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Variational aspects of singular Liouville systems

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Contents

Introduction

This thesis is concerned with the study of singular Liouville systems on closed surfaces, that is systems of second-order elliptic partial differential equations with exponential nonlinearities, which arise in many problems in both physics and geometry.
Such problems are attacked by a variational point of view, namely we consider solutions as critical points for a suitable *energy functional* defined on a suitable space.
We will first discuss the existence of *points of minima* for the energy functional, which solve the problem. Then, in the cases when the energy cannot have global minimum points, we will look for critical points of other kind, the so-called *min-max points*. Finally, we will also give some non-existence results for such problems.

Let (Σ, g) be a compact surface without boundary. We will consider the following system of PDEs:

$$-\Delta u_i = \sum_{j=1}^{N} a_{ij}\rho_j \left(h_j e^{u_j} - \frac{1}{|\Sigma|} \right) - 4\pi \sum_{m=1}^{M} \alpha_{im} \left(\delta_{p_m} - \frac{1}{|\Sigma|} \right), \qquad i = 1, \dots, N \tag{1}$$

Here, $-\Delta = -\Delta_g$ is the Laplace-Beltrami operator with respect to the metric g and the other quantities have the following properties:

- $A = (a_{ij})_{i,j=1,\dots,N} \in \mathbb{R}^{N\times N}$ is a positive definite symmetric $N \times N$ matrix,
- $\rho_1, \dots, \rho_N \in \mathbb{R}_{>0}$ are positive real parameters,
- $h_1, \dots, h_N \in C^\infty_{>0}(\Sigma)$ are positive smooth functions,
- $p_1, \dots, p_M \in \Sigma$ are given points,
- $\alpha_{im} > -1$ for $i = 1, \dots, N, m = 1, \dots, M$.

Recalling that $\displaystyle\int_\Sigma (-\Delta u) \mathrm{d}V_g = 0$ for any $u \in H^1(\Sigma)$, by integrating both sides of (1) in the whole surface Σ we deduce $\displaystyle\int_\Sigma h_i e^{u_i} \mathrm{d}V_g = 1$ for all $i = 1, \dots, N$.
Therefore, under the non-restrictive assumption that the surface area of $|\Sigma|$ of Σ equals 1, (1) can be re-written in the equivalent form

$$-\Delta u_i = \sum_{j=1}^{N} a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j} \mathrm{d}V_g} - 1 \right) - 4\pi \sum_{m=1}^{M} \alpha_{im}(\delta_{p_m} - 1), \qquad i = 1, \dots, N.$$

To better describe the properties of such systems, it is convenient to perform a change of variables. Consider the Green's function G_p of $-\Delta_g$ centered at a point $p \in \Sigma$, that is the solution of

$$\begin{cases} -\Delta G_p = \delta_p - 1 \\ \displaystyle\int_\Sigma G_p \mathrm{d}V_g = 0 \end{cases}, \tag{2}$$

and apply the following change of variable:

$$u_i \mapsto u_i + 4\pi \sum_{m=1}^{M} \alpha_{im} G_{p_m}.$$

The newly-defined u_i solve

$$-\Delta u_i = \sum_{j=1}^{N} a_{ij}\rho_j \left(\frac{\widetilde{h}_j e^{u_j}}{\int_\Sigma \widetilde{h}_j e^{u_j} \mathrm{d}V_g} - 1 \right), \qquad \text{with } \widetilde{h}_i := h_i e^{-4\pi \sum_{m=1}^{M} \alpha_{im} G_{p_m}}. \tag{3}$$

Basically, the new potentials $\widetilde{h}_i$ "absorbed" the Dirac deltas appearing in (1).
Since G_p blows up around p like $\dfrac{1}{2\pi} \log \dfrac{1}{d(\cdot, p)}$, then $\widetilde{h}_i$ will verify:

$$\widetilde{h}_i \in C^\infty_{>0}(\Sigma \setminus \{p_1, \dots, p_M\}), \qquad \widetilde{h}_i \sim d(\cdot, p_m)^{2\alpha_{im}} \quad \text{around } p_m. \tag{4}$$

Therefore, $\widetilde{h}_i$ will tend to $+\infty$ at p_m if and only if $\alpha_{im} < 0$ and it will tend to 0 at p_m if and only if $\alpha_{im} > 0$.

The form (3) is particularly useful because it admits a variational formulation. In fact, all and only its solutions are the critical points of the following energy functional defined on $H^1(\Sigma)^N$:

$$J_\rho(u) := \frac{1}{2} \sum_{i,j=1}^{N} a^{ij} \int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g - \sum_{i=1}^{N} \rho_i \left(\log \int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g - \int_\Sigma u_i \mathrm{d}V_g \right). \tag{5}$$

Here, $\nabla = \nabla_g$ is the gradient given by the metric g, $\cdot$ is the Riemannian scalar product and a^{ij} are the entries of the inverse matrix A^{-1} of A. Sometimes, to denote the dependence on the matrix A, we will denote the functional as $J_{A,\rho}$. We will also denote as $Q_A(u)$, or simply $Q(u)$, the quadratic expression $\dfrac{1}{2} \displaystyle\sum_{i,j=1}^{N} a^{ij} \nabla u_i \cdot \nabla u_j$.
The functional J_ρ is well defined on the space $H^1(\Sigma)^N$ because of the classical Moser-Trudinger inequalities by [73, 63, 38], which ensure exponential integrability in such a space.

The system (3) is a natural generalization of the scalar Liouville equation

$$-\Delta u = \rho \left(h e^u - 1 \right) - 4\pi \sum_{m=1}^{M} \alpha_m (\delta_{p_m} - 1),$$

which is equivalent, by manipulations similar to the ones described before, to

$$-\Delta u = \rho \left(\frac{\widetilde{h} e^u}{\int_\Sigma \widetilde{h} e^u \mathrm{d}V_g} - 1 \right). \tag{6}$$

Equation (6) arises in many well-known problem from different areas of mathematics.
In statistical mechanics, it is a mean field equation for the Euler flow in the Onsager's theory (see [17, 18, 46]). In theoretical physics, it is used in the description of abelian Chern-Simons vortices theory (see [70, 76]).
In geometry, (6) is the equation of Gaussian curvature prescription problem on surfaces with conical singularity (see [22, 23]). Here, each of the points p_m will have a conical singularity of angle $2\pi(1 + \alpha_m)$, whereas h is the Gaussian curvature of the new metric and the parameter ρ is determined by the Gauss-Bonnet theorem, that is by the Euler characteristic $\chi(\Sigma)$ of Σ.
The scalar Liouville equations has been very widely studied in literature, with many results concerning existence and multiplicity of solutions, compactness properties, blow-up analysis et al., which have been summarized for instance in the surveys [71, 58].

Liouville systems like (3) have several applications: in biology they appear in some models describing chemotaxis ([27]), in physics they arise in kinetic models of plasma ([47, 45]).

Particularly interesting are the cases where A is the Cartan matrix of a Lie algebra, such as

$$A = A_N = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix},$$

which is the Cartan Matrix of $SU(N+1)$. This particular system is known as the $SU(N+1)$ or the A_N Toda system.
The importance of the $SU(3)$ Toda system is due to its application in algebraic geometry, in the description of the holomorphic curves of $\mathbb{CP}^N$ (see e.g. [19, 15, 26]), and in mathematical physics in the non-abelian Chern-Simon vortices theory (see [37, 76, 70]).
The singularities represent, respectively, the ramification points of the complex curves and the *vortices* of the wave functions.
Two further important examples are given by the following 2×2 systems

$$B_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

which are known respectively as B_2 and G_2 Toda systems and can be seen as particular cases of the A_3 and A_6 Toda system, respectively. Just like the A_2 Toda, their study is closely related to holomorphic curves in projective spaces.
Although the matrices B_2, G_2 are not symmetric, their associated Liouville system is equivalent to one with associated to a symmetric matrix and a re-scaled parameter, through the elementary substitution:

$$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \frac{\rho_2}{2} \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \frac{\rho_2}{3} \end{pmatrix}.$$

Their energy functional will therefore be

$$J_{B_2,\rho}(u) := \int_\Sigma Q_{B_2}(u)\mathrm{d}V_g - \rho_1 \left(\log \int_\Sigma h_1 e^{u_1}\mathrm{d}V_g - \int_\Sigma u_1 \mathrm{d}V_g\right) - \frac{\rho_2}{2}\left(\log \int_\Sigma h_2 e^{u_2}\mathrm{d}V_g - \int_\Sigma u_2 \mathrm{d}V_g\right),$$

$$J_{G_2,\rho}(u) := \int_\Sigma Q_{G_2}(u)\mathrm{d}V_g - \rho_1 \left(\log \int_\Sigma h_1 e^{u_1}\mathrm{d}V_g - \int_\Sigma u_1 \mathrm{d}V_g\right) - \frac{\rho_2}{3}\left(\log \int_\Sigma h_2 e^{u_2}\mathrm{d}V_g - \int_\Sigma u_2 \mathrm{d}V_g\right),$$

with

$$Q_{B_2}(u) = \frac{|\nabla u_1|^2}{2} + \frac{\nabla u_1 \cdot \nabla u_2}{2} + \frac{|\nabla u_2|^2}{4} \qquad Q_{G_2}(u) = |\nabla u_1|^2 + \nabla u_1 \cdot \nabla u_2 + \frac{|\nabla u_2|^2}{3}. \tag{7}$$

A first tool to attack variationally problem (6) is given by the Moser-Trudinger inequality, from the aforementioned references [73, 63, 38] and, in the singular case, by [24, 72].
Such an inequality basically state that the energy functional, which in the scalar case has the form

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 \mathrm{d}V_g - \rho \left(\int_\Sigma \widetilde{h} e^u \mathrm{d}V_g - \int_\Sigma u \mathrm{d}V_g\right), \tag{8}$$

is bounded from below if and only if $\rho \leq 8\pi \min\left\{1, 1 + \min_{m=1,\dots,M} \alpha_m\right\}$. Moreover, if ρ is strictly smaller than this threshold, then I_ρ is (weakly) coercive, that is all of his sub-levels are bounded. Since I_ρ is also lower semi-continuous, as can be easily verified, if this occurs then direct methods from calculus of variations yield the existence of minimizers for I_ρ, which solve (6).

The first main goal of of this thesis is to prove Moser-Trudinger inequalities for singular Liouville system like (3), that is to establish sufficient and necessary conditions for the boundedness from below and for the coercivity of its energy functional J_ρ.
Such issues were addressed in the papers [12] and [9]. The former considers the $SU(3)$ Toda system, that is the following system:

$$\begin{cases} -\Delta u_1 = 2\rho_1\left(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} \mathrm{d}V_g} - 1\right) - \rho_2\left(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} \mathrm{d}V_g} - 1\right) \\ -\Delta u_2 = 2\rho_2\left(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} \mathrm{d}V_g} - 1\right) - \rho_1\left(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} \mathrm{d}V_g} - 1\right) \end{cases}; \tag{9}$$

the latter studies the general case. Here, they are presented in Chapter 2.
They are inspired by some results obtained for particular systems with no singularities ([44, 74]) and for similar problems on Euclidean domains ([28, 29]), on the sphere $\mathbb{S}^2$ ([68]) and on general compact manifolds ([67]).

The arguments used to this purpose are roughly the following.
As a first thing, an easy application of the scalar Moser-Trudinger inequality gives boundedness from below and coercivity for small values of ρ.
We then consider, for such values, the minimizing solutions u_ρ of J_ρ and perform a blow-up analysis. To this purpose, we first prove a concentration-compactness theorem for solutions of (3) and then show that compactness must occur under some algebraic conditions on ρ and α_{im}, which are satisfied in particular as long as ρ is in the neighborhood of 0.
Therefore, J_ρ must be coercive for all ρ's which satisfy this condition. On the other hand, through suitable test functions, we can show that such conditions are indeed also sufficient for the coercivity.

Next, we discuss whether J_ρ can still be bounded from below when it is not coercive.
In particular, we will consider the case of fully *competitive* systems, that is when $a_{ij} \le 0$ for any $i \ne j$ (hence also for the A_N, B_2, G_2 Toda system described before).
The conditions for coercivity, which are in general pretty lengthy to state (which will be done in Chapter 2), are much simpler under this assumption. In this case, coercivity occurs if and only if $\rho_i < \dfrac{8\pi \min\{1, 1 + \min_{m=1,\dots,M} \alpha_{im}\}}{a_{ii}}$, a condition which is also very similar to the one in the scalar case.
This is basically due to the following fact: under the assumption of A being non-positive outside the diagonal, the blow-up of minimizing sequences u_ρ is locally one-dimensional, that it, roughly speaking, each blowing-up component do not interact with any other. This means that a sharper blow-up analysis can be done using a local version of the scalar Moser-Trudinger inequality, thus enabling to prove that J_ρ is bounded from below even in the borderline case $\rho_i = \dfrac{8\pi \min\{1, 1 + \min_{m=1,\dots,M} \alpha_{im}\}}{a_{ii}}$ for $i = 1, \dots, N$.

The next major problem considered in this work is the existence of non-minimizing solutions, in case the parameter ρ exceeds the range of parameters which gives coercivity.

A first big issue which one encounters when looking for non-minimizing critical points is the lack of the Palais-Smale condition, which is needed to apply most of the standard min-max theorems.
Actually, despite the Palais-Smale condition is not known to hold true for functionals like J_ρ, we can exploit a *monotonicity trick* by Struwe ([69], see also [34]). Basically, because of the specific structure of J_ρ, and in particular the fact that $t \mapsto \dfrac{J_{t\rho}}{t}$ is non-increasing, we get the existence of some converging Palais-Smale sequences at mountain-pass critical level.
Due to this result, to apply standard min-max methods for a generic value of ρ we only need a compactness result for solutions of (3). In fact, if we had compactness of solutions, then we could

take $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho$ such that bounded Palais-Smale sequences exist for such values, getting a mountain pass solutions u_{ρ^n} and then, by compactness, considering $u_\rho = \lim_{n\to+\infty} u_{\rho^n}$, which would solve (3).

Non-compactness phenomena for the Liouville equation (6) have been pretty well understood.
The only possible scenario is a blow-up around a finite number of points, with no residual mass (see [16]).
Local quantization values, that is the portions of the integral of $\widetilde{h}e^u$ which accumulate around each blow-up point, are also fully known: they equal 8π for blow-up at a regular point (see [49]) and $8\pi(1+\alpha_m)$ in the case of blow-up at a singular point p_m (see [7, 5]).
Therefore, the only values of ρ which could generate non-compactness are all the possible finite sum of such values. We get a discrete set on the positive half-line, outside of which we get compactness of solutions. Min-max methods can thus be applied for a generic choice of ρ.

Concerning general Liouville systems, local quantization and blow-up analysis are still widely open problems.
A classification of local blow-up values has been given only for very specific systems, namely the A_2 Toda system (9) ([43, 53]) and, in the case of no singularities, the B_2 and (partially) G_2 Toda systems ([51]):

$$\begin{cases} -\Delta u_1 = 2\rho_1\left(\frac{h_1e^{u_1}}{\int_\Sigma h_1e^{u_1}\mathrm{d}V_g}-1\right)-\rho_2\left(\frac{h_2e^{u_2}}{\int_\Sigma h_2e^{u_2}\mathrm{d}V_g}-1\right)\\ -\Delta u_2 = 2\rho_2\left(\frac{h_2e^{u_2}}{\int_\Sigma h_2e^{u_2}\mathrm{d}V_g}-1\right)-2\rho_1\left(\frac{h_1e^{u_1}}{\int_\Sigma h_1e^{u_1}\mathrm{d}V_g}-1\right)\end{cases} \tag{10}$$

$$\begin{cases} -\Delta u_1 = 2\rho_1\left(\frac{h_1e^{u_1}}{\int_\Sigma h_1e^{u_1}\mathrm{d}V_g}-1\right)-\rho_2\left(\frac{h_2e^{u_2}}{\int_\Sigma h_2e^{u_2}\mathrm{d}V_g}-1\right)\\ -\Delta u_2 = 2\rho_2\left(\frac{h_2e^{u_2}}{\int_\Sigma h_2e^{u_2}\mathrm{d}V_g}-1\right)-3\rho_1\left(\frac{h_1e^{u_1}}{\int_\Sigma h_1e^{u_1}\mathrm{d}V_g}-1\right)\end{cases}. \tag{11}$$

Anyway, the quantization results do not suffice, by themselves, to ensure a generic compactness results on ρ, due to the possibility of residual mass. Actually, in [30] it was proved that non-vanishing residual may indeed occur for the regular Toda system.
The issue of residual has been rule out in the paper [14], where we showed that if it occurs, then it does only for one component of the A_2 Toda. Similarly, for $N\times N$ systems, there is at least one component which has not residual mass. This result is presented in Section 2.2.
Ruling out the chance of a double residual implies that, for blowing up sequences of solutions of the A_2, B_2, G_2 Toda systems, at least one between ρ_1 and ρ_2 must be a finite combination of the local blow-up values. Therefore, the set of parameters to be excluded for the purpose of compactness is just a union of horizontal and vertical half-lines on the first quadrant so, similarly as before, conditions to apply min-max methods are satisfied for almost every $\rho\in\mathbb{R}^2_{>0}$.

We are therefore allowed to search for min-max solutions, a goal to which Chapter 3 will be devoted.
The strategy we will follow will be based on analysis of energy sub-levels and application of Morse theory, rather than usual mountain pass or linking theorems, as was usually done by many authors who studied similar problems. The reason of such a choice is that, whereas the two arguments are perfectly equivalent to prove existence of solutions, Morse theory gives also information about the number of solutions, provided the energy is a Morse functional, which a generic assumption (in a sense which will be clarified later).
We will actually show that a change of topology occurs between very high sub-levels of energy functional, which are contractible, and very low sub-levels. The compactness assumptions discussed in the previous paragraph ensure, thanks to [56], that a change of topology between sub-levels implies existence of solutions.

Roughly speaking, if $J_\rho(u)$ is very negative, then the L^1-mass of $\widetilde{h}_ie^{u_i}$ accumulates, for one or both i's, around a finite number of points, depending on the parameters ρ_i and α_{im}.
This can be made rigorous by introducing a space $\mathcal{X}$, a subspace of finitely-supported unit measures

on Σ, and by building two maps Φ, Ψ from very low sub-levels to the space $\mathcal{X}$ and vice-versa, such that their composition is homotopically equivalent to the identity on $\mathcal{X}$. If $\mathcal{X}$ is not contractible, then low sub-levels of J_ρ will also be non-contractible, hence we will deduce existence of solutions. Moreover, by estimating the ranks of the homology groups of $\mathcal{X}$ we will get an estimate on the multiplicity of solutions.
Such a method has been introduced in [36] for a fourth-order elliptic problem and has been widely used to study the singular Liouville equation. Through this argument, general existence results have been proved for problem (6) in the case of no singularities ([35]) and in the case of positive singularities on surfaces of non-positive Euler-Poincaré characteristic ([3]), as well as partial existence results in the case of negative singularities ([21, 20]) and of positive singularities on general surfaces ([60, 4]). It has also been used in [59, 61, 42] to attack the regular $SU(3)$ Toda system in the cases when one or both between ρ_1, ρ_2 are less then 8π, and even in similar problems with exponential nonlinearities, such as the Sinh-Gordon equation ([77, 41]) and the Nirenberg problem ([55, 33]).
Here, we will present the results obtained in the papers [11, 10, 13, 8], the last of which is in preparation.

In the first of such papers we study the $SU(3)$ Toda system (9). We assume Σ to have non-positive Euler characteristic, that is neither homeomorphic to the sphere $\mathbb{S}^2$ nor to the projective plane $\mathbb{RP}^2$, and we assume that coefficients α_{im} to be non-negative.
Here, following [3], we exploit the topology of Σ to retract the surface on a bouquet of circles. By taking two of such retractions on disjoint curves we can by-pass a major issue which occurs in the study of Liouville systems of two or more equations, that is the interaction between the concentration of two (or more) components. In fact, through the push-forward of measures, we can restrict the study of u_1 on a curve γ_1 and of u_2 on the other γ_2. Moreover, by choosing γ_1, γ_2 not containing any of the singular points p_m, we also avoid the issue of singularities.
Performing such a retraction clearly causes a loss of topological information, but the partial characterization we give on sub-levels suffices to get a generic existence result.
Furthermore, we can apply Morse inequalities to get an estimate from below on the number of solutions. This can be done for a generic choice of the potentials h_1, h_2 and of the metric g, since J_ρ is a Morse functional for such a generic choice, as was proved in [32] for the scalar case. In particular, if the characteristic of Σ is greater or equal than 2, namely its Euler characteristic is negative, the number of solutions goes to $+\infty$ as either ρ_1 or ρ_2 goes to $+\infty$.

In the paper [9] we give a partial extension of the results from [11] to the case of singularity of arbitrary sign. The main difference is that negatively-signed vortices actually affect the best constant in Moser-Trudinger inequality, as will be shown in detail in Chapter 2, therefore they cannot be simply "ignored" as was done before. On the contrary, we will have to take into account each point p_m on γ_i if $\alpha_{im} < 0$.
This means that, since we need γ_1 and γ_2 to be disjoint, we have to assume $\max\{\alpha_{1m}, \alpha_{2m}\} \geq 0$ for any m, as well as the characteristic of Σ to be non-positive.
Moreover, we also need some algebraic condition on ρ and α_{im} to let low sub-levels be not contractible, much like [20].
By Morse theory, we also get another generic multiplicity result similar to the one before [11].

In [13], we consider the singular $SU(3)$ system on arbitrary surfaces and we allow both α_{1m} and α_{2m} to be negative for the same p_m. Here, the methods which were briefly described before cannot be applied anymore and we need a sharper analysis of sub-levels.
To this purpose, we introduce a center of mass and scale of concentration, inspired by [61] but strongly adapted to take into account the presence of singularities. We basically show that, for functions with same center and scale, Moser-Trudinger inequality holds with a higher constant. In other words, we get a so-called *improved Moser-Trudinger inequality.*
Such *improved inequalities* allow, for sufficiently small values of ρ, to give a precise characterization of sublevels, hence existence of min-max solutions also in this case.

Finally, in [8] the results from [11] are adapted to the regular B_2 and G_2 Toda systems (10), (11). We get similar general existence and multiplicity results for surfaces with non-positive Euler characteristic.

In Chapter 4 we give some non-existence result for singular Liouville systems (3), contained in the paper [13].

The first two results, inspired by the ones in [4] for the scalar equation, are for general systems defined on particular surfaces. The former holds on the unit Euclidean ball, equipped with the standard metric, with a unique singularity in its center; the latter holds on the standard unit sphere with two antipodal singularities.
Concerning the result on the ball, we show, through a Pohožaev identity, that if a solution exists then the parameters ρ, α_i must satisfy an algebraic relation.
The argument used for the case of the sphere is similar. We exploit the stereographic projection to transform the solution of (3) on $\mathbb{S}^2$ in an entire solution on the plane. Then, we use another Pohožaev identity for entire solutions to get again necessary algebraic conditions for the existence of solutions.
These result show that in the general existence results stated before the assumptions on $\chi(\Sigma)$ is somehow sharp.

The last result, inspired by [20], is given only for the $SU(3)$ Toda system, but it holds for any surfaces. It basically states that the system has no solutions if a couple of coefficients $(\alpha_{1m}, \alpha_{2m})$ is too close to -1.
The result is proven by contradiction, using blow-up analysis. We assume that a solution exists for α_{1m}, α_{2m} arbitrarily close to -1 and apply the Concentration-Compactness alternative from Chapter 2 to this sequence. By ruling out all the alternatives we get a contradiction.
By comparing this result with the ones in Chapter 3 we deduce that, to have such a general existence result, we need to assume *all* the coefficients α_{im} to be positive.

Before stating the main result of this thesis, we devote Chapter 1 to some preliminaries.
First of all, we introduce the notation we will use throughout the whole paper. Then, we present some known facts which will be used in the rest, mostly from analysis and topology, and some of their consequences which need very short proofs.

We will postpone some proofs in an Appendix: the proof of a Pohožaev identity for entire solutions of singular Liouville systems and of the fact that being a Morse function is a generic condition for the energy functional.
The reason of this choice is that such proofs are very similar to the ones for the scalar case, though quite lengthy.

Chapter 1

Preliminaries

In this chapter we present some notation and some preliminary facts which will be useful in the following.

1.1 Notation

The indicator function of a set $\Omega \subset \Sigma$ will be denoted as

$$\mathbf{1}_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

Given two points $x, y \in \Sigma$, we will indicate the metric distance on Σ between them as $d(x, y)$. Similarly, for any $\Omega, \Omega' \subset \Sigma$ we will write:

$$d(x, \Omega) := \inf\{d(x, y) : x \in \Omega\}, \qquad d(\Omega, \Omega') := \inf\{d(x, y) : x \in \Omega, \, y \in \Omega'\}.$$

The diameter of a set Ω will be indicated as

$$\operatorname{diam}(\Omega) := \sup\{d(x, y) : x, y \in \Omega\}.$$

We will indicate the open metric ball centered in p having radius r as

$$B_r(x) := \{y \in \Sigma : d(x, y) < r\}.$$

Similarly, for $\Omega \subset \Sigma$ we will write

$$B_r(\Omega) := \{y \in \Sigma : d(y, \Omega) < r\}.$$

For any subset of a topological space $A \subset X$ we indicate its closure as $\overline{A}$ and its interior part as $\mathring{A}$. For $r_2 > r_1 > 0$ we denote the open annulus centered at p with radii r_1, r_2 as

$$A_{r_1,r_2}(p) := \{x \in \Sigma : r_1 < d(x, p) < r_2\} = B_{r_2}(p) \setminus \overline{B_{r_1}(p)}.$$

If $\Omega \subset \Sigma$ has a smooth boundary, for any $x \in \partial\Omega$ we will denote the outer normal at x as $\nu(x)$. If $u \in C^1(\partial\Omega)$ we will indicate its normal derivative at x as $\partial_\nu u(x) := \nabla u(x) \cdot \nu(x)$.

Standard notation will be used for the usual numeric set, like $\mathbb{N}, \mathbb{R}, \mathbb{R}^N$. Here, $\mathbb{N}$ contains 0. A similar notation will be used for the space of $N \times M$ matrices, which we will denote as $\mathbb{R}^{N\times M}$.
The positive and negative part of real number t will be denoted respectively as $t^+ := \max\{0, t\}$ and $t^- := \max\{0, -t\}$.

The usual functional spaces will be denoted as $L^p(\Omega), C^\infty(\Sigma), C^\infty(\Sigma)^N, \ldots$. A subscript will be added to indicate vector with positive component or (almost everywhere) positive functions, like

$\mathbb{R}_{>0}, C^\infty_{>0}(\Sigma)$. Subscript may be also added to stress the dependence on the metric g defined on Σ. For a continuous map $f : \Sigma \to \Sigma$ and a measure μ defined on Σ, we define the push-forward of μ with respect to f as the measure defined by

$$f_*\mu(B) = \mu\left(f^{-1}(B)\right).$$

If μ has finite support, its push-forward has a particularly simple form:

$$\mu = \sum_{k=1}^{K} t_k\delta_{x_k} \qquad \Rightarrow \qquad f_*\mu = \sum_{k=1}^{K} t_k\delta_{f(x_k)}.$$

Given a function $u \in L^1(\Sigma)$ and a measurable $\Omega \subset \Sigma$ with positive measure, the average of u on Ω will be denoted as

$$⨍_\Omega u\mathrm{d}V_g = \frac{1}{|\Omega|}\int_\Omega u\mathrm{d}V_g.$$

In particular, since we assume $|\Sigma| = 1$, we can write

$$\int_\Sigma u\mathrm{d}V_g = ⨍_\Sigma u\mathrm{d}V_g.$$

We will indicate the subset of $H^1(\Sigma)$ which contains the functions with zero average as

$$\overline{H}^1(\Sigma) := \left\{u \in H^1(\Sigma) : \int_\Sigma u = 0\right\}.$$

Since the functional J_ρ defined by (5) is invariant by addition of constants, as well as the system (3), it will not be restrictive to study both of them on $\overline{H}^1(\Sigma)^N$ rather than on $H^1(\Sigma)^N$.
On the other hand, for a planar Euclidean domain $\Omega \subset \mathbb{R}^2$ (or $\Omega \subset \Sigma$) with smooth boundary and a function $u \in H^1(\Omega)$ we will indicate with the symbol $u|_{\partial\Omega}$ the trace of u on the boundary of Ω. The space of functions with zero trace will be denoted by

$$H^1_0(\Omega) := \left\{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\right\}. \tag{1.1}$$

The sub-levels of J_ρ, which will play, as anticipated, an essential role throughout most of the paper, will be denoted as

$$J_\rho^a = \left\{u \in H^1(\Sigma)^N : J_\rho(u) \le a\right\}. \tag{1.2}$$

We will denote with the symbol $X \simeq Y$ a homotopy equivalence between two topological spaces X and Y.
The composition of two homotopy equivalences $F_1 : X\times[0, 1] \to Y$ and $F_2 : Y \times [0, 1] \to Z$ satisfying $F_1(\cdot, 1) = F_2(\cdot, 0)$ is the map $F_2 * F_1 : X \times [0, 1] \to Z$ defined by

$$F_2 * F_1 : (x, s) \mapsto \begin{cases} F_1(x, 2s) & \text{if } s \le \frac{1}{2} \\ F_2(x, 2s-1) & \text{if } s > \frac{1}{2} \end{cases}.$$

The identity map on X will be denoted as Id_X.
We will denote the q^{th} homology group with coefficient in $\mathbb{Z}$ of a topological space X as $H_q(X)$. An isomorphism between two homology groups will be denoted just by equality sign.
Reduced homology groups will be denoted as $\widetilde{H}_q(X)$, namely

$$H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}, \qquad H_q(X) = \widetilde{H}_q(X) \quad \text{if } q \ge 1.$$

The q^{th} Betti number of X, namely the dimension of its q^{th} group of homology, will be indicated by $b_q(X) := \mathrm{rank}(H_q(X))$.
The symbol $\widetilde{b}_q(X)$ will stand for the q^{th} reduced Betti number, namely the dimension of $\widetilde{H}_q(X)$, that is

$$\widetilde{b}_0(X) = b_0(X) - 1, \qquad \widetilde{b}_q(X) = b_q(X) \quad \text{if } q \ge 1.$$

If J_ρ is a Morse function, we will denote as $\mathcal{C}_q(a,b)$ the number of critical points u of J_ρ with Morse index q satisfying $a \le J_\rho(u) \le b$. The total number of critical points of index q will be denoted as $\mathcal{C}_q$; in other words, $\mathcal{C}_q := \mathcal{C}_q(+\infty, -\infty)$.

We will indicate with the letter C large constants which can vary among different lines and formulas. To underline the dependence of C on some parameter α, we indicate with C_α and so on.
We will denote as $o_\alpha(1)$ quantities which tend to 0 as α tends to 0 or to $+\infty$ and we will similarly indicate bounded quantities as $O_\alpha(1)$, omitting in both cases the subscript(s) when it is evident from the context.

1.2 Compactness results

First of all, we need two results from Brezis and Merle [16].
The first is a classical estimate about exponential integrability of solutions of some elliptic PDEs.

Lemma 1.1. *([16], Theorem 1, Corollary 1)*
Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $f \in L^1(\Omega)$ be with $\|f\|_{L^1(\Omega)} < 4\pi$ and u be the solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}.$$

Then, for any $q < \frac{4\pi}{\|f\|_{L^1(\Omega)}}$ there exists a constant $C = C_{q,\mathrm{diam}(\Omega)}$ such that $\int_\Omega e^{q|u(x)|}\mathrm{d}x \le C$. Moreover, $e^{|u|} \in L^q(\Omega)$ for any $q < +\infty$.

The second result we need, which has been extended in [5, 6], is a concentration-compactness theorem for scalar Liouville-type equations, which can be seen as a particular case of the one which will be proved in Chapter 2:

Theorem 1.2. *([16], Theorem 3; [6], Theorem 5; [5], Theorem 2.1)*
Let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of solutions of (6) with $\rho^n \underset{n\to+\infty}{\longrightarrow} \mathbb{R}_{>0}$ and $\widetilde{h}^n = V^n\widetilde{h}$ with $V^n \underset{n\to+\infty}{\longrightarrow} 1$ in $C^1(\Sigma)$ and $\mathcal{S}$ be defined by

$$\mathcal{S} := \left\{ x \in \Sigma : \exists x^n \underset{n\to+\infty}{\longrightarrow} x \text{ such that } u^n(x^n) - \log\int_\Sigma \widetilde{h}^n e^{u^n} \underset{n\to+\infty}{\longrightarrow} +\infty \right\}. \tag{1.3}$$

Then, up to subsequences, one of the following occurs:

- *(Compactness) If $\mathcal{S} = \emptyset$, then $u^n - \log\int_\Sigma \widetilde{h}^n e^{u^n}\mathrm{d}V_g$ converges to some u in $W^{2,q}(\Sigma)$.*
- *(Concentration) If $\mathcal{S} \neq \emptyset$, then it is finite and $u^n - \log\int_\Sigma \widetilde{h}^n e^{u^n}\mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} -\infty$ in $L^\infty_{\mathrm{loc}}(\Sigma\setminus\mathcal{S})$.*

Let us now report the known local blow-up quantization results for the systems (6), (9), (10), (11).

