb{1}-\mathcal{M}^{[n-1]}(x_\ell)\right)^{-1}\\
&\qquad+\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n]}(x_\ell)M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)\\
&\qquad+\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\Psi_{n+1}(x_\ell)\left((\mathrm{i}x_\ell)\mathbb{1}-\mathcal{M}^{[n-1]}(x_\ell)\right)^{-1}M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)
\end{aligned}$$

and hence

$$\begin{aligned}
&\partial_{\beta_0}\mathcal{G}^{[n]}(x_\ell)\left(\mathbb{1}+M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)\right)\\
&\quad=\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n]}(x_\ell)+\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)\\
&\qquad-\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\chi_n(x_\ell)\left((\mathrm{i}x_\ell)\mathbb{1}-\mathcal{M}^{[n-1]}(x_\ell)\right)^{-1}M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)\\
&\qquad+\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n]}(x_\ell)M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)\\
&\qquad+\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\Psi_{n+1}(x_\ell)\left((\mathrm{i}x_\ell)\mathbb{1}-\mathcal{M}^{[n-1]}(x_\ell)\right)^{-1}M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)\\
&\quad=\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n]}(x_\ell)+\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)\\
&\qquad+\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n]}(x_\ell)M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell),
\end{aligned}$$

where we have used that

$$\big(\mathbb{1}-\Psi_{n+1}(x_\ell)\big((ix_\ell)\mathbb{1}-\mathcal{M}^{[n-1]}(x_\ell)\big)^{-1}M^{[n]}(x_\ell)\big)^{-1}\big((\mathrm{i}x_\ell)\mathbb{1}-\mathcal{M}^{[n-1]}(x_\ell)\big)^{-1}=\big((\mathrm{i}x_\ell)\mathbb{1}-\mathcal{M}^{[n]}(x_\ell)\big)^{-1}.$$

Also in this case, if we split $\partial_{\beta_0}=\partial_N+\partial_L$, all the terms with $\partial_N\mathcal{M}^{[n-1]}$ are contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0)$ – see Remark A.3. Again, we can represent graphically the three contributions obtained inserting (A.15) in (A.14); see Figure A.4. □

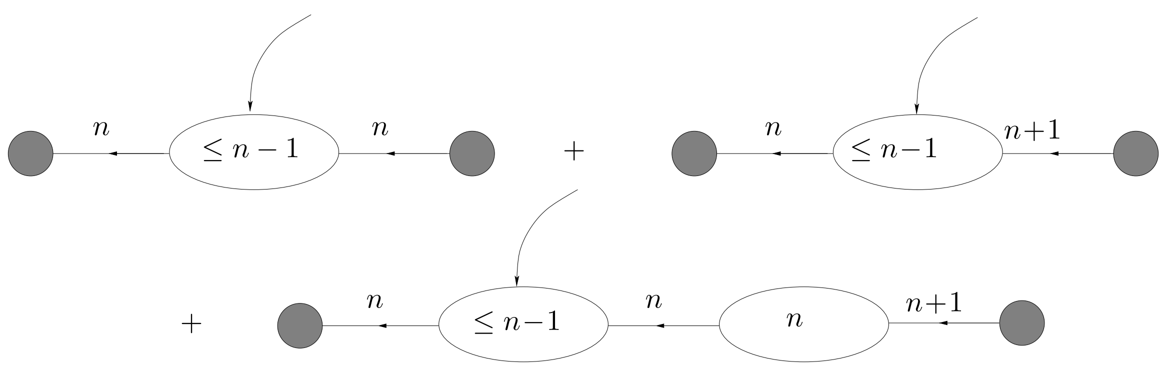

Figure A.4: Graphical representation of the three contributions in the last two lines of (A.15).

4. Assume now that ℓ is not the exiting line of a left-fake cluster and the insertion of a left-fake cluster, together with its entering line, produces a self-energy cluster. Note that this can happen only if ℓ is the entering line of a right-fake cluster T. Let $\overline{\ell}$ be the exiting line (on scale $n+1$) of the right-fake cluster T, call $\overline{\theta}$ the tree obtained from θ by removing T and ℓ and call $\tau_2(\overline{\theta},\overline{\ell})$ the set of trees θ' obtained from $\overline{\theta}$ by inserting a right-fake cluster, together with its entering line, before $\overline{\ell}$; see Figure A.5.

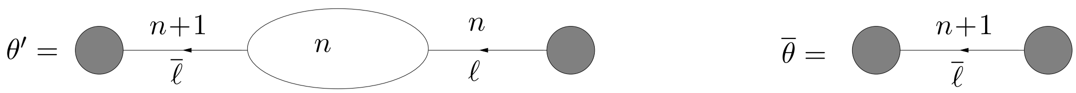

Figure A.5: The trees θ' of the set $\tau_2(\overline{\theta},\overline{\ell})$ obtained from $\overline{\theta}$ when $\ell \in L(\theta)$ enters a right-fake cluster.

By construction one has

$$\begin{aligned}\mathscr{V}(\overline{\theta}) &= \mathcal{A}_{\overline{\ell}}(\overline{\theta},x_\ell)\,\mathcal{G}^{[n+1]}_{e_{\overline{\ell}},u_{\overline{\ell}}}(x_{\overline{\ell}})\,\mathcal{B}_{\overline{\ell}}(\overline{\theta})\\ \sum_{\theta'\in\tau_2(\overline{\theta},\overline{\ell})}\mathscr{V}(\theta') &= \mathcal{A}_{\overline{\ell}}(\overline{\theta},x_\ell)\left(\mathcal{G}^{[n+1]}(x_{\overline{\ell}})\,M^{[n]}(x_{\overline{\ell}})\,\mathcal{G}^{[n]}(x_{\overline{\ell}})\right)_{e_{\overline{\ell}},u_{\overline{\ell}}}\mathcal{B}_{\overline{\ell}}(\overline{\theta}),\end{aligned}$$

where $u_{\overline{\ell}}$ denotes the u-component of $\overline{\ell}$ as line in $\overline{\theta}$ and we have used that $x_\ell = x_{\overline{\ell}}$.

Consider the contribution to $\partial_{\overline{\ell}}\mathscr{V}(\overline{\theta})$ given by

$$\mathcal{A}_{\overline{\ell}}(\overline{\theta},x_{\overline{\ell}})\Big(\mathcal{G}^{[n+1]}(x_{\overline{\ell}})\partial_L M^{[n]}(x_{\overline{\ell}})\mathcal{G}^{[n+1]}(x_{\overline{\ell}})\Big)_{e_{\overline{\ell}},u_{\overline{\ell}}}\mathcal{B}_{\overline{\ell}}(\overline{\theta}), \tag{A.15}$$

arising from (A.13). For $u,e,e'\in\{\beta,B\}$ and $T\in\mathfrak{RF}_{n,u,e'}$ call $\mathfrak{R}_{n,u,e}(T)$ the subset of $\mathfrak{R}_{n,u,e}$ such that if $T'\in\mathfrak{R}_{n,u,e}(T)$ the exiting line $\ell_{T'}$ exits also the renormalised right-fake cluster T; note that the entering line ℓ of T must be also the exiting line of some renormalised left-fake cluster T'' contained in T'; see Figure A.6.

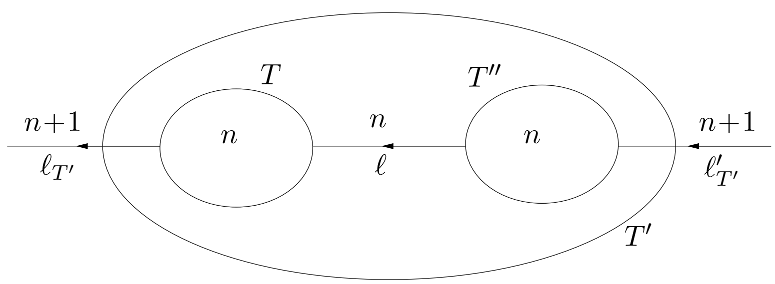

Figure A.6: A self-energy cluster $T'\in\mathfrak{R}_n(T)$.

Define $\mathfrak{M}^{[n]}(x_{\overline{\ell}})$ as the 2×2 matrix with components

$$\mathfrak{M}^{[n]}_{u,e}(x_{\overline{\ell}}) = \sum_{e'=\beta,B}\ \sum_{T\in\mathfrak{RF}_{n,u,e'}}\ \sum_{T'\in\mathfrak{R}_{n,u,e}(T)}\varepsilon^{k(T')}\,\mathscr{V}_{T'}(x_{\overline{\ell}}) \tag{A.16}$$

and consider the contribution $\mathfrak{M}^{[n]}(x_{\overline{\ell}})$ to $M^{[n]}(x_{\overline{\ell}})$ in (A.15). Let us pick up the term with the derivative acting on the line ℓ: one has

$$\begin{aligned}&\partial_\ell\sum_{\theta'\in\tau_2(\overline{\theta},\overline{\ell})}\mathscr{V}(\theta') + \mathcal{A}_{\overline{\ell}}(\overline{\theta},x_\ell)\left(\mathcal{G}^{[n+1]}(x_\ell)\,\partial_\ell\mathfrak{M}^{[n]}(x_{\overline{\ell}})\mathcal{G}^{[n+1]}(x_\ell)\right)_{e_{\overline{\ell}},u_{\overline{\ell}}}\mathcal{B}_{\overline{\ell}}(\overline{\theta})\\ &= \mathcal{A}_{\overline{\ell}}(\overline{\theta},x_\ell)\left(\mathcal{G}^{[n+1]}(x_\ell)M^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{G}^{[n]}(x_\ell)\left(\mathbb{1}+M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell)\right)\right)_{e_{\overline{\ell}},u_{\overline{\ell}}}\mathcal{B}_{\overline{\ell}}(\overline{\theta}),\end{aligned} \tag{A.17}$$

where we have used again that $x_\ell = x_{\overline{\ell}}$. Thus, one can reason as in (A.15), so as to obtain the sum of three contributions, as represented in Figure A.7. $\square$

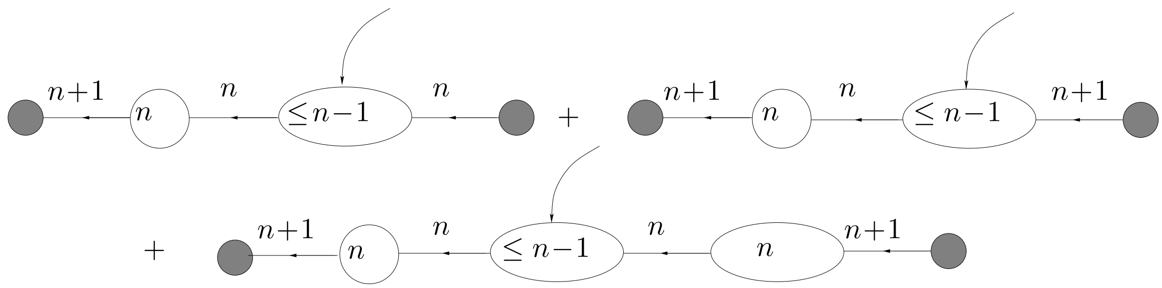

Figure A.7: Graphical representation of the three contributions arising from (A.17).

5. Finally, consider the case in which ℓ is the exiting line of a left-fake cluster, T_0 and the removal of T_0 and ℓ'_{T_0} (see Remark A.5) creates a self-energy cluster.