Theorem 1.3. *([49]; [48], Theorem 0.2; [6], Theorem 6; [5], Theorem 2.3)*
Let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of solutions of (6) with $\rho = \rho^n$, let $\mathcal{S}$ be defined by (1.3) and let, for $x \in \mathcal{S}$, $\sigma(x)$ be defined (up to subsequences) by

$$\sigma(x) := \lim_{r\to0}\lim_{n\to+\infty} \rho^n \frac{\int_{B_r(x)} \widetilde{h}^n e^{u^n}\mathrm{d}V_g}{\int_\Sigma \widetilde{h}^n e^{u^n}\mathrm{d}V_g}.$$

If $x \notin \{p_1, \dots, p_M\}$, then $\sigma(x) = 8\pi$, whereas $\sigma(p_m) = 8\pi(1+\alpha_m)$.

Corollary 1.4.
Let $\Gamma = \Gamma_{\underline{\alpha}}$ be defined by

$$\Gamma := 8\pi \left\{ n + \sum_{m \in \mathcal{M}} (1+\alpha_m),\ n \in \mathbb{N},\ \mathcal{M} \subset \{1, \dots, M\} \right\}.$$

Then, the family of solutions $\{u_\rho\}_{\rho \in \mathcal{K}} \subset \overline{H}^1(\Sigma)$ of (6) is uniformly bounded in $W^{2,q}(\Sigma)$ for some $q > 1$ for any given $\mathcal{K} \Subset \mathbb{R}_{>0} \setminus \Gamma$.

Proof.
Take $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho \in \mathcal{K}$ and apply Lemma 1.2 to $u^n = u_{\rho^n}$.
If *Concentration* occurred, then we can easily see that $\rho^n \dfrac{\widetilde{h}^n e^{u^n}}{\int_\Sigma \widetilde{h}^n e^{u^n} \mathrm{d}V_g} \underset{n\to+\infty}{\rightharpoonup} \sum_{x \in \mathcal{S}} \sigma(x)\delta_x$, hence $\rho = \sum_{x \in \mathcal{S}} \sigma(x) \in \Gamma$, which is a contradiction since we assumed $\rho \in \mathcal{K} \subset \mathbb{R}_{>0} \setminus \Gamma$.
Therefore, we must have *Compactness* for $u^n - \log \int_\Sigma \widetilde{h}^n e^{u^n} \mathrm{d}V_g$.
If u^n were not bounded in $W^{2,q}(\Sigma)$, then $\left| \log \int_\Sigma \widetilde{h}^n e^{u^n} \mathrm{d}V_g \right| \underset{n\to+\infty}{\longrightarrow} \pm\infty$.
Anyway, Jensen inequality gives $\log \int_\Sigma \widetilde{h}^n e^{u^n} \mathrm{d}V_g \ge \int_\Sigma \log \widetilde{h}^n \mathrm{d}V_g \ge -C$. Moreover, $\log \int_\Sigma \widetilde{h}^n e^{u^n} \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} +\infty$ would imply $\inf_\Sigma u^n \underset{n\to+\infty}{\longrightarrow} +\infty$, in contradiction with $\int_\Sigma u^n \mathrm{d}V_g = 0$. $\square$

Definition 1.5.
Let (α_1, α_2) be a couple of numbers greater than -1 and let $\Delta_{\alpha_1,\alpha_2} \subset \mathbb{R}^2$ as the piece of ellipse defined by

$$\Delta_{\alpha_1,\alpha_2} = \left\{ (\sigma_1, \sigma_2) \subset \mathbb{R}^2_{\ge 0} :\ \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 - 4\pi(1+\alpha_1)\sigma_2 - 4\pi(1+\alpha_2)\sigma_2 = 0 \right\}$$

We then define iteratively the finite set $\Xi_{\alpha_1,\alpha_2} \subset \Delta_{\alpha_1,\alpha_2}$ via the following rules:

- *Ξ_{α_1,α_2} contains the points*

$$\begin{array}{cccc} (0,0) & 4\pi((1+\alpha_1),0) & (0, 4\pi(1+\alpha_2)) & (4\pi(2+\alpha_1+\alpha_2), 4\pi(1+\alpha_2)) \\ & (4\pi(1+\alpha_1), 4\pi(2+\alpha_1+\alpha_2)) & & (4\pi(2+\alpha_1+\alpha_2), 4\pi(2+\alpha_1+\alpha_2)). \end{array} \tag{1.4}$$

- *If $(\sigma_1, \sigma_2) \in \Xi_{\alpha_1,\alpha_2}$, then any $(\sigma_1', \sigma_2') \in \Delta_{\alpha_1,\alpha_2}$ with $\sigma_1' = \sigma_1 + 4\pi n$ for some $n \in \mathbb{N}$ and $\sigma_2' \ge \sigma_2$ belongs to Ξ_{α_1,α_2}.*
- *If $(\sigma_1, \sigma_2) \in \Xi_{\alpha_1,\alpha_2}$, then any $(\sigma_1', \sigma_2') \in \Delta_{\alpha_1,\alpha_2}$ with $\sigma_2' = \sigma_2 + 4\pi n$ for some $n \in \mathbb{N}$ and $\sigma_1' \ge \sigma_1$ belongs to Ξ_{α_1,α_2}.*

Theorem 1.6. *([43], Proposition 2.4; [53], Theorem 1.1)*
Let $\{u^n = (u_1^n, u_2^n)\}_{n\in\mathbb{N}}$ be a sequence of solutions of (9) with $(\rho_1, \rho_2) = (\rho_1^n, \rho_2^n)$ and let, for $x \in \Sigma$, $\sigma(x) = (\sigma_1(x), \sigma_2(x))$ be defined (up to subsequences) by

$$\sigma_i(x) := \lim_{r\to 0} \lim_{n\to+\infty} \rho_i^n \frac{\int_{B_r(x)} \widetilde{h}_i^n e^{u_i^n} \mathrm{d}V_g}{\int_\Sigma \widetilde{h}_i^n e^{u_i^n} \mathrm{d}V_g}. \tag{1.5}$$

If $x \notin \{p_1, \dots, p_M\}$, then $\sigma(x) \in \Xi_{0,0}$, whereas $\sigma(p_m) \in \Xi_{\alpha_{1m},\alpha_{2m}}$.

Remark 1.7.
Notice that, if either $\alpha_1 = \alpha_2 = 0$ or they are both small enough, then Ξ_{α_1,α_2} contains only the set Ξ'_{α_1,α_2} of six points defined in (1.4).
The authors announced ([75]) that they refined the previous result by proving that $\sigma(p_m) \in \Xi'_{\alpha_{1m},\alpha_{2m}}$ if $\alpha_{1m}, \alpha_{2m} \le C$ for some $C > 0$. They also conjectured that $\sigma(p_m) \in \Xi'_{\alpha_{1m},\alpha_{2m}}$ for any α_{1m}, α_{2m}.
In [64], the authors prove that, in the regular case, all the values in $\Xi_{0,0} = \Xi'_{0,0}$ can be attained in case of blow-up (see also [31, 30]).

Theorem 1.8. *([51])*
Let $\{u^n = (u_1^n, u_2^n)\}_{n\in\mathbb{N}}$ be a sequence of solutions of (10) *with $(\rho_1, \rho_2) = (\rho_1^n, \rho_2^n)$ and let $\sigma(x)$ be defined by* (1.5).
For any $x \in \Sigma$:

$$\sigma(x) \in 4\pi\{(0,0),\ (1,0),\ (0,1),\ (1,3),\ (2,1),\ (3,3),\ (2,4),\ (3,4)\}.$$

Let $\{u^n = (u_1^n, u_2^n)\}_{n\in\mathbb{N}}$ be a sequence of solutions of (11) *with $(\rho_1, \rho_2) = (\rho_1^n, \rho_2^n)$ and let $\sigma(x)$ be defined by* (1.5).
If $\rho_1^n < 4\pi\left(2 + \sqrt{2}\right)$, $\rho_2^n < 4\pi\left(5 + \sqrt{7}\right)$, then for any $x \in \Sigma$:

$$\sigma(x) \in 4\pi\{(0,0),\ (1,0),\ (0,1),\ (1,4),\ (2,1),\ (2,6)\}.$$

Let us now state a couple of Lemmas from [56] concerning deformations of sub-levels.

Lemma 1.9. *([56], Proposition 1.1)*
Let $\rho \in \mathbb{R}^N_{>0}$, $a, b \in \mathbb{R}$ be given with $a < b$ and let J_ρ^a, J_ρ^b be defined by (1.2).
Then, one of the following alternatives occurs:

- *There exists a sequence $\{u^n\}_{n\in\mathbb{N}}$ of solutions of* (3) *with $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho$ and $a \le J_{\rho^n}(u^n) \le b$,*
- *J_ρ^a is a deformation retract of J_ρ^b.*

Corollary 1.10.
Let $\rho \in \mathbb{R}^N_{>0}$ be given and let J_ρ^L be defined by (1.2).
Then, one of the following alternatives occurs:

- *There exists a sequence $\{u^n\}_{n\in\mathbb{N}}$ of solutions of* (3) *with $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho$ and $J_{\rho^n}(u^n) \underset{n\to+\infty}{\longrightarrow} +\infty$,*
- *There exists $L > 0$ such that J_ρ^L is a deformation retract of $\overline{H}^1(\Sigma)^N$. In particular, J_ρ^L is contractible.*

In the scalar case the global compactness result (1.4) holds, hence for $\rho \notin \Gamma$ we can act as if Palais-Smale condition holds:

Corollary 1.11.
Let Γ be as in Corollary 1.4, $\rho \in \mathbb{R}_{>0} \setminus \Gamma, a, b \in \mathbb{R}$ be given with $a < b$ and such that (6) *has no solutions with $a \le I_\rho \le b$.*
Then, I_ρ^a is a deformation retract of I_ρ^b.
Moreover, there exists $L > 0$ such that I_ρ^L is contractible.

Proof.
If $\rho \notin \Gamma$ then the second alternative must occur in Lemma 1.9, since the first alternative would give, by Corollary 1.4, $u^n \underset{n\to+\infty}{\longrightarrow} u$ which solves (6) and satisfies $a \le I_\rho(u) \le b$.
Moreover, by Corollary 1.4, we have $\|u^n\|_{H^1(\Sigma)} \le C$ for any solution $u^n \in \overline{H}^1(\Sigma)$ of (6) with $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho$, therefore by Jensen's inequality every solution of (6) verifies

$$I_{\rho^n}(u^n) \le \frac{1}{2}\int_\Sigma |\nabla u^n|^2 \mathrm{d}V_g - \rho\int_\Sigma \log\widetilde{h}\mathrm{d}V_g \le \frac{C^2}{2} - \rho\int_\Sigma \log\widetilde{h}\mathrm{d}V_g =: L;$$

Corollary 1.10 gives the last claim. $\square$

1.3 Analytical preliminaries and Moser-Trudinger inequalities

To study the concentration phenomena of solutions of (3) we will use the following simple but useful calculus Lemma:

Lemma 1.12. *([44], Lemma 4.4)*
Let $\{a^n\}_{n\in\mathbb{N}}$ and $\{b^n\}_{n\in\mathbb{N}}$ be two sequences of real numbers satisfying

$$a^n \underset{n\to+\infty}{\longrightarrow} +\infty, \qquad \lim_{n\to+\infty}\frac{b^n}{a^n} \le 0.$$

Then, there exists a smooth function $F : [0, +\infty) \to \mathbb{R}$ which satisfies, up to subsequences,

$$0 < F'(t) < 1 \quad \forall\, t > 0, \qquad F'(t) \underset{t\to+\infty}{\longrightarrow} 0, \qquad F(a^n) - b^n \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Now we recall the Moser-Trudinger inequality for the scalar Liouville equation.

Theorem 1.13. *([38], Theorem 1.7; [63], Theorem 2; [72], Corollary 9)*
There exists $C > 0$ such that for any $u \in H^1(\Sigma)$

$$\log\int_\Sigma \widetilde{h}e^u \mathrm{d}V_g - \int_\Sigma u\mathrm{d}V_g \le \frac{1}{16\pi\min\{1, 1+\min_m \alpha_m\}}\int_\Sigma |\nabla u|^2\mathrm{d}V_g + C. \tag{1.6}$$

In other words, the functional I_ρ defined in (8) is bounded from below if and only if $\rho \le 8\pi\min\left\{1, 1+\min_m \alpha_m\right\}$ and it is coercive on $\overline{H}^1(\Sigma)$ if and only if $\rho < 8\pi\min\left\{1, 1+\min_m \alpha_m\right\}$.
In particular, in the latter case I_ρ has a global minimizer which solves (6).

We will also need a similar inequality by Adimurthi and Sandeep [1], holding on Euclidean domains, and its straightforward corollary.

Theorem 1.14. *([1], Theorem 2.1)*
Let $r > 0$, $\alpha \in (-1, 0]$ be given.
Then, there exists a constant $C = C_{\alpha,r}$ such that for any $u \in H^1_0(B_r(0))$

$$\int_{B_r(0)} |\nabla u(x)|^2\mathrm{d}x \le 1 \quad \Rightarrow \quad \int_{B_r(0)} |x|^{2\alpha}e^{4\pi(1+\alpha)u(x)^2}\mathrm{d}x \le C$$

Corollary 1.15.
Let $r > 0$, $\alpha \in (-1, 0]$ be given.
Then, there exists a constant $C = C_{\alpha,r}$ such that for any $u \in H_0^1(B_r(0))$

$$(1+\alpha)\log \int_{B_r(0)} |x|^{2\alpha} e^{u(x)} \mathrm{d}x \le \frac{1}{16\pi} \int_{B_r(0)} |\nabla u(x)|^2 \mathrm{d}x + C$$

Proof.
By the elementary inequality $u \le \theta u^2 + \frac{1}{4\theta}$ with $\theta = \frac{4\pi(1+\alpha)}{\int_\Omega |\nabla u(y)|^2 \mathrm{d}y}$ we get

$$\begin{aligned}
& (1+\alpha)\log \int_\Omega |x|^{2\alpha} e^{u(x)} \mathrm{d}x \\
\le \quad & (1+\alpha)\log \int_\Omega |x|^{2\alpha} e^{\theta u(x)^2 + \frac{1}{4\theta}} \mathrm{d}x \\
\le \quad & \frac{1}{16\pi} \int_\Omega |\nabla u(y)|^2 \mathrm{d}y + (1+\alpha)\log \int_\Omega |x|^{2\alpha} e^{4\pi(1+\alpha)\left(\frac{u(x)}{\sqrt{\int_\Omega |\nabla u(y)|^2 \mathrm{d}y}}\right)^2} \mathrm{d}x \\
\le \quad & \frac{1}{16\pi} \int_\Omega |\nabla u(y)|^2 \mathrm{d}y + C.
\end{aligned}$$

□

Let us now state the Moser-Trudinger inequality for the regular $SU(3)$ Toda system:

Theorem 1.16. *([44], Theorem 1.3)*
There exists $C > 0$ such that for any $u = (u_1, u_2) \in H^1(\Sigma)$:

$$\sum_{i=1}^{2} \left(\log \int_\Sigma e^{u_i} \mathrm{d}V_g - \int_\Sigma u_i \mathrm{d}V_g \right) \le \frac{1}{4\pi} \int_\Sigma Q_{A_2}(u) \mathrm{d}V_g + C.$$

In other words, the functional J_ρ defined by (5), in the case $A = A_2$, $\alpha_{im} \ge 0$, is bounded from below on $H^1(\Sigma)$ if and only if $\rho_1, \rho_2 \le 4\pi$ and it is coercive on $\overline{H}^1(\Sigma)^2$ if and only if $\rho_1, \rho_2 < 4\pi$. In particular, in the latter case I_ρ has a global minimizer which solves (9).

From this result, we deduce a Moser-Trudinger inequality for the $SU(3)$ Toda system on domains with boundary. This can be seen as a generalization of the well-known scalar Moser-Trudinger inequalities on Euclidean domains from [23], which we report:

Theorem 1.17.
Let $\Omega \subset \mathbb{R}^2$ be a smooth simply connected domain.
Then, there exists $C > 0$ such that, for any $u \in H^1(\Omega)$,

$$\log \int_\Omega e^{u(x)} \mathrm{d}x - \fint_\Omega u(x) \mathrm{d}x \le \frac{1}{8\pi} \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + C. \tag{1.7}$$

Before stating the inequality for the $SU(3)$ Toda, we introduce a class of smooth open subset of Σ which satisfy an *exterior and interior sphere condition* with radius $\delta > 0$:

$$\mathfrak{A}_\delta := \left\{ \Omega \subset \Sigma : \forall x \in \partial\Omega \, \exists x' \in \Omega, \, x'' \in \Sigma \setminus \overline{\Omega} : \, x = \overline{B_\delta(x')} \cap \partial\Omega = \overline{B_\delta(x'')} \cap \partial\Omega \right\} \tag{1.8}$$

Theorem 1.18.
There exists $C > 0$ such that, for any $u \in H^1(B_1(0))^2$,

$$\sum_{i=1}^{2}\left(\log\int_{B_1(0)} e^{u_i(x)}\mathrm{d}x - \fint_{B_1(0)} u_i(x)\mathrm{d}x\right) \le \frac{1}{2\pi}\int_{B_1(0)} Q_{A_2}(u(x))\mathrm{d}x + C.$$

The same result holds if $B_1(0)$ is replaced with a simply connected domain belonging to $\mathfrak{A}_\delta$ for some $\delta > 0$, with the constant C is replaced with some $C_\delta > 0$.

Sketch of the proof.
Consider a conformal diffeomorphism from $B_1(0)$ to the unit upper half-sphere and reflect the image of u through the equator.
Now, apply the Moser-Trudinger inequality to the reflected u', which is defined on $\mathbb{S}^2$. The Dirichlet integral of u' will be twice the one of u on $B_1(0)$, while the average and the integral of $e^{u'}$ will be the same, up to the conformal factor. Therefore the constant 4π is halved to 2π.
Starting from a simply connected domain, one can exploit the Riemann mapping theorem to map it conformally on the unit ball and repeat the same argument. The exterior and interior sphere condition ensures the boundedness of the conformal factor. $\square$

In Chapter 3, we will need to combine different type of Moser-Trudinger inequalities.
To do this, we will need the following technical estimates concerning averages of functions on balls and their boundary:

Lemma 1.19.
There exists $C > 0$ such that for any $u \in H^1(\Sigma)$, $x \in \Sigma$, $r > 0$ one has

$$\left|\fint_{B_r(x)} u\mathrm{d}V_g - \fint_{\partial B_r(x)} u\mathrm{d}V_g\right| \le C\sqrt{\int_{B_r(x)} |\nabla u|^2\mathrm{d}V_g}.$$

Moreover, for any $R > 1$ there exists $C = C_R$ such that

$$\left|\fint_{B_r(x)} u\mathrm{d}V_g - \fint_{B_{Rr}(x)} u\mathrm{d}V_g\right| \le C\sqrt{\int_{B_r(x)} |\nabla u|^2\mathrm{d}V_g}.$$

The same inequalities hold if $B_r(x)$ is replaced by a domain $\Omega \subset B_{Rr}(x)$ such that $\Omega \in \mathfrak{A}_{\delta r}$ for some $\delta > 0$, with the constants C and C_R replaced by some $C_\delta, C_{R,\delta} > 0$, respectively.

The proof of the above Lemma follows by the Poincaré-Wirtinger and trace inequalities, which are invariant by dilation. Details can be found, for instance, in [39].

We will also need the following estimate on harmonic liftings.

Lemma 1.20.
Let $r_2 > r_1 > 0$, $f \in H^1(B_{r_2}(0))$ with $\displaystyle\int_{B_{r_1}(0)} f(x)dx = 0$ be given and u be the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } A_{r_1,r_2}(0) \\ u = f & \text{on } \partial B_{r_1}(0) \\ u = 0 & \text{on } \partial B_{r_2}(0) \end{cases}.$$

Then, there exists $C = C_{\frac{r_2}{r_1}} > 0$ such that

$$\int_{A_{r_1,r_2}(0)} |\nabla u(x)|^2\mathrm{d}x \le C\int_{A_{r_1,r_2}(0)} |\nabla f(x)|^2\mathrm{d}x$$

Finally, we give a result concerning entire solution of singular Liouville systems, which will be used to prove the non-existence results in Chapter 4.
Unlike the previously stated results, this one has not been published, up to our knowledge, nor it follows straightforwardly from any known results. Anyway, it can be proved similarly as in the scalar case ([25], Theorem 1).
As anticipated in the introduction, the proof will be postponed to the Appendix.

Theorem 1.21.
Let $H_1,\dots,H_N \in C^1_{\mathrm{loc}}\left(\mathbb{R}^2 \setminus \{0\}\right)$ be such that, for suitable $a, c \geq 0$, $b > -2$, $C > 0$,

$$\frac{|x|^a}{C} \leq H_i(x) \leq C|x|^b \quad \forall\, x \in B_1(0) \setminus \{0\} \qquad\qquad 0 < H_i(x) \leq C|x|^c \quad \forall\, x \in \mathbb{R}^2 \setminus B_1(0);$$

let $U = (U_1,\dots,U_N)$ be a solution of

$$\begin{cases} -\Delta U_1 = \displaystyle\sum_{j=1}^N a_{ij} H_j e^{U_j} & \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} \left(|x|^b + |x|^c\right) e^{U_i(x)} \mathrm{d}x < +\infty \end{cases}, \qquad i = 1,\dots,N \tag{1.9}$$

and define

$$\rho_i := \int_{\mathbb{R}^2} H_i(x) e^{U_i(x)} \mathrm{d}x, \qquad \tau_i := \int_{\mathbb{R}^2} (x \cdot \nabla H_i(x)) e^{U_i(x)} \mathrm{d}x, \qquad i = 1, 2.$$

Then,

$$\sum_{i,j=1}^N a_{ij}\rho_i\rho_j - 4\pi \sum_{i=1}^N (2\rho_i + \tau_i) = 0. \tag{1.10}$$

1.4 Topological preliminaries, homology and Morse theory

We start with a simple fact from general topology, which anyway will be essential in most of Chapter 3:

Lemma 1.22.
Let Σ be a compact surface with $\chi(\Sigma) \leq 0$ and $\left\{p'_{01},\dots,p'_{0M'_0},p'_{11},\dots,p'_{1M'_1},p'_{21},\dots,p'_{2M'_1}\right\}$ be given points of Σ.
Then, there exist two curves γ_1, γ_2, each of which is homeomorphic to a bouquet of $1 + \left[\frac{-\chi(\Sigma)}{2}\right]$ circles and two global projections $\Pi_i : \Sigma \to \gamma_i$ such that:

- $\gamma_1 \cap \gamma_2 = \emptyset$.
- $p'_{im} \in \gamma_i$ *for all* $m = 1,\dots,M'_i$, $i = 1, 2$.
- $p'_{0m} \notin \gamma_i$ *for all* $m = 1,\dots,M'_0$, $i = 1, 2$.

Sketch of the proof.
If $\Sigma = \mathbb{T}^g$ is a g-torus, two retractions on disjoint bouquets of g circles can be easily built.
For instance, as in [3], Σ can be assumed to be embedded in $\mathbb{R}^3$ in such a way that each hole contains a line parallel to the x_3 axis and that the projection P_i on each plane $\left\{x_3 = (-1)^{i+1}\right\}$ is a disk with

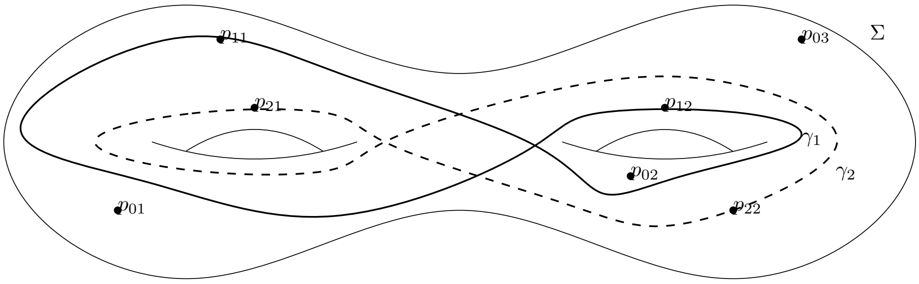

Figure 1.1: The curves γ_1, γ_2

g holes. Then, there exists two bouquets of circles $\gamma_i \subset \Sigma$ such that $P_i|_{\gamma_i}$ are homeomorphisms and there exists retractions $r_i : P_i(\Sigma) \to P_i(\gamma_i)$, and we suffice to define $\Pi_i := {P_i|_{\gamma_i}}^{-1} \circ r_i \circ P_i$.
One can argue similarly with a connected sum $\Sigma = \mathbb{P}^{2k}$ of an even number of copies of the projective plane, since this is homeomorphic to a connected sum of a $\mathbb{T}^{k-1}$ and a Klein bottle, which in turn retracts on a circle; therefore, $\mathbb{P}^{2k}$ retracts on two disjoint bouquets of k circles.
If instead Σ is a connected sum of an odd number of projective planes, one can argue as before setting the retractions constant on the last copy of $\mathbb{P}$.
In all these cases,

$$g = 1 + \frac{-\chi(\mathbb{T}^g)}{2}, \qquad k = 1 + \frac{-\chi\left(\mathbb{P}^{2k}\right)}{2} = 1 + \left[\frac{-\chi\left(\mathbb{P}^{2k+1}\right)}{2}\right].$$

Finally, with a small deformation, the curves γ_i can be assumed to contain all the points p'_{im} and they will not contain any of the other singular points. We can apply those deformations to γ_1 without intersecting γ_2 (or vice versa) because $\Sigma \setminus \gamma_2$ is pathwise connected.
See Figure 1.1 for an example. $\square$

In Chapter 3 we will often have to deal with the space $\mathcal{M}(\Sigma)$ of Radon measures defined on Σ, especially unit measures.
Such a space will be endowed with the Lip' topology, that is with the norm of the dual space of Lipschitz functions:

$$\|\mu\|_{\mathrm{Lip}'(\Sigma)} := \sup_{\phi \in \mathrm{Lip}(\Sigma), \|\phi\|_{\mathrm{Lip}(\Sigma)} \le 1} \left| \int_\Sigma \phi \mathrm{d}\mu \right|. \tag{1.11}$$

As a choice of this motivation, notice that by choosing, in (1.11), $\phi = d(\cdot, y)$, one gets $d_{\mathrm{Lip}'}(\delta_x, \delta_y) \sim d(x, y)$ for any $x, y \in \Sigma$. This means that $\mathcal{M}(\Sigma)$ contains a homeomorphic copy of Σ.
Moreover, one can see immediately that $L^1_{>0}(\Sigma)$ embeds into $\mathcal{M}(\Sigma)$.

Concerning $u \in H^1(\Sigma)^N$, there is a natural way to associate to any u, through the system (3), a N-tuple of positive normalized L^1 functions, that is N elements of the space

$$\mathcal{A} := \left\{ f \in L^1(\Sigma) : f > 0 \text{ a.e. and } \int_\Sigma f \mathrm{d}V_g = 1 \right\}. \tag{1.12}$$

Precisely, we define

$$(u_1, \dots, u_N) \mapsto \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} \mathrm{d}V_g}, \dots, \frac{\widetilde{h}_N e^{u_N}}{\int_\Sigma \widetilde{h}_N e^{u_N} \mathrm{d}V_g} \right) =: (f_{1,u}, \dots, f_{N,u}). \tag{1.13}$$

Lemma 1.23.
Let $\mathcal{A}$ be defined by (1.12) and $f_{i,u}$ be defined by (1.13).
Then, the map $u \mapsto f_{i,u}$ is weakly continuous for $i = 1, \dots, N$.

Proof.
It will suffice to give the proof for the index $i = 1$, which we will omit. We will just need to prove the continuity of the map $u \mapsto \widetilde{h}e^u$, since dividing a non-zero element by his norm is a continuous operation in any normed space.
Let $u^n \in H^1$ be converging weakly, and strongly in any $L^p(\Sigma)$, to u, and fix $q > 1$ in such a way that $q\alpha_m > -1\,\forall\, m$.
From the elementary inequality $e^t - 1 \le |t|e^{|t|}$ we get, by Lemma 1.13 and Hölder's inequality,

$$
\begin{aligned}
& \left|\int_\Sigma \widetilde{h}e^{u^n}\mathrm{d}V_g - \int_\Sigma \widetilde{h}e^{u}\mathrm{d}V_g\right| \\
\le\ & \int_\Sigma \widetilde{h}e^u\left|e^{u^n-u}-1\right|\mathrm{d}V_g \\
\le\ & \int_\Sigma \widetilde{h}e^u\left|u^n-u\right|e^{|u^n-u|}\mathrm{d}V_g \\
\le\ & \left(\int_\Sigma \widetilde{h}^q e^{qu}\mathrm{d}V_g\right)^{\frac{1}{q}}\left(\int_\Sigma |u^n-u|^{\frac{2q}{q-1}}\mathrm{d}V_g\right)^{1-\frac{1}{2q}}\left(\int_\Sigma e^{\frac{2q}{q-1}|u^n-u|}\mathrm{d}V_g\right)^{1-\frac{1}{2q}} \\
\le\ & \left(Ce^{q\int_\Sigma u\mathrm{d}V_g+\frac{q^2}{16\pi\min\{1,1+q\min_m\alpha_m\}}\int_\Sigma|\nabla u|^2\mathrm{d}V_g}\right)^{\frac{1}{q}}\left(\int_\Sigma |u^n-u|^{\frac{2q}{q-1}}\mathrm{d}V_g\right)^{1-\frac{1}{2q}}\cdot \\
\cdot\ & \left(Ce^{\frac{2q}{q-1}\int_\Sigma|u^n-u|\mathrm{d}V_g+\frac{q^2}{4\pi(q-1)^2}\int_\Sigma|\nabla(u^n-u)|^2\mathrm{d}V_g}\right)^{1-\frac{1}{2q}} \\
\le\ & C\left(\int_\Sigma |u^n-u|^{\frac{2q}{q-1}}\mathrm{d}V_g\right)^{1-\frac{1}{2q}} \\
\underset{n\to+\infty}{\longrightarrow}\ & 0.
\end{aligned}
$$

□

From the proof of the previous Lemma we deduce the following useful Corollary:

Corollary 1.24.
The functional $J_\rho : H^1(\Sigma)^N \to \mathbb{R}$ defined in (5) is of class C^1 and weakly lower semi-continuous.

Speaking about unit measures, a fundamental role will be played by the so-called *K-barycenters*, that is unit measures supported in at most K points of Σ, for some given K. They will be used in Chapter (3) to express the fact that $f_{i,u}$ concentrates around at most K points.
For a subset $X \subset \Sigma$, we define:

$$
(X)_K := \left\{\sum_{k=1}^K t_k\delta_{x_k} : x_k \in X,\, t_k \ge 0,\, \sum_{k=1}^K t_k = 1\right\}. \tag{1.14}
$$

If we choose X to be homeomorphic to a bouquet of g circles, such as for instance the curves γ_1, γ_2 defined in Lemma 1.22, the homology of the K- barycenters on X is well-known:

Proposition 1.25. *([3], Proposition 3.2)*
Let γ be a bouquet of g circles and let $(\gamma)_K$ be defined by (1.14).
Then, its homology groups are the following:

$$
H_q\left((\gamma)_K\right) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}^{\binom{K+g-1}{g-1}} & \text{if } q = 2K-1 \\ 0 & \text{if } q \ne 0, 2K-1 \end{cases}.
$$

Remark 1.26.
In Proposition 1.25, when $g = 1$ we get not only a homology equivalence but a homotopy equivalence between $\left(\mathbb{S}^1\right)_K$ and $\mathbb{S}^{2K-1}$.