Set (for a reason that will become clear later) $\theta_0 = \theta$ and $\ell_0 = \ell$. Then there is a maximal $m \geq 1$ such that there are $2m$ lines $\ell_1, \ldots, \ell_m$ and $\ell'_1, \ldots \ell'_m$, with the following properties:
(i) $\ell_i \in \mathcal{P}(\ell_{\theta_0}, \ell_{i-1})$, for $i = 1, \ldots, m$,
(ii) $n_{\ell_i} = n+i < \max\{p : \Psi_p(x_{\ell_i}) \neq 0\} = n+i+1$, for $i = 0, \ldots, m-1$, while $n_m := n_{\ell_m} = n+m+\sigma$, with $\sigma \in \{0, 1\}$,
(iii) $\boldsymbol{\nu}_{\ell_i} \neq \boldsymbol{\nu}_{\ell_{i-1}}$ and the lines preceding ℓ_i but not ℓ_{i-1} are on scale $\leq n+i-1$, for $i = 1, \ldots, m$,
(iv) $\boldsymbol{\nu}_{\ell'_i} = \boldsymbol{\nu}_{\ell_i}$, for $i = 1, \ldots, m$,
(v) if $m \geq 2$, ℓ'_i is the exiting line of a left-fake cluster T_i, for $i = 1, \ldots, m-1$,
(vi) $\ell'_i \prec \ell'_{T_{i-1}}$ and all the lines preceding $\ell'_{T_{i-1}}$ but not ℓ'_i are on scale $\leq n+i-1$, for $i = 1, \ldots, m$,
(vii) $n'_m := n_{\ell'_m} = n+m+\sigma'$ with $\sigma' \in \{0, 1\}$.

Note that one cannot have $\sigma = \sigma' = 1$, otherwise the subgraph between ℓ_m and ℓ'_m would be a self-energy cluster. Note also that (ii), (iv) and (v) imply $n_{\ell'_i} = n+i$ for $i = 1, \ldots, m-1$ if $m \geq 2$. Call S_i the subgraph between ℓ_{i+1} and ℓ_i and S'_i the cluster between ℓ'_{T_i} and ℓ'_{i+1}, for all $i = 0, \ldots, m-1$. For $i = 1, \ldots, m$, call θ_i the tree obtained from θ_0 by removing everything between ℓ_i and the part of θ_0 preceding ℓ'_i Note that, if $m \geq 2$, properties (i)–(vii) hold for θ_i but with $m-i$ instead of m, for all $i = 1, \ldots, m-1$.

For $i = 1, \ldots, m$, call R_i the self-energy cluster obtained from the subgraph of θ_{i-1} between ℓ_i and ℓ'_i, by removing the left-fake cluster T_{i-1} together with ℓ'_{T_i}. Note that $L(R_i) = L(S_{i-1}) \cup \{\ell_{i-1}\} \cup L(S'_{i-1})$ and $N(R_i) = N(S_{i-1}) \cup N(S'_{i-1})$; see Figure A.8.

For $i = 0, \ldots, m-1$, given $\ell', \ell \in L(\theta_i)$, with $\ell' \prec \ell$, call $\mathcal{P}^{(i)}(\ell, \ell')$ the path of lines in θ_i connecting ℓ' to ℓ (hence $\mathcal{P}^{(i)}(\ell, \ell') = \mathcal{P}(\ell, \ell') \cap L(\theta_i)$). For any $i = 0, \ldots, m-1$ and any $\ell \in \mathcal{P}^{(i)}(\ell_i, \ell'_m)$, let $\tau_3(\theta_i, \ell)$ be the set of all renormalised trees which can be obtained from θ_i by replacing each left-fake cluster preceding ℓ but not ℓ'_m with all possible left-fake clusters. Set also $\tau_3(\theta_{m-1}, \ell'_m) = \theta_{m-1}$.

Note that, by construction,

$$
\begin{aligned}
&\mathcal{A}_{\ell_m}(\theta_m, x_{\ell_m}) \mathcal{G}^{[n_m]}_{e_{\ell_m}, u_{\ell_m}}(x_{\ell_m}) \mathscr{V}(S_{m-1}) = \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}}), \\
&\mathscr{V}(S'_{m-1}) \mathcal{G}^{[n'_m]}_{e_{\ell'_m}, u_{\ell'_m}}(x_{\ell_m}) \mathcal{B}_{\ell_m}(\theta_m) = \mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}).
\end{aligned}
\tag{A.18}
$$

One among cases **1**–**4** holds for $\ell_m \in L(\theta_m)$, so that we can consider the contribution to $\partial_{\ell_m} \mathscr{V}(\theta_m)$ (together with other contributions as in **3** and **4**, if necessary) given by – see (A.11), (A.13) and (A.15)

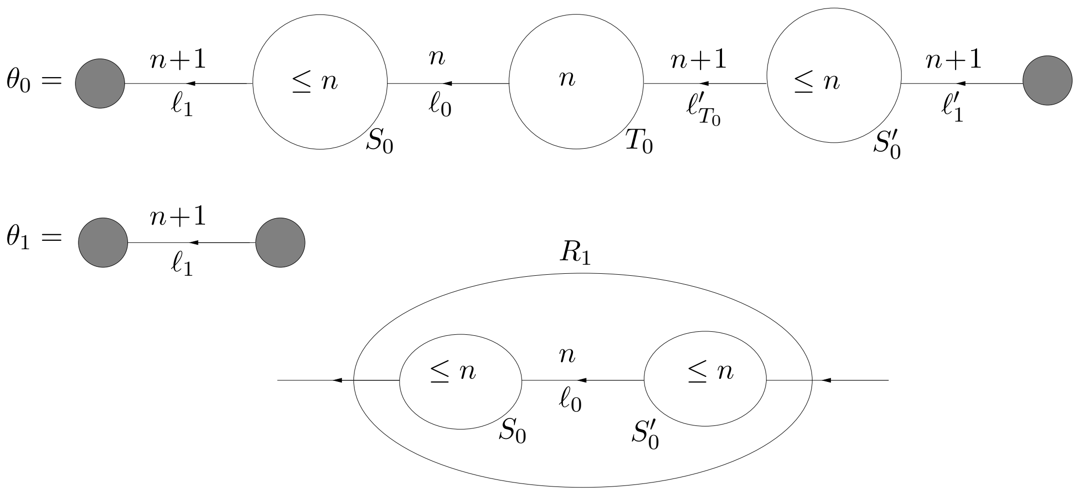

Figure A.8: The renormalised trees θ_0 and θ_1 and the self-energy cluster R_1 in case **5** with $m = 1$ and $\sigma = \sigma' = 0$. Note that the set S'_0 is a cluster, but not a self-energy cluster.

–

$$\mathcal{A}_{\ell_m}(\theta_m, x_{\ell_m})\Big(\mathcal{G}^{[n_m]}(x_{\ell_m})\partial_{\ell_{m-1}}\mathscr{V}_{R_m}(x_{\ell_m})\mathcal{G}^{[n'_m]}(x_{\ell_m})\Big)_{e_{\ell_m},u_{\ell'_m}}\mathcal{B}_{\ell_m}(\theta_m).$$

Then one has

$$\begin{aligned}
&\mathcal{A}_{\ell_m}(\theta_m, x_{\ell_m})\Big(\mathcal{G}^{[n_m]}(x_{\ell_m})\partial_{\ell_{m-1}}\mathscr{V}_{R_m}(x_{\ell_m})\mathcal{G}^{[n'_m]}(x_{\ell_m})\Big)_{e_{\ell_m},u_{\ell'_m}}\mathcal{B}_{\ell_m}(\theta_m) + \partial_{\ell_{m-1}}\sum_{\theta'\in\tau_3(\theta_{m-1},\ell_{m-1})}\mathscr{V}(\theta';\varepsilon,\beta_0)\\
&= \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\Big(\partial_{\beta_0}\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}}) \times \\
&\quad\times\Big(\mathbb{1} + M^{[n+m-1]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m]}(x_{\ell_{m-1}})\Big)\Big)_{e,u'}\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}),
\end{aligned}\tag{A.19}$$

where we have shortened $e, u' = e_{\ell_{m-1}}, u_{\ell'_{T_{m-1}}}$ to simplify notation. By reasoning as in (A.15), this gives

$$\begin{aligned}
&\mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\Big(\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{M}^{[n+m-2]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}})\Big)_{e,u'}\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1})\\
&\quad + \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\Big(\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{M}^{[n+m-2]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m]}(x_{\ell_{m-1}})\Big)_{e,u'}\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1})\\
&\quad + \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\Big(\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{M}^{[n+m-2]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}}) \times \\
&\qquad\times M^{[n+m-1]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m]}(x_{\ell_{m-1}})\Big)_{e,u'}\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}).
\end{aligned}\tag{A.20}$$

where again $e, u' = e_{\ell_{m-1}}, u_{\ell'_{T_{m-1}}}$.

Then, for $i = m-1, \ldots, 1$ we recursively reason as follows. Set

$$\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i, \ell'_{i+1})) := \sum_{\theta'\in\tau_3(\theta_i,\ell'_{i+1})}\mathcal{B}_{\ell'_{T_i}}(\theta')$$

and note that

$$
\begin{aligned}
&\mathcal{A}_{\ell_i}(\theta_i, x_{\ell_i})\mathcal{G}^{[n+i]}_{e_{\ell_i},u_{\ell_i}}(x_{\ell_i})\mathscr{V}(S_{i-1}) = \mathcal{A}_{\ell_{i-1}}(\theta_{i-1}, x_{\ell_{i-1}}),\\
&\mathscr{V}(S'_{i-1})\Big(\mathcal{G}^{[n+i]}(x_{\ell_i})M^{[n+i]}(x_{\ell_i})\mathcal{G}^{[n+i+1]}(x_{\ell_i})\Big)_{e_{\ell'_{i-1}},u_{\ell'_{T_i}}}\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i,\ell'_{i+1})) = \mathcal{B}_{\ell'_{T_{i-1}}}(\tau_3(\theta_{i-1},\ell'_i)).
\end{aligned}
\tag{A.21}
$$

Consider the contribution

$$
\begin{aligned}
\mathcal{A}_{\ell_i}(\theta_i, x_{\ell_i})\Big(&\mathcal{G}^{[n+i]}(x_{\ell_i})\partial_{\ell_{i-1}}\mathscr{V}_{R_i}(x_{\ell_i})\mathcal{G}^{[n+i]}(x_{\ell_i})\times\\
&\times M^{[n+i]}(x_{\ell_i})\mathcal{G}^{[n+i+1]}(x_{\ell_i})\Big)_{e_{\ell_i},u_{\ell'_{T_i}}}\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i,\ell'_{i+1})),
\end{aligned}
\tag{A.22}
$$

obtained at the $(i+1)$-th step of the recursion. By (A.21) one has (see Figure A.9)

$$
\begin{aligned}
&\mathcal{A}_{\ell_i}(\theta_i, x_{\ell_i})\Big(\mathcal{G}^{[n+i]}(x_{\ell_i})\partial_{\ell_{i-1}}\mathscr{V}_{R_i}(x_{\ell_i})\mathcal{G}^{[n+i]}(x_{\ell_i})\times\\
&\qquad\times M^{[n+i]}(x_{\ell_i})\mathcal{G}^{[n+i+1]}(x_{\ell_i})\Big)_{e_{\ell_i},u_{\ell'_{T_i}}}\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i,\ell'_{i+1})) + \partial_{\ell_{i-1}}\sum_{\theta'\in\tau_3(\theta_{i-1},\ell_{i-1})}\mathscr{V}(\theta')\\
&\quad= \mathcal{A}_{\ell_{i-1}}(\theta_{i-1}, x_{\ell_{i-1}})\,\Big(\partial_{\beta_0}\mathcal{G}^{[n+i-1]}(x_{\ell_{i-1}})\times\\
&\qquad\times\Big(\mathbb{1} + M^{[n+i-1]}(x_{\ell_{i-1}})\mathcal{G}^{[n+i]}(x_{\ell_{i-1}})\Big)\Big)_{e_{\ell_{i-1}},u_{\ell'_{T_{i-1}}}}\mathcal{B}_{\ell'_{T_{i-1}}}(\tau_3(\theta_{i-1},\ell'_i)),
\end{aligned}
\tag{A.23}
$$

which produces, as in (A.20), the contribution

$$
\begin{aligned}
\mathcal{A}_{\ell_{i-1}}(\theta_{i-1}, x_{\ell_{i-1}})\Big(&\mathcal{G}^{[n+i-1]}(x_{\ell_{i-1}})\partial_{\ell_{i-2}}\mathscr{V}_{R_{i-1}}(x_{\ell_{i-1}})\mathcal{G}^{[n+i-1]}(x_{\ell_{i-1}})\times\\
&\times M^{[n+i-1]}(x_{\ell_{i-1}})\mathcal{G}^{[n+i]}(x_{\ell_{i-1}})\Big)_{e_{\ell_{i-1}},u_{\ell'_{T_{i-1}}}}\mathcal{B}_{\ell'_{T_{i-1}}}(\tau_3(\theta_{i-1},\ell'_i)).
\end{aligned}
\tag{A.24}
$$