We will also need a similar definition, which extends the K-barycenters defined before. We still consider unit measure with finite support, but we do not give a constraint on the number of the points, but rather on a *weight* defined on such points.
Given a set $X \subset \Sigma$, a finite number of points $p_1, \dots, p_M \in X$ and a multi-index $\underline{\alpha} = (\alpha_1, \dots, \alpha_M)$ with $-1 < \alpha_m < 0$ for any $m = 1, \dots, M$, we define the *weighted cardinality* $\omega_{\underline{\alpha}}$ as

$$\omega_{\underline{\alpha}}(\{x\}) := \begin{cases} 1 + \alpha_m & \text{if } x = p_m \\ 1 & \text{if } x \notin \{p_1, \dots, p_M\} \end{cases}, \qquad \omega_{\underline{\alpha}}\left(\bigcup_{x_k \in \mathcal{J}} \{x_k\}\right) := \sum_{x_k \in \mathcal{J}} \omega_{\underline{\alpha}}(\{x_k\}).$$

We then define the *weighted barycenters* on X with respect to the a parameter $\rho > 0$ and the multi-index $\underline{\alpha}$ as

$$(X)_{\rho,\underline{\alpha}} := \left\{ \sum_{x_k \in \mathcal{J}} t_k \delta_{x_k} : x_k \in X,\, t_k \geq 0,\, \sum_{x_k \in \mathcal{J}} t_k = 1,\, 4\pi\omega_{\underline{\alpha}}(\mathcal{J}) < \rho \right\}. \tag{1.15}$$

As a motivation for this weight, introduced in [21] to study the singular Liouville equation, consider inequality (1.6). In the case of no singularities, the constant multiplying $\int_\Sigma |\nabla u|^2 \mathrm{d}V_g$ is $\frac{1}{16\pi}$, and in case of one singular point p_m with $\alpha_m < 0$, that constant is $\frac{1}{16\pi(1 + \alpha_m)}$.
Roughly speaking, the weight of each point represents how much that point affects the Moser-Trudinger inequality (1.6).

The space of weighted barycenters can be in general more complicated than the non-weighted barycenters, which are a particular case given by defining $\omega_{\underline{\alpha}}(\{x\}) = 1$ for any $x \in X$ and K as the largest integer strictly smaller than $\frac{\rho}{4\pi}$.
Anyway, both the weighted and the non-weighted barycenters are stratified set, that is, roughly speaking, union of manifolds of different dimensions with possibly non-smooth gluings.
For this reason, they have the fundamental property of being a Euclidean Neighborhood Retract, namely a deformation retract of an open neighborhood of theirs.

Lemma 1.27. *([21], Lemma 3.12)*
Let, for $\rho \in \mathbb{R}_{>0}$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_M)$, $(\Sigma)_{\rho,\underline{\alpha}}$ be defined as in (1.15).
Then, there exists $\varepsilon_0 > 0$ and a continuous retraction

$$\psi_{\rho,\underline{\alpha}} : \left\{\mu \in \mathcal{M}(\Sigma) : d_{\mathrm{Lip}'}\left(\mu, (\Sigma)_{\rho,\underline{\alpha}}\right) < \varepsilon_0\right\} \to (\Sigma)_{\rho,\underline{\alpha}}$$

In particular, if $\mu^n \underset{n\to\infty}{\rightharpoonup} \sigma$ for some $\sigma \in (\Sigma)_{\rho,\underline{\alpha}}$, then $\psi_{\rho,\underline{\alpha}}(\mu^n) \underset{n\to\infty}{\longrightarrow} \sigma$.

Another tool which we will take from general topology is the *join* between two spaces X and Y, defined by

$$X \star Y := \frac{X \times Y \times [0,1]}{\sim} \tag{1.16}$$

where $\sim$ is the identification given by

$$(x, y, 0) \sim (x, y', 0) \quad \forall x \in X, \forall y, y' \in Y, \qquad (x, y, 1) \sim (x', y, 1) \quad \forall x, x' \in X, \forall y \in Y.$$

Basically, the join expresses a (non-exclusive) alternative between X and Y: if $t = 0$ we only "see" X and not Y, if $t = 1$ we see only Y and if $0 < t < 1$ we see both X and Y (for more details, see [40], page 9). This will be used in Chapter 3 as a model for the alternative between concentration of $f_{1,u}$ and $f_{2,u}$.
The homology of $X \star Y$ depends from the homology of X and Y through the following:

Proposition 1.28. *([40], Theorem 3.21)*
Let X and Y be two CW-complexes and $X \star Y$ be their join as in (1.16).
Then, its homology group are defined by

$$\widetilde{H}_q(X \star Y) = \bigoplus_{q'=0}^{q} \widetilde{H}_{q'}(X) \otimes \widetilde{H}_{q-q'-1}(Y).$$

In particular,

$$\widetilde{b}_q(X \star Y) = \sum_{q'=0}^{q} \widetilde{b}_{q'}(X) \widetilde{b}_{q-q'-1}(Y)$$

and

$$\sum_{q=0}^{+\infty} \widetilde{b}_q(X \star Y) = \sum_{q'=0}^{+\infty} \widetilde{b}_{q'}(X) \sum_{q''=0}^{+\infty} \widetilde{b}_{q''}(Y).$$

Remark 1.29.
By taking, in the previous Proposition, two wedge sum of spheres $X = \left(\mathbb{S}^{D_1}\right)^{\vee N_1}$ and $Y = \left(\mathbb{S}^{D_2}\right)^{\vee N_2}$, we find that $X \star Y$ has the homology of another wedge sum of spheres $\left(\mathbb{S}^{D_1+D_2+1}\right)^{\vee N_1 N_2}$.
In the same book [40] it is shown that actually a homotopically equivalence $\left(\mathbb{S}^{D_1}\right)^{\vee N_1} \star \left(\mathbb{S}^{D_2}\right)^{\vee N_2} \simeq \left(\mathbb{S}^{D_1+D_2+1}\right)^{\vee N_1 N_2}$ holds. In particular, from Remark 1.26, $\left(\mathbb{S}^1\right)_{K_1} \star \left(\mathbb{S}^1\right)_{K_2} \simeq \mathbb{S}^{2K_1+1} \star \mathbb{S}^{2K_2+1} \simeq \mathbb{S}^{2K_1+2K_2-1}$.

Remark 1.30.
Proposition 1.28 shows, in particular, that if both X and Y have some non-trivial homology, then the same is true for $X \star Y$.
It is easy to see that a partial converse holds, concerning contractibility rather than homology: if either X or Y is contractible, that $X \star Y$ is also contractible.
In fact, if F is a homotopy equivalence between X and a point, then

$$((x, y, t), s) \mapsto (F(x, s), y, t)$$

is a homotopical equivalence between $X \star Y$ and the cone based on Y, which is contractible.

Morse inequalities yield the following estimate on the number of solutions, through the Betti numbers of low sub-levels:

Lemma 1.31.
Let $\rho \in \mathbb{R}^2_{>0}$ be such that $J_{A_2,\rho}$ is a Morse functional and $\|u\|_{H^1(\Sigma)^2} \leq C$ for any solution $u \in \overline{H}^1(\Sigma)^2$ of (9).
Then, there exists $L > 0$ such that

$$\#\ \textit{solutions of } (9) \geq \sum_{q=0}^{+\infty} \widetilde{b}_q\left(J_{A_2,\rho}^{-L}\right).$$

The same result holds if A_2 is replace by B_2 or G_2.

Proof.
By Corollary 1.24, $-L < J_\rho(u) \leq L$ for some $L > 0$. In particular, $-L$ is a regular value for J_ρ, hence the exactness of the sequence

$$\cdots \to \widetilde{H}_q\left(J_\rho^{-L}\right) \to \widetilde{H}_q\left(J_\rho^{L}\right) \to H_q\left(J_\rho^{-L}, J_\rho^{L}\right) \to \widetilde{H}_{q-1}\left(J_\rho^{-L}\right) \to \widetilde{H}_{q-1}\left(J_\rho^{L}\right) \to \dots$$

reduces to

$$H_{q+1}\left(J_\rho^{-L}, J_\rho^{L}\right) = \widetilde{H}_q\left(J_\rho^{-L}\right).$$

Therefore, Morse inequalities give:

$$\#\text{ solutions of (10)} = \sum_{q=0}^{+\infty} \mathcal{C}_q\left(J_\rho\right) = \sum_{q=0}^{+\infty} \mathcal{C}_q\left(J_\rho; -L, L\right) \geq \sum_{q=0}^{+\infty} b_q\left(J_\rho^{-L}, J_\rho^{L}\right) = \sum_{q=0}^{+\infty} \widetilde{b}_q\left(J_\rho^{-L}\right).$$

□

Finally, we need a density result for J_ρ, given in [32] for I_ρ, which will be proved in the Appendix.

Theorem 1.32.
Let $\mathcal{M}^2(\Sigma)$ be the space of Riemannian metrics on Σ, equipped with the C^2 norm, and $\mathcal{M}_1^2(\Sigma)$ its subspaces of the metrics g satisfying $\int_\Sigma \mathrm{d}V_g = 1$.
Then, there exists dense open set $\mathcal{D} \subset \mathcal{M}^2(\Sigma) \times C^2_{>0}(\Sigma) \times C^2_{>0}(\Sigma)$, $\mathcal{D}_1 \subset \mathcal{M}_1^2(\Sigma) \times C^2_{>0}(\Sigma) \times C^2_{>0}(\Sigma)$ such that for any $(g, h_1, h_2) \in \mathcal{D} \cup \mathcal{D}_1$ the three of $J_{A_2,\rho}$, $J_{B_2,\rho}$, $J_{G_2,\rho}$ are all Morse functions from $\overline{H}^1(\Sigma)^2$ to $\mathbb{R}$.

Chapter 2

New Moser-Trudinger inequalities and minimizing solutions

This chapter will be devoted to proving two Moser-Trudinger inequalities for systems (3), namely to give conditions for the energy functional J_ρ to be coercive and bounded from below.

The first result gives a characterization of the values of ρ which yield coercivity for J_ρ and some necessary conditions for boundedness from below:

Theorem 2.1.
Define, for $\rho \in \mathbb{R}^N_{>0}$, $x \in \Sigma$ and $i \in \mathcal{I} \subset \{1, \dots, N\}$:

$$\alpha_i(x) = \begin{cases} \alpha_{im} & \text{if } x = p_m \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

$$\Lambda_{\mathcal{I},x}(\rho) := 8\pi \sum_{i\in\mathcal{I}} (1 + \alpha_i(x))\rho_i - \sum_{i,j\in\mathcal{I}} a_{ij}\rho_i\rho_j, \tag{2.2}$$

$$\Lambda(\rho) := \min_{\mathcal{I}\subset\{1,\dots,N\}, x\in\Sigma} \Lambda_{\mathcal{I},x}(\rho) \qquad . \tag{2.3}$$

Then, J_ρ is bounded from below if $\Lambda(\rho) > 0$ and it is unbounded from below if $\Lambda(\rho) < 0$.
Moreover, J_ρ is coercive in $\overline{H}^1(\Sigma)^N$ if and only if $\Lambda(\rho) > 0$. In particular, if this occurs, then it has a minimizer u which solves (3).

By this theorem, the values of ρ which yield coercivity belong to a region of the positive *orthant* which is delimited by hyperplanes and hypersurfaces, whose role will be clearer in the blow-up analysis which will be done in this chapter.
The coercivity region is shown in Figure 2.1:

Theorem 2.1 leaves an open question about what happens when $\Lambda(\rho) = 0$. In this case one encounters blow-up phenomena which are not yet fully known for general systems.
Anyway, we can say something more precise if we assume the matrix A to be non-positive outside its main diagonal. First of all, it is not hard to see that notice that in this case

$$\Lambda(\rho) = \min_{i\in\{1,\dots,N\}} \left(8\pi(1 + \widetilde{\alpha}_i)\rho_i - a_{ii}\rho_i^2\right),$$

where

$$\widetilde{\alpha}_i := \min_{m\in\{1,\dots,M\}, x\in\Sigma} \alpha_i(x) = \min\left\{0, \min_{m\in\{1,\dots,M\}} \alpha_{im}\right\}. \tag{2.4}$$

In particular, only the negative α_{im} play a role in the coercivity of J_ρ, like for the scalar case and unlike the general case in Theorem 2.1.

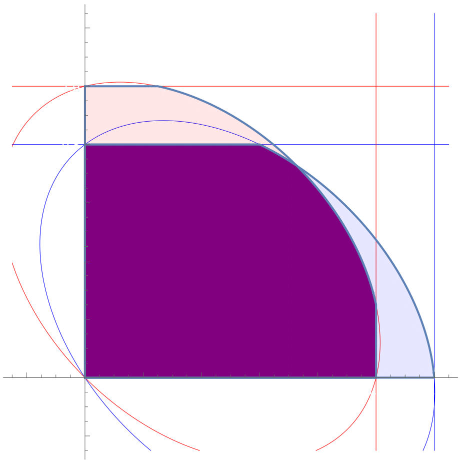

Figure 2.1: The set $\Lambda(\rho) > 0$, in the case $N = 2$.

Therefore, under these assumptions, the coercivity region is actually a rectangle and the sufficient condition in Theorem 2.1 is equivalent to assuming $\rho_i < \frac{8\pi(1+\widetilde{\alpha}_i)}{a_{ii}}$ for any i:

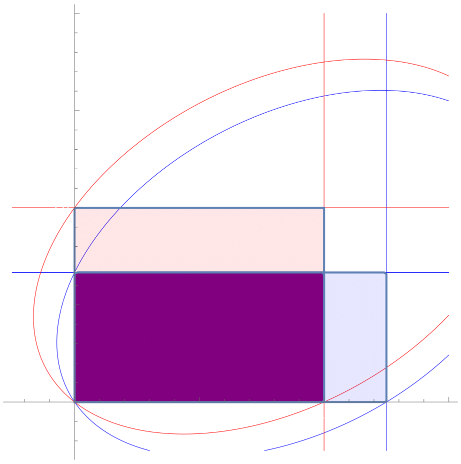

Figure 2.2: The set $\Lambda(\rho) > 0$, in the case $N = 2$, $a_{12} \leq 0$.

With this assumption, the blow-up analysis needed to study what happens when $\Lambda(\rho) = 0$ is locally one-dimensional, hence can be treated by using well-known scalar inequalities like Lemma 1.15. Therefore, we get the following sharp result:

Theorem 2.2.
Let $\Lambda(\rho)$ as in (2.3), $\widetilde{\alpha}_i$ as in (2.4) and suppose $a_{ij} \leq 0$ for any $i, j = 1, \dots, N$ with $i \neq j$. Then, J_ρ is bounded from below on $H^1(\Sigma)^N$ if and only if $\Lambda(\rho) \geq 0$, namely if and only if $\rho_i \leq \frac{8\pi(1+\widetilde{\alpha}_i)}{a_{ii}}$ for any $i = 1, \dots, N$. In other words, there exists $C > 0$ such that

$$\sum_{i=1}^{2} \frac{1+\widetilde{\alpha}_i}{a_{ii}} \left(\log \int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g - \int_\Sigma u_i \mathrm{d}V_g \right) \leq \frac{1}{8\pi} \int_\Sigma Q(u) \mathrm{d}V_g + C \tag{2.5}$$

Remark 2.3.
We remark that assuming A to be positive definite is necessary. If A is invertible but not symmetric

definite, then J_ρ is unbounded from below for any ρ.

In fact, suppose there exists $v \in \mathbb{R}^N$ such that $\sum_{i,j=1}^N a^{ij} v_i v_j \le -\theta |v|^2$ for some $\theta > 0$. Then, consider the family of functions $u^\lambda(x) := \lambda v \cdot x$.

By Jensen's inequality we get:

$$J_\rho\left(u^\lambda\right) \le \frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i^\lambda \cdot \nabla u_j^\lambda \mathrm{d}V_g - \sum_{i=1}^N \rho_i \int_\Sigma \log \widetilde{h}_i \mathrm{d}V_g \le -\frac{\theta}{2}\lambda^2|v|^2 + C \underset{n\to+\infty}{\longrightarrow} \infty.$$

The proofs of Theorems 2.1 and 2.2 will be given respectively in Sections 2.3 and 2.4.

2.1 Concentration-compactness theorem

The aim of this section is to prove a result which describes the concentration phenomena for the solutions of (3), extending what was done for the two-dimensional Toda system in [12, 57].

We actually have to normalize such solutions to bypass the issue of the invariance by translation by constants and to have the parameter ρ multiplying only the constant term.
In fact, for any solution u of (3) the functions

$$v_i := u_i - \log \int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g + \log \rho_i \tag{2.6}$$

solve

$$\begin{cases} -\Delta v_i = \displaystyle\sum_{j=1}^N a_{ij}\left(\widetilde{h}_j e^{v_j} - \rho_j\right) \\ \displaystyle\int_\Sigma \widetilde{h}_i e^{v_i} \mathrm{d}V_g = \rho_i \end{cases}, \qquad i = 1, \dots, N. \tag{2.7}$$

Moreover, we can rewrite in a shorter way the local blow-up masses defined in (1.5) as

$$\sigma_i(x) = \lim_{r\to 0}\lim_{n\to+\infty}\int_{B_r(x)} \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g.$$

For such functions, we get the following concentration-compactness alternative:

Theorem 2.4.
Let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of solutions of (3) with $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho \in \mathbb{R}^N_{>0}$ and $\widetilde{h}_i^n = V_i^n \widetilde{h}_i$ with $V_i^n \underset{n\to+\infty}{\longrightarrow} 1$ in $C^1(\Sigma)^N$, $\{v^n\}_{n\in\mathbb{N}}$ be defined as in (2.6) and $\mathcal{S}_i$ be defined, for $i = 1, \dots, N$, by

$$\mathcal{S}_i := \left\{ x \in \Sigma : \exists x^n \underset{n\to+\infty}{\longrightarrow} x \text{ such that } v_i^n\left(x^n\right) \underset{n\to+\infty}{\longrightarrow} +\infty \right\}. \tag{2.8}$$

Then, up to subsequences, one of the following occurs:

- *(Compactness) If $\mathcal{S} := \cup_{i=1}^N \mathcal{S}_i = \emptyset$, then $v^n \underset{n\to+\infty}{\longrightarrow} v$ in $W^{2,q}(\Sigma)^N$ for some $q > 1$ and some v which solves (2.7).*
- *(Concentration) If $\mathcal{S} \ne \emptyset$, then it is finite and*

$$\widetilde{h}_i^n e^{v_i^n} \underset{n\to+\infty}{\rightharpoonup} \sum_{x\in\mathcal{S}} \sigma_i(x)\delta_x + f_i$$

  as measures, with $\sigma_i(x)$ defined as in (2.8) and some $f_i \in L^1(\Sigma)$.
  In this case, for any given i, either $v_i^n \underset{n\to+\infty}{\longrightarrow} -\infty$ in $L^\infty_{\mathrm{loc}}(\Sigma \setminus \mathcal{S})$ and $f_i \equiv 0$, or $v_i^n \underset{n\to+\infty}{\longrightarrow} v_i$ in $W^{2,q}_{\mathrm{loc}}(\Sigma \setminus \mathcal{S})$ for some $q > 1$ and some suitable v_i and $f_i = \widetilde{h}_i e^{v_i} > 0$ a.e. on Σ.

Since $\widetilde{h}_j$ is smooth outside the points p_m's, the estimates in $W^{2,q}(\Sigma)$ are actually in $C^{2,\alpha}_{\mathrm{loc}}\left(\Sigma \setminus \bigcup_{m=1}^{M} p_m\right)$ and the estimates in $W^{2,q}_{\mathrm{loc}}(\Sigma \setminus \mathcal{S})$ are actually in $C^{2,\alpha}_{\mathrm{loc}}\left(\Sigma \setminus \left(\mathcal{S} \cup \bigcup_{m=1}^{M} p_m\right)\right)$.
Anyway, estimates in $W^{2,q}$ will suffice throughout all this Chapter.

To prove Theorem 2.4 we need two preliminary lemmas.
The first is a Harnack-type alternative for sequences of solutions of PDEs. It is inspired by [16, 57].

Lemma 2.5.
Let $\Omega \subset \Sigma$ be a connected open subset, $\{f^n\}_{n\in\mathbb{N}}$ a bounded sequence in $L^q_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$ for some $q > 1$ and $\{w^n\}_{n\in\mathbb{N}}$ bounded from above and solving $-\Delta w^n = f^n$ in Ω.
Then, up to subsequences, one of the following alternatives holds:

- *w^n is uniformly bounded in $L^\infty_{\mathrm{loc}}(\Omega)$.*
- *$w^n \underset{n\to+\infty}{\longrightarrow} -\infty$ in $L^\infty_{\mathrm{loc}}(\Omega)$.*

Proof.
Take a compact set $\mathcal{K} \Subset \Omega$ and cover it with balls of radius $\frac{\delta}{2}$, with δ smaller than the injectivity radius of Σ. By compactness, we can write $\mathcal{K} \subset \bigcup_{l=1}^{L} B_{\frac{\delta}{2}}(x_l)$.
If the second alternative does not occur, then up to re-labeling we get $\sup_{B_\delta(x_1)} w^n \geq -C$.
Then, we consider the solution z^n of

$$\begin{cases} -\Delta z^n = f^n & \text{in } B_\delta(x_1) \\ z^n = 0 & \text{on } \partial B_\delta(x_1) \end{cases},$$

which is bounded in $L^\infty(B_\delta(x_1))$ by elliptic estimates.
This means that, for a large constant C, the function $C - w^n + z^n$ is positive, harmonic and bounded from below on $B_\delta(x_1)$, and moreover its infimum is bounded from above. Therefore, applying the Harnack inequality for harmonic function (which is allowed since r is small enough) we get that $C - w^n + z^n$ is uniformly bounded in $L^\infty\left(B_{\frac{\delta}{2}}(x_1)\right)$, hence w^n is.
At this point, by connectedness, we can re-label the index l in such a way that $B_{\frac{\delta}{2}}(x_l) \cap B_{\frac{\delta}{2}}(x_{l+1}) \neq \emptyset$ for any $l = 1, \dots, L-1$ and we repeat the argument for $B_{\frac{\delta}{2}}(x_2)$. Since it has nonempty intersection with $B_{\frac{\delta}{2}}(x_1)$, we have $\sup_{B_\delta(x_2)} w^n \geq -C$, hence we get boundedness in $L^\infty\left(B_{\frac{\delta}{2}}(x_2)\right)$.
In the same way, we obtain the same result in all the balls $B_{\frac{\delta}{2}}(x_l)$, whose union contains $\mathcal{K}$; therefore w^n must be uniformly bounded on $\mathcal{K}$ and we get the conclusion. $\square$

The second Lemma basically says that if all the concentration values in a point are under a certain threshold, and in particular if all of them equal zero, then compactness occurs around that point.
On the other hand, if a point belongs to some set $\mathcal{S}_i$, then at least a fixed amount of mass has to accumulate around it; hence, being the total mass uniformly bounded from above, this can occur only for a finite number of points, so we deduce the finiteness of the $\mathcal{S}_i$'s.
Precisely, we have the following, inspired again by [57], Lemma 4.4:

Lemma 2.6.
Let $\{v^n\}_{n\in\mathbb{N}}$ and $\mathcal{S}_i$ be as in (2.8) and σ_i as in (1.5), and suppose $\sigma_i(x) < \sigma_i^0$ for any $i = 1, \dots, N$,

where

$$\sigma_i^0 := \frac{4\pi \min\left\{1, 1+\min_{j\in\{1,\dots,N\}, m\in\{1,\dots,M\}} \alpha_{jm}\right\}}{\sum_{j=1}^N a_{ij}^+}.$$

Then, $x \notin \mathcal{S}_i$ for any $i \in \{1,\dots,N\}$.

Proof.

First of all we notice that σ_i^0 is well-defined for any i because $a_{ii} > 0$, hence $\sum_{j=1}^N a_{ij}^+ > 0$.

Under the hypotheses of the Lemma, for large n and small δ we have

$$\int_{B_\delta(x)} \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g < \sigma_i^0. \tag{2.9}$$

Let us consider w_i^n and z_i^n defined by

$$\begin{cases} -\Delta w_i^n = -\sum_{j=1}^N a_{ij}\rho_j^n & \text{in } B_\delta(x) \\ w_i^n = 0 & \text{on } \partial B_\delta(x) \end{cases}, \qquad \begin{cases} -\Delta z_i^n = \sum_{j=1}^N a_{ij}^+ \widetilde{h}_j^n e^{v_j^n} & \text{in } B_\delta(x) \\ z_i^n = 0 & \text{on } \partial B_\delta(x) \end{cases}. \tag{2.10}$$

Is it evident that the w_i^n's are uniformly bounded in $L^\infty(B_\delta(x))$.
As for the z_i^n's, we can suppose to be working on a Euclidean disc, up to applying a perturbation to $\widetilde{h}_i^n$ which is smaller as δ is smaller, hence for δ small enough we still have the strict estimate (2.9). Therefore, we get

$$\|-\Delta z_i^n\|_{L^1(B_\delta(x))} = \sum_{j=1}^N a_{ij}^+ \int_{B_\delta(x)} \widetilde{h}_j^n e^{v_j^n} \mathrm{d}V_g < \sum_{j=1}^N a_{ij}^+ \sigma_j^0 \le 4\pi \min\{1, 1+\alpha_i(x)\},$$

and we can apply Lemma 1.1 to obtain $\int_{B_\delta(x)} e^{q|z_i^n|} \mathrm{d}V_g \le C$ for some $q > \frac{1}{\min\{1, 1+\alpha_i(x)\}}$.

If $\alpha_i(x) \ge 0$, then taking $q \in \left(1, \frac{4\pi}{\|-\Delta z_i^n\|_{L^1(B_\delta(x))}}\right)$ we have

$$\int_{B_\delta(x)} \left(\widetilde{h}_i^n e^{z_i^n}\right)^q \mathrm{d}V_g \le C_\delta \int_{B_\delta(x)} e^{q|z_i^n|} \mathrm{d}V_g \le C.$$

On the other hand, if $\alpha_i(x) < 0$, we choose

$$q \in \left(1, \frac{4\pi}{\|-\Delta z_i^n\|_{L^1(B_\delta(x))} - 4\pi\alpha_i(x)}\right), \qquad q' \in \left(\frac{4\pi}{4\pi - q\,\|-\Delta z_i^n\|_{L^1(B_\delta(x))}}, \frac{1}{-\alpha_i(x) q}\right),$$

and, applying Hölder's inequality,

$$\begin{aligned} & \int_{B_\delta(x)} \left(\widetilde{h}_i^n e^{z_i^n}\right)^q \mathrm{d}V_g \\ \le\ & C_\delta \int_{B_\delta(x)} d(\cdot, x)^{2q\alpha_i(x)} e^{q z_i^n} \mathrm{d}V_g \\ \le\ & C \left(\int_{B_\delta(x)} d(\cdot, x)^{2qq'\alpha_i(x)} \mathrm{d}V_g\right)^{\frac{1}{q'}} \left(\int_{B_\delta(x)} e^{q\frac{q'}{q'-1}|z_i^n|} \mathrm{d}V_g\right)^{1-\frac{1}{q'}} \\ \le\ & C, \end{aligned}$$

because $qq'\alpha_i(x) > -1$ and $q\frac{q'}{q'-1}\alpha_i(x) < \frac{4\pi}{\|-\Delta z_i^n\|_{L^1(B_\delta(x))}}$.

Now, let us consider $v_i^n - z_i^n - w_i^n$: it is a subharmonic sequence by construction, so for $y \in B_{\frac{\delta}{2}}(x)$ we get

$$
\begin{aligned}
& v_i^n(y) - z_i^n(y) - w_i^n(y) \\
\leq \; & \fint_{B_{\frac{\delta}{2}}(y)} \left(v_i^n - z_i^n - w_i^n\right) \mathrm{d}V_g \\
\leq \; & C \int_{B_{\frac{\delta}{2}}(y)} \left(v_i^n - z_i^n - w_i^n\right)^+ \mathrm{d}V_g \\
\leq \; & C \int_{B_\delta(x)} \left(\left(v_i^n - z_i^n\right)^+ + \left(w_i^n\right)^-\right) \mathrm{d}V_g \\
\leq \; & C \left(1 + \int_{B_\delta(x)} \left(v_i^n - z_i^n\right)^+ \mathrm{d}V_g\right).
\end{aligned}
$$

Moreover, since the maximum principle yields $z_i^n \geq 0$, taking $\theta = \begin{cases} 1 & \text{if } \alpha_i(x) \leq 0 \\ \in \left(0, \dfrac{1}{1+\alpha_i(x)}\right) & \text{if } \alpha_i(x) > 0 \end{cases}$,

we get

$$
\begin{aligned}
& \int_{B_\delta(x)} \left(v_i^n - z_i^n\right)^+ \mathrm{d}V_g \\
\leq \; & \int_{B_\delta(x)} \left(v_i^n\right)^+ \mathrm{d}V_g \\
\leq \; & \frac{1}{\theta} \int_{B_\delta(x)} e^{\theta v_i^n} \mathrm{d}V_g \\
\leq \; & C \left\| \left(\widetilde{h}_i^n\right)^{-\theta} \right\|_{L^{\frac{1}{1-\theta}}(B_\delta(x))} \left(\int_{B_\delta(x)} \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g\right)^\theta \\
\leq \; & C.
\end{aligned}
$$

Therefore, we showed that $v_i^n - z_i^n - w_i^n$ is bounded from above in $B_{\frac{\delta}{2}}(x)$, that is $e^{v_i^n - z_i^n - w_i^n}$ is uniformly bounded in $L^\infty\left(B_{\frac{\delta}{2}}(x)\right)$. Since the same holds for $e^{w_i^n}$, and $\widetilde{h}_i^n e^{z_i^n}$ is uniformly bounded in $L^q\left(B_{\frac{\delta}{2}}(x)\right)$ for some $q > 1$, we also deduce that

$$
\widetilde{h}_i^n e^{v_i^n} = \widetilde{h}_i^n e^{z_i^n} e^{v_i^n - z_i^n - w_i^n} e^{w_i^n}
$$

is bounded in the same $L^q\left(B_{\frac{\delta}{2}}(x)\right)$.
Thus, we have an estimate on $\left\|-\Delta z_i^n\right\|_{L^q\left(B_{\frac{\delta}{2}}(x)\right)}$ for any $i = 1, \ldots, N$, hence by standard elliptic estimates we deduce that z_i^n is uniformly bounded in $L^\infty\left(B_{\frac{\delta}{2}}(x)\right)$.
Therefore, we also deduce that

$$
v_i^n = \left(v_i^n - z_i^n - w_i^n\right) + z_i^n + w_i^n
$$

is bounded from above on $B_{\frac{\delta}{2}}(x)$, which is equivalent to saying $x \notin \mathcal{S}$. $\square$

From this proof, we notice that, under the assumptions of Theorem 2.2, the same result holds for any single index $i = 1, \ldots, N$.
In other words, for such systems, the upper bound on one σ_i implies that $x \notin \mathcal{S}_i$. In particular, $\sigma_i(x) = 0$ implies $x \in \mathcal{S}_i$, whereas in the general case we could have blow-up of one component at a point without that component accumulates any mass.

Corollary 2.7.
Suppose $a_{ij} \leq 0$ for any $i \neq j$.
Then, for any given $i = 1, \dots, N$ the following conditions are equivalent:

- $x \in \mathcal{S}_i$.
- $\sigma_i(x) \neq 0$.
- $\sigma_i(x) \geq \sigma_i' = \dfrac{4\pi \min\{1, 1 + \min_m \alpha_{im}\}}{a_{ii}}$.

Proof.
The third statement obviously implies the second and the second implies the first, since if v_i^n is bounded from above in $B_\delta(x)$ then $\widetilde{h}_i^n e^{v_i^n}$ is bounded in $L^q(B_\delta(x))$.
Finally, if $\sigma_i(x) < \sigma_i'$ then the sequence $\widetilde{h}_i^n e^{z_i^n}$ defined by (2.10) is bounded in L^q for $q > 1$, so one can argue as in Lemma 2.6 to get boundedness from above of v_i^n around x, that is $x \notin \mathcal{S}_i$. $\square$

We can now prove the main theorem of this Section.

Proof of Theorem 2.4.
If $\mathcal{S} = \emptyset$, then $e^{v_i^n}$ is bounded in $L^\infty(\Sigma)$, so $-\Delta v_i^n$ is bounded in $L^q(\Sigma)$ for any $q \in \left[1, \dfrac{1}{\max_{j,m} \alpha_{jm}^-}\right)$.
Therefore, we can apply Lemma 2.5 to v_i^n on Σ, where we must have the first alternative for every i, since otherwise the dominated convergence would give $\displaystyle\int_\Sigma \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} 0$ which is absurd; standard elliptic estimates allow to conclude compactness in $W^{2,q}(\Sigma)$.

Suppose now $\mathcal{S} \neq \emptyset$. From Lemma 2.6 we deduce, for any i,

$$|\mathcal{S}_i| \sigma_i^0 \leq \sum_{x \in \mathcal{S}_i} \max_j \sigma_j(x) \leq \sum_{j=1}^N \sum_{x \in \mathcal{S}_i} \sigma_j(x) \leq \sum_{j=1}^N \rho_j,$$

hence $\mathcal{S}_i$ is finite.