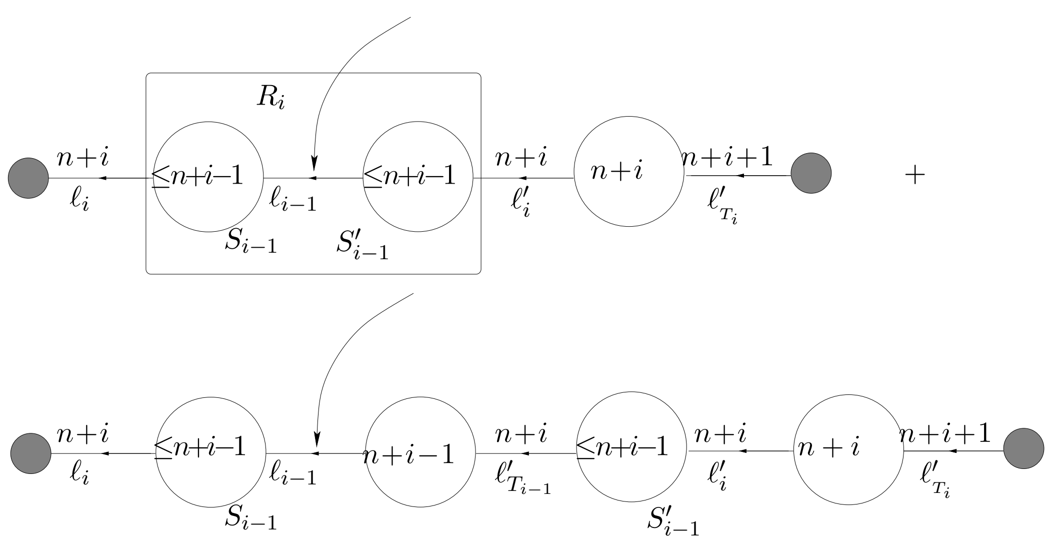

Figure A.9: Graphical representation of the left hand side of (A.23).

Hence we can proceed recursively from θ_m up to θ_0, until we obtain

$$\begin{aligned} &\mathcal{A}_{\ell_0}(\theta_0, x_{\ell_0})\Big(\mathcal{G}^{[n]}(x_{\ell_0})\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_{\ell_0})\mathcal{G}^{[n]}(x_{\ell_0})\Big)_{e_{\ell_0},u_{\ell'_{T_0}}}\mathcal{B}_{\ell'_{T_0}}(\tau_3(\theta_0,\ell'_1)) \\ &\quad + \mathcal{A}_{\ell_0}(\theta_0, x_{\ell_0})\Big(\mathcal{G}^{[n]}(x_{\ell_0})\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_{\ell-0})\mathcal{G}^{[n+1]}(x_{\ell_0})\Big)_{e_{\ell_0},u_{\ell'_{T_0}}}\mathcal{B}_{\ell'_{T_0}}(\tau_3(\theta_0,\ell'_1)) \\ &\quad + \mathcal{A}_{\ell_0}(\theta_0, x_{\ell_0})\Big(\mathcal{G}^{[n]}(x_{\ell_0})\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_{\ell_0})\mathcal{G}^{[n]}(x_{\ell_0})M^{[n]}(x_{\ell_0})\mathcal{G}^{[n+1]}(x_{\ell_0})\Big)_{e_{\ell_0},u_{\ell'_{T_0}}}\mathcal{B}_{\ell'_{T_0}}(\tau_3(\theta_0,\ell'_1)). \end{aligned} \tag{A.25}$$

Once again, if we split $\partial_{\beta_0} = \partial_N + \partial_L$, all the terms with $\partial_N \mathcal{M}^{[n-1]}$ are contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0)$. □

6. We are left with the derivatives $\partial_L M^{[q]}(x)$, $q \le n$, when the differentiated propagator is not one of those used along the cases **4** or **5**; see for instance (A.17), (A.19) and (A.23). One can reason as in the case $\partial_L \mathscr{V}(\theta)$, by studying the derivatives $\partial_\ell \mathscr{V}_T(x_\ell)$ and proceed iteratively along the lines of cases **1** to **5** above, until only lines on scales 0 are left. In that case the derivatives $\partial_{\beta_0}\mathcal{G}^{[0]}(x_\ell)$ produce derivatives

$$\partial_{\beta_0} M^{[-1]}(x) = \begin{pmatrix} \varepsilon\partial^2_{\beta_0} F_{\mathbf{0}}(\beta_0, B_0) & \varepsilon\partial^2_{\beta_0,B_0} F_{\mathbf{0}}(\beta_0, B_0) \\ \varepsilon\partial^2_{\beta_0} G_{\mathbf{0}}(\beta_0, B_0) & \varepsilon\partial^2_{\beta_0,B_0} G_{\mathbf{0}}(\beta_0, B_0) \end{pmatrix}$$

(see Remarks 3.5 and 3.6). Therefore, for $n = -1$, in the splitting (A.10), there are no terms with the derivatives ∂_ℓ and the derivatives ∂_v can be interpreted as said in Remark A.2. □

7. By construction, each contribution to $\mathcal{M}^{[p-1]}_{\beta,\beta}(0)$ appears as one term among those considered in the discussion above, that is among the contributions to $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)$ arising from the trees $\theta \in \Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ satisfying the condition (A.2). Of course, when computing $\partial_{\beta_0}\mathscr{V}(\theta)$ for such trees, also some contributions to $M^{[p]}_{\beta,\beta}(0)$ have been produced. Call $W^{[p]}$ the contributions to $M^{[p]}_{\beta,\beta}(0)$ which are not obtained in the previous steps. Define also $R^{[p]}$ as the sum of the contributions to $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}$ such that

$$\partial_{\beta_0}\Phi^{\mathcal{R},p-1}_{\mathbf{0}} + R^{[p]} = \mathcal{M}^{[p-1]}_{\beta,\beta}(0) + \Big(M^{[p]}_{\beta,\beta}(0) - W^{[p]}\Big), \tag{A.26}$$

where we have used that $\mathcal{M}^{[p]}_{\beta,\beta}(0) = \mathcal{M}^{[p-1]}_{\beta,\beta}(0) + M^{[p]}_{\beta,\beta}(0)$ – see definition (3.9) and use that $\chi_q(0) = 1$ for all $q \ge -1$. Hence $\partial_{\beta_0}\Phi^{\mathcal{R},p-1}_{\mathbf{0}} + R^{[p]}$ represents the sum of all contributions to $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}$ used in **1**–**6**. One can write

$$\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}} = \partial_{\beta_0}\Phi^{\mathcal{R},p-1}_{\mathbf{0}} + R^{[p]} + S^{[p]}, \tag{A.27}$$

for a suitable $S^{[p]}$: by construction $S^{[p]}$ takes into account all contributions arising from the trees $\theta \in \Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ which do not satisfy the condition (A.2), i.e. such that

$$\max_{\ell\in\Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}} \{n \in \mathbb{Z}_+ : \Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell) \neq 0\} = p+1. \tag{A.28}$$

Such trees have been excluded in the discussion above, because on the one hand they would produce the remaining contributions to $M^{[p]}_{\beta,\beta}(0)$, on the other hand they would equally produce contributions to $M^{[p+1]}_{\beta,\beta}(0)$. Therefore, by combining (A.26) and (A.27), we obtain $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}} = \mathcal{M}^{[p]}_{\beta,\beta}(0) + \big(S^{[p]} - W^{[p]}\big)$, where both $W^{[p]}$ and $S^{[p]}$ arise from trees containing at least one line ℓ on scale p and such that $\Psi_{p+1}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell) \neq 0$: for such a line ℓ one has $|\boldsymbol{\nu}_\ell| > 2^{m_{p+1}-1}$ by Remark 3.8. Therefore, one has $\max\big\{|S^{[p]}|, |W^{[p]}|\big\} \le |\varepsilon|\, D_1 e^{-D_2 2^{m_{p+1}}}$, for some constants D_1, D_2 and this is enough to prove the bound (A.1). □

B Proof of Lemmas 6.4 and 6.5

We want to prove by induction that

$$\mathfrak{N}_n^\bullet(\theta) \le \max\{2^{-(m_n-3)}K(\theta) - 2, 0\}. \tag{B.1}$$

First of all note that $\mathfrak{N}_n^\bullet(\theta) \ge 1$ implies $\mathfrak{N}_n^*(\theta) \ge 1$ and hence $K(\theta) \ge 2^{m_n-1}$.

Set $\zeta_0 := \zeta_{\ell_\theta}$, $n_0 := n_{\ell_\theta}$ and $\boldsymbol{\nu} := \boldsymbol{\nu}_{\ell_\theta}$, and note that either $n_0 = \zeta_0$ or $n_0 = \zeta_0 + 1$. If $\zeta_0 < n$ the bound (B.1) follows from the inductive hypothesis. If $\zeta_0 \ge n$, call $\ell_1, \dots, \ell_r$ the lines with minimum scale $\ge n$ closest to ℓ_θ and $\theta_1, \dots, \theta_r$ the subtrees with root lines $\ell_1, \dots, \ell_r$, respectively. If $r = 0$ the bound trivially holds. If $r \ge 2$, by the inductive hypothesis one has $\mathfrak{N}_n^\bullet(\theta) = 1 + \mathfrak{N}_n^\bullet(\theta_1) + \dots + \mathfrak{N}_n^\bullet(\theta_r) \le 1 + 2^{-(m_n-3)}K(\theta) - 2r \le 2^{-(m_n-3)}K(\theta) - 3$, so that the bound follows once more.

If $r = 1$ call T the subgraph with exiting line ℓ_θ and entering line ℓ_1. Then either T is a self-energy cluster or $K(T) \ge 2^{m_n-1}$. This can be proved as follows. Set $\boldsymbol{\nu}_1 := \boldsymbol{\nu}_{\ell_1}$. If T is not a cluster, then it must contain at least one line ℓ' on scale $n_{\ell'} = n$, so that if $\ell' \notin \mathcal{P}_T$ and θ' is the subtree with root line ℓ' one has $K(T) \ge K(\theta') \ge 2^{m_n-1}$, while if $\ell \in \mathcal{P}_T$ then $\boldsymbol{\nu}_{\ell'} \ne \boldsymbol{\nu}_1$ (because $\zeta_{\ell'} = n-1$ and $\zeta_{\ell_1} \ge n$), so that $|\boldsymbol{\omega} \cdot (\boldsymbol{\nu}_{\ell'} - \boldsymbol{\nu}_1)| < \alpha_{m_n-1}(\boldsymbol{\omega})$ implies $K(T) \ge |\boldsymbol{\nu}_{\ell'} - \boldsymbol{\nu}_1| \ge 2^{m_n-1}$. If T is a cluster then either (i) $\boldsymbol{\nu}_1 \ne \boldsymbol{\nu}$ so that $K(T) \ge |\boldsymbol{\nu} - \boldsymbol{\nu}_1| \ge 2^{m_n-1}$ or (ii) $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$ and there is a line $\ell \in \mathcal{P}_T$ with $n_\ell = -1$ so that $K(T) \ge |\boldsymbol{\nu}_\ell^0| = |\boldsymbol{\nu}_1| \ge 2^{m_n-1}$, or (iii) $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$ and T is a self-energy cluster, otherwise there would be a line $\ell' \in \mathcal{P}_T$ with $\boldsymbol{\nu}_{\ell'} = \boldsymbol{\nu}_1$, which is incompatible with $\zeta_{\ell'} \le n-1$ and $\zeta_{\ell_1} \ge n$.