For any $i = 1, \dots, N$, we can apply Lemma 2.5 on $\Sigma \setminus \mathcal{S}$ with $f^n = \displaystyle\sum_{j=1}^N a_{ij} \left(\widetilde{h}_j^n e^{v_j^n} - \rho_j^n\right)$, since the last function is bounded in $L^q_{\mathrm{loc}}(\Sigma \setminus \mathcal{S})$.
Therefore, either v_j^n goes to $-\infty$ or it is bounded in L^∞_{loc}, and in the last case we get compactness in $W^{2,q}_{\mathrm{loc}}$ by applying again standard elliptic regularity.
Now, set $f_i := 0$ in the former case and $f_i := \widetilde{h}_i e^{v_i}$ in the latter case and take $r^n \underset{n\to+\infty}{\longrightarrow} 0$ such that

$$\int_{B_{r^n}(x)} \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} \sigma_i(x), \qquad \sup_{\Sigma \setminus \bigcup_{x \in \mathcal{S}} B_{r^n}(x)} \left| \widetilde{h}_i^n e^{v_i^n} - \widetilde{h}_i e^{v_i} \right| \underset{n\to+\infty}{\longrightarrow} 0.$$

For any $\phi \in C(\Sigma)$ we have

$$\begin{aligned} & \int_\Sigma \widetilde{h}_i^n e^{v_i^n} \phi \mathrm{d}V_g - \sum_{x \in \mathcal{S}} \sigma_i(x) \phi(x) + \int_\Sigma f_i \phi \mathrm{d}V_g \\ = & \sum_{x \in \mathcal{S}} \int_{B_{r^n}(x)} \widetilde{h}_i^n e^{v_i^n} (\phi - \phi(x)) \mathrm{d}V_g + \int_{B_{r^n}(x)} \left(\widetilde{h}_i^n e^{v_i^n} - f_i\right) \phi \mathrm{d}V_g + o(1) \\ \underset{n\to+\infty}{\longrightarrow} & \; 0. \end{aligned}$$

The proof is now complete. $\square$

Remark 2.8.
We can easily give a localized *version of Theorem 2.4, namely to take an open* $\Omega \subset \Sigma$, $V_i^n \underset{n\to+\infty}{\longrightarrow} 1$ *in* $C^1\left(\overline{\Omega}\right)^N$ *and to study* $v^n|_\Omega$.
The main difference with respect to the original form of Theorem 2.4 is the following: in case of Compactness *we could have, for any* $i = 1, \dots, N$, $v_i^n \underset{n\to+\infty}{\longrightarrow} -\infty$ *in* $L^\infty_{\mathrm{loc}}(\Omega)$. *This is because, at the beginning of the proof of the Theorem,* $\displaystyle\int_\Omega \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} 0$ *is allowed. All the rest of the proof can be adapted step-by-step.*
Such a localized version will be useful in Chapter 4.

2.2 Pohožaev identity and quantization for the Toda system

The main goal of this Section is to prove a Pohožaev identity for solutions of (3), namely an algebraic conditions which must be satisfied by the quantities $\sigma_i(x)$. This was already done in [44, 53] for some special cases.
Such a result is very important for two reasons. First of all, it is essential to prove Theorem 2.1 through blow-up analysis. Moreover, it allows to deduce a global compactness theorem for systems (9), (10), (11) from the local quantization theorems (1.6), (1.8).
This global compactness result will be proved in the end of this Section.

The content of this Section is mostly from the paper [14].

Theorem 2.9.
Let $\{u^n\}_{n\in\mathbb{N}}$ *be a sequence of solutions of* (3), $\alpha_i(x)$ *and* $\Lambda_{\mathcal{I},x}$ *as in* (2.1) *and* $\sigma(x) = (\sigma_1(x), \dots, \sigma_N(x))$ *as in* (1.5).
Then,

$$\Lambda_{\{1,\dots,N\},x}(\sigma(x)) = 8\pi\sum_{i=1}^N (1+\alpha_i(x))\sigma_i(x) - \sum_{i,j=1}^N a_{ij}\sigma_i(x)\sigma_j(x) = 0.$$

As a first step, we prove that blowing up sequences $\{u^n\}$ resemble suitable combination of Green's functions plus a remainder term.

Lemma 2.10.
Let $\{v^n\}_{n\in\mathbb{N}}$, $\mathcal{S}_i$ *be as in Theorem 2.4,* $\sigma_i(x)$ *as in* (2.8) *and* G_x *be the Green's function of* $-\Delta$ *as in* (2); *assume* Concentration *occurs in Theorem 2.4*
Then, there exist $w_1, \dots, w_N$ *such that* $\displaystyle\int_\Sigma e^{qw_i}\mathrm{d}V_g < +\infty$ *for any* $q < +\infty$, $i = 1, \dots, N$ *and*

$$v_i^n - \int_\Sigma v_i^n \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} \sum_{j=1}^N a_{ij}\sum_{x\in\mathcal{S}}\sigma_j(x)G_x + w_i,$$

weakly in $W^{1,q'}(\Sigma)$ *for any* $q' < 2$ *and strongly in* $W^{2,q}(\Sigma\setminus\mathcal{S})$ *for some* $q > 1$.

Proof.
Define w_i as the solution of

$$\begin{cases} -\Delta w_i = \displaystyle\sum_{j=1}^N a_{ij}\left(f_j - \int_\Sigma f_j \mathrm{d}V_g\right) \\ \displaystyle\int_\Sigma w_i \mathrm{d}V_g = 0 \end{cases}$$

Since $f_i \in L^1(\Sigma)$, then $\int_\Sigma e^{qw_i}\mathrm{d}V_g < +\infty$ by Lemma 1.1.
Take now $q' \in (1,2)$ and $\phi \in W^{1,\frac{q'}{q'-1}}(\Sigma)$. Since $\frac{q'}{q'-1} > 2$, $\phi \in C(\Sigma)$, therefore:

$$\begin{aligned}
& \int_\Sigma \nabla\left(v_i^n - \int_\Sigma v_i^n \mathrm{d}V_g - \sum_{j=1}^N a_{ij}\sum_{x\in\mathcal{S}}\sigma_j(x)G_x - w_i\right)\cdot\nabla\phi\mathrm{d}V_g \\
= \quad & \int_\Sigma -\Delta\left(v_i^n - \sum_{j=1}^N a_{ij}\sum_{x\in\mathcal{S}}\sigma_j(x)G_x - w_i\right)\phi\mathrm{d}V_g \\
= \quad & \int_\Sigma \sum_{j=1}^N a_{ij}\left(\widetilde{h}_i^n e^{v_i^n} - \sum_{x\in\mathcal{S}}\sigma_j(x)\delta_x - f_j + \rho_j^n - \rho_j\right)\phi\mathrm{d}V_g \\
\underset{n\to+\infty}{\longrightarrow} \quad & 0,
\end{aligned}$$

which gives weak convergence in $W^{1,q'}(\Sigma)$.
Strong convergence in $W^{2,q}(\Sigma\setminus\mathcal{S})$ follows from standard elliptic estimates. $\square$

At this point, the main issue is given by the residual f_i defined in Theorem 2.4.
If we had $w_i \equiv 0$, then Theorem 2.9 would follow quite easily by an integration by parts. Anyway, the w_i's could in principle play a role in the double limit which is taken in the definition of $\sigma(x)$, because we do not know whether it belongs to $L^\infty(\Sigma)$ or $H^1(\Sigma)$.
Some information on the residuals is given by the following Lemma.

Lemma 2.11.
Let v^n and f_i be as in Theorem 2.4, $\sigma_i(x)$ as in (1.5) and w_i as in Lemma 2.10 and assume Concentration *occurs.*
Then, for any $i = 1,\dots,N$, one of the following holds true:

- *$f_i \equiv 0$ and $\int_\Sigma v_i^n\mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} -\infty$.*
- *$f_i > 0$ and $\int_\Sigma v_i^n\mathrm{d}V_g$ is uniformly bounded.*

Moreover, in the latter case, there exists $\widehat{h}_i \in L^q_{\mathrm{loc}}(\Sigma\setminus\mathcal{S})$ for some $q>1$ such that $f_i = \widehat{h}_i e^{w_i}$ and $\widehat{h}_i \sim d(\cdot,x)^{2\alpha_i(x) - \frac{\sum_{j=1}^N a_{ij}\sigma_j(x)}{2\pi}}$ and $\widehat{h}_i \sim \widetilde{h}_i$ around any $x \in \Sigma\setminus\mathcal{S}$.

Proof.
First of all, by Jensen's inequality

$$\int_\Sigma v_i^n\mathrm{d}V_g \le \int_\Sigma v_i^n\mathrm{d}V_g + \log\int_\Sigma \widetilde{h}_i^n\mathrm{d}V_g + C \le \log\int_\Sigma \widetilde{h}_i^n e^{v_i^n}\mathrm{d}V_g + C \le C.$$

Therefore, up to subsequences, there exists $L := \lim_{n\to+\infty} e^{\int_\Sigma v_i^n\mathrm{d}V_g}$.
Now, fix $\Omega \Subset \Sigma\setminus\mathcal{S}$ and write:

$$\begin{aligned}
& \int_\Omega f_i\mathrm{d}V_g \\
\underset{n\to+\infty}{\longleftarrow} \quad & \int_\Omega \widetilde{h}_i^n e^{v_i^n}\mathrm{d}V_g \\
= \quad & e^{\int_\Sigma v_i^n\mathrm{d}V_g}\int_\Omega \widetilde{h}_i^n e^{v_i^n - \int_\Sigma v_i^n\mathrm{d}V_g}\mathrm{d}V_g
\end{aligned}$$

$$\underset{n\to+\infty}{\longrightarrow}\quad \underbrace{L\int_\Omega \widetilde{h}_i e^{\sum_{j=1}^N a_{ij}\sum_{x\in\mathcal{S}}\sigma_j(x)G_x+w_i}\mathrm{d}V_g}_{\in(0,+\infty)}.$$

Therefore, $r\equiv 0$ if and only if $L=0$, that is if and only if $\int_\Sigma v_i^n \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} -\infty$.
For the last statement, just set $\widehat{h}_i := L\widetilde{h}_i e^{\sum_{j=1}^N a_{ij}\sum_{x\in\mathcal{S}}\sigma_j(x)G_x}$. $\square$

A key step in the proof of Theorem 2.9 is given by the following result:

Lemma 2.12.
Let v^n and f_i be as in Theorem 2.4, $\sigma_i(x)$ as in (1.5) and w_i as in Lemma 2.10 and assume Concentration *occurs.*
If $f_i\not\equiv 0$, then it belongs to $L^q(\Sigma)$ for some $q>1$ and $\sum_{j=1}^N a_{ij}\sigma_j(x)<4\pi(1+\alpha_i(x))$ for any $x\in\mathcal{S}$.
Moreover, $w_j\in W^{2,q}(\Sigma)$ for any $j=1,\dots,N$.

To proof this Lemma, we need a couple of results about matrices.
We believe such results are both well-known, but we could not find any references, so we will give a new proof.

Lemma 2.13.
Let $B=(b_{ij})_{i,j=1,\dots,M}\in\mathbb{R}^{L\times L}$ be such that $b_{ij}>0$ for any $i\neq j$.
Then, there exists $\upsilon=(\upsilon_1,\dots,\upsilon_L)\in\mathbb{R}^L_{>0}$ such that $B\upsilon\in\mathbb{R}^L_{>0}\cup\mathbb{R}^L_{<0}\cup\{0\}$.

Proof.
We proceed by induction in L.
In the case $L=1$ there is nothing to prove.
If $L\geq 2$, the lemma can be proved easily if $b_{ii}\geq 0$ for some i. In fact, if $b_{11}\geq 0$, it suffices to take $\upsilon_i=1$ for $i\geq 2$ and $\upsilon_1>\max\limits_{i=2,\dots,L}\frac{b_{ii}^-}{b_{i1}}$: $\sum_{j=1}^L b_{1j}\upsilon_j>0$ holds for any $\upsilon_j>0$, and for $i\geq 2$

$$\sum_{j=1}^L b_{ij}\upsilon_j\geq b_{i1}\upsilon_1+b_{ii}\upsilon_i=b_{i1}\upsilon_1+b_{ii}>0.$$

The same argument obviously works if instead $b_{ii}\geq 0$ for some $i\geq 2$.
Suppose now that $b_{ii}<0$ for all i's.
Then, the system

$$\sum_{j=1}^M b_{ij}\upsilon_j \gtreqqless 0,\qquad \forall\, i=1=\dots,L \tag{2.11}$$

is equivalent to

$$\begin{cases}\upsilon_L\lesseqqgtr\sum_{j=1}^{L-1}\frac{b_{Lj}}{-b_{LL}}\upsilon_j\\ \upsilon_L\gtreqqless-\sum_{j=1}^{L-1}\frac{b_{ij}}{b_{iL}}\upsilon_j\qquad\forall\, i\in\{1,\dots,L-1\}\end{cases},$$

which in turn is solvable if and only if

$$\sum_{j=1}^{L-1}(b_{iL}b_{Lj}-b_{ij}b_{LL})\upsilon_j\gtreqqless 0,\qquad\text{for any } i=1,\dots,L-1. \tag{2.12}$$

Consider now the matrix $B' \in \mathbb{R}^{(L-1)\times(L-1)}$ defined by $b'_{ij} = b_{iL}b_{Lj} - b_{ij}b_{LL}$.
For $i \neq j$ it verifies $b'_{ij} > b_{iL}b_{Lj} > 0$, therefore by inductive hypothesis the system (2.12) is solvable for at least one between $<, =, >$.
This means that the system (2.11) is also solvable for that choice of the sign, hence the Lemma is proved. $\square$

Lemma 2.14.
Let $B = (b_{ij})_{i,j=1,\dots,L} \in \mathbb{R}^{L\times L}$ be a positive definite symmetric matrix.
Then, there exists $\upsilon = (\upsilon_1, \dots, \upsilon_L) \in \mathbb{R}^L_{>0}$ such that $B\upsilon \in \mathbb{R}^L_{>0}$.

Proof.
We will argue, as in the previous Lemma, by induction and, as before, we have nothing to prove if $L = 1$.
If $L \geq 2$, we consider, for any $l = 1, \dots, L$, the sub-matrix $B_l \in \mathbb{R}^{(L-1)\times(L-1)} = (b_{ij})_{i,j\neq l}$ obtained by removing to B the l^{th} row and column.
By inductive hypothesis, for every l there exists $\upsilon_l = (\upsilon_{1l}, \dots, \upsilon_{l-1,l}, \upsilon_{l+1,l}, \dots, \upsilon_{L,l})$ such that $\sum_{k\neq l} b_{ik}\upsilon_{kl} > 0$ for all $i \neq l$.
Now, define the matrix $\Upsilon \in \mathbb{R}^{L\times L}$ as $(\Upsilon)_{ij} = \upsilon_{ij}$ for $i \neq j$ and $(\Upsilon)_{ii} = 0$. By what we showed before, the matrix $B' := B\Upsilon$ verifies $b'_{ij} > 0$ for any $i \neq j$, so it satisfies the hypotheses of Lemma 2.13.
We then get $\upsilon' \in \mathbb{R}^L_{>0}$ such that $B'\upsilon' \in \mathbb{R}^L_{>0} \cup \mathbb{R}^L_{<0} \cup \{0\}$. Actually, it must be $B'\upsilon' \in \mathbb{R}^L_{>0}$; in fact, since $\Upsilon\upsilon' \in \mathbb{R}^L_{>0}$, $B'\upsilon' \in \mathbb{R}^L_{\leq 0}$ would imply $B(\Upsilon\upsilon')\cdot(\Upsilon\upsilon') = B'\upsilon' \cdot \Upsilon\upsilon' \leq 0$, in contradiction with the fact that B is positive definite.
Therefore, we conclude by setting $\upsilon := \Upsilon\upsilon'$. $\square$

Proof of Lemma 2.12.
We first notice that the Lemma will follow by showing that $\widehat{\alpha}_i(x) := \alpha_i(x) - \frac{\sum_{j=1}^N a_{ij}\sigma_j(x)}{4\pi}$ is greater than -1 for all i, x such that $f_i \neq 0$.
In fact, this would imply $\widehat{h}_i \in L^{q'}(\Sigma)$ for $q' \in \left(1, \frac{1}{\max_i \left\{\max\left\{\widehat{\alpha}_i^-, \max_m \alpha_{im}^-\right\}\right\}}\right)$ and, since $e^{w_i} \in L^{q''}$ for any $q'' < +\infty$, then by Hölder's inequality $f_i \in L^q(\Sigma)$ for $q \in (1, q')$.
Moreover, we would get $-\Delta w_j \in L^q(\Sigma)$ for any j, hence $w_j \in W^{2,q}(\Sigma)$.
Assume, by contradiction, that $\widehat{\alpha}_i(x) \leq -1$ for some i, x. Up to re-labeling indexes, this will occur if and only if $i \in \{1, \dots, L\}$ for some $L \geq 1$.

Consider now the matrix B given by inverting the first L rows and columns of A, namely $b_{ij} = A|^{ij}_{(1,\dots,L)\times(1,\dots,L)}$.
Then, for $i = 1, \dots, L$,

$$\begin{aligned}
-\Delta\left(\sum_{j=1}^L b_{ij}w_j\right) &= f_i - \int_\Sigma f_i \mathrm{d}V_g + \sum_{j=1}^L \sum_{k=L+1}^N b_{ij}a_{jk}\left(f_k - \int_\Sigma f_k \mathrm{d}V_g\right) \\
&\geq -\int_\Sigma f_i \mathrm{d}V_g + \sum_{j=1}^L \sum_{k=L+1}^N b_{ij}a_{jk}\left(f_k - \int_\Sigma f_k \mathrm{d}V_g\right) \in L^q(\Sigma)
\end{aligned}$$

for some $q > 1$ as at the in the beginning of this proof. Therefore, by the Green's representation formula, $\sum_{j=1}^L b_{ij}w_j \geq -C$.
Now, apply Lemma 2.14 to B and take $\upsilon_1, \dots, \upsilon_L$ given by the Lemma. Up to multiplying by a

suitable positive constant, we may assume $\sum_{i,j=1}^{L} b_{ij}v_j = 1$, and clearly $\sum_{i,j=1}^{L} b_{ij}w_i v_j \geq -C$.
Therefore, for $x \in \mathcal{S}$ and $\delta > 0$ small enough, by the convexity of $t \mapsto e^t$ we get:

$$
\begin{aligned}
& \int_{B_\delta(x)} d(\cdot,x)^{2\max_{i=1,\dots,L}\widehat{\alpha}_i(x)} \mathrm{d}V_g \\
\leq \; & C \int_{B_\delta(x)} d(\cdot,x)^{2\max_{i=1,\dots,L}\widehat{\alpha}_i(x)} e^{\sum_{i=1}^{L}\left(\sum_{j=1}^{L} b_{ij}v_j\right)w_i} \mathrm{d}V_g \\
\leq \; & C \sum_{i=1}^{L}\left(\sum_{j=1}^{L} b_{ij}v_j\right) \int_{B_\delta(x)} d(\cdot,x)^{2\max_{i=1,\dots,L}\widehat{\alpha}_i(x)} e^{w_i} \mathrm{d}V_g \\
\leq \; & C \sum_{i=1}^{L} \int_{\Sigma} f_i \mathrm{d}V_g \\
< \; & +\infty;
\end{aligned}
$$

which means $\widehat{\alpha}_i > -1$ for some $i \in \{1,\dots,L\}$. This gives a contradiction and proves the Lemma. $\square$

Proof of Theorem 2.9.
To compute the limits in the definition of σ_i it is convenient, and not restrictive, to work on a small Euclidean ball $B_r(x)$. We will therefore write $|y-x|$ in place of $d(y,x)$.
Since r is small, we can write $\widetilde{h}_i^n = |\cdot - x|^{2\alpha_i(x)} {h'_i}^n$ for some smooth ${h'_i}^n$ converging to h'_i in $C^1\left(\overline{B_r(x)}\right)$.
From Lemma 2.12 and the Green's representation formula we deduce $d(\cdot,x)|\nabla w_i| = o(1)$, therefore it is negligible when integrating the gradient terms on ∂B_r:

$$
\begin{aligned}
& \lim_{r\to 0}\lim_{n\to+\infty} r \int_{\partial B_r(x)} \partial_\nu v_i^n \partial_\nu v_j^n \mathrm{d}\sigma \\
= \; & \lim_{r\to 0}\lim_{n\to+\infty} r \int_{\partial B_r(x)} \nabla v_i^n \cdot \nabla v_j^n \mathrm{d}\sigma \\
= \; & \lim_{r\to 0}\lim_{n\to+\infty} r \int_{\partial B_r(x)} \left(\sum_{k=1}^{N} a_{ik}\sigma_k(x)\nabla G_x\right) \cdot \left(\sum_{l=1}^{N} a_{jl}\sigma_l(x)\nabla G_x\right) \mathrm{d}\sigma \\
= \; & \frac{1}{2\pi}\left(\sum_{i,j=1}^{N} a_{ij}\sigma_j(x)\right)^2 .
\end{aligned}
$$

From Lemma 2.12 we also know that $w_i \in L^\infty(\Sigma)$ and $\widehat{h}_i \sim d(\cdot,x)^{2\widehat{\alpha}_i(x)}$ around x, with $\widehat{\alpha}_i(x) > -1$. Therefore, for a fixed $r > 0$, we will have $\widetilde{h}_i^n e^{v_i^n} \leq C r^{2\widehat{\alpha}_i(x)}$ on $\partial B_r(x)$. This implies that

$$
\lim_{r\to 0}\lim_{n\to+\infty} r \int_{\partial B_r(x)} \widetilde{h}_i^n(y) e^{v_i^n(y)} \mathrm{d}\sigma(y) = 0.
$$

Moreover, since v_i^n is bounded in $W^{1,q'}(\Sigma)$ for $q' \in (1,2)$ and $\widetilde{h}_i^n e^{v_i^n}$ is bounded in $L^1(\Sigma)$, we find:

$$
\lim_{r\to 0}\lim_{n\to+\infty} \int_{B_r(x)} (y-x)\cdot\nabla v_i^n(y) = 0,
$$

$$
\lim_{r\to 0}\lim_{n\to+\infty} \int_{B_r(x)} (y-x)\cdot\nabla {h'_i}^n(y)|y-x|^{2\alpha_i(x)} e^{v_i^n(y)} \mathrm{d}y = 0.
$$

After these consideration, Theorem 2.9 follows by an integration by parts and some computations:

$$
\frac{1}{4\pi}\sum_{i,j=1}^{N} a_{ij}\sigma_i(x)\sigma_j(x)
$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{N} a^{ij} \frac{1}{4\pi} \left(\sum_{i,j=1}^{N} a_{ij}\sigma_j(x) \right)^2 \\
&= \lim_{r\to 0} \lim_{n\to+\infty} \sum_{i,j=1}^{N} a^{ij} r \int_{\partial B_r(x)} \left(\partial_\nu v_i^n \partial_\nu v_j^n - \frac{\nabla v_i^n \cdot \nabla v_j^n}{2} \mathrm{d}\sigma \right) \\
&= \lim_{r\to 0} \lim_{n\to+\infty} \sum_{i,j=1}^{N} a^{ij} \int_{\partial B_r(x)} \left(((y-x)\cdot\nabla v_i^n)\nabla v_j^n - \frac{\nabla v_i^n \cdot \nabla v_j^n}{2}(y-x) \right) \cdot \nu(y) \mathrm{d}\sigma(y) \\
&= \lim_{r\to 0} \lim_{n\to+\infty} \sum_{i,j=1}^{N} a^{ij} \int_{B_r(x)} \mathrm{div} \left(((y-x)\cdot\nabla v_i^n)\nabla v_j^n - \frac{\nabla v_i^n \cdot \nabla v_j^n}{2}(y-x) \right) \mathrm{d}y \\
&= \lim_{r\to 0} \lim_{n\to+\infty} \sum_{i,j=1}^{N} a^{ij} \int_{B_r(x)} \Bigg(((y-x)\cdot\nabla v_i^n(y))\Delta v_j^n(y) \\
&+ (y-x)\cdot \frac{D^2 v_i^n(y)\nabla v_j^n(y) - D^2 v_j^n(y)\nabla v_i^n(y)}{2} \Bigg) \mathrm{d}y \\
&= \lim_{r\to 0} \lim_{n\to+\infty} \sum_{i,j=1}^{N} a^{ij} \int_{B_r(x)} ((y-x)\cdot\nabla v_i^n(y))\Delta v_j^n(y)\mathrm{d}y \\
&= \lim_{r\to 0} \lim_{n\to+\infty} \sum_{i=1}^{N} \left(\rho_i^n \int_{B_r(x)} (y-x)\cdot\nabla v_i^n(y) - \int_{B_r(x)} ((y-x)\cdot\nabla v_i^n(y))\widetilde{h}_i^n(y) e^{v_i^n(y)}\mathrm{d}y \right) \\
&= \lim_{r\to 0} \lim_{n\to+\infty} \sum_{i=1}^{N} \Bigg(\int_{B_r(x)} \mathrm{div}(y-x)\widetilde{h}_i^n(y)e^{v_i^n(y)}\mathrm{d}y + \int_{B_r(x)} \left((y-x)\cdot\nabla\left(\widetilde{h}_i^n(y)\right)\right) e^{v_i^n(y)}\mathrm{d}y \\
&+ r \int_{\partial B_r(x)} \widetilde{h}_i^n(y) e^{v_i^n(y)} \mathrm{d}\sigma(y) \Bigg) \\
&= \lim_{r\to 0} \lim_{n\to+\infty} \sum_{i=1}^{N} \Bigg(2\int_{B_r(x)} \widetilde{h}_i^n(y)e^{v_i^n(y)}\mathrm{d}y + 2\alpha_i(x)\int_{B_r(x)} \widetilde{h}_i^n(y)e^{v_i^n(y)}\mathrm{d}y \\
&+ \int_{B_r(x)} \left((y-x)\cdot\nabla {h'}_i^n(y)\right) |y-x|^{2\alpha_i(x)} e^{v_i^n(y)}\mathrm{d}y \Bigg) \\
&= 2\sum_{i=1}^{N} (1+\alpha_i(x))\sigma_i(x).
\end{aligned}
$$

□

From Lemma 2.12 and Theorem 2.9 we deduce a simple but very important fact about the residuals f_i.
It is described by Figure 2.2:

Corollary 2.15.
Let v^n, f_i as in Theorem 2.4 and assume Concentration occurs.
Then, there exists $i = 1, \dots, N$ such that $f_i \equiv 0$.

Proof.
Assume, by contradiction, that $f_i \not\equiv 0$ for all i's, and take $x \in \mathcal{S}$. By Lemma 2.6, $\sigma_{i_0}(x) > 0$ for some i_0.

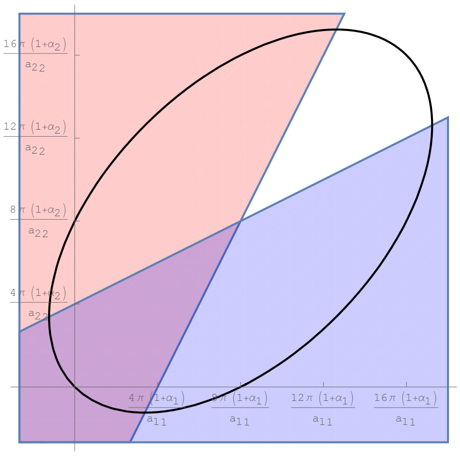

Figure 2.3: The algebraic conditions satisfied by $\sigma(x)$, in the case $N = 2$.

Moreover, by Lemma 2.12, for any $i = 1, \dots, N$ we must have $\sum_{j=1}^{N} a_{ij}\sigma_j(x) < 4\pi(1+\alpha_i(x))$. Multiply each of these inequality by $\sigma_i(x)$ and sum over $i = 1, \dots, N$. Since the $\sigma(x)$ is not identically 0, the following strict inequality is preserved:

$$\sum_{i,j=1}^{N} a_{ij}\sigma_i(x)\sigma_j(x) < 4\pi \sum_{i=1}^{N}(1 + \alpha_i(x))\sigma_i(x);$$

this is in contradiction with Theorem 2.9. $\square$

Now we are in position to prove a global compactness result for the A_2, B_2, G_2 Toda systems.

Theorem 2.16.
Let Ξ_{α_1,α_2} be as in Definition 1.5 and $\pi_i : \mathbb{R}^2 \to \mathbb{R}$ the projection on the i^{th} component, for $i = 1, 2$, and define:

$$\Gamma_i := \left\{ 4\pi n + \sum_{m\in\mathcal{M}} \sigma_m;\ n \in \mathbb{N}, \mathcal{M} \subset \{1, \dots, M\},\ \sigma_m \in \pi_i(\Xi_{\alpha_{1m},\alpha_{2m}}) \right\},$$

$$\Gamma = \Gamma_{\underline{\alpha}_1,\underline{\alpha}_2} := \Gamma_1 \times \mathbb{R} \cup \mathbb{R} \times \Gamma_2, \qquad \Gamma_0 := 4\pi\mathbb{N} \times \mathbb{R} \cup \mathbb{R} \times 4\pi\mathbb{N}. \tag{2.13}$$

Then, the family of solutions $\{u_\rho\} \subset \overline{H}^1(\Sigma)^2$ of (9) is uniformly bounded in $W^{2,q}(\Sigma)^2$ for some $q > 1$, for any given $\mathcal{K} \Subset \mathbb{R}^2_{>0} \setminus \Gamma$.
The same holds true for solutions of (10), under assuming $\mathcal{K} \Subset \mathbb{R}^2_{>0} \setminus \Gamma_0$, and for solutions of (11), provided $\mathcal{K} \Subset \left(0, 4\pi\left(2 + \sqrt{2}\right)\right) \times \left(0, 4\pi\left(5 + \sqrt{7}\right)\right) \setminus \Gamma_0$.

Proof.
We will just show the proof for (9).
Take $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho \in \mathcal{K}$, apply Theorem 2.4 and assume, by contradiction, that *Concentration* occurs.
Then, by Corollary 2.15, we must have $\widetilde{h}_i^n e^{v_i^n} \underset{n\to\infty}{\rightharpoonup} \sum_{x\in\mathcal{S}} \sigma_i(x)\delta_x$ for either $i = 1$ or $i = 2$, therefore $\rho_i^n \underset{n\to+\infty}{\longrightarrow} \sum_{x\in\mathcal{S}} \sigma_i(x)$ for that i. By construction, $\sum_{x\in\mathcal{S}} \sigma_i(x) \in \Gamma_i$ for any possible $\mathcal{S}$, hence we would

get $\rho_i \in \Gamma_i$ for some i, which is a contradiction.
Therefore, *Compactness* must occur for v^n. Now, write

$$u_i^n = v_i^n + \log \int_\Sigma \widetilde{h}_i^n e^{u_i^n} \mathrm{d}V_g + \log \rho_i^n =: v_i^n + c_i^n,$$

with $c_i^n \geq -C$ by Jensen's inequality.
If u^n were not bounded in $W^{2,q}(\Sigma)^2$, then we had $c_i^n \underset{n\to+\infty}{\longrightarrow} +\infty$ for some i, but this would mean $\inf_\Sigma u_i^n \underset{n\to+\infty}{\longrightarrow} +\infty$, contradicting $\int_\Sigma u_i \mathrm{d}V_g = 0$. □

Corollary 2.17.
Let Γ, Γ_0 be as in Theorem 2.16, $\rho \in \mathbb{R}^2_{>0} \setminus \Gamma, a, b \in \mathbb{R}$ be given with $a < b$ and such that (9) has no solutions with $a \leq J_\rho \leq b$.
Then, J_ρ^a is a deformation retract of I_ρ^b.
Moreover, there exists $L > 0$ such that I_ρ^L is contractible.
The same result holds true for solutions of (10) if $\rho \in \mathbb{R}^2_{>0} \setminus \Gamma_0$ and for solutions of (11) if $\rho \in \left(0, 4\pi\left(2+\sqrt{2}\right)\right) \times \left(0, 4\pi\left(5+\sqrt{7}\right)\right) \setminus \Gamma_0$.

Proof.
If $\rho \notin \Gamma$ then the second alternative must occur in Lemma 1.9, since the first alternative would give, by Corollary 2.16, $u^n \underset{n\to+\infty}{\longrightarrow} u$ which solves (6) and satisfies $a \leq I_\rho(u) \leq b$.
Moreover, by Corollary 1.4, we have $\|u^n\|_{H^1(\Sigma)} \leq C$ for any solution $u^n \in \overline{H}^1(\Sigma)^2$ of (9) with $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho$, therefore by Jensen's inequality every solution of (6) verifies

$$J_{\rho^n}(u^n) \leq \frac{1}{2}\int_\Sigma \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) \mathrm{d}V_g - \sum_{i=1}^2 \rho_i \int_\Sigma \log \widetilde{h}_i \mathrm{d}V_g \leq \frac{C^2}{2} - \sum_{i=1}^2 \rho_i \int_\Sigma \log \widetilde{h}_i \mathrm{d}V_g =: L;$$

Corollary 1.10 gives the last claim.
The same argument works for the cases of (10), (11). □

2.3 Proof of Theorem 2.1

Here we will prove the theorem which gives conditions for the functional J_ρ to be bounded from below and coercive.
Setting

$$E := \left\{\rho \in \mathbb{R}^N_{>0} : J_\rho \text{ is bounded from below on } H^1(\Sigma)^N\right\}, \tag{2.14}$$

$$E' := \left\{\rho \in \mathbb{R}^N_{>0} : J_\rho \text{ is coercive on } \overline{H}^1(\Sigma)^N\right\}, \tag{2.15}$$

then $E' \subset E$, so we will suffice to prove that $E' = \{\Lambda > 0\}$ and $E \subset \{\Lambda \geq 0\}$.

As a first thing, we notice that E is not empty and it verifies a simple monotonicity condition.

Lemma 2.18.
The set E defined by (2.14) is nonempty.
Moreover, for any $\rho \in E$ then $\rho' \in E$ provided $\rho'_i \leq \rho_i$ for any $i \in \{1, \ldots, N\}$.

Proof.
Let $\theta > 0$ be the biggest eigenvalue of the matrix (a_{ij}). Then,

$$J_\rho(u) \geq \sum_{i=1}^N \left(\frac{1}{2\theta} \int_\Sigma |\nabla u_i|^2 \mathrm{d}V_g - \rho_i \left(\log \int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g - \int_\Sigma u_i \mathrm{d}V_g \right) \right).$$

Therefore, from scalar Moser-Trudinger inequality (1.6), we deduce that J_ρ is bounded from below if $\rho_i \leq \dfrac{8\pi(1+\widetilde{\alpha}_i)}{\theta}$, hence $E \neq \emptyset$.
Suppose now $\rho \in E$ and $\rho'_i \leq \rho_i$ for any i.
Then, through Jensen's inequality, we get for any $u \in H^1(\Sigma)^N$

$$\begin{aligned} & J_{\rho'}(u) \\ = \; & J_\rho(u) + \sum_{i=1}^N (\rho_i - \rho'_i) \left(\log \int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g - \int_\Sigma u_i \mathrm{d}V_g \right) \\ \geq \; & -C + \sum_{i=1}^N (\rho_i - \rho'_i) \int_\Sigma \log \widetilde{h}_i \mathrm{d}V_g \\ \geq \; & -C, \end{aligned}$$

hence the claim. □

It is interesting to observe that a similar monotonicity condition is also satisfied by the set $\{\Lambda > 0\}$ (although one can easily see that it is not true if we replace Λ with $\Lambda_{\mathcal{I},x}$).

Lemma 2.19.
Let $\rho, \rho' \in \mathbb{R}^N_{>0}$ be such that $\Lambda(\rho) > 0$ and $\rho'_i \leq \rho_i$ for any $i \in \{1, \dots, N\}$.
Then, $\Lambda(\rho') > 0$.

Proof.
Suppose by contradiction $\Lambda(\rho') \leq 0$, that is $\Lambda_{\mathcal{I},x}(\rho') \leq 0$ for some $\mathcal{I}, x$.
This cannot occur for $\mathcal{I} = \{i\}$ because it would mean $\rho'_i \geq \dfrac{8\pi(1+\alpha_i(x))}{a_{ii}}$, so the same inequality would for ρ_i, hence $\Lambda(\rho) \leq \Lambda_{\mathcal{I},x}(\rho) \leq 0$.
Therefore, there must be some $\mathcal{I}, x$ such that $\Lambda_{\mathcal{I},x}(\rho') \leq 0$ and $\Lambda_{\mathcal{I}\setminus\{i\},x}(\rho') > 0$ for any $i \in \mathcal{I}$; this implies

$$\begin{aligned} & 0 \\ < \; & \Lambda_{\mathcal{I}\setminus\{i\},x}(\rho') - \Lambda_{\mathcal{I},x}(\rho') \\ = \; & 2\sum_{j\in\mathcal{I}} a_{ij}\rho'_i\rho'_j - a_{ii}{\rho'_i}^2 - 8\pi(1+\alpha_i(x))\rho'_i \\ < \; & \rho'_i \left(2\sum_{j\in\mathcal{I}} a_{ij}\rho'_j - 8\pi(1+\alpha_i(x)) \right). \end{aligned} \tag{2.16}$$

It will be not restrictive to suppose, from now on, $\rho'_1 \leq \rho_1$ and $\rho'_i = \rho_i$ for any $i \geq 2$, since the general case can be treated by exchanging the indices and iterating.
Assuming this, we must have $1 \in \mathcal{I}$, therefore we obtain:

$$\begin{aligned} & 0 \\ < \; & \Lambda_{\mathcal{I},x}(\rho) - \Lambda_{\mathcal{I},x}(\rho') \\ = \; & 8\pi(1+\alpha_1(x))(\rho_1 - \rho'_1) - a_{11}\left({\rho'_1}^2 - \rho_1^2\right) - 2\sum_{j\in\mathcal{I}\setminus\{1\}} a_{1j}(\rho'_1 - \rho_1)\rho_j \end{aligned}$$

$$
\begin{aligned}
&= (\rho_1 - \rho_1')\left(8\pi(1+\alpha_1(x)) - a_{11}(\rho_1' + \rho_1) - 2\sum_{j\in\mathcal{I}\setminus\{1\}} a_{1j}\rho_j\right) \\
&< (\rho_1 - \rho_1')\left(8\pi(1+\alpha_1(x)) - 2\sum_{j\in\mathcal{I}} a_{1j}\rho_j'\right),
\end{aligned}
$$

which is negative by (2.16). We found a contradiction. $\square$

We will now show that relation between E and E': the interior part of E is contained in E'. On the other hand, if $\rho \in \partial E$, then J_ρ has a different behavior.

Lemma 2.20.
Let E, E' be as in (2.14), (2.15) and take $\rho \in \mathring{E}$.
Then, there exists a constant $C = C_\rho$ such that

$$
J_\rho(u) \geq \frac{1}{C}\sum_{i=1}^{N} \int_\Sigma |\nabla u_i|^2 \mathrm{d}V_g - C.
$$

In particular, $\mathring{E} \subset E'$.

Proof.
Take $\delta \in \left(0, \dfrac{d_{\mathbb{R}^N}(\rho, \partial E)}{\sqrt{N}|\rho|}\right)$ so that $(1+\delta)\rho \in E$. Then,

$$
\begin{aligned}
&J_\rho(u) \\
&= \frac{\delta}{2(1+\delta)} \sum_{i,j=1}^{N} a^{ij} \int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g + \frac{1}{1+\delta} J_{(1+\delta)\rho}(u) \\
&\geq \frac{\delta}{2\theta(1+\delta)} \sum_{i=1}^{N} \int_\Sigma |\nabla u_i|^2 \mathrm{d}V_g - C,
\end{aligned}
$$

hence $\mathring{E} \subset E'$.

$\square$

Lemma 2.21.
Let E be as in (2.14) and take $\rho \in \partial E$.
Then, there exists a sequence $\{u^n\}_{n\in\mathbb{N}} \subset H^1(\Sigma)^N$ such that

$$
\sum_{i=1}^{N} \int_\Sigma |\nabla u_i^n|^2 \, \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} +\infty, \qquad \lim_{n\to+\infty} \frac{J_\rho(u^n)}{\sum_{i=1}^{N} \int_\Sigma |\nabla u_i^n|^2 \, \mathrm{d}V_g} \leq 0,
$$

Proof.
We first notice that $(1-\delta)\rho \in E$ for any $\delta \in (0,1)$. In fact, otherwise, from Lemma 2.18 we would get $\rho' \notin E$ as soon as $\rho_i' \geq (1-\delta)\rho_i$ for some i, hence $\rho \notin \partial E$.
Now, suppose by contradiction that for any sequence u^n one gets

$$
\sum_{i=1}^{N} \int_\Sigma |\nabla u_i^n|^2 \, \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} +\infty \qquad \Rightarrow \qquad \frac{J_\rho(u^n)}{\sum_{i=1}^{N} \int_\Sigma |\nabla u_i^n|^2 \, \mathrm{d}V_g} \geq \varepsilon > 0.
$$

Therefore, we would have

$$J_\rho(u) \geq \frac{\varepsilon}{2}\sum_{i=1}^N \int_\Sigma |\nabla u_i|^2 \mathrm{d}V_g - C.$$

Hence, indicating as θ' the smallest eigenvalue of the matrix A, for small δ we would get

$$\begin{aligned} & J_\rho(u) \\ = \ & (1+\delta)J_{(1+\delta)\rho}(u) - \frac{\delta}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g \\ \geq \ & \left((1+\delta)\frac{\varepsilon}{2} - \frac{\delta}{2\theta'}\right)\sum_{i=1}^N \int_\Sigma |\nabla u_i|^2 - C \\ \geq \ & -C, \end{aligned}$$

therefore $(1+\delta)\rho \in E$.
Being also, by Lemma 2.18, $(1-\delta)\rho \in E$, we get a contradiction with $\rho \in \partial E$. □

To see what happens when $\rho \in \partial E$, we build an auxiliary functional using Lemma 1.12.

Lemma 2.22.
Define, for $\rho' \in \partial E$:

$$a_{\rho'}^n := \frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i^n \cdot \nabla u_j^n \mathrm{d}V_g, \qquad b_{\rho'}^n := J_{\rho'}(u^n),$$

$$J'_{\rho',\rho}(u) = J_\rho(u) - F_{\rho'}\left(\frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g\right),$$

where u^n is given by Lemma 2.21 and $F_{\rho'}$ by Lemma 1.12.
If $\rho \in \mathring{E}$, then $J'_{\rho',\rho}$ is bounded from below on $H^1(\Sigma)^N$ and its infimum is achieved by a solution of

$$-\Delta\left(u_i - \sum_{i,j=1}^N a^{ij} f u_j\right) = \sum_{j=1}^N a_{ij}\rho_j\left(\frac{\widetilde{h}_j e^{u_j}}{\int_\Sigma \widetilde{h}_j e^{u_j}\mathrm{d}V_g} - 1\right), \qquad i = 1, \dots, N,$$

with $f = (F_{\rho'})'\left(\frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g\right)$.
On the other hand, $J'_{\rho',\rho'}$ is unbounded from below.

Proof.
For $\rho \in \mathring{E}$, we can argue as in Lemma 2.20, since the continuity follows from the regularity of F and the coercivity from the behavior of F' at the infinity.
For $\rho = \rho'$, if we take u^n as in Lemma 2.21 we get

$$J'_{\rho',\rho'}(u^n) = b_{\rho'}^n - F_{\rho'}\left(a_{\rho'}^n\right) \underset{n\to+\infty}{\longrightarrow} -\infty.$$

□

Now we can prove the first part of Theorem 2.1, that is J_ρ is bounded from below if $\Lambda(\rho) > 0$. Moreover, being the set $\{\Lambda > 0\}$ open, we also get coercivity.

Proof of $\{\Lambda > 0\} \subset E'$.
We will show that $\{\Lambda > 0\} \subset \mathring{E}$.
Suppose by contradiction there is some $\rho' \in \partial E$ with $\Lambda(\rho) > 0$ and take a sequence $\rho^n \in E$ with $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho'$.
Then, by Lemma 2.22, the auxiliary functional J_{ρ',ρ^n} has a minimizer u^n, so the functions v_i^n defined as in (2.6) solve

$$\begin{cases} -\Delta v_i^n = \sum_{j,k=1}^N a_{ij} b^{jk,n} \left(\widetilde{h}_j e^{v_j^n} - \rho_j^n\right) \\ \int_\Sigma \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g = \rho_i^n \end{cases}, \qquad i = 1, \dots, N$$

where $b^{ij,n}$ is the inverse matrix of $b_{ij}^n := \delta_{ij} - a^{ij} f^n$, hence $b^{ij,n} \underset{n\to+\infty}{\longrightarrow} \delta_{ij}$.
We can then apply Theorem 2.4. *Compactness* is excluded, since otherwise we would get, for any $u \in H^1(\Sigma)^N$,

$$J'_{\rho',\rho'}(u) = \lim_{n\to+\infty} J'_{\rho',\rho^n}(u) \geq \lim_{n\to+\infty} J'_{\rho',\rho^n}(v^n) = J'_{\rho',\rho'}(v) > -\infty,$$

thus contradicting Lemma 2.22.
Therefore, *Concentration* must occur. This means, by Lemma 2.6, that $\sigma_i(x) \neq 0$ for some $i \in \{1, \dots, N\}$ and some $x \in \Sigma$.
By Theorem 2.9 follows $\Lambda(\sigma(x)) \leq 0$. On the other hand, since by its definition $\sigma_i(x) \leq \rho'_i$ for any i, Lemma 2.19 yields $\Lambda(\rho') \leq 0$, which contradicts our assumptions. $\square$

To prove the unboundedness from below of J_ρ in the case $\Lambda(\rho) < 0$ we will use suitable test functions. Their profile is inspired by the well-known entire solution of the Liouville equation on $\mathbb{R}^2$; here we use *truncated* versions of the standard bubbles, rather than the smooth ones, because they yield the same estimates with simpler calculations.
Similar test functions are considered in Section 3.3.

The properties of such test functions are described by the following:

Lemma 2.23.
Define, for $x \in \Sigma$ *and* $\lambda > 0$, $\varphi = \varphi^{\lambda,x}$ *as*

$$\varphi_i := -2(1 + \alpha_i(x)) \log \max\{1, \lambda d(\cdot, x)\}. \tag{2.17}$$

Then, as $\lambda \to +\infty$*, one has*

$$\int_\Sigma \nabla\varphi_i \cdot \nabla\varphi_j \mathrm{d}V_g = 8\pi(1 + \alpha_i(x))(1 + \alpha_j(x)) \log\lambda + O(1)$$

$$\int_\Sigma \varphi_i \mathrm{d}V_g = -2(1 + \alpha_i(x)) \log\lambda + O(1)$$

$$\int_\Sigma \widetilde{h}_i e^{\sum_{j=1}^N \theta_j \varphi_j} \mathrm{d}V_g \sim \lambda^{-2(1+\alpha_i(x))} \qquad \text{if} \quad \sum_{j=1}^N \theta_j(1 + \alpha_j(x)) > 1 + \alpha_i(x).$$

Remark 2.24.
When using normal coordinates near the peaks of the test functions, the metric coefficients will slightly deviate from the Euclidean ones. We will then have coefficients of order $(1 + o_\lambda(1))$ *in front of the logarithmic terms appearing below. To keep the formulas shorter, we will omit them, as they will be harmless for the final estimates.*
The same convention will be adopted in Chapter 3, Section 3.3.

Proof.
It holds

$$\nabla\varphi_i = \begin{cases} 0 & \text{if } d(\cdot,x) < \frac{1}{\lambda} \\ -2(1+\alpha_i(x))\frac{\nabla d(\cdot,x)}{d(\cdot,x)} & \text{if } d(\cdot,x) > \frac{1}{\lambda} \end{cases}.$$

Therefore, being $|\nabla d(\cdot,x)| = 1$ almost everywhere on Σ:

$$\begin{aligned} & \int_\Sigma \nabla\varphi_i \cdot \nabla\varphi_j \mathrm{d}V_g \\ = \ & 4(1+\alpha_i(x))(1+\alpha_j(x)) \int_{\Sigma\setminus B_{\frac{1}{\lambda}}(x)} \frac{\mathrm{d}V_g}{d(\cdot,x)^2} \\ = \ & 8\pi(1+\alpha_i(x))(1+\alpha_j(x))\log\lambda + O(1). \end{aligned}$$

For the average of φ_i, we get

$$\int_\Sigma \varphi_i \mathrm{d}V_g = -2(1+\alpha_i(x)) \int_{\Sigma\setminus B_{\frac{1}{\lambda}}(x)} (\log\lambda + \log d(\cdot,x))\mathrm{d}V_g + O(1) = -2(1+\alpha_i(x))\log\lambda + O(1).$$

For the last estimate, choose $r > 0$ such that $\overline{B_\delta(x)}$ does not contain any of the points p_m for $m = 1, \dots, M$, except possibly x.
Then, outside such a ball, $e^{\sum_{j=1}^N \theta_j\varphi_j} \le C\lambda^{-2\sum_{j=1}^N \theta_j(1+\alpha_j(x))}$.
Therefore, under the assumptions of the Lemma,

$$\int_{\Sigma\setminus B_\delta(x)} \widetilde{h}_i e^{\sum_{i=1}^N \theta_j\varphi_j} \mathrm{d}V_g = o\left(\lambda^{-2(1+\alpha_i(x))}\right),$$

hence

$$\begin{aligned} & \int_\Sigma \widetilde{h}_i e^{\sum_{i=1}^N \theta_j\varphi_j} \mathrm{d}V_g \\ \sim \ & \int_{B_\delta(x)} \widetilde{h}_i e^{\sum_{i=1}^N \theta_j\varphi_j} \mathrm{d}V_g \\ \sim \ & \int_{B_{\frac{1}{\lambda}}(x)} d(\cdot,x)^{2\alpha_i(x)} \mathrm{d}V_g + \frac{1}{\lambda^{2\sum_{j=1}^N \theta_j(1+\alpha_j(x))}} \int_{A_{\frac{1}{\lambda},\delta}(x)} d(\cdot,x)^{2\alpha_i(x) - 2\sum_{i=1}^N \theta_j(1+\alpha_j(x))} \mathrm{d}V_g \\ \sim \ & \lambda^{-2(1+\alpha_i(x))}, \end{aligned}$$

which concludes the proof. □

Proof of $E \subset \{\Lambda \ge 0\}$ and $E' \subset \{\Lambda > 0\}$.
Let us start by the first assertion.
Take $\rho, \mathcal{I}, x$ such that $\Lambda_{\mathcal{I},x}(\rho) < 0$ and $\Lambda_{\mathcal{I}\setminus\{i\},x}(\rho) \ge 0$ for any $i \in \mathcal{I}$, and consider the family of functions $\{u^\lambda\}_{\lambda>0}$ defined by

$$u_i^\lambda := \sum_{j\in\mathcal{I}} \frac{a_{ij}\rho_j}{4\pi(1+\alpha_i(x))} \varphi_j^{\lambda,x}.$$

By Jensen's inequality we get

$$\begin{aligned} & J_\rho\left(u^\lambda\right) \\ \le \ & \frac{1}{2}\sum_{i,j=1}^N a^{ij} \int_\Sigma \nabla u_i^\lambda \cdot \nabla u_j^\lambda \mathrm{d}V_g + \sum_{i\in\mathcal{I}} \rho_i \left(\log\int_\Sigma \widetilde{h}_i e^{u_i^\lambda} \mathrm{d}V_g - \int_\Sigma u_i^\lambda \mathrm{d}V_g\right) + C \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i,j\in\mathcal{I}} \frac{a_{ij}\rho_i\rho_j}{16\pi^2(1+\alpha_i(x))(1+\alpha_j(x))}\int_\Sigma \nabla\varphi_i\cdot\nabla\varphi_j \mathrm{d}V_g \\
&+ \sum_{i,j\in\mathcal{I}} \frac{a_{ij}\rho_i\rho_j}{4\pi(1+\alpha_j(x))}\int_\Sigma \varphi_j \mathrm{d}V_g - \sum_{i\in\mathcal{I}}\rho_i\log\int_\Sigma \widetilde{h}_i e^{\sum_{j\in\mathcal{I}}\frac{a_{ij}\rho_j}{4\pi(1+\alpha_j(x))}\varphi_j}\mathrm{d}V_g + C.
\end{aligned}
$$

At this point, we would like to apply Lemma 2.23 to estimate $J_\rho\left(u^\lambda\right)$.
To be able to do this, we have to verify that

$$
\frac{1}{4\pi}\sum_{j\in\mathcal{I}} a_{ij}\rho_j > 1+\alpha_i(x), \qquad \forall\, i\in\mathcal{I}.
$$

If $\mathcal{I}=\{i\}$, then $\rho_i > \frac{8\pi(1+\alpha_i(x))}{a_{ii}}$, so it follows immediately. For the other cases, it follows from (2.16).
So we can apply Lemma 2.23 and we get:

$$
\begin{aligned}
& J_\rho\left(u^\lambda\right) \\
\leq\ & \left(\frac{1}{4\pi}\sum_{i,j\in\mathcal{I}} a_{ij}\rho_i\rho_j - \frac{1}{2\pi}\sum_{i,j\in\mathcal{I}} a_{ij}\rho_i\rho_j + 2\sum_{i\in\mathcal{I}}\rho_i(1+\alpha_i(x))\right)\log\lambda + C \\
=\ & -\frac{\Lambda_{\mathcal{I},x}(\rho)}{4\pi}\log\lambda + C \\
\underset{n\to+\infty}{\longrightarrow}\ & -\infty.
\end{aligned}
$$

To prove the second assertion, we still use family $\left\{u^\lambda\right\}_{\lambda>0}$.
If $\Lambda(\rho)\geq 0$, then by the previous estimate we get:

$$
\sum_{i=1}^N \int_\Sigma \left|\nabla u_i^\lambda\right|^2 \mathrm{d}V_g \underset{\lambda\to+\infty}{\longrightarrow} +\infty, \qquad J_\rho\left(u^\lambda\right) \leq -\frac{\Lambda_{\mathcal{I},x}(\rho)}{4\pi}\log\lambda + C \leq C.
$$

□

2.4 Proof of Theorem 2.2

Here we will finally prove a sharp inequality in the case when the matrix a_{ij} has non-positive entries outside its main diagonal.

As already pointed out, $\Lambda(\rho)$ can be written in a much shorter form under these assumptions, so the condition $\Lambda(\rho)\geq 0$ is equivalent to $\rho_i \leq \frac{8\pi(1+\widetilde{\alpha}_i)}{a_{ii}}$ for any $i\in\{1,\dots,N\}$.
Moreover, thanks to Lemma 2.18, in order to prove Theorem 2.2 for all such ρ's it will suffice to consider

$$
\rho^0 := \left(\frac{8\pi(1+\widetilde{\alpha}_1)}{a_{11}},\dots,\frac{8\pi(1+\widetilde{\alpha}_N)}{a_{NN}}\right). \tag{2.18}
$$

By what we proved in the previous Section, for any sequence $\rho^n \underset{n\to+\infty}{\nearrow} \rho^0$ one has

$$
\inf_{H^1(\Sigma)^N} J_{\rho^n} = J_{\rho^n}(u^n) \geq -C_{\rho^n},
$$

so Theorem 2.2 will follow by showing that, for a given sequence $\{\rho^n\}_{n\in\mathbb{N}}$, the constant $C_n = C_{\rho^n}$ can be chosen independently of n.

As a first thing, we provide a Lemma which shows the possible blow-up scenarios for such a sequence u^n.
Here, the assumption on a_{ij} is crucial since it reduces largely the possible cases.

Lemma 2.25.
Let ρ^0 be as in (2.18), $\{\rho^n\}_{n\in\mathbb{N}}$ such that $\rho^n \nearrow \rho^0$, u^n a minimizer of J_{ρ^n} and v^n as in (2.6). Then, up to subsequences, there exists a set $\mathcal{I} \subset \{1,\dots,N\}$ such that:

- *If $i \in \mathcal{I}$, then $\mathcal{S}_i = \{x_i\}$ for some $x_i \in \Sigma$ which satisfy $\widetilde{\alpha}_i = \alpha_i(x_i)$ and $\sigma_i(x_i) = \rho_i^0$, and $v_i^n \underset{n\to+\infty}{\longrightarrow} -\infty$ in $L^\infty_{\mathrm{loc}}\left(\Sigma \setminus \bigcup_{j\in\mathcal{I}}\{x_j\}\right)$.*
- *If $i \notin \mathcal{I}$, then $\mathcal{S}_i = \emptyset$ and $v_i^n \underset{n\to+\infty}{\longrightarrow} v_i$ in $W^{2,q}_{\mathrm{loc}}\left(\Sigma \setminus \bigcup_{j\in\mathcal{I}}\{x_j\}\right)$ for some $q > 1$ and some suitable v_i.*

Moreover, if $a_{ij} < 0$ then $x_i \neq x_j$.

Proof.
From Theorem 2.4 we get a $\mathcal{I} \subset \{1,\dots,N\}$ such that $\mathcal{S}_i \neq \emptyset$ for $i \in \mathcal{I}$.
If $\mathcal{S}_i \neq \emptyset$, then by Corollary 2.7 one gets

$$0 < \sigma_i(x) \le \rho_i^0 \le \frac{8\pi(1+\alpha_i(x))}{a_{ii}}$$

for all $x \in \mathcal{S}_i$, hence

$$\begin{aligned} & 0 \\ = \;& \Lambda_{\{1,\dots,N\},x}(\sigma(x)) \\ \ge \;& \sum_{j=1}^N \left(8\pi(1+\alpha_j(x))\sigma_j(x) - a_{jj}\sigma_j(x)^2\right) \\ \ge \;& 8\pi(1+\alpha_i(x))\sigma_i(x) - a_{ii}\sigma_i(x)^2 \\ \ge \;& 0. \end{aligned} \tag{2.19}$$

Therefore, all these inequalities must actually be equalities.
From the last, we have $\sigma_i(x) = \rho_i^0 = \dfrac{8\pi(1+\alpha_i(x))}{a_{ii}}$, hence $\alpha_i(x) = \widetilde{\alpha}_i$. On the other hand, since $\sum_{x\in\mathcal{S}_i}\sigma_i(x) \le \rho_i^0$, it must be $\sigma_i(x) = 0$ for all but one $x_i \in \mathcal{S}_i$, so Corollary 2.7 yields $\mathcal{S}_i = \{x_i\}$.
Let us now show that $v_i^n \underset{n\to+\infty}{\longrightarrow} -\infty$ in L^∞_{loc}.
It this were not the case, Theorem 2.4 would imply $v_i^n \underset{n\to+\infty}{\longrightarrow} v_i$ almost everywhere, therefore by Fatou's Lemma we would get the following contradiction:

$$\sigma_i(x_i) < \int_\Sigma \widetilde{h}_i e^{v_i} \mathrm{d}V_g + \sigma_i(x_i) \le \int_\Sigma \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g = \rho_i^n \le \rho_i = \sigma_i(x_i).$$

Since also inequality (2.19) has to be an equality, we get $a_{ij}\sigma_i(x_i)\sigma_j(x_i)$ for any $i,j \in \mathcal{I}$, so whenever $a_{ij} < 0$ there must be $\sigma_j(x_i) = 0$, so $x_i \neq x_j$.
Finally, if $\mathcal{S}_i = \emptyset$, the convergence in $W^{2,q}_{\mathrm{loc}}$ follows from what we just proved and Theorem 2.4. □

We basically showed that if a component of the sequence v^n blows up, then all its mass concentrates at a single point which has the lowest singularity coefficient.

We will now consider particular combinations of the v_i^n which have some good blow-up properties:

Lemma 2.26.
Let v^n and x_i be as in Lemma 2.25 and w^n be defined by $w_i^n = \sum_{j=1}^N a^{ij}\left(v_j^n - \int_\Sigma v_j^n \mathrm{d}V_g\right)$ for $i \in \{1, \dots, N\}$.
If $i \in \mathcal{I}$, then w_i^n is uniformly bounded in $W^{1,q'}(\Sigma)$ and in $W^{2,q}_{\mathrm{loc}}(\Sigma \setminus \{x_i\})$, for any $q' \in (1,2)$ and some $q > 1$, and if $i \notin \mathcal{I}$ then it is bounded in $W^{2,q}(\Sigma)$.

Proof.
The boundedness in $W^{1,q'}(\Sigma)$ follows from the boundedness of $v_i^n - \int_\Sigma v_i^n \mathrm{d}V_g$ in the same space, which was proved in Lemma 2.10.
Moreover, w_i^n solves
$$\begin{cases} -\Delta w_i^n = \widetilde{h}_i^n e^{v_i^n} - \rho_i^n \\ \int_\Sigma w_i^n \mathrm{d}V_g = 0 \end{cases},$$
with $\widetilde{h}_i^n e^{v_i^n} \in L^q_{\mathrm{loc}}(\Sigma \setminus \{x_i\})$ if $i \in \mathcal{I}$, or $\widetilde{h}_i^n e^{v_i^n} \in L^q(\Sigma)$ if $i \notin \mathcal{I}$, for some $q > 1$.
Therefore, boundedness in $W^{2,q}$ or $W^{2,q}_{\mathrm{loc}}$ follows by standard elliptic estimates. $\square$

The last Lemma we need is a localized scalar Moser-Trudinger inequality for the blowing-up sequence.