Therefore, if $K(T) \ge 2^{m_n-1}$, the inductive hypothesis yields the bound (B.1). If $K(T) < 2^{m_n-1}$ then T is a self-energy cluster (and hence $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$). In such a case call θ_1 the tree with root line ℓ_1; by construction $\mathfrak{N}_n^\bullet(\theta) = 1 + \mathfrak{N}^\bullet(\theta_1)$. We can repeat the argument above: call $\ell'_1, \dots, \ell'_{r'}$ the lines with minimum scale $\ge n$ closest to ℓ_1 and $\theta'_1, \dots, \theta'_{r'}$ the subtrees with root lines $\ell'_1, \dots, \ell'_{r'}$, respectively. Again the case $r' = 0$ is trivial. If $r' \ge 2$ then $\mathfrak{N}_n^\bullet(\theta) = 2 + \mathfrak{N}_n^\bullet(\theta'_1) + \dots + \mathfrak{N}_n^\bullet(\theta'_{r'}) \le 2 + 2^{-(m_n-3)}K(\theta) - 2r \le 2^{-(m_n-3)}K(\theta) - 2$, so yielding the bound. Therefore the only case which does not imply immediately the bound (B.1) through the inductive hypothesis is when ℓ_1 exits a subgraph T' with only one entering line ℓ'_1 on minimum scale $\ge n$. Set $\boldsymbol{\nu}'_1 := \boldsymbol{\nu}_{\ell'_1}$ and call θ'_1 the tree with root line ℓ'_1. As before we have that either $K(T') \ge 2^{m_n-1}$ or T' is a self-energy cluster. If $K(T') \ge 2^{m_n-1}$ then $\mathfrak{N}_n^\bullet(\theta) = 2 + \mathfrak{N}_n^\bullet(\theta'_1)$ and one can reason as before to obtain the bound by relying on the inductive hypothesis. If T' is a self-energy cluster then ℓ_1 is a resonant line and $\mathfrak{N}_n^\bullet(\theta) = 1 + \mathfrak{N}_n^\bullet(\theta'_1)$.

One can iterate again the argument until either one reaches a case which can be dealt with through the inductive hypothesis or one obtains $\mathfrak{N}_n^\bullet(\theta) = 1 + \mathfrak{N}_n^\bullet(\theta'')$, for some tree θ'' which has no line ℓ with $\zeta_\ell \ge n$. Thus $\mathfrak{N}_n^\bullet(\theta'') = 0$ and the bound (B.1) follows. This completes the proof of Lemma 6.4.

Now, we pass to the proof of Lemma 6.5. Consider a self-energy cluster $T \in \mathfrak{S}_{n,u,e}^k$. First of all we prove that $K(T) \ge 2^{m_n-1}$: the argument proceeds as in the proof of Lemma 3.10, by noting that if there is a line $\ell \in \mathcal{P}_T$ on scale n then $\boldsymbol{\nu}_\ell \ne \boldsymbol{\nu}_{\ell'_T}$ (otherwise $\boldsymbol{\nu}_\ell^0 = 0$ and hence T would not be a self-energy cluster).

Define $\mathcal{C}(n,p)$ as the set of subgraphs T of θ with only one entering line ℓ'_T and one exiting line ℓ_T both on minimum scale $\ge p$, such that $L(T) \ne \emptyset$ and $n_\ell \le n$ for any line $\ell \in L(T)$. We prove by induction on the order the bound

$$\mathfrak{N}_p^\bullet(T) \le 2^{-(m_p-3)}K(T) \tag{B.2}$$

for all $T \in \mathcal{C}(n,p)$ and all $0 \le p \le n$. Consider $T \in \mathcal{C}(n,p)$, $p \le n$: call $\ell_1, \dots, \ell_r$ the lines with minimum scale $\ge p$ closest to ℓ_T. The case $r = 0$ is trivial. If $r \ge 1$ and none of such lines is along the path $\mathcal{P}_T$ then the bound follows from (B.1). If one of such lines, say ℓ_1, is along the path $\mathcal{P}_T$, then denote by $\theta_2, \ldots, \theta_r$ the subtrees with root lines $\ell_2, \ldots, \ell_r$, respectively, and by T_1 the subgraph with exiting line ℓ_1 and entering line ℓ'_T. One has $\mathfrak{N}^\bullet_p(T) \le 1 + \mathfrak{N}^\bullet_p(T_1) + \mathfrak{N}^\bullet_p(\theta_2) + \ldots + \mathfrak{N}^\bullet_p(\theta_r)$. By construction $T_1 \in \mathcal{C}(n,p)$, so that the bound (B.2) follows by the inductive hypothesis for $r \ge 2$.

If $r = 1$ then call T_0 the subgraph with exiting line ℓ_T and entering line ℓ_1. By reasoning as in the proof of Lemma 6.4 we find that either $K(T_0) \ge 2^{m_p-1}$ or T_0 is a self-energy cluster. Since $\mathfrak{N}^\bullet_p(T) \le 1+\mathfrak{N}^\bullet_p(T_1)$, if $K(T_0) \ge 2^{m_p-1}$ the bound follows once more. If on the contrary T_0 is a self-energy cluster we can iterate the construction: call $\ell'_1, \ldots, \ell'_{r'}$ the lines with minimum scale $\ge p$ closest to ℓ_1. If either $r' = 0$ or no line among $\ell'_1, \ldots, \ell'_{r'}$ is along the path $\mathcal{P}_{T_1}$, the bound follows easily. Otherwise if a line, say ℓ'_1 is along the path $\mathcal{P}_{T_1}$ and $r' \ge 2$ one has $\mathfrak{N}^\bullet_p(T) \le 2 + \mathfrak{N}^\bullet_p(T'_1) + \mathfrak{N}^\bullet_p(\theta'_2) + \ldots + \mathfrak{N}^\bullet_p(\theta'_{r'})$, where T'_1 is the subgraph with exiting line ℓ'_1 and entering line ℓ'_T and hence $\mathfrak{N}^\bullet_p(T) \le 2+2^{-(m_p-3)}K(T)-2$, by the inductive hypothesis, so that (B.2) follows.

If $r' = 1$ let T'_0 be the subgraph with exiting line ℓ_1 and entering line ℓ'_1. If $K(T'_0) \ge 2^{m_p-1}$ then the inductive hypothesis implies once more the bound (B.2), while if $K(T'_0) < 2^{m_p-1}$ then, by the same argument as above, T'_0 must be a self-energy cluster, so that ℓ_1 does not contribute to $\mathfrak{N}^\bullet_p(T)$, i.e. $\mathfrak{N}^\bullet_p(T) \le 1 + \mathfrak{N}_p(T''_1)$ where T''_1 is the subgraph with exiting line ℓ'_1 and entering line ℓ'_T. Again we can iterate the argument until either one finds a subgraph T'' with $K(T'') \ge 2^{m_p-1}$, so that the inductive hypothesis compels the bound (B.2) for T, or one obtains $\mathfrak{N}^\bullet_p(T) \le 1 + \mathfrak{N}_p(T'')$ for some subgraph T'' which has no line on minimum scale $\ge p$, so that $\mathfrak{N}^\bullet_p(T) \le 1$.

C Proof of Lemma 6.12

We say that a self-energy cluster T is *isolated* if both its external lines are non-resonant and that is *relevant* if it is not isolated. As will emerge from the proof, it is convenient to introduce a further label $\mathfrak{d}_T \in \{0,1\}$ to be associated with each relevant self-energy cluster T. We shall see later how to fix such a label: for the time being we consider it as an abstract label and we define the *subchains* as follows. Given $p \ge 2$ relevant self-energy clusters $T_1, \ldots, T_p$ of a tree θ, with $\ell'_{T_i} = \ell_{T_{i+1}}$ for all $i = 1, \ldots, p$, we say that $C = \{T_1, \ldots, T_p\}$ is a subchain if $\mathfrak{d}_{T_i} = 1$ for $i = 1, \ldots, p$, the line ℓ_{T_1} either is non-resonant or enters a relevant self-energy cluster T_0 with $\mathfrak{d}_{T_0} = 0$ and the line ℓ'_{T_p} either is non-resonant or exits a relevant self-energy cluster T_{p+1} with $\mathfrak{d}_{T_{p+1}} = 0$. We say that a relevant self-energy cluster T is a *link* if $\mathfrak{d}_T = 1$.

Given a subchain $C = \{T_1, \ldots, T_p\}$ of a tree θ, the relevant self-energy clusters T_i are called the *links* of C. Define $\ell_0(C) := \ell_{T_1}$ and $\ell_i(C) := \ell'_{T_i}$ for $i = 1, \ldots, p$ and set $n_i(C) = n_{\ell_i(C)}$ for $i = 0, \ldots, p$. The lines $\ell_0(C), \ldots, \ell_p(C)$ are the *chain-lines* of C: we call $\ell_1(C), \ldots, \ell_{p-1}(C)$ the *internal chain-lines* of C and $\ell_0(C), \ell_p(C)$ the *external chain-lines* of C. For future convenience we also set $\ell_C = \ell_0(C)$ and $\ell'_C = \ell_p(C)$. We also call $k(C) := k(T_1) + \ldots + k(T_p)$ the *total order* of the subchain C and $p(C) = p$ the *length* of C. The value of a subchain C is defined as in (6.8). Note that for all $i = 1, \ldots, p-1$ one has $\zeta_{\ell_i(C)} = \zeta_{\ell_C} = \zeta_{\ell'_C}$ if $\mathscr{V}(\theta) \ne 0$.

We denote by $\mathfrak{C}_1(k; h, h'; n_0, \ldots, n_p)$ the set of all subchains $C = \{T_1, \ldots, T_p\}$ with total order k and with fixed labels $h_{\ell_0(C)} = h$, $h_{\ell_p(C)} = h'$ and $n_i(C) = n_i$ for $i = 0, \ldots, p$.

If all relevant self-energy clusters T of θ carried a label $\mathfrak{d}_T = 1$ the definition of subchain would reduce to that of chain in Section 6. We want to prove the bound (6.9). The sum is over all chains $C = \{T_1, \ldots, T_p\}$ in $\mathfrak{C}(k; h, h'; n_0, \ldots, n_p)$; then we set $\mathfrak{d}_{T_i} = 1$ for $i = 1, \ldots, p$, so that we can replace

$\mathfrak{C}(k;h,h';n_0,\ldots,n_p)$ with $\mathfrak{C}_1(k;h,h';n_0,\ldots,n_p)$. Thus in (6.9) we can write

$$\sum_{C\in\mathfrak{C}(k;h,h';\overline{n}_0,\ldots,\overline{n}_p)} \mathscr{V}_C(x) = \sum_{C\in\mathfrak{C}_1(k;h,h';\overline{n}_0,\ldots,\overline{n}_p)} \mathscr{V}_C(x) = \sum_{\substack{h_1,\ldots,h_{p-1}\in\{\beta,B\}\\ k_1+\ldots+k_p=k}} \prod_{i=1}^{p} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i), \tag{C.1}$$

where $h_0=h$, $h_p=h'$ and $n_i=\min\{\overline{n}_{i-1},\overline{n}_i\}-1$ for $i=1,\ldots,p$; of course $|n_i-n_j|\le 1$ for all $i,j=1,\ldots,p$ and $k_i\ge 0$ for $i=1,\ldots,p$; see Figure C.1.