Lemma 2.27.
Let w_i^n be as in Lemma 2.26 and x_i as in the previous Lemmas.
Then, for any $i \in \mathcal{I}$ and any small $\delta > 0$ one has
$$\frac{a_{ii}}{2}\int_{B_\delta(x_i)} |\nabla w_i^n|^2 \mathrm{d}V_g - \rho_i^n \log \int_{B_\delta(x_i)} \widetilde{h}_i e^{a_{ii} w_i^n} \mathrm{d}V_g \geq -C_\delta.$$

Proof.
Since Σ is locally conformally flat, we can choose δ small enough so that we can apply Corollary 1.15 up to modifying $\widetilde{h}_i^n$. We also take δ so small that $\overline{B_\delta(x_i)}$ contains neither any x_j for $x_j \neq x_i$ nor any p_m for $m = 1, \dots, M$ (except possibly x_i).
Let z^n be the solution of
$$\begin{cases} -\Delta z_i^n = \widetilde{h}_i^n e^{v_i^n} - \rho_i^n & \text{in } B_\delta(x_i) \\ z_i^n = 0 & \text{on } \partial B_\delta(x_i) \end{cases}.$$
Then, $w_i^n - z_i^n$ is harmonic and it has the same value as w_i^n on $\partial B_\delta(x_i)$, so from standard estimates
$$\|w_i^n - z_i^n\|_{C^1(B_\delta(x_i))} \leq C\|w_i^n\|_{C^1(\partial B_\delta(x_i))} \leq C.$$
From Lemma 2.26 we get
$$\begin{aligned} & \left|\int_{B_\delta(x_i)} |\nabla w_i^n|^2 \mathrm{d}V_g - \int_{B_\delta(x_i)} |\nabla z_i^n|^2 \mathrm{d}V_g\right| \\ = & \left|\int_{B_\delta(x_i)} |\nabla (w_i^n - z_i^n)|^2 \mathrm{d}V_g + 2\int_{B_\delta(x_i)} \nabla w_i^n \cdot \nabla (w_i^n - z_i^n) \mathrm{d}V_g\right| \\ \leq & \int_{B_\delta(x_i)} |\nabla (w_i^n - z_i^n)|^2 \mathrm{d}V_g + 2\|\nabla w_i^n\|_{L^1(\Sigma)} \|\nabla (w_i^n - z_i^n)\|_{L^\infty(B_\delta(x_i))} \\ \leq & \; C_\delta. \end{aligned}$$
Moreover,
$$\int_{B_\delta(x_i)} \widetilde{h}_i e^{a_{ii} w_i^n} \mathrm{d}V_g \leq e^{a_{ii}\|w_i^n - z_i^n\|_{L^\infty(B_\delta(x_i))}} \int_{B_\delta(x_i)} \widetilde{h}_i e^{a_{ii} z_i^n} \mathrm{d}V_g \leq C_\delta \int_{B_\delta(x_i)} d(\cdot, x_i)^{2\widetilde{\alpha}_i} e^{a_{ii} z_i^n} \mathrm{d}V_g.$$

Therefore, since $\widetilde{\alpha}_i \leq 0$ and $a_{ii}\rho_i^n \leq 8\pi(1+\widetilde{\alpha}_i)$, we can apply Corollary 1.15 to get the claim:

$$\begin{aligned}
&\frac{a_{ii}}{2}\int_{B_\delta(x_i)} |\nabla w_i^n|^2 \,\mathrm{d}V_g - \rho_i^n \log \int_{B_\delta(x_i)} \widetilde{h}_i e^{a_{ii}w_i^n} \,\mathrm{d}V_g \\
\geq\ & \frac{1}{2a_{ii}}\int_{B_\delta(x_i)} |\nabla (a_{ii}z_i^n)|^2 \,\mathrm{d}V_g - \rho_i^n \log \int_{B_\delta(x_i)} d(\cdot,x_i)^{2\widetilde{\alpha}_i} e^{a_{ii}z_i^n} \,\mathrm{d}V_g - C_\delta \\
\geq\ & -C_\delta.
\end{aligned}$$

□

Proof of Theorem 2.2.
As noticed before, it suffices to prove the boundedness from below of $J_{\rho^n}(u^n)$ for a sequence $\rho^n \underset{n\to+\infty}{\nearrow} \rho^0$ and a sequence of minimizers u^n for J_{ρ^n}. Moreover, due to the invariance by addition of constants, one can consider v^n in place of u^n.
Let us start by estimating the term involving the gradients.
From Lemma 2.26 we deduce that the integral of $|\nabla w_i^n|^2$ outside a neighborhood of x_i is uniformly bounded for any $i \in \mathcal{I}$, and the integral on the whole Σ is bounded if $i \notin \mathcal{I}$.
For the same reason, the integral of $a_{ij}\nabla w_i^n \cdot \nabla w_j^n$ on the whole surface is uniformly bounded. In fact, if $a_{ij} \neq 0$, then $x_i \neq x_j$, then

$$\begin{aligned}
&\left|\int_\Sigma \nabla w_i^n \cdot \nabla w_j^n \mathrm{d}V_g\right| \\
\leq\ & \int_{\Sigma\setminus B_\delta(x_j)} |\nabla w_i^n|\left|\nabla w_j^n\right| \mathrm{d}V_g + \int_{\Sigma\setminus B_\delta(x_i)} |\nabla w_i^n|\left|\nabla w_j^n\right| \mathrm{d}V_g \\
\leq\ & \|\nabla w_i^n\|_{L^{q'}(\Sigma)} \left\|\nabla w_j^n\right\|_{L^{q''}(\Sigma\setminus B_\delta\{x_j\})} + \|\nabla w_i^n\|_{L^{q''}(\Sigma\setminus B_\delta\{x_i\})} \left\|\nabla w_j^n\right\|_{L^{q'}(\Sigma)} \\
\leq\ & C_\delta,
\end{aligned}$$

with q as in Lemma 2.26, $q' = \begin{cases} \frac{2q}{3q-2} < 2 & \text{if } q<2 \\ 1 & \text{if } q \geq 2 \end{cases}$ and $q'' = \begin{cases} \frac{2q}{2-q} & \text{if } q<2 \\ \infty & \text{if } q \geq 2 \end{cases}$.
Therefore, we can write

$$\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla v_i^n \cdot \nabla v_j^n \mathrm{d}V_g = \sum_{i,j=1}^N a_{ij}\int_\Sigma \nabla w_i^n \cdot \nabla w_j^n \mathrm{d}V_g \geq \sum_{i\in\mathcal{I}} a_{ii}\int_{B_\delta(x_i)} |\nabla w_i^n|^2 \,\mathrm{d}V_g - C_\delta.$$

To deal with the other term in the functional, we use the boundedness of w_i^n away from x_i: choosing r as in Lemma 2.27, we get

$$\begin{aligned}
&\int_\Sigma \widetilde{h}_i^n e^{v_i^n - \int_\Sigma v_i^n \mathrm{d}V_g} \mathrm{d}V_g \\
\leq\ & 2\int_{B_\delta(x_i)} \widetilde{h}_i^n e^{v_i^n - \int_\Sigma v_i^n \mathrm{d}V_g} \mathrm{d}V_g \\
=\ & 2\int_{B_\delta(x_i)} \widetilde{h}_i e^{\sum_{j=1}^N a_{ij}w_j^n} \mathrm{d}V_g \\
\leq\ & C_\delta \int_{B_\delta(x_i)} \widetilde{h}_i e^{a_{ii}w_i^n} \mathrm{d}V_g.
\end{aligned}$$

Therefore, using Lemma 2.27 we obtain

$$\begin{aligned}
&J_{\rho^n}(v^n) \\
=\ & \frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla v_i^n \cdot \nabla v_j^n \mathrm{d}V_g - \sum_{i=1}^N \rho_i^n \left(\log \int_\Sigma \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g - \int_\Sigma v_i^n \mathrm{d}V_g\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i\in\mathcal{I}}\left(\frac{a_{ii}}{2}\int_{B_\delta(x_i)}|\nabla w_i^n|^2\,\mathrm{d}V_g-\rho_i^n\log\int_{B_\delta(x_i)}\widetilde{h}_i e^{a_{ii}w_i^n}\mathrm{d}V_g\right)-C_\delta\\
&\geq -C_\delta
\end{aligned}
$$

Since the choice of δ does not depend on n, the proof is complete. □

Remark 2.28.
The same arguments used to prove Theorems 2.1 and 2.2 can be applied to get the same results in the case of a compact surface with boundary Σ for the functional

$$J_\rho(u)=\frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i\cdot\nabla u_j\mathrm{d}V_g-\sum_{i=1}^N\rho_i\log\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g$$

on the space $H_0^1(\Sigma)$ defined by (1.1).
Its critical points solve

$$\begin{cases} -\Delta u_i=\displaystyle\sum_{j=1}^N a_{ij}\rho_j\frac{\widetilde{h}_j e^{u_j}}{\int_\Sigma \widetilde{h}_j e^{u_j}\mathrm{d}V_g} & \text{in } \Sigma \\ u_i=0 & \text{on } \partial\Sigma\end{cases}, \qquad i=1,\dots,N.$$

As in Remark 2.8, we could also have Vanishing *in Theorem 2.4, but this can easily be excluded for minimizing sequences.*
The main issue in adapting the argument seems to be the blow up at a point $x\in\partial\Sigma$. Anyway, in [52] this phenomenon has been ruled out for the $SU(3)$ Toda system and the same can also be done in the general case.
This can be seen by arguing as in Theorem 2.9 and applying to $B_r(x)\cap\Sigma$ a conformal diffeomorphism which flattens $B_r(x)\cap\partial\Sigma$, as was done in [65].

Chapter 3

Existence and multiplicity of min-max solutions

The largest chapter of this work is devoted to the existence of min-max solutions for some particular systems, namely the A_2 Toda system (9) and the regular B_2 and G_2 Toda systems (10), (11).

As anticipated in the introductions, the results we present are based on a variational analysis of the sub-levels J_ρ^{-L} of the energy functional.
We recall that the functional $J_{A_2,\rho}$ is defined by

$$J_{A_2,\rho}(u) := \int_\Sigma Q(u) \mathrm{d}V_g - \sum_{i=1}^{2} \rho_i \left(\log \int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g - \int_\Sigma u_i \mathrm{d}V_g \right),$$

with

$$Q(u) = \frac{1}{3} \left(|\nabla u_1|^2 + \nabla u_1 \cdot \nabla u_2 + |\nabla u_2|^2 \right),$$

whereas J_{B_2}, J_{G_2} have been defined by (7).
To use such variational techniques we need a compactness theorem like 2.16 (or 1.4). This is the reason why we can consider only these three specific systems. We will also have to assume $\rho \notin \Gamma$ or $\rho \notin \Gamma_0$, with Γ, Γ_0 as in Theorem 2.16, which is a generic assumptions due to the construction of Γ, Γ_0.
We will follow a standard scheme which has been widely used for problems with exponential non-linearities. Roughly speaking, we need a non-contractible space $\mathcal{X}$ which roughly resembles very low sub-levels. Then, we build, for large L, two maps $\Phi : \mathcal{X} \to J_\rho^{-L}$, $\Psi : J_\rho^{-L} \to \mathcal{X}$ such that $\Phi \circ \Psi \simeq \mathrm{Id}_\mathcal{X}$. This will prove that J_ρ^{-L} is not contractible, hence, by Corollary 2.17, existence of solutions.

The first result we present is from [11]. We consider the A_2 Toda system on compact surfaces with $\chi(\Sigma) \leq 0$ and non-negative coefficients $\alpha_{im} \geq 0$.
The assumption on the topology of Σ allows to retract it on two disjoint bouquets of circles which do not contain any of the singular points p_m (see Lemma 1.22). Roughly speaking, this permits to study u_1 only on γ_1 and u_2 only on γ_2, thus avoiding both the issue of concentration around singular points and interaction between concentration of u_1 and u_2.
We then "compare" (in the sense describe above) low sub-levels J_ρ^{-L} with the join of the barycenters on γ_1 and γ_2 $\mathcal{X} = \mathcal{X}_{K_1,K_2} := (\gamma_1)_{K_1} \star (\gamma_2)_{K_2}$, which are well-known to be non-contractible (see Lemma 1.25 and Remark 1.29).
In this way, we get existence of solution without any further assumption, and also multiplicity in dependence of the homology groups of $\mathcal{X}$ (see Lemma 1.31).

Theorem 3.1.
Let Σ be a closed surface with $\chi(\Sigma) \le 0$, Γ be as in (2.13) and assume $\alpha_{im} \ge 0$ for all i, m and

$$\rho \in (4K_1\pi, 4(K_1+1)\pi) \times (4K_2\pi, 4(K_2+1)\pi) \setminus \Gamma.$$

Then, the problem (9) has solutions and, for a generic choice of (g, h_1, h_2), it has at least

$$\binom{K_1 + \left[\frac{-\chi(\Sigma)}{2}\right]}{\left[\frac{-\chi(\Sigma)}{2}\right]}\binom{K_2 + \left[\frac{-\chi(\Sigma)}{2}\right]}{\left[\frac{-\chi(\Sigma)}{2}\right]}$$

solutions.

This result was generalized in the paper [10] to allow the coefficients α_{im} to attain negative values. The main issue is due to the fact that negatively-signed singularities actually affect the Moser-Trudinger inequality (2.5). Notice that, in the case of the $SU(3)$ Toda, such an inequality holds if and only if $\rho_i \le 4\pi(1+\widetilde{\alpha}_i)$ for both i's, with $\widetilde{\alpha}_i$ as in (2.4), whereas for the B_2 and G_2 Toda systems it holds for $\rho_1, \rho_2 \le 4\pi$. Precisely, we have

$$4\pi \sum_{i=1}^{2}(1+\widetilde{\alpha}_i)\left(\log\int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g - \int_\Sigma u_i \mathrm{d}V_g\right) \le \int_\Sigma Q_{A_2}(u)\mathrm{d}V_g + C, \tag{3.1}$$

$$4\pi\left(\log\int_\Sigma e^{u_1}\mathrm{d}V_g - \int_\Sigma u_1\mathrm{d}V_g\right) + 2\pi\left(\log\int_\Sigma e^{u_2}\mathrm{d}V_g - \int_\Sigma u_2\mathrm{d}V_g\right) \le \int_\Sigma Q_{B_2}(u)\mathrm{d}V_g + C, \tag{3.2}$$

$$4\pi\left(\log\int_\Sigma e^{u_1}\mathrm{d}V_g - \int_\Sigma u_1\mathrm{d}V_g\right) + \frac{4}{3}\pi\left(\log\int_\Sigma e^{u_2}\mathrm{d}V_g - \int_\Sigma u_2\mathrm{d}V_g\right) \le \int_\Sigma Q_{G_2}(u)\mathrm{d}V_g + C. \tag{3.3}$$

For this reason, negative singularities cannot be “forgotten” as was done in the previous theorem. Conversely, we have to take them into account when we retract of each of the two curves. We want $p \in \gamma_1$ if $\alpha_{1m} < 0$ and $p \in \gamma_2$ if $\alpha_{2m} < 0$.
Clearly, since we must assume $\gamma_1 \cap \gamma_2 = \emptyset$, we cannot have both α_{1m} and α_{2m} negative for the same m.

Following these considerations, it is convenient to divide the points p_m in three subsets, depending on whether α_{1m}, α_{2m} or none of them is negative, and we order each subset so that the respective α_{im} are not decreasing.
Precisely, we write

$$\{p_1, \dots, p_M\} = \left\{p'_{01}, \dots, p'_{0M'_0}, p'_{11}, \dots, p'_{1M'_1}, p'_{21}, \dots, p'_{2M'_2}\right\} \tag{3.4}$$

with $p_m = p'_{im_i}$ for some $i = 1, 2$, $m' = 1, \dots, M'_i$ if and only if $\alpha'_{im'} := \alpha_{im} < 0$ and $\alpha'_{i1} \le \dots \le \alpha'_{iM'_i}$.
We therefore modify the curves γ_i so that each contains, among the singular points, all and only the p'_{im}, as in Lemma 1.22. To take into account such points, we consider the weighted barycenters defined by (1.15). Precisely, we replace $\mathcal{X}_{K_1,K_2}$ with

$$\mathcal{X} = \mathcal{X}_{\rho_1,\underline{\alpha}'_1,\rho_2,\underline{\alpha}'_2} := (\gamma_1)_{\rho_1,\underline{\alpha}'_1} \star (\gamma_2)_{\rho_2,\underline{\alpha}'_2}, \tag{3.5}$$

where the multi-indexes $\underline{\alpha}'_i$ are defined by $\underline{\alpha}'_1 := \left(\alpha'_{11}, \dots, \alpha'_{1M'_1}\right)$, $\underline{\alpha}'_2 := \left(\alpha'_{21}, \dots, \alpha'_{2M'_2}\right)$.
With respect to Theorem 3.1, the weighted barycenters may be contractible, therefore we have to make some extra assumptions to get existence of solutions.

Theorem 3.2.
Let Σ be a closed surface with $\chi(\Sigma) \le 0$, Γ be as in (2.13) and p'_{im}, α'_{im} as before, and $\max\{\alpha_{1m}, \alpha_{2m}\} \ge$

0 for any $m = 1, \dots, M$.
Then, the problem (9) has solutions provided ρ satisfies

$$4\pi\left(K_i + \sum_{m\in\mathcal{M}_i}(1+\alpha'_{im})\right) < \rho_i < 4\pi\left(K_i + \sum_{m\in\mathcal{M}_i\cup\{1\}}(1+\alpha'_{im}),\right) \qquad i=1,2 \tag{3.6}$$

for some $K_i \in \mathbb{N}$ and $\mathcal{M}_i \subset \{2,\dots,M'_i\}$.

Through Morse theory, we again get multiplicity of solutions. Although the statement of next theorem looks quite complicated, it basically says that, the more are the quadruples $(K_1, \mathcal{M}_1, K_2, \mathcal{M}_2)$ for which (3.6) is verified, the higher is the number of solutions.

Theorem 3.3.
Assume the hypotheses of Theorem 3.2 hold, and suppose that for $i = 1, 2$ there exist $L_i, K_{i1}, \dots, K_{iL_i} \in \mathbb{N}$ and $\mathcal{M}_{i1}, \dots, \mathcal{M}_{iL_i} \subset \{2, \dots, M'_i\}$ such that any $l = 1, \dots, L_i$ verifies

$$4\pi\left(K_{im} + \sum_{m\in\mathcal{M}_{im}}(1+\alpha'_{im})\right) < \rho_i < 4\pi\min\left\{K_{im} + \sum_{m\in\mathcal{M}_{im}\cup\{1\}}(1+\alpha'_{im}), K_{im}+1+\sum_{m\in\mathcal{M}_{im}\setminus\{\max\mathcal{M}_{im}\}}(1+\alpha'_{im})\right\}.$$

Then, for a generic choice of $(g, h_1, h_2) \in \mathcal{D}$ (in the sense of Theorem 1.32), the problem (9) has at least

$$\sum_{l_1,l_2}\begin{pmatrix} K_{1l_1} + |\mathcal{M}_{1l_1}| + \left[\frac{-\chi(\Sigma)}{2}\right] \\ |\mathcal{M}_{1l_1}| + \left[\frac{-\chi(\Sigma)}{2}\right]\end{pmatrix}\begin{pmatrix} K_{2l_2} + |\mathcal{M}_{2l_2}| + \left[\frac{-\chi(\Sigma)}{2}\right] \\ |\mathcal{M}_{2l_2}| + \left[\frac{-\chi(\Sigma)}{2}\right]\end{pmatrix}$$

solutions.

Since Theorem 3.1 is a particular case of Theorems 3.2, 3.3, obtained setting $\underline{\alpha'_1} = \underline{\alpha'_2} = \emptyset$, we will just prove the latter two.

The same argument used in the proof of Theorems 3.2, 3.3 will allow to treat some other cases, both again from [10].
First of all, we can remove the hypotheses $\max\{\alpha_{1m}, \alpha_{2m}\} \geq 0$ if we suppose one of the parameter ρ_i to be small enough so that concentration of both component around p_m with $\alpha_{1m}, \alpha_{2m} < 0$ is excluded (hence, in particular, if it is under the coercivity threshold $4\pi(1+\widetilde{\alpha}_2)$). Precisely, this occurs when $\rho_2 < 4\pi(1+\alpha_{2\max})$, with

$$\alpha_{1\max} := \min\{\alpha_{1m} : \max\{\alpha_{1m}, \alpha_{2m}\} < 0\}, \qquad \alpha_{2\max} := \min\{\alpha_{2m} : \max\{\alpha_{1m}, \alpha_{2m}\} < 0\}. \tag{3.7}$$

In this case we have to keep the assumption on $\chi(\Sigma)$ to retract on γ_i.

Theorem 3.4.
Let Σ be a closed surface with $\chi(\Sigma) \leq 0$, Γ be as in (2.13), $\alpha_{i\max}$ as in (3.7) and assume $\rho \notin \Gamma$, $\rho_2 < 4\pi(1+\alpha_{2\max})$ and

$$4\pi\left(K + \sum_{m\in\mathcal{M}_1}(1+\alpha'_{1m})\right) < \rho_1 < 4\pi\left(K + \sum_{m\in\mathcal{M}_1\cup\{1\}}(1+\alpha'_{1m})\right)$$
$$4\pi\sum_{m\in\mathcal{M}_2}(1+\alpha'_{2m}) < \rho_2 < 4\pi\sum_{m\in\mathcal{M}_2\cup\{1\}}(1+\alpha'_{2m}) \tag{3.8}$$

for some $K \in \mathbb{N}$ and $\mathcal{M}_i \subset \{1, \dots, M'_i\}$.
Then, the problem (9) has solutions.

If moreover the condition (3.8) *is satisfied by* $\mathcal{M}_{21}, \ldots, \mathcal{M}_{2L_2}$ *and there exist* $L_1, K_1, \ldots, K_{L_1} \in \mathbb{N}$ *and* $\mathcal{M}_{11}, \ldots, \mathcal{M}_{1L_1} \subset \{2, \ldots, M_1'\}$ *satisfying, for any* $l = 1, \ldots, L_1$,

$$4\pi\left(K_l + \sum_{m\in\mathcal{M}_{1l}}(1+\alpha'_{1m})\right) < \rho_1 < 4\pi\min\left\{K_l + \sum_{m\in\mathcal{M}_{1l}\cup\{1\}}(1+\alpha'_{1m}), K_l + 1 + \sum_{m\in\mathcal{M}_{1l}\setminus\{\max\mathcal{M}_{1l}\}}(1+\alpha'_{1m})\right\},$$

then a generic choice of (g, h_1, h_2) *yields at least*

$$L_2\sum_l \binom{K_l + |\mathcal{M}_{1l}| + \left[\frac{-\chi(\Sigma)}{2}\right]}{|\mathcal{M}_{1l}| + \left[\frac{-\chi(\Sigma)}{2}\right]}$$

solutions.

A similar statement can be proved in the case both ρ_i under $4\pi(1+\alpha_{i\max})$, even without the assumption on $\chi(\Sigma)$. Anyway, such configurations will be covered by a more general theorem which will be stated later on.

In [13], we remove the restriction on the topology of Σ and the coefficients α_{im}.
We perform a sharper analysis of sub-levels, focusing on the case when both u_1 and u_2 concentrate at a point p_m with $\alpha_{1m}, \alpha_{2m} < 0$.
Inspired by the regular Toda system [61], we define a suitable center of mass β_i and scale of concentration ς_i for each component. We then get an *improved Moser-Trudinger inequality* (see Section 3.5), namely we proved that, if $(\beta_1, \varsigma_1) = (\beta_2, \varsigma_2)$, then $J_\rho(u)$ is bounded from below for $\rho_1, \rho_2 < 4\pi(2+\alpha_{1m}+\alpha_{2m})$, namely for values which are higher than the usual coercivity threshold.
For simplicity, we will consider only relatively low values of ρ, in such a way that the space of weighted barycenter is finite and it contains only Dirac deltas centered at points p_m. This will be the case under the following assumptions:

$$\overline{\rho}_1 := 4\pi\min\left\{1, \min_{m\neq m'}(2+\alpha_{1m}+\alpha_{1m'})\right\} \qquad \overline{\rho}_2 := 4\pi\min\left\{1, \min_{m\neq m'}(2+\alpha_{2m}+\alpha_{2m'})\right\}. \quad (3.9)$$

With respect to the previously stated results, we will consider weighted barycenters on the whole surface Σ. Anyway, in view of the improved Moser-Trudinger inequalities, we will have to "puncture" their join in some points.
We will consider the following object:

$$\mathcal{X}' := \Sigma_{\rho_1,\underline{\alpha}_1} \star \Sigma_{\rho_2,\underline{\alpha}_2} \setminus \left\{\left(p_m, p_m, \frac{1}{2}\right) : \rho_1, \rho_2 < 4\pi(2+\alpha_{1m}+\alpha_{2m})\right\}. \quad (3.10)$$

Notice that, by the upper bound (3.9) we are assuming on ρ, only the negative coefficients α_{im} actually play a role, therefore the multi-indexes $\underline{\alpha}_1$, $\underline{\alpha}_2$ could be replaced by $\underline{\alpha}'_1$, $\underline{\alpha}'_2$ introduced before. Anyway, we are no longer allowed to split the set of singular points like (3.4).
As for (3.5), we will have to make some assumptions to ensure $\mathcal{X}'$ is not contractible. In particular, this will depend on the number of points in the two $\Sigma_{\rho_i,\underline{\alpha}_i}$ and on the number of punctures.
We get the following existence result:

Theorem 3.5.
Let Γ *as in* (2.13), $(\overline{\rho}_1, \overline{\rho}_2)$ *be as in* (3.9), *and let* $\rho \in \mathbb{R}^2_{>0} \setminus \Gamma$ *satisfy* $\rho_i < \overline{\rho}_i$ *for both* $i = 1, 2$.
Define integer numbers M_1, M_2, M_3 *by:*

$$M_1 := \#\{m : 4\pi(1+\alpha_{1m}) < \rho_1\} \qquad M_2 := \#\{m : 4\pi(1+\alpha_{2m}) < \rho_2\}$$
$$M_3 := \#\{m : 4\pi(1+\alpha_{im}) < \rho_i \text{ and } \rho_i < 4\pi(2+\alpha_{1m}+\alpha_{2m}) \text{ for both } i = 1, 2\}. \quad (3.11)$$

Then system (9) *has solutions provided the following condition holds*

$$(M_1, M_2, M_3) \notin \{(1, m, 0), (m, 1, 0), (2, 2, 1), (2, 3, 2), (3, 2, 2), m \in \mathbb{N}\}.$$

Finally, in the work in progress [8], we extended, with few modifications, both the existence and multiplicity result of Theorem 3.1 to the regular B_2 and G_2 systems.

Theorem 3.6.
Let Σ be a closed surface with $\chi(\Sigma) \le 0$, Γ_0 be as in (2.13) and assume $\rho \notin 4\pi\mathbb{N} \times \mathbb{R} \cup \mathbb{R} \times 4\pi\mathbb{N}$. Then, the problem (10) admits at least a solution and, if

$$\rho \in (4K_1\pi, 4(K_1+1)\pi) \times (4K_2\pi, 4(K_2+1)\pi), \tag{3.12}$$

for a generic choice of (g, h_1, h_2) it has at least

$$\binom{K_1 + \left[\frac{-\chi(\Sigma)}{2}\right]}{\left[\frac{-\chi(\Sigma)}{2}\right]} \binom{K_2 + \left[\frac{-\chi(\Sigma)}{2}\right]}{\left[\frac{-\chi(\Sigma)}{2}\right]}$$

solutions.
The same results hold true for the system (11), provided $\rho_1 < 4\pi\left(2+\sqrt{2}\right)$, $\rho_2 < 4\pi\left(5+\sqrt{7}\right)$.

This chapter is sub-divided as follows.
In Sections 3.1, 3.2 we prove that the the spaces $\mathcal{X}, \mathcal{X}'$ defined by (3.5), (3.10) are not contractible. In Section 3.3 we build a family of test functions Φ^λ from $\mathcal{X}$ and $\mathcal{X}'$ to arbitrarily low sub-levels of J_ρ. In Sections 3.4 and 3.5 we proved the improved Moser-Trudinger inequalities which lead to the construction of the map $\Phi : J_\rho^{-L} \to \mathcal{X}, \mathcal{X}'$. Finally, in Section 3.6, we put together all these result to prove the theorems.

3.1 Topology of the space $\mathcal{X}$

In this section, we will provide information about the topology and the homology of the space $\mathcal{X} = (\gamma_1)_{\rho_1,\underline{\alpha}'_1} \star (\gamma_2)_{\rho_2,\underline{\alpha}'_2}$.
First of all, we notice that most information can be deduced by studying the weighted barycenters spaces $(\gamma_i)_{\rho_i,\underline{\alpha}'_i}$. Proposition 1.16 shows how the homology groups of the join depend on the ones of the spaces which form it.
Some of the results contained in this section will be inspired by [20], where weighted barycenters centered at Σ are studied.
As pointed out in Remark 1.30, the join of two spaces, one of which is contractible, is itself contractible. Therefore, for our purposes, we will just need to give conditions under which both the spaces $(\gamma_i)_{\rho_i,\underline{\alpha}'_i}$ are contractible.

In the following, we will omit the indices $i = 1, 2$ and consider a generic weighted barycenters set $(\gamma)_{\rho,\underline{\alpha}'}$ with the multi-indices $\alpha' = (\alpha'_1, \dots, \alpha'_{M'})$ such that $\alpha'_m \le \alpha'_{m+1}$ and singular points $p'_1, \dots, p'_{M'}$ satisfy $\omega_{\underline{\alpha}'}(p'_{m'}) = 1 + \alpha'_m < 1$.
To start with, following [20] we consider $(\gamma)_{\rho,\underline{\alpha}'}$ as a union of strata of the kind

$$(\gamma)^{K,\mathcal{M}} = \left\{ \sum_{k=1}^{K} t_k \delta_{q_k} + \sum_{m\in\mathcal{M}} t'_m \delta_{p'_m} : q_k \in \Sigma,\, t_k \ge 0,\, t'_m \ge 0,\, \sum_{k=1}^{K} t_k + \sum_{m\in\mathcal{M}} t'_m = 1 \right\},$$

for $K \in \mathbb{N} \cup \{0\}$, $\mathcal{M} \subset \{1, \dots, M'\}$.
One can easily notice that each of these strata is a union of manifolds whose maximal dimension is $2K + |\mathcal{M}| - 1$. Considering only the strata which are maximal with respect to the inclusion, we write a unique decomposition

$$(\gamma)_{\rho,\underline{\alpha}'} = \bigcup_{l=1}^{L} (\gamma)^{K_l,\mathcal{M}_l}. \tag{3.13}$$

It is easy to see how the strata depend on the position of ρ with respect to the α'_m's. A stratum $(\gamma)^{K,\mathcal{M}}$ is contained in $(\gamma)_{\rho,\underline{\alpha}'}$ if and only if

$$\rho > 4\pi\left(K + \sum_{m\in\mathcal{M}}(1+\alpha'_m)\right). \tag{3.14}$$

Moreover, we notice that a stratum $(\gamma)^{K,\mathcal{M}}$ is contained in $(\gamma)^{K',\mathcal{M}'}$ if and only if $|\mathcal{M}\backslash\mathcal{M}'| \le K'-K$. Therefore, the maximality of an existing stratum is equivalent to the condition

$$\rho \le 4\pi \min\left\{K+1+\sum_{m\in\mathcal{M}\backslash\{\max\mathcal{M}\}}(1+\alpha'_m), K+\sum_{m\in\mathcal{M}\cup\{\min(\{1,\dots,M\}\backslash\mathcal{M})\}}(1+\alpha'_m)\right\},$$

and the equality sign is excluded if we take $\rho \notin \Gamma$.
Notice that in the regular case the decomposition in maximal strata is just $(\gamma)_{\rho,\emptyset} = (\gamma)^{K,\emptyset} = (\gamma)_K$, with K such that $\rho \in (4K\pi, 4(K+1)\pi)$, and all the strata are of the kind $(\gamma)^{K',\emptyset} = (\gamma)^{K'}$ for $K' = 1,\dots,K$.
However, in the regular case Proposition 1.25 gives already full information about homology of the barycenters.
In the general case the decomposition in strata makes more difficult the computation of the homology groups. Nonetheless, we can still obtain information on the homology of $(\gamma)_{\rho,\underline{\alpha}'}$ with an estimate from below of its Betti numbers.
Precisely, we will prove the following result:

Theorem 3.7.
Suppose $(\gamma)_{\rho,\underline{\alpha}'}$ has the following decomposition in maximal strata:

$$(\gamma)_{\rho,\underline{\alpha}'} = \bigcup_{l=1}^{L}(\gamma)^{K_l,\mathcal{M}_l} \cup \bigcup_{l'=1}^{L'}(\gamma)^{K'_{l'},\mathcal{M}'_{l'}}, \tag{3.15}$$

with $1 \notin \mathcal{M}_l$ for any $l = 1,\dots,L$. Then,

$$\widetilde{b}_q\left((\gamma)_{\rho,\underline{\alpha}'}\right) \ge \sum_{l=1}^{L}\binom{K_l + |\mathcal{M}_l| + \left[\frac{-\chi(\Sigma)}{2}\right]}{|\mathcal{M}_l| + \left[\frac{-\chi(\Sigma)}{2}\right]}\delta_{q,2K_l+\mathcal{M}_l-1}.$$

In particular, if $l \ge 1$, then $\widetilde{b}_q\left((\gamma)_{\rho,\underline{\alpha}'}\right) \ne 0$ for some $q \ne 0$.

We will start by analyzing the cases which are not covered by the previous theorem, that is when every maximal stratum is defined by a multi-index containing the index 1.
In this case, we find out that $(\gamma)_{\rho,\underline{\alpha}'}$ is contractible, so in conclusion we get a necessary and sufficient condition for the contractibility of $(\gamma)_{\rho,\underline{\alpha}'}$.

Lemma 3.8.
Suppose $(\gamma)_{\rho,\underline{\alpha}'}$ has the decomposition (3.13) in maximal strata, with $p_1,\dots,p_{M'}$ such that $\alpha'_1 \le \dots \le \alpha'_{M'}$. Then, the following conditions are equivalent:

(1) $(\gamma)_{\rho,\underline{\alpha}'}$ is star-shaped with respect to δ_{p_1}.

(2) There exists some $m \in \{1,\dots,M'\}$ such that $(\gamma)_{\rho,\underline{\alpha}'}$ is star-shaped with respect to δ_{p_m}.

(3) $(\gamma)^{K_l,\mathcal{M}_l}$ is star-shaped with respect to δ_{p_1} for any $l \in \{1,\dots,L\}$.

(4) There exists some $m \in \{1,\dots,M'\}$ such that $(\gamma)^{K_l,\mathcal{M}_l}$ is star-shaped with respect to δ_{p_m} for any $l \in \{1,\dots,L\}$.

Moreover, each of these conditions implies that $(\gamma)_{\rho,\underline{\alpha}'}$ is contractible.

Proof.
The contractibility of $(\gamma)_{\rho,\underline{\alpha}'}$ follows trivially from its star-shapedness, so it suffices to prove the equivalences between the conditions.
The following implications are immediate:

$$(1) \Rightarrow (2), \qquad (3) \Rightarrow (1), \qquad (3) \Rightarrow (4), \qquad (4) \Rightarrow (2);$$

therefore, we suffice to show that (2) implies (1) and (1) implies (3).
We will start by showing that if $(\gamma)_{\rho,\underline{\alpha}'}$ is star-shaped with respect to some $p'_{\widetilde{m}}$, then the same holds with p_1.
We notice immediately that star-shapedness of $(\gamma)_{\rho,\underline{\alpha}'}$ is equivalent to saying that for any stratum $(\gamma)^{K,\mathcal{M}} \subset (\gamma)_{\rho,\underline{\alpha}'}$ we have $(\gamma)^{K,\mathcal{M}\cup\widetilde{m}} \subset (\gamma)_{\rho,\underline{\alpha}'}$; moreover, we recall that the existence of a stratum within $(\gamma)_{\rho,\underline{\alpha}'}$ means (3.14). Let us now suppose condition *2* occurs for $\widetilde{m} > 1$, that is

$$\rho > 4\pi\left(K + \sum_{m\in\mathcal{M}}(1+\alpha'_m)\right) \qquad \Rightarrow \qquad \rho > 4\pi\left(K + \sum_{m\in\mathcal{M}\cup\{\widetilde{m}\}}(1+\alpha'_m)\right),$$

and let us recall that we are assuming $\alpha'_m \le \alpha'_{m+1}$ for any m. This implies

$$\rho > 4\pi\left(K + \sum_{m\in\mathcal{M}\cup\{\widetilde{m}\}}(1+\alpha'_m)\right) \ge 4\pi\left(K + \sum_{m\in\mathcal{M}\cup\{1\}}(1+\alpha'_m)\right),$$

that is star-shapedness of $(\gamma)_{\rho,\underline{\alpha}'}$ with respect to p_1.
Suppose now, by contradiction, that condition (3) holds but condition (1) does not, that is $(\gamma)_{\rho,\underline{\alpha}'}$ is star-shaped with respect to p_1 but it contains a maximal stratum $(\gamma)^{K,\mathcal{M}}$ which is not.
Then, star-shapedness of $(\gamma)_{\rho,\underline{\alpha}'}$ with respect to δ_{p_1} implies the existence of a stratum $(\gamma)^{K,\mathcal{M}\cup\{1\}} \subset (\gamma)_{\rho,\underline{\alpha}'}$, which contains properly $(\gamma)^{K,\mathcal{M}}$, thus contradicting its maximality. $\square$

Let us now see what happens if we are in a scenario which is opposite to the previous lemma, that is some index j is not contained in any multi-index which defines the strata.
The following lemma shows that this situation produces some non-trivial homology.

Lemma 3.9.
Suppose $K \in \mathbb{N}$, $\mathcal{M} \subset \{1,\dots,M'\}$ and $\widetilde{m} \notin \mathcal{M}$ and define

$$(\gamma)^{K,\mathcal{M},\widetilde{m}} := \bigcup_{\mathcal{M}'\subset\mathcal{M}\cup\{\widetilde{m}\},\,|\mathcal{M}'|=|\mathcal{M}|}(\gamma)^{K,\mathcal{M}}.$$

Then, it holds

$$\widetilde{H}_q\left((\gamma)^{K,\mathcal{M},\widetilde{m}}\right) = \begin{cases} \mathbb{Z}^{\binom{K+|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]}{|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]}} & \text{if } q = 2K+|\mathcal{M}|-1 \\ 0 & \text{if } q \neq 2K+|\mathcal{M}|-1 \end{cases}.$$

The proof of the lemma will use the Mayer-Vietoris exact sequence.
Actually, when applying the Mayer-Vietoris sequence the sets A and B should be open. If they are not, we are implicitly considering two suitable open neighborhoods in their stead.
The existence of such neighborhoods follows from the properties of the weighted barycenters, which can be deduced by arguing as in [20], Section 2 and [21], Section 3.

Proof.
We proceed by double induction on K and $|\mathcal{M}|$.
If $\mathcal{M}=\emptyset$ we have $(\gamma)^{K,\emptyset,m}=(\gamma)^{K,\emptyset}=(\gamma)_K$ so the claim follows by Proposition 1.25.
If $K=0$, any stratum $(\gamma)^{0,\mathcal{M}'}$ is actually the $(|\mathcal{M}'|-1)$-simplex $\left[\delta_{p_{m_1}},\ldots,\delta_{p_{m_{|\mathcal{M}'|}}}\right]$ if we can write $\mathcal{M}'=\left\{m_1,\ldots,m_{|\mathcal{M}'|}\right\}$. Therefore, $(\gamma)^{0,\mathcal{M},\widetilde{m}}$ is the boundary of the $|\mathcal{M}|$-simplex with vertices in δ_{p_m} for $m\in\mathcal{M}\cup\{\widetilde{m}\}$; hence, it is homeomorphic to the sphere $\mathbb{S}^{|\mathcal{M}|-1}$ and the claim follows also in this case.
Suppose now that the lemma is true for $K-1,\mathcal{M}$ and for any $K,\mathcal{M}_0$ with $|\mathcal{M}_0|=|\mathcal{M}|-1$.
Being $(\gamma)^{K,\mathcal{M},\widetilde{m}}$ union of manifolds of dimension less or equal to $2K+|\mathcal{M}|-1$, all the higher homology groups are trivial.
To compute the other groups, we write $(\gamma)^{K,\mathcal{M},\widetilde{m}}=A\cup B$ with

$$A=(\gamma)^{K,\mathcal{M}},\qquad B=\bigcup_{m\in\mathcal{M}}(\gamma)^{K,\mathcal{M}\setminus m\cup\{\widetilde{m}\}}$$

and consider the Mayer-Vietoris sequence. The set B is star-shaped with respect to $\delta_{p'_{\widetilde{m}}}$ whereas A is star-shaped with respect to δ_{p_m} for any $m\in\mathcal{M}$, hence we can write

$$0=\widetilde{H}_q(A)\oplus\widetilde{H}_q(B)\to\widetilde{H}_q(A\cup B)\to\widetilde{H}_{q-1}(A\cap B)\to\widetilde{H}_{q-1}(A)\oplus\widetilde{H}_{q-1}(B)=0,$$

that is $\widetilde{H}_q(A\cup B)=\widetilde{H}_{q-1}(A\cap B)$. Moreover, this set can be written as

$$A\cap B=C\cup D,\qquad C:=(\gamma)^{K-1,\mathcal{M}\cup\{\widetilde{m}\}},\qquad D:=\bigcup_{m\in\mathcal{M}}(\gamma)^{K,\mathcal{M}\setminus\{m\}}.$$

As before, C is contractible, whereas we can write $D=(\gamma)^{K,\mathcal{M}\setminus\{m\},\{m\}}$ for any $m\in\mathcal{M}$ and $C\cap D=(\gamma)^{K-1,\mathcal{M},\widetilde{m}}$. Therefore, by inductive hypothesis we know the homology of these sets and we can apply again Mayer-Vietoris. If $q<2K+|\mathcal{M}|-1$ we get

$$0=\widetilde{H}_{q-1}(C)\oplus\widetilde{H}_{q-1}(D)\to\widetilde{H}_{q-1}(C\cup D)\to\widetilde{H}_{q-2}(C\cap D)\to\widetilde{H}_{q-2}(C)\oplus\widetilde{H}_{q-2}(D)=0,$$

that is

$$\widetilde{H}_q(A\cup B)=\widetilde{H}_{q-1}(A\cap B)=\widetilde{H}_{q-1}(C\cup D)=\widetilde{H}_{q-2}(C\cap D)=0.$$

Finally, for the last homology group we get

$$0=\widetilde{H}_{2K+|\mathcal{M}|-2}(C\cap D)\to\widetilde{H}_{2K+|\mathcal{M}|-2}(C)\oplus\widetilde{H}_{2K+|\mathcal{M}|-2}(D)\to\widetilde{H}_{2K+|\mathcal{M}|-2}(C\cup D)\to$$

$$\to\widetilde{H}_{2K+|\mathcal{M}|-3}(C\cap D)\to\widetilde{H}_{2K+|\mathcal{M}|-3}(C)\oplus\widetilde{H}_{2K+|\mathcal{M}|-3}(D)=0.$$

Hence, by the inductive hypothesis and the properties of binomial coefficients,

$$\begin{aligned}
&\widetilde{H}_{2K+|\mathcal{M}|-1}(A\cup B)\\
=\;&\widetilde{H}_{2K+|\mathcal{M}|-2}(C\cup D)\\
=\;&\widetilde{H}_{2K+|\mathcal{M}|-2}(D)\oplus\widetilde{H}_{2K+|\mathcal{M}|-3}(C\cap D)\\
=\;&\mathbb{Z}^{\binom{K+|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]-1}{|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]-1}}\oplus\mathbb{Z}^{\binom{K+|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]-1}{|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]}}\\
=\;&\mathbb{Z}^{\binom{K+|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]-1}{|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]-1}+\binom{K+|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]-1}{|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]}}\\
=\;&\mathbb{Z}^{\binom{K+|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]}{|\mathcal{M}|+\left[\frac{-\chi(\Sigma)}{2}\right]}},
\end{aligned}$$

which is what we wanted. ∎

Finally, we see how the sets defined in the previous lemma affect the homology of $(\gamma)_{\rho,\underline{\alpha}'}$.

Proof of Theorem 3.7.
We proceed by induction on L. If $L = 0$ there is nothing to prove.
Suppose now the theorem holds true for $L-1$. Then it also holds for H when $q \neq 2K_L + |\mathcal{M}_L| - 1$. For $q = 2K_L + |\mathcal{M}_L| - 1$, we notice that $(\gamma)^{K_L,\mathcal{M}_L,1} \subset (\gamma)_{\rho,\underline{\alpha}'}$, since the coefficients α'_M are non-increasing; hence we can apply Mayer-Vietoris sequence by writing $(\gamma)_{\rho,\underline{\alpha}'} = A \cup B$ with

$$A = (\gamma)^{K_L,\mathcal{M}_L,1}, \qquad B = \bigcup_{l=1}^{L-1} (\gamma)^{K_h,\mathcal{M}_l} \cup \bigcup_{l'=1}^{L'} (\gamma)^{K'_{l'},\mathcal{M}'_{l'}}.$$

By a dimensional argument we have $\widetilde{H}_{2K_L+|\mathcal{M}_L|-1}(A \cap B) = 0$, so we get

$$0 = \widetilde{H}_{2K_L+|\mathcal{M}_L|-1}(A \cap B) \to \widetilde{H}_{2K_L+|\mathcal{M}_L|-1}(A) \oplus \widetilde{H}_{2K_L+|\mathcal{M}_L|-1}(B) \to \widetilde{H}_{2K_L+|\mathcal{M}_L|-1}(A \cup B) \to \dots$$

which means, by the exactness of the Mayer-Vietoris sequence,

$$\widetilde{H}_{2K_L+|\mathcal{M}_L|-1}(A) \oplus \widetilde{H}_{2K_L+|\mathcal{M}_L|-1}(B) \hookrightarrow \widetilde{H}_{2K_L+|\mathcal{M}_L|-1}(A \cup B).$$

Therefore, applying the inductive hypothesis and Lemma 3.9, we get

$$\begin{aligned}
& \widetilde{b}_{2K_L+|\mathcal{M}_L|-1}(A \cup B) \\
\geq \; & \widetilde{b}_{2K_L+|\mathcal{M}_L|-1}(A) + \widetilde{b}_{2K_L+|\mathcal{M}_L|-1}(B) \\
\geq \; & \binom{K + |\mathcal{M}| + \left[\frac{-\chi(\Sigma)}{2}\right]}{|\mathcal{M}| + \left[\frac{-\chi(\Sigma)}{2}\right]} + \sum_{l=1}^{L-1} \binom{K_l + |\mathcal{M}_l| + \left[\frac{-\chi(\Sigma)}{2}\right]}{|\mathcal{M}_l| + \left[\frac{-\chi(\Sigma)}{2}\right]} \delta_{2K_L+|\mathcal{M}_L|-1, 2K_l+\mathcal{M}_l-1} \\
= \; & \sum_{l=1}^{L} \binom{K_l + |\mathcal{M}_l| + \left[\frac{-\chi(\Sigma)}{2}\right]}{|\mathcal{M}_l| + \left[\frac{-\chi(\Sigma)}{2}\right]} \delta_{2K_L+|\mathcal{M}_L|-1, 2K_l+\mathcal{M}_l-1},
\end{aligned}$$

hence the claim. $\square$

Finally, by Proposition 1.16, we get some information on the homology of the join.

Corollary 3.10.
Suppose $(\gamma_i)_{\rho_i,\underline{\alpha}'_i}$ has the decomposition (3.15) in maximal strata, with $L_i, K_1, \dots, K_{L_i} \in \mathbb{N}$ and $\mathcal{M}_{i1}, \dots, \mathcal{M}_{iL_i} \subset \{1, \dots, L_i\}$. Then, it holds

$$\sum_{q=0}^{+\infty} \widetilde{b}_q(\mathcal{X}) \geq \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \binom{K_{l_1} + |\mathcal{M}_{l_1}| + \left[\frac{-\chi(\Sigma)}{2}\right]}{|\mathcal{M}_{l_1}| + \left[\frac{-\chi(\Sigma)}{2}\right]} \binom{K_{l_2} + |\mathcal{M}_{l_2}| + \left[\frac{-\chi(\Sigma)}{2}\right]}{|\mathcal{M}_{l_2}| + \left[\frac{-\chi(\Sigma)}{2}\right]}.$$

In particular, if $L_1, L_2 \geq 1$, then $\widetilde{b}_q(\mathcal{X}) \neq 0$ for some $q \neq 0$.

3.2 Topology of the space $\mathcal{X}'$

In this Section, we will prove that, under the assumptions of Theorem 3.5, the space $\mathcal{X}'$ defined by (3.10) is not contractible. In particular, we will prove that it has a non-trivial homology group.
By the assumption $\rho_i \leq \overline{\rho}_i$, the weighted barycenters $\Sigma_{\rho_i,\underline{\alpha}_i}$ are actually discrete sets, hence their join will be just a finite union of segments. Therefore $\mathcal{X}'$ will be a quite simple object, especially compared with the spaces studied in the previous section. Some examples are pictured in Figures 3.1 and 3.2.
The main result of this section is the following:

Theorem 3.11.
Let M_1, M_2, M_3 be as in (3.11) and $\mathcal{X}'$ be as in (3.10) and suppose

$$(M_1, M_2, M_3) \notin \{(1, m, 0), (m, 1, 0), (2, 2, 1), (2, 3, 2), (3, 2, 2),\ m \in \mathbb{N}\}. \tag{3.16}$$

Then, the space $\mathcal{X}'$ has non-trivial homology groups. In particular, it is not contractible.

The assumptions on the M_1, M_2, M_3, that is, respectively on the cardinality of $\Sigma_{\rho_1,\underline{\alpha}_1}$, $\Sigma_{\rho_2,\underline{\alpha}_2}$ and on the number of midpoints to be removed, are actually sharp.
This can be seen clearly from the Figure 3.1: the configurations $M_1 = 1$, $M_3 = 0$ are star-shaped, and even in the two remaining case it is easy to see $\mathcal{X}'$ has trivial topology. On the other hand, Figure 2 shows a non-contractible configuration.

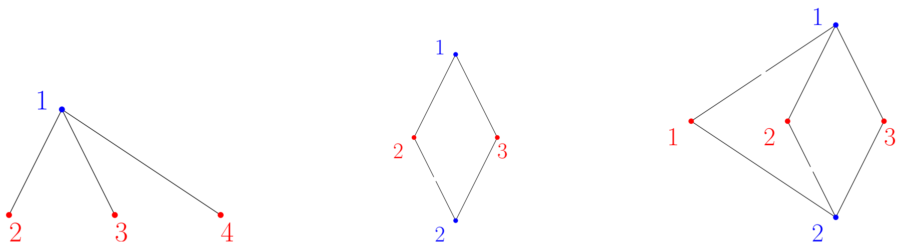

Figure 3.1: The space $\mathcal{X}'$ in the cases $(M_1, M_2, M_3) \in \{(1, 3, 0), (2, 2, 1), (2, 3, 2)\}$ (contractible).

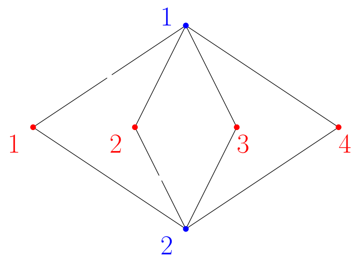

Figure 3.2: The space $\mathcal{X}'$ in the case $M_1 = 2$, $M_2 = 4$, $M_3 = 2$ (not contractible).

Proof of Theorem 3.11.
The spaces $\Sigma_{\rho_i,\underline{\alpha}_i}$ are discrete sets of M_i points, for $i = 1, 2$, that is a wedge sum of $M_i - 1$ copies of $\mathbb{S}^0$. Therefore, by Theorem 1.28, $\Sigma_{\rho_1,\underline{\alpha}_1} \star \Sigma_{\rho_2,\underline{\alpha}_2}$ has the same homology as $\left(\mathbb{S}^1\right)^{\vee(M_1-1)(M_2-1)}$.
The set we have to remove from the join is made up by M_3 singular points $\{p_{m_1}, \dots, p_{m_{M_3}}\}$ for some $\{m_1, \dots, m_{M_3}\} \subset \{1, \dots, M\}$.
Defining then, for some fixed $\delta < \frac{1}{2}$, $\mathcal{Y} := \bigcup_{j=1}^{M_3} B_\delta\left(p_{m_j}, p_{m_j}, \frac{1}{2}\right)$, $\mathcal{Y}$ retracts on $\{p_{m_1}, \dots, p_{m_{M_3}}\}$.
On the other hand, $\mathcal{X}' \cap \mathcal{Y}$ is a disjoint union of M_3 punctured intervals, that is a discrete set of $2M_3$ points, and $\mathcal{X}' \cup \mathcal{Y}$ is the whole join. Therefore, the Mayer-Vietoris sequence yields

$$\underbrace{H_1(\mathcal{X}' \cap \mathcal{Y})}_{0} \to H_1(\mathcal{X}') \oplus \underbrace{H_1(\mathcal{Y})}_{0} \to \underbrace{H_1(\mathcal{X}' \cup \mathcal{Y})}_{\mathbb{Z}^{(M_1-1)(M_2-1)}} \to \underbrace{\widetilde{H}_0(\mathcal{X}' \cap \mathcal{Y})}_{\mathbb{Z}^{2M_3-1}} \to \widetilde{H}_0(\mathcal{X}') \oplus \underbrace{\widetilde{H}_0(\mathcal{Y})}_{\mathbb{Z}^{M_3-1}} \to \underbrace{\widetilde{H}_0(\mathcal{X}' \cup \mathcal{Y})}_{0}.$$

The exactness of the sequence implies that $b_1(\mathcal{X}') - \widetilde{b}_0(\mathcal{X}') = (M_1 - 1)(M_2 - 1) - M_3$, so if the latter number is not zero we get at least a non-trivial homology group.
Simple algebraic computations show that, under the assumption $M_1, M_2 \geq M_3$, $(M_1 - 1)(M_2 - 1) \neq M_3$ is equivalent to (3.16), therefore the proof is complete. $\square$

3.3 Test functions

In this section, we will introduce some test function from the spaces $\mathcal{X}, \mathcal{X}'$ to arbitrarily low sub-levels of J_ρ.

Such test functions will be mostly inspired to the standard bubbles (2.17) introduced in Chapter 2, though with several modifications.
As a first thing, in both the case of $\mathcal{X}$ and $\mathcal{X}'$, we have to be careful about the two endpoints of the join, that is when one of the two weighted barycenters is identified to a point. In this case, the test functions must depend only on the elements of the other barycenters set.
When we are dealing with $(\gamma_i)_{\rho_i,\underline{\alpha}_i}$, we have to interpolate the parameter $\alpha_i(x)$, which cannot switch suddenly from 0 to α'_{im} in presence of a singular point p'_{im}. Moreover, here we will take smooth bubbles rather than truncated because, since they are centered in more than one point, truncating does not simplify calculations.

Concerning the cases considered in Theorem 3.5, we will need two more profiles for the construction of test functions. These profile are quite a natural choice, since they resemble the entire solutions of the singular Liouville equation and of the A_2 Toda system:

$$\varphi'^{\lambda,x} = -2\log\max\left\{1, (\lambda d(\cdot,x))^{2(2+\alpha_1(x)+\alpha_2(x))}\right\}$$

$$\varphi''^{\lambda,x} = -2\log\max\left\{1, \lambda^{2(2+\alpha_1(x)+\alpha_2(x))} d(\cdot,x)^{2(1+\alpha_1(x))}\right\}.$$

This is because, when u_1 and u_2 are centered in the same points, a higher amount of energy is due to the expression of $Q(u)$ which penalizes parallel gradients.

The test functions needed in Theorem 3.6 are very similar to the ones used in Theorem 3.2, though simpler in their definition because of the lack of singularities.

Since the explicit definition of such test functions is quite lengthy, it will be postponed in the proof of the theorems, rather than in its statement.

Theorem 3.12.
Let $\mathcal{X}$ be defined by (3.5).
Then, there exists a family of maps $\left\{\Phi^\lambda\right\}_{\lambda>2} : \mathcal{X} \to H^1(\Sigma)^2$ such that

$$J_{A_2,\rho}\left(\Phi^\lambda(\zeta)\right) \underset{\lambda\to+\infty}{\longrightarrow} -\infty \qquad \textit{uniformly for } \zeta \in \mathcal{X}.$$

Proof.
To define Φ^λ we first fix a $\delta > 0$ sufficiently small to make negligible the interaction between the singular points p_m. A suitable choice is:

$$\delta = \min\left\{\min_{i=1,2,m=1,\dots,M'_0} d(\gamma_i, p'_{0m}), \frac{\min_{i=1,2,m\neq m'=1,\dots,M'_i} d(p'_{im}, p'_{im'})}{2}\right\}. \tag{3.17}$$

We then need to define, as stated before, an exponent $\beta_i(x)$ which interpolates between 0 and α'_{im} for $x \in \gamma_i$ near a point p'_{im}: we define

$$\beta_1(x) = \begin{cases} 0 & \text{if } d := \min_m d(x, p'_{1m}) \geq \delta \\ \dfrac{\alpha'_{1\widetilde{m}} \log\frac{\delta}{d}}{\log\max\{2,\lambda(1-t)\} - \alpha'_{1\widetilde{m}}\log\frac{\delta}{d}} & \text{if } d = d(x, p'_{1\widetilde{m}}) \in \left[\max\{2,\lambda(1-t)\}^{-\frac{1}{1+\alpha'_{1\widetilde{m}}}}\delta, \delta\right) \\ \alpha'_{1\widetilde{m}} & \text{if } d < \max\{2,\lambda(1-t)\}^{-\frac{1}{1+\alpha'_{1\widetilde{m}}}}\delta \end{cases}. \tag{3.18}$$

and similarly $\beta_2(x)$.
Such a choice will verify the condition

$$\max\{2,\lambda(1-t)\}^{\frac{\beta_1(x)}{\alpha_1\widetilde{m}(1+\beta_1(x))}}=\frac{\delta}{d}\qquad\qquad \max\{2,\lambda t\}^{\frac{\beta_2(x)}{\alpha_2\widetilde{m}(1+\beta_2(x))}}=\frac{\delta}{d}.$$

Given an element

$$\sigma_i=\sum_{x_{ik}\in\mathcal{J}_i}t_{ik}\delta_{x_{ik}}\in(\gamma_i)_{\rho_i,\underline{\alpha}_i'}\qquad\text{for } i=1,2,\qquad \zeta=(\sigma_1,\sigma_2,t)\in\mathcal{X}$$

we define, for $\lambda>0$,

$$\Phi^\lambda(\zeta)=\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac{\varphi_1}{2}\right),$$

with

$$\begin{aligned}\varphi_1=\varphi_1^\lambda(\zeta)&=\log\sum_{x_{1k}\in\mathcal{J}_1}\frac{t_{1k}}{\left(1+(\lambda(1-t))^2d(\cdot,x_{1k})^{2(1+\beta_{1k})}\right)^2}\\ \varphi_2=\varphi_2^\lambda(\zeta)&=\log\sum_{x_{2k}\in\mathcal{J}_2}\frac{t_{2k}}{\left(1+(\lambda t)^2d(\cdot,x_{2k})^{2(1+\beta_{2k})}\right)^2},\end{aligned}\tag{3.19}$$

with $\beta_{ik}:=\beta_i(x_{ik})$.
The proof of this theorem will be a consequence of the three following lemmas, each of which provides estimates for a different part of the functional $J_{A_2,\rho}$. □

Lemma 3.13.
Let ζ, $\Phi^\lambda(\zeta)$ be as in the proof of Theorem 3.12. Then,

$$\int_\Sigma Q_{A_2}\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac{\varphi_1}{2}\right)\mathrm{d}V_g\le 8\pi\omega_{\underline{\alpha}_1'}(\mathcal{J}_1)\log\max\{1,(\lambda(1-t))\}+8\pi\omega_{\underline{\alpha}_2'}(\mathcal{J}_2)\log\max\{1,\lambda t\}+C.$$

Proof.
First of all, we write

$$Q_{A_2}\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac{\varphi_1}{2}\right)=\frac{1}{4}\left(|\nabla\varphi_1|^2-\nabla\varphi_1\cdot\nabla\varphi_2+|\nabla\varphi_2|^2\right).\tag{3.20}$$

Since $|\nabla d(\cdot,x_{ik})|=1$ almost everywhere, then

$$\begin{aligned}&|\nabla\varphi_1|\\ =&\left|\frac{\sum_k\frac{-4(1+\beta_{1k})t_{1k}(\lambda(1-t))^2d(\cdot,x_{1k})^{1+2\beta_{1k}}\nabla d(\cdot,x_{1k})}{\left(1+(\lambda(1-t))^2d(\cdot,x_{1k})^{2(1+\beta_1(x_{ik}))}\right)^2}}{\sum_k\frac{t_{1k}}{1+(\lambda(1-t))^2d(\cdot,x_{1k})^{2(1+\beta_{1k})}}}\right|\\ \le&\frac{\sum_k\frac{4(1+\beta_{1k})t_{1k}(\lambda(1-t))^2d(\cdot,x_{1k})^{1+2\beta_{1k}}}{\left(1+(\lambda(1-t))^2d(\cdot,x_{1k})^{2(1+\beta_1(x_{ik}))}\right)^2}}{\sum_k\frac{t_{1k}}{1+(\lambda(1-t))^2d(\cdot,x_{1k})^{2(1+\beta_1(x_k))}}}\\ \le&\max_k\underbrace{\frac{4(1+\beta_{1k})(\lambda(1-t))^2d(\cdot,x_{1k})^{1+2\beta_{1k}}}{1+(\lambda(1-t))^2d(\cdot,x_{1k})^{2(1+\beta_{1k})}}}_{=:M_{1k}}.\end{aligned}\tag{3.21}$$

In view of these estimates, we divide Σ into a finite number of regions depending on which of the M_{1k}'s attains the maximum:

$$\Omega_{1k}:=\left\{x\in\Sigma:M_{1k}(x)=\max_{k'}M_{1k'}(x)\right\}.$$

By similar estimates on $|\nabla\varphi_2|$ we get $|\nabla\varphi_2|\le M_{2k}$ and we will define:

$$\Omega_{2k}:=\left\{x\in\Sigma: M_{2k}(x)=\max_{k'} M_{2k'}(x)\right\}.$$

Moreover, we can easily see that the following estimates hold for M_{ik}:

$$M_{1k}\le\begin{cases}\dfrac{4\,(1+\beta_{1k})}{d\,(\cdot,x_{1k})}\\ 4(1+\beta_{1k})(\lambda(1-t))^2 d(\cdot,x_{1k})^{1+2\beta_{1k}}\end{cases}, \tag{3.22}$$

$$M_{2k}\le\begin{cases}\dfrac{4\,(1+\beta_{2k})}{d\,(\cdot,x_{2k})}\\ 4(1+\beta_{2k})(\lambda t)^2 d(\cdot,x_{2k})^{1+2\beta_{2k}}\end{cases}.$$

We will estimate the mixed term first. Basically, since the points x_{ik} belong to γ_i and the curves γ_i's are disjoint, we only have summable singularities and therefore the integral of $\nabla\varphi_1\cdot\nabla\varphi_2$ is uniformly bounded.
Therefore, from (3.21) and the first inequality in (3.22), one finds

$$\begin{aligned}&\left|\int_\Sigma \nabla\varphi_1\cdot\nabla\varphi_2 \mathrm{d}V_g\right|\\ \le\ &\sum_{k,k'}\int_{\Omega_{1k}\cap\Omega_{2k'}}|\nabla\varphi_1||\nabla\varphi_2|\mathrm{d}V_g\\ \le\ &\sum_{k,k'}16(1+\beta_{1k})(1+\beta_{2k})\int_{\Omega_{1k}\cap\Omega_{2k'}}\frac{\mathrm{d}V_g}{d(\cdot,x_{1k})d(\cdot,x_{2k'})}.\end{aligned}$$

We then notice that, by the definition of (3.17), the distance between γ_1 and γ_2 is at least 2δ, so $B_\delta(x_{1k})\cap B_\delta(x_{2k'})=\emptyset$ for any choice of k,k'. Therefore,

$$\begin{aligned}&\int_{\Omega_{1k}\cap\Omega_{2k'}}\frac{\mathrm{d}V_g}{d(\cdot,x_{1k})d(\cdot,x_{2k'})}\\ \le\ &\int_{\Omega_{1k}\cap\Omega_{2k'}\setminus B_\delta(x_{1k})}\frac{\mathrm{d}V_g}{\delta d(\cdot,x_{2k'})}+\int_{\Omega_{1k}\cap\Omega_{2k'}\setminus B_\delta(x_{2k'})}\frac{\mathrm{d}V_g}{\delta d(\cdot,x_{1k})}\\ \le\ &\frac{1}{\delta}\int_\Sigma\left(\frac{1}{d(\cdot,x_{2k'})}+\frac{1}{d(\cdot,x_{1k})}\right)\mathrm{d}V_g\\ \le\ &C_\delta,\end{aligned}$$

hence, being the number of k,k' bounded from above depending on ρ and α'_{im}'s only, we obtain

$$\left|\int_\Sigma \nabla\varphi_1\cdot\nabla\varphi_2 \mathrm{d}V_g\right|\le C. \tag{3.23}$$

Now, we consider the term involving $|\nabla\varphi_1|^2$. We split the integral into the sets Ω_{1k} defined above.

$$\begin{aligned}&\int_\Sigma|\nabla\varphi_1|^2\mathrm{d}V_g\\ \le\ &\sum_k\int_{\Omega_{1k}}M_{1k}^2\mathrm{d}V_g\\ \le\ &\sum_k\left(\int_{\Sigma\setminus B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})}M_{1k}^2\mathrm{d}V_g+\int_{B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})}M_{1k}^2\mathrm{d}V_g\right)\end{aligned} \tag{3.24}$$

Outside the balls we will apply the first estimate in (3.21):

$$\int_{\Sigma\setminus B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})}|\nabla\varphi_1|^2\mathrm{d}V_g$$

$$
\begin{aligned}
&\leq 16(1+\beta_{1k})^2 \int_{\Sigma\setminus B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})} \frac{\mathrm{d}V_g}{d(\cdot,x_{1k})^2} \\
&\leq 32\pi(1+\beta_{1k})^2 \log\max\left\{1,(\lambda(1-t))^{\frac{1}{1+\beta_{1k}}}\right\}+C \\
&\leq 32\pi(1+\beta_{1k})\log\max\{1,(\lambda(1-t))\}+C. \qquad (3.25)
\end{aligned}
$$

The integral inside the balls is actually uniformly bounded, as can be seen using now the second estimate in (3.21):

$$
\begin{aligned}
&\int_{B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})} |\nabla\varphi_1|^2 \mathrm{d}V_g \\
\leq\ & 16(1+\beta_{1k})^2(\lambda(1-t))^4 \int_{B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})} d(\cdot,x_{1k})^{2(1+2\beta_{1k})}\mathrm{d}V_g \\
\leq\ & C_{\beta_{1k}}(\lambda(1-t))^4\left((\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}\right)^{4(1+\beta_{1k})} \\
\leq\ & C \qquad (3.26)
\end{aligned}
$$

Observing that, from the definitions of (1.15) and (3.18), one has $\sum_k(1+\beta_{1k})\leq\omega_{\underline{\alpha}'_1}(\mathcal{J}_1)$, one can now deduce from (3.24), (3.25), (3.26):

$$
\int_\Sigma |\nabla\varphi_1|^2 \leq 32\pi\omega_{\underline{\alpha}'_1}(\mathcal{J}_1)\log\max\{1,(\lambda(1-t))\}+C. \qquad (3.27)
$$

The same argument gives a similar estimate for $\int_\Sigma|\nabla\varphi_2|^2$, therefore putting together (3.27) with (3.23) and (3.20) we get the conclusion. □

Lemma 3.14.
Let ζ, φ_i be as in the proof of Theorem 3.12. Then,

$$
\int_\Sigma \varphi_1 \mathrm{d}V_g = -4\log\max\{1,\lambda(1-t)\}+O(1), \qquad \int_\Sigma \varphi_2\mathrm{d}V_g = -4\log\max\{1,\lambda t\}+O(1).
$$

Proof.
We will give the proof for $i=1$, since the argument for the case $i=2$ is the same.
From the elementary inequality $\max\{1,x\}\leq 1+x\leq 2\max\{1,x\}$ for $x\geq 0$ we deduce

$$
v_1=\log\sum_{x_{1k}\in\mathcal{J}_1}\frac{t_{1k}}{\max\left\{1,\lambda(1-t)d(\cdot,x_{1k})^{1+\beta_{1k}}\right\}^4}+O(1).
$$