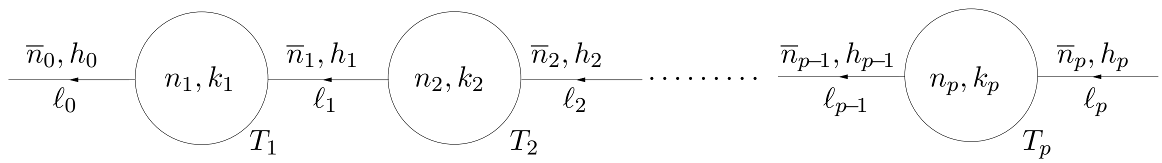

Figure C.1: A subchain C of length p with links $T_1,\ldots,T_p$ and chain-lines $\ell_0,\ldots,\ell_p$; summing over all possible C with $h_0=h$, $h_p=h'$, $k_1+\ldots+k_p=k$ and $\overline{n}_1,\ldots,\overline{n}_p$ fixed, one obtains a graphical representation of (C.1).

For all $k\ge 1$, all $n\ge -1$ and all $h,h'\in\{\beta,B\}$ let us write

$$\mathcal{M}^{(k)}_{h,h'}(x,n) = \sum_{\delta\in\Delta} \mathcal{M}^{(k)}_{h,h'}(x,n,\delta) \tag{C.2}$$

where $\Delta:=\{\mathcal{L},\partial,\partial^2,\mathcal{R}\}$ is a set of labels and

$$\begin{aligned} &\mathcal{M}^{(k)}_{h,h'}(x,n,\mathcal{L}) := \mathcal{M}^{(k)}_{h,h'}(0) \qquad \mathcal{M}^{(k)}_{h,h'}(x,n,\partial) := x\partial\mathcal{M}^{(k)}_{h,h'}(0),\\ &\mathcal{M}^{(k)}_{h,h'}(x,n,\partial^2) := x^2\int_0^1 \mathrm{d}\tau(1-\tau)\partial^2\mathcal{M}^{(k)}_{h,h'}(\tau x),\\ &\mathcal{M}^{(k)}_{h,h'}(x,n,\mathcal{R}) := \mathcal{M}^{(k)}_{h,h'}(x,n) - \mathcal{M}^{(k)}_{h,h'}(x), \end{aligned} \tag{C.3}$$

so that we can decompose the sum in (C.1) as

$$\sum_{\delta_1,\ldots,\delta_p\in\Delta} \sum_{\substack{h_1,\ldots,h_{p-1}\in\{\beta,B\}\\ k_1+\ldots+k_p=k}} \prod_{i=1}^{p} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i). \tag{C.4}$$

There are several contributions to (C.4) which sum up to zero. This holds for all contributions with $\delta_j=\delta_{j+1}=\mathcal{L}$ for some $j=1,\ldots,p-1$. Indeed one can write such contributions as

$$\begin{aligned} &\sum_{\delta_1,\ldots,\delta_{j-1},\delta_{j+1},\ldots,\delta_p\in\Delta} \sum_{\substack{h_1,\ldots,h_{j-1},h_{j+1},\ldots h_{p-1}\in\{\beta,B\}\\ k_1+\ldots+k_{j-1}+\overline{k}+k_{j+2}+\ldots+k_p=k}} \prod_{\substack{i=1\\ i\neq j}}^{p} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i) \;\times\\ &\times\left(\sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{h_{j-1},\beta}(0)\mathcal{M}^{(k_{j+1})}_{\beta,h_{j+1}}(0) + \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{h_{j-1},B}(0)\mathcal{M}^{(k_{j+1})}_{B,h_{j+1}}(0)\right), \end{aligned} \tag{C.5}$$

and by Lemma 6.8 one has (for instance)

$$\begin{aligned}
&\sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,\beta}(0)\mathcal{M}^{(k_{j+1})}_{\beta,\beta}(0) + \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,B}(0)\mathcal{M}^{(k_{j+1})}_{B,\beta}(0) \\
&\qquad = \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,\beta}(0)\mathcal{M}^{(k_{j+1})}_{\beta,\beta}(0) + \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,B}(0)\left(-\sum_{k'+k''=k_{j+1}} \mathcal{M}^{(k')}_{BB}(0)\partial_{\beta_0}B_0^{(k'')}\right) \\
&\qquad = \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,\beta}(0)\mathcal{M}^{(k_{j+1})}_{\beta,\beta}(0) + \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{BB}(0)\mathcal{M}^{(k_{j+1})}_{\beta,\beta}(0) = 0;
\end{aligned}$$

one can reason in the same way also for the cases $(h_{j-1},h_{j+1}) \neq (\beta,\beta)$. By (6.7a) of Lemma 6.9, also the contributions with $\delta_j = \partial$ and $h_{j-1} \neq h_j$ for some $j = 1,\dots,p$ sum up zero. Finally we obtain zero also when we sum together all contributions with $\delta_j = \ldots = \delta_{j+q} = \partial$, $h_{j-1} = \ldots = h_{j+q} = \overline{h}$ for some $\overline{h} \in \{\beta, B\}$ and $\delta_{j-1} = \delta_{j+q+1} = \mathcal{L}$ for some $j = 2,\dots,p-1$ and $q = 0,\dots,p-1-j$. Indeed we can write the sum of such contributions as

$$\begin{aligned}
&\sum_{\substack{h_1,\dots,h_{j-2},\overline{h},h_{j+q+1},\dots,h_{p-1}=\beta,B \\ k_1+\ldots+k_p=k}} \left(\prod_{i=1}^{j-2} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i)\right)\mathcal{M}^{(k_{j-1})}_{h_{j-2},\overline{h}}(x,n_{j-1},\mathcal{L})\times \\
&\qquad \times \left(\prod_{i=j}^{j+q} \mathcal{M}^{(k_i)}_{\overline{h},\overline{h}}(x,n_i,\partial)\right)\mathcal{M}^{(k_{j+q+1})}_{\overline{h},h_{j+q+1}}(x,n_{j+q+1},\mathcal{L})\left(\prod_{i=j+q+2}^{p} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i)\right) \\
&= \sum_{k''\le k}\sum_{k_j+\ldots+k_{j+q}=k''} \left(\prod_{i=j}^{j+q} \mathcal{M}^{(k_i)}_{\beta,\beta}(x,n_i,\partial)\right)\times \\
&\qquad \times \sum_{\substack{k_1+\ldots+k_{j-1}+k_{k+q+1}+\ldots+k_p=k-k' \\ h_1,\dots,h_{j-2},h_{j+q+1},\dots,h_{p-1}=\beta,B}} \left(\prod_{i=1}^{j-2} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i)\right)\times \\
&\qquad \times \sum_{\overline{h}=\beta,B} \mathcal{M}^{(k_{j-1})}_{h_{j-2},\overline{h}}(x,n_{j-1},\mathcal{L})\mathcal{M}^{(k_{j+q+1})}_{\overline{h},h_{j+q+2}}(x,n_{j+q+1},\mathcal{L})\left(\prod_{i=j+q+2}^{p} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i)\right),
\end{aligned}$$

and the last sum is zero by the same argument used for (C.5); note that we used (6.7b) to extract a common factor $\mathcal{M}^{(k_i)}_{\beta,\beta}(x,n_i,\partial)$ in the third line.

We say that a cluster T is a *fake cluster* on scale n if it is a connected subgraph of a tree with only one entering line ℓ'_T and one exiting line ℓ_T such that (i) all lines in T have scale $\le n$ and there is at least one line on T with scale n and (ii) the lines ℓ_T and ℓ'_T carry the same momentum; note that a fake cluster can fail to be a self-energy cluster only because there the scales of the external lines have no relation with n (and hence it can even fail to be a cluster). Denote by $\mathfrak{S}^{*k}_{m,u,e}$ the set of fake clusters with order k, scale m and such that $h_{\ell'_T} = e$ and $h_{\ell_T} = u$.

In (C.4) we can expand

$$\mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i) = \sum_{T_i \in \mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i,\delta_i)} \mathscr{V}_{T_i}(x,\delta_i), \qquad i = 1,\dots,p,$$

where we have set

$$\mathfrak{S}^{*k}_{u,e}(n,\delta) := \begin{cases} \displaystyle\bigcup_{m\geq -1} \mathfrak{S}^{*k}_{m,u,e} & \delta = \mathcal{L}, \partial, \partial^2, \\ \displaystyle\bigcup_{m>n} \mathfrak{S}^{k*}_{m,u,e}, & \delta = \mathcal{R}, \end{cases} \tag{C.6}$$

for all $k \geq 0$, $n \geq 0$ and $u, e \in \{\beta, B\}$, and defined

$$\mathscr{V}_T(x,\delta) := \begin{cases} \mathscr{V}_T(0), & \delta = \mathcal{L}, \\ x\,\partial\,\mathscr{V}_T(0), & \delta = \partial, \\ -\,\mathscr{V}_T(x), & \delta = \mathcal{R}, \\ x^2 \displaystyle\int_0^1 \mathrm{d}\tau (1-\tau)\partial_x^2\,\mathscr{V}_T(\tau x), & \delta = \partial^2, \end{cases} \tag{C.7}$$

with $\mathscr{V}_T(x)$ defined as for self-energy clusters in Section 6. Denote by $\mathfrak{C}^*(k;h,h';\overline{n}_0,\ldots,\overline{n}_p)$ the set of fake clusters $\{T_1,\ldots,T_p\}$ with $T_i \in \mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i,\delta_i)$ for any choice of the labels $\{k_i,n_i,\delta_i\}_{i=1}^p$ and $\{h_i\}_{i=0}^p$ with the following constraints (see Figure C.2):
(i) $k_1 + \ldots + k_p = k$,
(ii) $n_i < \min\{\overline{n}_{i-1},\overline{n}_i\}$ for $i = 1,\ldots,p$,
(iii) $h_0 = h$, $h_p = h'$,
(iv) if $\delta_i = \mathcal{L}$ for $i = 2,\ldots,p-1$, then $\delta_{i-1},\delta_{i+1} \neq \mathcal{L}$,
(v) if $\delta_i = \partial$ for $i = 1,\ldots,p$, then $h_{i-1} = h_i$,
(vi) if $\delta_j = \delta_{j+1} = \ldots = \delta_{j+q} = \partial$ for some $j \in \{2,\ldots,p-1\}$ and some $q \in \{0,\ldots,p-1-j\}$ and $\delta_{j-1} = \mathcal{L}$, then $\delta_{j+q+1} \neq \mathcal{L}$.

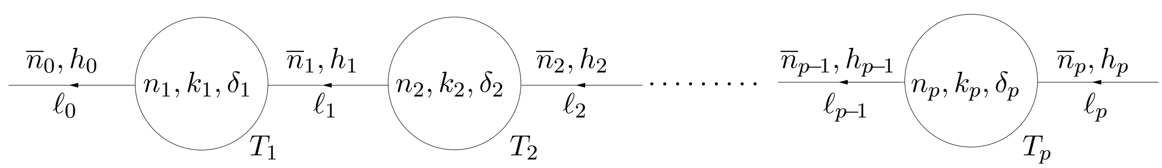

Figure C.2: A $*$-chain $C \in \mathfrak{C}^*(k;h,h';\overline{n}_0,\ldots,\overline{n}_p)$: the labels satisfy the constraints listed in the text.

We call $*$-*chain* any set $C \in \mathfrak{C}^*(k;h,h';\overline{n}_0,\ldots,\overline{n}_p)$ and $*$-*links* the sets $T_1,\ldots,T_p$. By the discussion between (C.4) and (C.6), we can write (C.4) – and hence the sum in (6.9) – as

$$\sum_{C\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\ldots,\overline{n}_p)} \mathscr{V}_C(x), \tag{C.8}$$

where

$$\mathscr{V}_C(x) := \prod_{i=1}^p \mathscr{V}_{T_i}(x,\delta_i). \tag{C.9}$$

With each $*$-chain C summed over in (C.8) we associate a *depth* label $D(C) = 0$; if $C = \{T_1,\ldots,T_p\}$ we associate with each T_i the same depth label as C, i.e. $D(T_i) = D(C) = 0$ for $i = 1,\ldots,p$; the introduction of such a label is due to the fact that we are performing an iterative construction and we want to keep track of the iteration step by means of the depth label.

Given a $*$-chain $C = \{T_1,\ldots,T_p\}$, for all $i = 1,\ldots,p$ and all $\ell \in L(T_i)$ there exist $q \geq 1$ relevant self-energy clusters $T_i = T_i^{(0)} \supset T_i^{(1)} \supset \ldots \supset T_i^{(q-1)}$, with $T_i^{(j)}$ a maximal relevant self-energy cluster inside $T_i^{(j-1)}$ for all $j = 0, \dots, q-1$ and $T_i^{(q-1)}$ is the minimal relevant self-energy cluster containing ℓ. Note that both q and the relevant self-energy clusters $T^{(1)}, \dots, T^{(q)}$ depend on ℓ, even though we are not making explicit such a dependence. We call $\{T_i^{(j)}\}_{j=0}^q$ the *cloud* of ℓ and $\{T_i^{(j)}\}_{j=1}^q$ the *internal cloud* of ℓ. Of course if $q = 0$ the internal cloud of ℓ is the empty set.

In (C.9) consider first a factor $\mathscr{V}_{T_i}(x, \delta_i)$ with $\delta_i = \mathcal{L}, \mathcal{R}$. Assign a label $\mathfrak{d}_T = 1$ with each maximal relevant self-energy cluster T contained inside T_i. Denote by $\mathfrak{C}_0(T_i)$ the set of maximal subchains C' contained inside T_i. For each $C_j \in \mathfrak{C}_0(T_i)$ there are labels $k_j^{(i)}, h_j^{(i)}, h_j^{(i)\prime}, \overline{n}_{j,0}^{(i)}, \dots, \overline{n}_{j,p_j}^{(i)}$ such that $C_j \in \mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \dots, \overline{n}_{j,p_j}^{(i)})$. Call $\mathring{T}_i$ the set of nodes and lines obtained from T_i by removing all nodes and lines of the subchains in $\mathfrak{C}_0(T_i)$ and $\mathfrak{F}(T_i)$ the family of all possible sets $T_i' \in \mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i, \delta_i)$ obtained from T_i by replacing, for all $j = 1, \dots, |\mathfrak{C}_0(T_i)|$, each subchain C_j with any subchain $C_j' \in \mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \dots, \overline{n}_{j,p_j}^{(i)})$. Note that $\mathring{T}_i' = \mathring{T}_i$ for all $T_i' \in \mathfrak{F}(T_i)$. If we sum together all contributions in $\mathfrak{F}(T_i)$ we obtain

$$\sum_{T_i' \in \mathfrak{F}(T_i)} \mathscr{V}_{T'}(x, \delta_i) = a(\delta_i) \sum_{\substack{C_j' \in \mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \dots, \overline{n}_{j,p_j}^{(i)}) \\ 1 \le j \le |\mathfrak{C}_0(T_i)|}} \mathscr{V}_{\mathring{T}_i}(x, \delta_i) \prod_{j=1}^{|\mathfrak{C}_0(T_i)|} \mathscr{V}_{C_j'}(x_{\ell_{C'}, \delta_i}), \tag{C.10}$$

where

$$\mathscr{V}_{\mathring{T}_i}(x, \delta_i) := \Bigg(\prod_{v \in N(\mathring{T}_i)} \mathcal{F}_v \Bigg) \Bigg(\prod_{\ell \in L(\mathring{T}_i)} \mathcal{G}_{n_\ell}(x_{\ell, \delta_i}) \Bigg) \tag{C.11}$$

and

$$a(\delta) = \begin{cases} 1, & \delta = \mathcal{L}, \\ -1, & \delta = \mathcal{R}, \end{cases} \qquad x_{\ell,\delta} = \begin{cases} \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell^0, & \delta = \mathcal{L}, \\ \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell, & \delta = \mathcal{R}, \end{cases} \tag{C.12}$$

for all $\ell \in L(T')$ with $T' \in \mathfrak{F}(T_i)$.

Now consider a set T_i in (C.9) with $\delta_i = \partial, \partial^2$. Write

$$\mathscr{V}_{T_i}(x, \partial) = x \sum_{\ell \in L(T_i)} \partial_x \mathcal{G}_\ell(x_\ell^0) \Bigg(\prod_{v \in N(T_i)} \mathcal{F}_v \Bigg) \Bigg(\prod_{\substack{\ell' \in L(T_i) \\ \ell' \neq \ell}} \mathcal{G}_{\ell'}(x_{\ell'}^0) \Bigg), \tag{C.13}$$

with $x_\ell^0 := \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell^0$, and $\mathscr{V}_{T_i}(x, \partial^2)$ as in the last line of (C.7), with

$$\begin{aligned} \partial_x^2 \mathscr{V}_{T_i}(\tau x) = & \sum_{\ell_1 \neq \ell_2 \in L(T)} \Big(\partial_x \mathcal{G}_{n_{\ell_1}}(x_{\ell_1}(\tau)) \Big) \Big(\partial_x \mathcal{G}_{n_{\ell_2}}(x_{\ell_2}(\tau)) \Big) \Bigg(\prod_{\ell \in L(T) \setminus \{\ell_1, \ell_2\}} \mathcal{G}_{n_\ell}(x_\ell(\tau)) \Bigg) \Bigg(\prod_{v \in N(T)} \mathcal{F}_v \Bigg) \\ & + \sum_{\ell_1 \in L(T)} \Big(\partial_x^2 \mathcal{G}_{n_{\ell_1}}(x_{\ell_1}(\tau)) \Big) \Bigg(\prod_{\ell \in L(T) \setminus \{\ell_1\}} \mathcal{G}_{n_\ell}(x_\ell(\tau)) \Bigg) \Bigg(\prod_{v \in N(T)} \mathcal{F}_v \Bigg), \end{aligned} \tag{C.14}$$

where $x_\ell(\tau) = x_\ell^0 + \tau x$ if $\ell \in \mathcal{P}_T$ and $x_\ell = x_\ell^0$ otherwise. To simplify the notations we associate with each line ℓ a label $d_\ell = 0, 1, 2$, which denotes the number of derivatives acting on the corresponding propagator, and set

$$\overline{\mathcal{G}}_\ell(x) = \partial_x^{d_\ell} \mathcal{G}_{n_\ell}(x); \tag{C.15}$$

then we rewrite

$$\mathscr{V}_{T_i}(x,\partial) = x\sum_{\ell_1\in L(T_i)}\Bigg(\prod_{\ell\in L(T_i)}\overline{\mathcal{G}}_\ell(x_\ell^0)\Bigg)\Bigg(\prod_{v\in N(T_i)}\mathcal{F}_v\Bigg), \tag{C.16a}$$

$$\partial_x^2\,\mathscr{V}_{T_i}(\tau x) = \sum_{\ell_1,\ell_2\in L(T_i)}\Bigg(\prod_{\ell\in L(T_i)}\overline{\mathcal{G}}_\ell(x_\ell(\tau))\Bigg)\Bigg(\prod_{v\in N(T_i)}\mathcal{F}_v\Bigg), \tag{C.16b}$$

with the constraint $\sum_{\ell\in L(T)} d_\ell = 1,2$ for $\delta_i = \partial,\partial^2$ respectively.

Each summand in (C.16a) can be regarded as the value of a fake cluster T_i in which the propagator of a line ℓ_1 has been differentiated. Given a relevant self-energy cluster T contained inside T_i, we set $\mathfrak{d}_T = 0$ if either (1) T belongs to the cloud of ℓ_1 or (2) T is a relevant self-energy cluster with either (2a) ℓ_T non-resonant and $\ell'_T = \ell_{T'}$ for some T' belonging to the cloud of ℓ_1 or (2b) ℓ'_T is non-resonant and $\ell_T = \ell'_{T''}$ for some T'' belonging to the cloud of ℓ_1. Moreover we set $\mathfrak{d}_T = 1$ if T is any other maximal relevant self-energy cluster in T_i or in T' with $\mathfrak{d}_{T'} = 0$. If $\mathfrak{C}_0(T_i) = \{C_1, C_2, \ldots\}$, each C_j belongs to $\mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \ldots, \overline{n}_{j,p_j}^{(i)})$ for suitable values of the labels. Call $\mathring{T}_i$ the set of nodes and lines obtained from T_i by removing all nodes and lines of the subchains in $\mathfrak{C}_0(T_i)$ and $\mathfrak{F}(T_i)$ the family of all possible sets $T'_i \in \mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i,\delta_i)$ obtained from T_i by replacing, for all $j = 1,\ldots,|\mathfrak{C}_0(T_i)|$, each subchain C_j with any subchain $C'_j \in \mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \ldots, \overline{n}_{j,p_j}^{(i)})$. Note that $\mathring{T}'_i = \mathring{T}_i$ for all $T'_i \in \mathfrak{F}(T_i)$. Note also that both $\mathfrak{F}(T_i)$ and $\mathring{T}_i$ depend on the choice of the line ℓ_1, although we are not writing explicitly such a dependence. By summing together all contributions obtained by choosing first the line ℓ_1 and hence the fake clusters belonging to the corresponding family $\mathfrak{F}(T_i)$, we obtain for $\delta_i = \partial$

$$\begin{aligned} & x\sum_{\ell_1\in L(T_i)}\sum_{T'_i\in\mathfrak{F}(T_i)}\Bigg(\prod_{\ell\in L(T'_i)}\overline{\mathcal{G}}_\ell(x_\ell^0)\Bigg)\Bigg(\prod_{v\in N(T'_i)}\mathcal{F}_v\Bigg) \\ &\quad = \sum_{\ell_1\in L(T_i)} x \sum_{\substack{C'_j\in\mathfrak{C}_1(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\ldots,\overline{n}_{j,p_j}^{(i)}) \\ 1\le j\le|\mathfrak{C}_0(T_i)|}} \mathscr{V}_{\mathring{T}_i}(x,\partial)\prod_{j=1}^{|\mathfrak{C}_0(T_i)|}\mathscr{V}_{C'_j}(x^0_{\ell_{C_j}}), \end{aligned} \tag{C.17}$$

with

$$\mathscr{V}_{\mathring{T}_i}(x,\partial) = \Bigg(\prod_{\ell\in L(\mathring{T}_i)}\overline{\mathcal{G}}_\ell(x_\ell^0)\Bigg)\Bigg(\prod_{v\in N(\mathring{T}_i)}\mathcal{F}_v\Bigg). \tag{C.18}$$

We deal in a similar way with (C.16b). Indeed, each summand can be regarded as the value of a fake cluster T_i in which the propagators of two lines ℓ_1,ℓ_2 (possibly coinciding) have been differentiated. As in the previous case we associate a label $\mathfrak{d}_T = 0$ with (1) the relevant self-energy clusters T of the clouds of both ℓ_1,ℓ_2 and (2) the relevant self-energy clusters T such that either (2a) ℓ_T is non-resonant and $\ell'_T = \ell_{T'}$ for some T' belonging to the cloud of ℓ_1 or ℓ_2 or (2b) ℓ'_T is non-resonant and $\ell_T = \ell'_{T'}$ for some T' belonging to the cloud of ℓ_1 or ℓ_2 or (2c) both ℓ_T, ℓ'_T are resonant and $\ell_T = \ell'_{T'}$, $\ell'_T = \ell_{T''}$ for some T', T'' belonging to the clouds of ℓ_1,ℓ_2. Moreover we associate a label $\mathfrak{d}_T = 1$ with the maximal relevant self-energy clusters T in T_i or in any T' with $\mathfrak{d}_{T'} = 0$. Then we reason as in the previous case, by defining $\mathring{T}_i$ as the set of nodes and lines obtained from T_i by removing all nodes and lines of the subchains in $\mathfrak{C}_0(T_i)$ and $\mathfrak{F}(T_i)$ the family of all possible sets

$T_i' \in \mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i,\delta_i)$ obtained from T_i by replacing, for all $j = 1,\dots,|\mathfrak{C}_0(T_i)|$, each subchain C_j with any subchain $C_j' \in \mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)},\dots,\overline{n}_{j,p_j}^{(i)})$. Then for $\delta_i = \partial^2$ we obtain

$$\begin{aligned} & x^2 \int_0^1 \mathrm{d}\tau (1-\tau) \sum_{\ell_1\ell_2\in L(T_i)} \sum_{T_i'\in\mathfrak{F}(T_i)} \Bigg(\prod_{\ell\in L(T_i')} \overline{\mathcal{G}}_\ell(x_\ell(\tau)) \Bigg) \Bigg(\prod_{v\in N(T_i')} \mathcal{F}_v \Bigg) \\ & = \sum_{\ell_1\ell_2\in L(T_i)} x^2 \int_0^1 \mathrm{d}\tau(1-\tau) \sum_{\substack{C_j'\in\mathfrak{C}_1(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\dots,\overline{n}_{j,p_j}^{(i)}) \\ 1\le j\le|\mathfrak{C}_0(T_i)|}} \mathscr{V}_{\mathring{T}_i}(x,\partial^2) \prod_{j=1}^{|\mathfrak{C}_0(T_i)|} \mathscr{V}_{C'_j}(x_{\ell_{C_j}}(\tau)), \end{aligned} \tag{C.19}$$

with

$$\mathscr{V}_{\mathring{T}_i}(x,\partial^2) = \Bigg(\prod_{\ell\in L(\mathring{T}_i)} \overline{\mathcal{G}}_\ell(x_\ell(\tau)) \Bigg) \Bigg(\prod_{v\in N(\mathring{T}_i)} \mathcal{F}_v \Bigg), \tag{C.20}$$

where $x_\ell(\tau) = x_\ell^0 + \tau x$ if $\ell \in \mathcal{P}_{T_i}$ and $x_\ell(\tau) = x_\ell^0$ otherwise.

By summarising we obtained

$$\begin{aligned} \sum_{C\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\dots,\overline{n}_p)} \mathscr{V}_C(x) &= \sum_{\{T_1,\dots,T_p\}\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\dots,\overline{n}_p)} \prod_{i=1}^p \mathscr{V}_{T_i}(x,\delta_i) \\ &= \sum_{\{T_1,\dots,T_p\}\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\dots,\overline{n}_p)} \Bigg(\prod_{\substack{i=1 \\ \delta_i=\partial}}^p \sum_{\ell_{i,1}\in L(T_i)} \Bigg) \Bigg(\prod_{\substack{i=1 \\ \delta_i=\partial^2}}^p \sum_{\ell_{i,1},\ell_{i,2}\in L(T_i)} \Bigg) \times \\ &\quad \times \Bigg(\prod_{i=1}^p a(x_{\ell_{T_i}},\delta_i) \Bigg) \Bigg(\prod_{\substack{i=1 \\ \delta_i=\partial^2}}^p \int_0^1 \mathrm{d}\tau_i\,(1-\tau_i) \Bigg) \times \\ &\quad \times \prod_{i=1}^p \frac{1}{|\mathfrak{F}(T_i)|} \mathscr{V}_{\mathring{T}_i}(x_{\ell_{T_i}},\delta_i) \sum_{C_j\in\mathfrak{C}_1(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\dots,\overline{n}_{j,p_j}^{(i)})} \mathscr{V}_{C_j}(x_{\ell_{C_j}}(\tau_i(\delta_i))), \end{aligned} \tag{C.21}$$

with $x_{\ell_{T_i}} = x$ by construction and

$$\mathscr{V}_{\mathring{T}_i}(x,\delta_i) = \Bigg(\prod_{\ell\in L(\mathring{T}_i)} \overline{\mathcal{G}}_\ell(x_\ell(\tau(\delta_i))) \Bigg) \Bigg(\prod_{v\in N(\mathring{T}_i)} \mathcal{F}_v \Bigg), \tag{C.22}$$

where we have defined $x_\ell(\tau) := x_\ell^0 + \tau x$ and

$$\tau_i(\delta) := \begin{cases} 0, & \delta=\mathcal{L}, \\ 1, & \delta=\mathcal{R}, \\ 0, & \delta=\partial, \\ \tau_i, & \delta=\partial^2, \end{cases} \qquad a(x,\delta) := \begin{cases} 1, & \delta=\mathcal{L}, \\ -1, & \delta=\mathcal{R}, \\ x, & \delta=\partial, \\ x^2, & \delta=\partial^2. \end{cases} \tag{C.23}$$

The factors $1/|\mathfrak{F}(T_i)|$ have been introduced in (C.21) to avoid overcountings. The last sums in (C.21) have the same form as the sum (C.1), so that we can iterate the procedure, by writing

$$\sum_{C_j\in\mathfrak{C}_1(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\dots,\overline{n}_{j,p_j}^{(i)})} \mathscr{V}_{C_j}(x_{\ell_{C_j}}(\tau_i(\delta_i))) = \sum_{C_j\in\mathfrak{C}^*(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\dots,\overline{n}_{j,p_j}^{(i)})} \mathscr{V}_{C_j}(x_{\ell_{C_j}}(\tau_i(\delta_i))) \tag{C.24}$$

for $i=1,\ldots,p$ and $j=1,\ldots,|\mathfrak{C}_0(T_i)|$, with $\mathscr{V}_T(x)$ defined as in (C.9). Now we associate with each $*$-chain C_j a depth label $D(C_j)=1$, and if $C_j=\{T_1^{(j)},\ldots,T_{p_j}^{(j)}\}$ we associate with each $T_i^{(j)}$ the same depth label as C_j, i.e. $D(T_i^{(j)})=D(C_j)$. More generally, by pursuing the construction, in order to keep track of the iteration step, we associate a depth label $D(C)=d$ with each $*$-chain C which appears at the d-th step and the same depth label $D(T)=d$ with each $*$-link T of C. Since at each step the order of the chains is decreased, sooner or later the procedure stops.

To make the notation more uniform, for any $*$-link T such that $\mathfrak{C}_0(T)=\emptyset$ we write $\mathring{T}=T$. Given any $*$-link T with $\mathring{T}\neq T$, if two lines $\ell,\ell'\in L(\mathring{T})$ are such that there exists a maximal link T' in T with $\ell_{T'}=\ell$ and $\ell'_{T'}=\ell'$, we say that the two lines are *consecutive* and we write $\ell'\prec\ell$.

At the end of the procedure described above we obtain a sum of terms of the form

$$\left(\prod_{\substack{i\in I\\ \delta_{T_i}=\partial^2}}\int_0^1 \mathrm{d}\tau_i(1-\tau_i)\right)\prod_{i\in I} a(x_{\ell_{T_i}}(\underline{\tau}),\delta_i)\,\mathscr{V}_{\mathring{T}_i}(x_{\ell_{T_i}}(\underline{\tau}),\delta_i), \tag{C.25}$$

where the following notations have been used:
(i) $I=\{1,2,\ldots,N\}$ for some $N\in\mathbb{N}$;
(ii) $\{T_i\}_{i\in I}$ are $*$-links such that (1) for all T_i with $D(T_i)=d$, $d\geq 1$, there exists a $*$-link T_j, $j\in I$, with $D(T_j)=d-1$ and two consecutive lines $\ell'\prec\ell\in L(\mathring{T}_j)$ with the same labels as ℓ'_{T_i},ℓ_{T_i}, respectively, and conversely (2) for all T_j with $D(T_j)=d$, $d\geq 0$, and all pairs of consecutive lines $\ell'\prec\ell\in L(\mathring{T}_j)$, there is a $*$-link T_i, $i\in I$, with $D(T_i)=d+1$, such that ℓ'_{T_i},ℓ_{T_i} have the same labels as ℓ',ℓ, respectively (roughly we can imagine to 'fill the holes' of all $\mathring{T}$ such that $D(T)=0$ with all $\mathring{T}'$ with $D(T')=1$, then 'fill the remaining holes' with all $\mathring{T}''$ with $D(T'')=2$ and so on up to the $*$-links of maximal depth which have no 'holes');
(iii) $\underline{\tau}=(\tau_1(\delta_1),\ldots,\tau_N(\delta_N))$, with $\tau_i(\delta_i)$ defined as in (C.23), and for all $\ell\in L(\mathring{T}_1)\cup\ldots\cup L(\mathring{T}_N)$ we have set

$$x_\ell(\underline{\tau}):=x_\ell^0+\tau_{i_d}(\delta_{T_{i_d}})\left(x^0_{\ell_d}+\tau_{i_{d-1}}(\delta_{T_{i_{d-1}}})\Big(x^0_{\ell_{d-1}}+\tau_{i_{d-2}}(\delta_{T_{i_{d-2}}})(\ldots+\tau_{i_0}(\delta_{T_{i_0}})x)\Big)\right), \tag{C.26}$$

where T_{i_d} is the minimal $*$-link (with depth d) containing ℓ and ℓ_j is the line in the $*$-link $T_{i_{j-1}}$ with depth $j-1$ corresponding to the entering line of T_{i_j}, for $j=1,\ldots,d$.

Recall that each propagator is differentiated at most twice and note that, for T such that $\delta_T=\partial$, there is a line $\ell\in L(\mathring{T})$ with $d_\ell=1$. Then, when bounding the product of propagators, instead of

$$\prod_{\ell\in L(T)}\frac{\gamma_0}{2}\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^{-1} \tag{C.27}$$

with γ_0 as in the proof of Lemma 4.3, we obtain the bound (C.27) times an extra factor

$$c_1\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^{-1}|x_{\ell_T}|, \tag{C.28}$$

for suitable constant c_1. Analogously, for T such that $\delta_T=\partial^2$, either there are two lines ℓ_1,ℓ_2 with $d_{\ell_1}=d_{\ell_2}=1$ or one line ℓ_1 with $d_{\ell_1}=2$; in both cases we obtain (C.27) times an extra factor

$$c_1\alpha_{m_{n_{\ell_1}}}(\boldsymbol{\omega})^{-1}\alpha_{m_{n_{\ell_2}}}(\boldsymbol{\omega})^{-1}|x_{\ell_T}|^2. \tag{C.29}$$

On the other hand we have no gain factor coming from the $*$-links with label $\delta = \mathcal{L}, \mathcal{R}$, or from the relevant self-energy clusters T' with $\mathfrak{d}_{T'} = 0$. In order to deal with such lines we need some preliminary results.

Given a $*$-link T, define $\mathring{T}$ as before and denote by $L_R(\mathring{T})$ the set of resonant lines in $\mathring{T}$. Set
(i) $L_{NR}(\mathring{T}) := L(\mathring{T}) \setminus L_R(\mathring{T})$,
(ii) $L_D(\mathring{T}) := \{\ell \in L_R(\mathring{T}) \,:\, d_\ell > 0\}$,
(iii) $L_0^1(\mathring{T}) := \{\ell \in L_R(\mathring{T}) \,:\, \ell = \ell_{T'}$ for some relevant self-energy cluster $T' \subset T$ with $\mathfrak{d}_{T'} = 0\}$,
(iv) $L_0^2(\mathring{T}) := \{\ell \in L_R(\mathring{T}) \,:\, \ell = \ell'_{T'}$ for some relevant self-energy cluster $T' \subset T$ with $\delta_{T'} = 0\}$;
(v) $L_0(\mathring{T}) := L_0^1(\mathring{T}) \cup L_0^2(\mathring{T})$;
(vi) $L_R^*(\mathring{T}) = L_D(\mathring{T}) \cup L_0(\mathring{T})$.
Of course $L_D(\mathring{T}) = L_0(\mathring{T}) = \emptyset$ if $\delta_T = \mathcal{L}, \mathcal{R}$. Given a $*$-link T we say that $\ell \in L(\mathring{T})$ is *maximal in* T' if T' is the minimal relevant self-energy cluster contained in $\mathring{T}$ (with $\mathfrak{d}_{T'} = 0$) such that $\ell \in L(T')$. If there is no such relevant self-energy cluster we say that ℓ is *maximal in* T. Given a $*$-link T with $\delta_T = \partial, \partial^2$, we denote by $L_M(T)$ the set of lines which are maximal in T; for any relevant self-energy cluster T' with $\mathfrak{d}_{T'} = 0$ we denote by $L_M(T')$ the set of lines which are maximal in T'.

Lemma C.1. *Let T be any $*$-link with $\delta_T = \partial, \partial^2$. For all relevant self-energy clusters T' contained in $\mathring{T}$ one has $q_0(T') := |L_M(T') \cap L_0(\mathring{T})| \le 4$. Moreover*

$$q_1(T') := \sum_{\ell \in L_M(T')} d_\ell \le \min\{4 - q_0(T'), 2\}.$$

The same hold if we replace T' with the $*$-link T.

Proof. If $\delta_T = \partial$ there is at most one line $\ell \in L(\mathring{T})$ such that $d_\ell = 1$. Let $\{T_j\}_{j=0}^m$ be the cloud of ℓ, where we denoted $T_0 = T$, so that $L_0(\mathring{T}) \subseteq \{\ell_{T_1}, \ell'_{T_1}, \ldots, \ell_{T_m}, \ell'_{T_m}\}$. Then T_m is the minimal relevant self-energy cluster containing ℓ and $L_M(T_j) \cap L_0(\mathring{T}) \subseteq \{\ell_{T_{j+1}}, \ell'_{T_{j+1}}\}$, $j = 0, \ldots, m-1$. Hence $q_0(T_j) \le 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, m-1$, while $q_0(T_m) = 0$ and $q_1(T_m) = 1$. If $\delta_T = \partial^2$ and there is one line $\ell \in L(\mathring{T})$ with $d_\ell = 2$ one can reason as in the previous case, denoting by $\{T_j\}_{j=0}^m$ the cloud of ℓ and hence obtaining $q_0(T_j) \le 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, m-1$, while $q_0(T_m) = 0$ and $q_1(T_m) = 2$. If there are two lines $\ell'_1, \ell'_2 \in L(\mathring{T})$ with $d_{\ell'_1} = d_{\ell'_2} = 1$ one proceeds as follows. Call $\{T_j^{(i)}\}_{j=0}^{m_i}$ the cloud of ℓ'_i, $i = 1, 2$, with $T_0^{(1)} = T_0^{(2)} = T_0 = T$, and set

$$r := \max\{j \ge 0 \,:\, T_j^{(1)} = T_j^{(2)} =: T_j\}.$$

If $r = m_1 = m_2$ then again $q_0(T_j) \le 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, r-1$, while $q_0(T_r) = 0$ and $q_1(T_r) = 2$. If $r = m_1 < m_2$ then $q_0(T_j) \le 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, r-1$, $q_0(T_r) \le 2$ and $q_1(T_r) = 1$, $q_0(T_j^{(2)}) \le 2$ and $q_1(T_j^{(2)}) = 0$ for $j = r+1, \ldots, m_2$, and $q_0(T_{m_2}) = 0$ and $q_1(T_{m_2}) = 1$. Finally if $r < \min\{m_1, m_2\}$ then $L_M(T_r) \cap L_0(\mathring{T}) \subseteq \{\ell_{T_{r+1}^{(1)}}, \ell'_{T_{r+1}^{(1)}}, \ell_{T_{r+1}^{(2)}}, \ell'_{T_{r+1}^{(2)}}\}$, so that $q_0(T_j) \le 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, r-1$, $q_0(T_r) \le 4$ and $q_1(T_r) = 0$, while $q_0(T_j^{(i)}) \le 2$ and $q_1(T_j^{(i)}) = 0$ for $j = r+1, \ldots, m_i$, and $q_0(T_{m_i}) = 0$ and $q_1(T_{m_i}) = 1$, $i = 1, 2$. ■

Define the multiplicity (function) of a non-injective map as the cardinality of its pre-image sets [26, 53].

Lemma C.2. *Let T be a $*$-link with $\delta_T = \partial, \partial^2$. There exists an application $\Lambda : L_R^*(\mathring{T}) \to L_{NR}(\mathring{T})$ with multiplicity at most 2 such that $\zeta_\ell = \zeta_{\Lambda(\ell)}$.*

Proof. By Lemma C.1 there are at most four lines $\ell_1, \ell_2, \ell_3, \ell_4 \in L_R^*(\mathring{T})$ such that, if T_i' denote the minimal relevant self-energy cluster containing ℓ_i, then $T_1' = \ldots = T_4'$. Moreover by Remark 6.11 if ℓ is a resonant line, then the minimal relevant self-energy cluster containing ℓ contains also two non-resonant lines ℓ_1', ℓ_2' with the same minimum scale as ℓ. Therefore the assertion follows. ■

Now consider a $*$-link T contributing to (C.25) with largest depth, say D. Since T does not contain any resonant line, by Lemma 6.6 and Remark 6.7 we have

$$|\mathscr{V}_{T_i}(x_{\ell_{T_i}}(\underline{\tau}), \mathcal{R})| \le c_2^{k(T_i)} \mathrm{e}^{-\xi K(T_i)/2} \le c_3 c_2^{k(T_i)} |x_{\ell_{T_i}}(\underline{\tau})|^2, \tag{C.30}$$

for some positive constants c_2 and c_3; we have also used that $K(T_i) \ge 2^{m_{n_{\ell_{T_i}}}-1}$ for $\delta_{T_i} = \mathcal{R}$ and $|x_{\ell_{T_i}}(\underline{\tau})| \ge \alpha_{m_{n_{\ell_{T_i}}}}(\boldsymbol{\omega})$ if $\Psi_{n_{\ell_{T_i}}}(x_{\ell_{T_i}}(\underline{\tau})) \ne 0$. Therefore we can bound

$$|a(x_{\ell_T}(\underline{\tau}), \delta_T)| |\mathscr{V}_T(x_{\ell_T}(\underline{\tau}), \delta_T)| \le \begin{cases} c_4^{k(T)}, & \delta = \mathcal{L}, \\ c_4^{k(T)} |x_{\ell_T}(\underline{\tau})|, & \delta_T = \partial, \\ c_4^{k(T)} |x_{\ell_T}(\underline{\tau})|^2, & \delta = \partial^2, \mathcal{R}, \end{cases} \tag{C.31}$$

for some constant c_4. Now consider a $*$-link T contributing to (C.25) with depth $D-1$. For each resonant line $\ell \in L_R(\mathring{T})$ denote by T' the minimal relevant self-energy cluster containing ℓ (set $T' = T$ if there is no minimal relevant self-energy cluster containing ℓ). For all resonant lines $\ell' \in L_M(T') \backslash L_0(\mathring{T})$, there is a subchain $C \in \mathfrak{C}_0(T)$ for which ℓ' is an internal chain-line, such that C is uniquely associated (see the comments after (C.25)) with a $*$-chain $C^* = \{T_{i_1}, \ldots, T_{i_{p(C)}}\}$ with depth D contributing to (C.25), whose value can be bounded by

$$\left| \prod_{j=1}^{p(C)} a(x_{\ell_{T_{j_i}}}(\underline{\tau}), \delta_i) \mathscr{V}_{\mathring{T}_i}(x_{\ell_{T_{j_i}}}(\underline{\tau}), \delta_{j_i}) \right| \le |x_{\ell_{T_{j_i}}}(\underline{\tau})|^{p(C)-1} c_5^{k(C)}, \tag{C.32}$$

for some $c_5 \ge 0$, and this can be obtained as follows. If either $\delta_{T_{i_j}} \ne \mathcal{L}$ for all $j = 1, \ldots, p(C)$ or there is only one $j = 1, \ldots, p(C)$ such that $\delta_{T_{i_j}} = \mathcal{L}$, then (C.32) trivially follows from (C.31). Otherwise if $1 \le j < j' \le p(C)$ are such that $\delta_{T_{i_j}} = \delta_{T_{i_{j'}}} = \mathcal{L}$ then there is at least one $j'' = j+1, \ldots, j'-1$ such that $\delta_{T_{i_{j''}}} = \partial^2, \mathcal{R}$; recall the constraints (i)–(vi) after (C.7). Then (C.32) follows.

Moreover by Lemma C.1, if there are $R_n^\bullet(T')$ resonant lines in $L_M(T')$ with minimum scale n, we have an overall gain $\sim \alpha_{m_n}(\boldsymbol{\omega})^{R_n^\bullet(T') - q_0(T')}$ and on the other hand the product of the propagators of such resonant lines can be bounded proportionally to $\alpha_{m_n}(\boldsymbol{\omega})^{-(R_n^\bullet(T') + q_1(T'))}$, with $q_0(T') + q_1(T') \le 4$. Therefore, by Lemma C.2, if we replace the bound for the propagators of each non-resonant line $\ell \in L_M(T')$, $\zeta_\ell = n$, with $c_6 \alpha_{m_n}(\boldsymbol{\omega})^{-3}$ for some positive constant c_6, we have exactly a gain factor which is enough to compensate each propagator of the resonant lines with minimum scale n in $L_M(T')$. But since we can reason in the same way for all n and all resonant lines in T, if we replace the bound for the propagators of each $\ell \in L_{NR}(\mathring{T})$ with $c_6 \alpha_{m_{n_\ell}}(\boldsymbol{\omega})^{-3}$ we obtain a gain proportional to $\alpha_{m_{n_{\ell'}}}(\boldsymbol{\omega})^{1+d_{\ell'}}$ for any $\ell' \in L_R(\mathring{T})$, and hence we can use again Lemma 6.5 in order to obtain the bound (C.31) also for the $*$-links with depth $D-1$.

Then we pass to the $*$-links with depth $D-2$ and reason in the same way as above and so on. When we arrive to the $*$-links with depth 0 we only have to recall that they are associated with the maximal relevant self-energy clusters T of θ, which all have label $\mathfrak{d}_T = 1$. Hence in (C.25) we can bound

$$\left| \prod_{i=1}^{N} a(x_{\ell_{T_i}}(\underline{\tau}), \delta_i)\, \mathscr{V}_{\mathring{T}_i}(x_{\ell_{T_i}}(\underline{\tau}), \delta_i) \right| \le c_7^{k(T_1)+\ldots+k(T_N)} |x|^{p-1} = c_7^{k} |x|^{p-1},$$

for some positive constant c_7. We have still to sum over all the possible contributions of the form (C.25). To take into account the scale labels n_ℓ, $\ell \in L(\mathring{T}_1) \cup \ldots \cup L(\mathring{T}_N)$ simply recall that for each momentum $\boldsymbol{\nu}_\ell$ only 2 scale labels are allowed; see Remark 3.8. To sum over the mode labels $\boldsymbol{\nu}_v$, $v \in N(\mathring{T}_1) \cup \ldots \cup N(\mathring{T}_N)$ we can neglect the constraints and use a factor $\mathrm{e}^{-(\xi/4)|\boldsymbol{\nu}_v|}$ for each v. Moreover the component labels h_ℓ are 2. Finally the sum over all possible unlabelled chains with order k is bounded by a constant to the power k. This completes the proof of the bound (6.9).

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