Therefore, we can give an estimate from above:

$$
\begin{aligned}
&\int_\Sigma \varphi_1\mathrm{d}V_g \\
\leq\ & \int_\Sigma \log\max_k\left\{\frac{1}{\max\left\{1,\lambda(1-t)d(\cdot,x_{1k})^{1+\beta_{1k}}\right\}^4}\right\}\mathrm{d}V_g+C \\
=\ & \int_\Sigma \log\min\left\{1,\frac{1}{(\lambda(1-t))^4\min_k d(\cdot,x_{1k})^{4(1+\beta_{1k})}}\right\}\mathrm{d}V_g+C \\
=\ & \int_{\Sigma\setminus\bigcup_k B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})} \log\frac{1}{(\lambda(1-t))^4\min_k d(\cdot,x_{1k})^{4(1+\beta_{1k})}}+C
\end{aligned}
$$

$$
\begin{aligned}
&= -4\log\left(la(1-t)\left|\Sigma\setminus\bigcup_k B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})\right|\right)\\
&- 4\int_{\Sigma\setminus\bigcup_k B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})} \min_k\{(1+\beta_{1k})\log d(\cdot,x_{1k})\}+C\\
&\le -4\log\max\{1,\lambda(1-t)\}+C.
\end{aligned}
$$

We conclude the proof getting, in a similar way, a lower estimate:

$$
\begin{aligned}
&\int_\Sigma \varphi_1 \mathrm{d}V_g\\
\ge\; &\int_\Sigma \log\min_k\left\{\frac{1}{\max\left\{1,\lambda(1-t)d(\cdot,x_{1k})^{1+\beta_{1k}}\right\}^4}\right\}\mathrm{d}V_g + C\\
=\; &\int_\Sigma \log\min\left\{1,\frac{1}{(\lambda(1-t))^4\max_k d(\cdot,x_{1k})^{4(1+\beta_{1k})}}\right\}\mathrm{d}V_g + C\\
=\; &\int_{\Sigma\setminus\bigcap_k B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})} \log\frac{1}{(\lambda(1-t))^4\max_k d(\cdot,x_{1k})^{4(1+\beta_{1k})}}+C\\
=\; &-4\log\left(\lambda(1-t)\left|\Sigma\setminus\bigcap_k B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})\right|\right)\\
-\; &4\int_{\Sigma\setminus\bigcap_k B_{(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}}(x_{1k})} \max_k\{(1+\beta_{1k})\log d(\cdot,x_{1k})\}+C\\
\ge\; &-4\log\max\{1,\lambda(1-t)\}+C.
\end{aligned}
$$

□

Lemma 3.15.
Let ζ, φ_i be as in the proof of Theorem 3.12. Then,

$$
\log\int_\Sigma \widetilde{h}_1 e^{\varphi_1-\frac{\varphi_2}{2}}\mathrm{d}V_g = -2\log\max\{1,\lambda(1-t)\}+2\log\max\{1,\lambda t\}+O(1).
$$

$$
\log\int_\Sigma \widetilde{h}_1 e^{\varphi_2-\frac{\varphi_1}{2}}\mathrm{d}V_g = -2\log\max\{1,\lambda t\}+2\log\max\{1,\lambda(1-t)\}+O(1).
$$

Proof.
As in the proof of Lemma 3.14, it is not restrictive to suppose $i=1$.
It is easy to notice that

$$
\int_{\Sigma\setminus\bigcup_k B_\delta(x_{1k})} \widetilde{h}_1 e^{\varphi_1}\mathrm{d}V_g \le \frac{C}{\max\{1,\lambda(1-t)\}^4}.
$$

Moreover, since

$$
\varphi_2 \ge -4\log\max\{1,\lambda t\}-C \quad \text{on } \Sigma, \qquad \varphi_2 \le -4\log\max\{1,\lambda t\}+C \quad \text{on } \bigcup_k B_\delta(x_{1k}),
$$

then we can write

$$
\int_\Sigma \widetilde{h}_1 e^{\varphi_1-\frac{\varphi_2}{2}}\mathrm{d}V_g \sim C\max\{1,\lambda t\}^2\sum_k t_{1k}\int_{B_\delta(x_{1k})}\frac{\widetilde{h}_1}{\left(1+(\lambda(1-t))^2 d(\cdot,x_{1k})^{2(1+\beta_{1k})}\right)^2}\mathrm{d}V_g.
$$

Therefore, the lemma will follow by showing that, for any k, the following holds true:

$$
\int_{B_\delta(x_{1k})}\frac{\widetilde{h}_1}{\left(1+(\lambda(1-t))^2 d(\cdot,x_{1k})^{2(1+\beta_{1k})}\right)^2}\mathrm{d}V_g \sim \frac{1}{\max\{1,\lambda(1-t)\}^2}.
$$

We will provide the estimate in three cases depending on how β_{1k} is defined in (3.18).
In the first case the ball $B_\delta(x_{1k})$ does not contain any singular point, therefore using normal coordinates and a change of variables we get

$$\begin{aligned}
& \int_{B_\delta(x_{1k})} \frac{\widetilde{h}_1}{\left(1+(\lambda(1-t))^2 d(\cdot,x_{1k})^{2(1+\beta_{1k})}\right)^2} \mathrm{d}V_g \\
\sim & \int_{B_\delta(0)} \frac{dx}{\left(1+(\lambda(1-t))^2|x|^2\right)^2} \\
\sim & \frac{1}{(\lambda(1-t))^2} \int_{B_{\lambda(1-t)\delta}(0)} \frac{dy}{(1+|y|^2)^2} \\
\sim & \frac{1}{\max\{1,\lambda(1-t)\}^2}.
\end{aligned}$$

If instead x_{1k} is distant at most δ from a point $p = p'_{1\widetilde{m}}$ with a singularity $\alpha = \alpha'_{1\widetilde{m}}$, by the definition of δ this will be the only singular point in the ball we are considering, so arguing as before we find:

$$\begin{aligned}
& \int_{B_\delta(x_{1k})} \frac{\widetilde{h}_1}{\left(1+(\lambda(1-t))^2 d(\cdot,x_{1k})^{2(1+\beta_{1k})}\right)^2} \mathrm{d}V_g \\
\sim & \int_{B_\delta(0)} \frac{|x-p|^{2\alpha}}{\left(1+(\lambda(1-t))^2|x|^{2(1+\beta_{1k})}\right)^2} dx \\
\sim & \frac{1}{(\lambda(1-t))^2} \int_{B_{(\lambda(1-t))^{\frac{1}{1+\beta_{1k}}}\delta}(0)} \frac{\left|(\lambda(1-t))^{\frac{\beta_{1k}-\alpha}{(1+\beta_{1k})\alpha}} y - (\lambda(1-t))^{\frac{\beta_{1k}}{(1+\beta_{1k})\alpha}} p\right|^{2\alpha}}{\left(1+|y|^{2(1+\beta_{1k})}\right)^2} dy
\end{aligned}$$

To conclude the proof, it will suffice to show that the last integral is bounded from above and below when $\lambda(1-t)$ is large, since the starting quantity is clearly bounded if $\lambda(1-t)$ is small. If we are in the second alternative in the definition of (3.18), we have

$$\int_{B_{\lambda(1-t)\left(\frac{\delta}{|p|}\right)^\alpha}(0)} \frac{\left|\frac{1}{\lambda(1-t)}\left(\frac{\delta}{|p|}\right)^{1+\alpha} y - \delta\frac{p}{|p|}\right|^{2\alpha}}{\left(1+|y|^{2(1+\beta_{1k})}\right)^2} dy,$$

which is uniformly bounded because in this case $\delta(\lambda(1-t))^{-\frac{1}{1+\alpha}} \le |p| \le \delta$, so the radius of the ball is greater or equal to $\lambda(1-t)$ and the quantity which multiplies y is less or equal to 1.
Finally, when $\beta_{1k} = \alpha$, we find

$$\int_{B_{(\lambda(1-t))^{\frac{1}{1+\alpha}}\delta}(0)} \frac{\left|y - (\lambda(1-t))^{\frac{1}{1+\alpha}} p\right|^{2\alpha}}{\left(1+|y|^{2(1+\alpha)}\right)^2} dy,$$

which again is bounded because this time the vector preceded by the minus sign has a norm smaller or equal than δ. □

Proof of Theorem 3.12, continued.
Since on $(\gamma_i)_{\rho_i,\underline{\alpha}'_i}$ one has $\rho_i < 4\pi\omega_{\underline{\alpha}'_i}(\mathcal{J}_i)$, and moreover $\max\{\lambda(1-t),\lambda t\} \ge \dfrac{\lambda}{2}$, Lemmas 3.13, 3.14 and 3.15 yield

$$\begin{aligned}
& J_\rho\left(\varphi_1 - \frac{\varphi_2}{2}, \varphi_2 - \frac{\varphi_1}{2}\right) \\
= & \int_\Sigma Q\left(\varphi_1 - \frac{\varphi_2}{2}, \varphi_2 - \frac{\varphi_1}{2}\right) \mathrm{d}V_g - \sum_{i=1}^2 \rho_i \left(\log \int_\Sigma \widetilde{h}_i e^{\varphi_i - \frac{\varphi_{3-i}}{2}} \mathrm{d}V_g - \int_\Sigma \left(\varphi_i - \frac{\varphi_{3-i}}{2}\right) \mathrm{d}V_g\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \left(8\pi\omega_{\underline{\alpha}_1'}(\mathcal{J}_1)-2\rho_1\right)\log\max\{1,\lambda(1-t)\}+\left(8\pi\omega_{\underline{\alpha}_2'}(\mathcal{J}_2)-\rho_2\right)\log\max\{1,\lambda t\}+C\\
&\le \max\{8\pi\omega_{\underline{\alpha}_1'}(\mathcal{J}_1)-2\rho_1,8\pi\omega_{\underline{\alpha}_2'}(\mathcal{J}_2)-2\rho_2\}\log\max\{1,\lambda(1-t),\lambda t\}+C\\
&\le \max\{8\pi\omega_{\underline{\alpha}_1'}(\mathcal{J}_1)-2\rho_1,8\pi\omega_{\underline{\alpha}_2'}(\mathcal{J}_2)-2\rho_2\}\log\max\{1,\lambda\}+C\\
&\underset{\lambda\to+\infty}{\longrightarrow} -\infty
\end{aligned}
$$

uniformly in $\zeta\in\mathcal{X}$, which is what we wished to prove. □

The following theorem will use truncated test functions centered at only one point, therefore it will use some estimates from Lemma 2.23.

Theorem 3.16.
Let $\mathcal{X}'$ be defined by (3.10).
Then, there exists a family of maps $\left\{\Phi'^{\lambda}\right\}_{\lambda>2}:\mathcal{X}'\to H^1(\Sigma)^2$ such that

$$J_{A_2,\rho}\left(\Phi'^{\lambda}(\zeta)\right)\underset{\lambda\to+\infty}{\longrightarrow}-\infty \qquad \textit{uniformly for } \zeta\in\mathcal{X}'.$$

Proof.
Let us start by defining $\Phi^{\lambda}(\zeta)=\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac{\varphi_1}{2}\right)$ when $\zeta=(p_m,p_m,t)$ for some m. Φ^{λ} will be defined in different ways, depending on the relative positions of $\rho_1,\rho_2,\alpha_{1m},\alpha_{2m}$ in $\mathbb{R}$.

(<<) $\rho_1,\rho_2<4\pi(2+\alpha_{1m}+\alpha_{2m})$:

$$
\varphi_1:=\begin{cases}-2\log\max\left\{1,(\lambda d(\cdot,p_m))^{2(1+\alpha_{1m})}\right\} & \text{if } t<\frac12\\ 0 & \text{if } t>\frac12\end{cases}
$$
$$
\varphi_2:=\begin{cases}0 & \text{if } t<\frac12\\ -2\log\max\left\{1,(\lambda d(\cdot,p_m))^{2(1+\alpha_{2m})}\right\} & \text{if } t>\frac12.\end{cases}
$$

(<>) $\rho_1<4\pi(2+\alpha_{1m}+\alpha_{2m})<\rho_2$:

$$
\begin{aligned}
\varphi_1&:=-2\log\max\left\{1,\max\left\{1,(\lambda t)^{2(1+\alpha_{2m})}\right\}(\lambda d(\cdot,p_m))^{2(1+\alpha_{1m})}\right\}\\
\varphi_2&:=-2\log\max\left\{1,(\lambda t d(\cdot,p_m))^{2(2+\alpha_{1m}+\alpha_{2m})}\right\}.
\end{aligned}
$$

(><) $\rho_2<4\pi(2+\alpha_{1m}+\alpha_{2m})<\rho_1$:

$$
\begin{aligned}
\varphi_1&:=-2\log\max\left\{1,(\lambda(1-t)d(\cdot,p_m))^{2(2+\alpha_{1m}+\alpha_{2m})}\right\}\\
\varphi_2&:=-2\log\max\left\{1,\max\left\{1,(\lambda(1-t))^{2(1+\alpha_{1m})}\right\}(\lambda d(\cdot,p_m))^{2(1+\alpha_{2m})}\right\}.
\end{aligned}
$$

(>>) $\rho_1,\rho_2>4\pi(2+\alpha_{1m}+\alpha_{2m})$:

$$
\begin{aligned}
\varphi_1&:=-2\log\max\left\{1,\left(\lambda\frac{\max\{1,\lambda t\}}{\max\{1,\lambda(1-t)\}}\right)^{2+\alpha_{1m}+\alpha_{2m}}d(\cdot,p_m)^{2(1+\alpha_{1m})},(\lambda d(\cdot,p_m))^{2(2+\alpha_{1m}+\alpha_{2m})}\right\}\\
\varphi_2&:=-2\log\max\left\{1,\left(\lambda\frac{\max\{1,\lambda(1-t)\}}{\max\{1,\lambda t\}}\right)^{2+\alpha_{1m}+\alpha_{2m}}d(\cdot,p_m)^{2(1+\alpha_{2m})},(\lambda d(\cdot,p_m))^{2(2+\alpha_{1m}+\alpha_{2m})}\right\}.
\end{aligned}
$$

By arguing as in the proof of Lemma 2.23, we deduce the following estimates, which will be presented in three separates lemmas. $\square$

Lemma 3.17.
Let φ_1, φ_2 be as in Theorem 3.16.
Then, in each case the following estimates hold true for $\mathcal{Q} := \int_\Sigma Q_{A_2}\left(\varphi_1 - \frac{\varphi_2}{2}, \varphi_2 - \frac{\varphi_1}{2}\right) \mathrm{d}V_g$:

(<<)

$$\mathcal{Q} = \begin{cases} 8\pi(1+\alpha_{1m})^2 \log\lambda + O(1) & \text{if } t < \frac{1}{2} \\ 8\pi(1+\alpha_{2m})^2 \log\lambda + O(1) & \text{if } t > \frac{1}{2} \end{cases}.$$

(<>)

$$\mathcal{Q} = 8\pi(2+\alpha_{1m}+\alpha_{2m})^2 \log\max\{1, \lambda t\} + 8\pi(1+\alpha_{1m})^2 \log\min\left\{\lambda, \frac{1}{t}\right\} + O(1).$$

(><)

$$\mathcal{Q} = 8\pi(2+\alpha_{1m}+\alpha_{2m})^2 \log\max\{1, \lambda(1-t)\} + 8\pi(1+\alpha_{2m})^2 \log\min\left\{\lambda, \frac{1}{1-t}\right\} + O(1).$$

(>>)

$$\mathcal{Q} = 8\pi(2+\alpha_{1m}+\alpha_{2m})^2 \log\lambda + O(1).$$

Lemma 3.18.
Let φ_1, φ_2 be as above. Then, in each case we have:

(<<)

$$\int_\Sigma \varphi_1 \mathrm{d}V_g = \begin{cases} -4(1+\alpha_{1m}) \log\lambda + O(1) & \text{if } t < \frac{1}{2} \\ 0 & \text{if } t > \frac{1}{2} \end{cases}$$

$$\int_\Sigma \varphi_2 \mathrm{d}V_g = \begin{cases} 0 & \text{if } t < \frac{1}{2} \\ -4(1+\alpha_{2m}) \log\lambda + O(1) & \text{if } t > \frac{1}{2}. \end{cases}$$

(<>)

$$\int_\Sigma \varphi_1 \mathrm{d}V_g = -4(1+\alpha_{1m}) \log\lambda - 4(1+\alpha_{2m}) \log\max\{1, \lambda t\} + O(1),$$
$$\int_\Sigma \varphi_2 \mathrm{d}V_g = -4(2+\alpha_{1m}+\alpha_{2m}) \log\max\{1, \lambda t\} + O(1).$$

(><)

$$\int_\Sigma \varphi_1 \mathrm{d}V_g = -4(2+\alpha_{1m}+\alpha_{2m}) \log\max\{1, \lambda(1-t)\} + O(1),$$
$$\int_\Sigma \varphi_2 \mathrm{d}V_g = -4(1+\alpha_{2m}) \log\lambda - 4(1+\alpha_{1m}) \log\max\{1, \lambda(1-t)\} + O(1).$$

(>>)

$$\int_\Sigma \varphi_1 \mathrm{d}V_g = \int_\Sigma \varphi_2 \mathrm{d}V_g + O(1) = -4(2+\alpha_{1m}+\alpha_{2m})\log\lambda + O(1).$$

Lemma 3.19. *Let φ_1, φ_2 be as above. Then, in each case we have:*

(<<)

$$\log \int_\Sigma \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g = \begin{cases} -2(1+\alpha_{1m})\log\lambda + O(1) & \textit{if } t < \frac{1}{2} \\ 2(1+\alpha_{2m})\log\lambda + O(1) & \textit{if } t > \frac{1}{2} \end{cases},$$

$$\log \int_\Sigma \widetilde{h}_2 e^{\varphi_2 - \frac{\varphi_1}{2}} \mathrm{d}V_g = \begin{cases} 2(1+\alpha_{1m})\log\lambda + O(1) & \textit{if } t < \frac{1}{2} \\ -2(1+\alpha_{2m})\log\lambda + O(1) & \textit{if } t > \frac{1}{2}. \end{cases}.$$

(<>)

$$\log \int_\Sigma \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g = -2(1+\alpha_{1m})\log\lambda - 2(1+\alpha_{2m})\log\max\{1,\lambda t\},$$

$$\log \int_\Sigma \widetilde{h}_2 e^{\varphi_2 - \frac{\varphi_1}{2}} \mathrm{d}V_g = 2(1+\alpha_{1m})\min\left\{\lambda, \frac{1}{t}\right\}.$$

(><)

$$\log \int_\Sigma \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g = 2(1+\alpha_{2m})\min\left\{\lambda, \frac{1}{1-t}\right\},$$

$$\log \int_\Sigma \widetilde{h}_2 e^{\varphi_2 - \frac{\varphi_1}{2}} \mathrm{d}V_g = -2(1+\alpha_{2m})\log\lambda - 2(1+\alpha_{1m})\log\max\{1,\lambda(1-t)\}.$$

(>>)

$$\log \int_\Sigma \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g = -(2+\alpha_{1m}+\alpha_{2m})\log\left(\lambda\frac{\max\{1,\lambda t\}}{\max\{1,\lambda(1-t)\}}\right) + O(1)$$

$$\log \int_\Sigma \widetilde{h}_2 e^{\varphi_2 - \frac{\varphi_1}{2}} \mathrm{d}V_g = -(2+\alpha_{1m}+\alpha_{2m})\log\left(\lambda\frac{\max\{1,\lambda(1-t)\}}{\max\{1,\lambda t\}}\right) + O(1)$$

Proof of Theorem 3.16, continued.
We can now easily prove the Theorem in the case $x_1 = x_2$. In fact, by the explicit expression of J_ρ, we get, in each case,

(<<)

$$J_\rho\left(\varphi_1 - \frac{\varphi_2}{2}, \varphi_2 - \frac{\varphi_1}{2}\right) = \begin{cases} 2(1+\alpha_{1m})(4\pi(2+\alpha_{1m}) - \rho_1)\log\lambda + O(1) & \text{if } t < \frac{1}{2} \\ 2(1+\alpha_{2m})(4\pi(2+\alpha_{2m}) - \rho_2)\log\lambda + O(1) & \text{if } t > \frac{1}{2} \end{cases},$$

(<>)

$$\begin{aligned} & J_\rho\left(\varphi_1 - \frac{\varphi_2}{2}, \varphi_2 - \frac{\varphi_1}{2}\right) \\ = \; & 2(1+\alpha_{1m})(4\pi(1+\alpha_{1m}) - \rho_1)\log\min\left\{\lambda, \frac{1}{t}\right\} \\ + \; & 2(2+\alpha_{1m}+\alpha_{2m})(4\pi(2+\alpha_{1m}+\alpha_{2m}) - \rho_2)\log\max\{1,\lambda t\} + O(1), \end{aligned}$$

(><)

$$\begin{aligned}
&J_\rho\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac{\varphi_1}{2}\right)\\
=\;& 2(2+\alpha_{1m}+\alpha_{2m})(4\pi(2+\alpha_{1m}+\alpha_{2m})-\rho_1)\log\max\{1,\lambda(1-t)\}\\
+\;& 2(1+\alpha_{2m})(4\pi(1+\alpha_{2m})-\rho_2)\log\min\left\{\lambda,\frac{1}{1-t}\right\}+O(1),
\end{aligned}$$

(>>)

$$\begin{aligned}
&J_\rho\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac{\varphi_1}{2}\right)\\
=\;& (2+\alpha_{1m}+\alpha_{2m})(4\pi(2+\alpha_{1m}+\alpha_{2m})-\rho_1)\log\left(\lambda\frac{\max\{1,\lambda(1-t)\}}{\max\{1,\lambda t\}}\right)\\
+\;& (2+\alpha_{1m}+\alpha_{2m})(4\pi(2+\alpha_{1m}+\alpha_{2m})-\rho_2)\log\left(\lambda\frac{\max\{1,\lambda t\}}{\max\{1,\lambda(1-t)\}}\right)+O(1),
\end{aligned}$$

which all tend to $-\infty$ independently of t.

Let us now consider the case $x_1\neq x_2$.
Here, we define Φ^λ just by interpolating linearly between the test functions defined before:

$$\Phi^\lambda(x_1,x_2,t)=\Phi^{\lambda(1-t)}(x_1,x_1,0)+\Phi^{\lambda t}(x_2,x_2,1).$$

Since $d(p_m,p_{m'})\geq\delta>0$, then the bubbles centered at p_m and $p_{m'}$ do not interact, therefore the estimates from Lemmas 3.17, 3.18, 3.19 also work for such test functions.
We will show this fact in detail in the case $\rho_1,\rho_2<4\pi(2+\alpha_{1m}+\alpha_{2m}),4\pi(2+\alpha_{1m'}+\alpha_{2m'})$.
Writing

$$(\varphi_1,\varphi_2)=\left(-2\log\max\left\{1,(\lambda(1-t)d(\cdot,p_m))^{2(1+\alpha_{1m})}\right\},-2\log\max\left\{1,(\lambda t d(\cdot,p_{m'}))^{2(1+\alpha_{2m'})}\right\}\right),$$

by the previous explicit computation of $\nabla\varphi_1,\nabla\varphi_2$ we get

$$\begin{aligned}
&\mathcal{Q}\\
=\;&\frac14\int_{B_\delta(p_m)}|\nabla\varphi_1|^2\mathrm{d}V_g+\frac14\int_{B_\delta(p_{m'})}|\nabla\varphi_2|^2\mathrm{d}V_g+O(1)\\
=\;&8\pi(1+\alpha_{1m})^2\log\max\{1,\lambda(1-t)\}+8\pi(2+\alpha_{2m'})^2\log\max\{1,\lambda t\}+O(1).
\end{aligned}\tag{3.28}$$

Moreover, by linearity,

$$\int_\Sigma\varphi_1\mathrm{d}V_g=-4(1+\alpha_{1m})\log\max\{1,\lambda(1-t)\}+O(1)$$

$$\int_\Sigma\varphi_2\mathrm{d}V_g=-4(1+\alpha_{2m'})\log\max\{1,\lambda t\}+O(1).\tag{3.29}$$

Finally, as before the integral of $\widetilde{h}_1e^{\varphi_1-\frac{\varphi_2}{2}}$ is negligible outside $B_\delta(p_m)$, and inside the ball we have $\frac{1}{C_\delta}\leq\left|\varphi_2-\int_\Sigma\varphi_2\mathrm{d}V_g\right|\leq C$ on $B_\delta(p_m)$, hence

$$\begin{aligned}
&\log\int_\Sigma\widetilde{h}_1e^{\varphi_1-\frac{\varphi_2}{2}}\mathrm{d}V_g\\
=\;&\log\left(\max\{1,\lambda t\}^{2(1+\alpha_{2m'})}\int_{B_{\frac{1}{\max\{1,\lambda(1-t)\}}}(p_m)}d(\cdot,p_m)^{2\alpha_{1m}}\mathrm{d}V_g\right.\\
+\;&\left.\max\{1,\lambda t\}^{2(1+\alpha_{2m'})}\max\{1,\lambda(1-t)\}^{2(1+\alpha_{1m})}\int_{A_{\frac{1}{\lambda},\delta}(p_m)}\frac{\mathrm{d}V_g}{d(\cdot,p_m)^{2(2+\alpha_{1m})}}\right)+O(1)
\end{aligned}$$

$$= \quad 2(1+\alpha_{2m'})\log\max\{1,\lambda t\} - 2(1+\alpha_{1m})\log\max\{1,\lambda(1-t)\} + O(1) \tag{3.30}$$

and similarly

$$\log\int_\Sigma \widetilde{h}_2 e^{\varphi_2-\frac{\varphi_1}{2}}\mathrm{d}V_g = 2(1+\alpha_{1m})\log\max\{1,\lambda(1-t)\} - 2(1+\alpha_{1m'})\log\max\{1,\lambda t\} + O(1).$$

Therefore, by (3.28), (3.29) and (3.30) we deduce

$$\begin{aligned} J_\rho\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac{\varphi_1}{2}\right) &= 2(1+\alpha_{1m})(4\pi(1+\alpha_{1m})-\rho_1)\log\max\{1,\lambda(1-t)\} \\ &+ 2(1+\alpha_{2m'})(4\pi(1+\alpha_{2m'})-\rho_2)\log\max\{1,\lambda t\} + O(1). \end{aligned}$$

This concludes the proof. □

To prove the last main result of this section we will not need new auxiliary lemmas, but rather we will use the ones used to prove Theorem 3.12.

Theorem 3.20.
Let $\mathcal{X}''$ be defined by

$$\mathcal{X}'' = \mathcal{X}''_{K_1,K_2} := (\gamma_1)_{K_1} \star (\gamma_2)_{K_2}, \tag{3.31}$$

with K_1, K_2 such that (3.12) holds.
Then, there exist two a family of maps $\{\Phi^\lambda_{B_2}\}_{\lambda>2}$, $\{\Phi^\lambda_{G_2}\}_{\lambda>2} : \mathcal{X}'' \to H^1(\Sigma)^2$ such that

$$\left.\begin{aligned} J_{B_2,\rho}\left(\Phi^\lambda_{B_2}(\zeta)\right) &\underset{\lambda\to+\infty}{\longrightarrow} -\infty \\ J_{G_2,\rho}\left(\Phi^\lambda_{G_2}(\zeta)\right) &\underset{\lambda\to+\infty}{\longrightarrow} -\infty \end{aligned}\right. \qquad \textit{uniformly in } \zeta\in\mathcal{X}''.$$

Proof.
We define

$$\Phi^\lambda_{B_2}(\zeta) = \left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\varphi_1\right) \qquad \Phi^\lambda_{G_2}(\zeta) = \left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac{3}{2}\varphi_1\right),$$

with

$$\varphi_1 = \varphi_1^\lambda(\zeta) = \log\sum_{k=1}^{K_1}\frac{t_{1k}}{(1+(\lambda(1-t)d(\cdot,x_{1k}))^2)^2} \qquad \varphi_2 = \varphi_2^\lambda(\zeta) = \log\sum_{k=1}^{K_2}\frac{t_{2k}}{(1+(\lambda t d(\cdot,x_{2k}))^2)^2}.$$

Notice that the φ_1, φ_2 are defined in the very same way as in Theorem 3.12 when there are no singularities.
Since there holds

$$Q_{B_2}\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\varphi_1\right) = \frac14\int_\Sigma|\nabla\varphi_1|^2\mathrm{d}V_g - \frac14\int_\Sigma\nabla\varphi_1\cdot\nabla\varphi_2\mathrm{d}V_g + \frac18\int_\Sigma|\nabla\varphi_2|^2\mathrm{d}V_g$$

$$Q_{G_2}\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac32\varphi_1\right) = \frac14\int_\Sigma|\nabla\varphi_1|^2\mathrm{d}V_g - \frac14\int_\Sigma\nabla\varphi_1\cdot\nabla\varphi_2\mathrm{d}V_g + \frac{1}{12}\int_\Sigma|\nabla\varphi_2|^2\mathrm{d}V_g,$$

then arguing as in Lemma 3.13 gives

$$\int_\Sigma Q_{B_2}\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\varphi_1\right)\mathrm{d}V_g \le 8K_1\pi\log\max\{1,\lambda(1-t)\} + 4K_2\pi\log\max\{1,\lambda t\} + C$$

$$\int_\Sigma Q_{G_2}\left(\varphi_1-\frac{\varphi_2}{2},\varphi_2-\frac32\varphi_1\right)\mathrm{d}V_g \le 8K_1\pi\log\max\{1,\lambda(1-t)\} + \frac83K_2\pi\log\max\{1,\lambda t\} + C.$$

Moreover, from Lemma 3.14 we deduce

$$\int_\Sigma\varphi_1\mathrm{d}V_g = -4\log\max\{1,\lambda(1-t)\} + O(1), \qquad \int_\Sigma\varphi_2\mathrm{d}V_g = -4\log\max\{1,\lambda t\} + O(1).$$

Finally, the same argument as Lemma 3.15 yields

$$\log \int_\Sigma h_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g = -2\log\max\{1, \lambda(1-t)\} + 2\log\max\{1,\lambda t\} + O(1),$$

$$\log \int_\Sigma h_2 e^{\varphi_2 - \varphi_1} \mathrm{d}V_g = -2\log\max\{1, \lambda t\} + 4\log\max\{1,\lambda (1-t)\} + O(1),$$

$$\log \int_\Sigma h_2 e^{\varphi_2 - \frac{3}{2}\varphi_1} \mathrm{d}V_g = -2\log\max\{1, \lambda t\} + 6\log\max\{1,\lambda (1-t)\} + O(1).$$

From these estimates the theorem follows easily. □

3.4 Macroscopic improved Moser-Trudinger inequalities

Here we will perform an analysis of the sub-levels of the functional J_ρ, which will permit to prove the existence of a map $\Psi = J_\rho^{-L} \to \mathcal{X}$ in the cases covered by Theorems 3.2, 3.6.

We prove an improved Moser-Trudinger inequality, that is we show that, under certain conditions on the spreading of u_1 and u_2, the constant in Moser-Trudinger inequalities (3.1), (3.2), (3.3) can be improved. This fact will give, after some technical work, some information about the sub-levels.

Theorem 3.21.
Let Γ, Γ_0 as in (2.13), $f_{i,u}$ as in (1.13) and $\rho \in \mathbb{R}^2_{>0} \setminus \Gamma$ be given.
Then, for any $\varepsilon > 0$ there exists $L = L_\varepsilon > 0$ such that any $u \in J^{-L}_{A_2,\rho}$ verifies, for some $i = 1, 2$,

$$d_{\mathrm{Lip}'}\left(f_{i,u}, \Sigma_{\rho_i, \underline{\alpha}'_i}\right) < \varepsilon.$$

The same holds true if $\rho \in \mathbb{R}^2_{>0} \setminus \Gamma_0$ and $u \in J^{-L}_{B_2,\rho}$ or $u \in J^{-L}_{G_2,\rho}$.

To adapt the original argument to the case of Toda system we first need a covering lemma, inspired by [59], [61].
With respect to the previous works, we have to take into account the singularities and consider sets which contain at most one negative singularity.

Lemma 3.22.
Let $\delta > 0, J_1, K_1, J_2, K_2 \in \mathbb{N}$ be given numbers, $\{m_{i1}, \ldots, m_{iJ_i}\} \subset \{1, \ldots, M'_i\}$ selections of indices, $f_1, f_2 \in L^1(\Sigma)$ be non-negative functions with $\int_\Sigma f_i \mathrm{d}V_g = 1$ and $\{\Omega_{ij}\}_{i=1,2}^{j=1,\ldots,J_i+K_i}$ be measurable subsets of Σ such that

$$d(\Omega_{ij}, \Omega_{ij'}) \geq \delta \qquad \forall\, i = 1,2,\ \forall\, j, j' = 1, \ldots, J_i + K_i,\ j \neq j',$$

$$d(p'_{im}, \Omega_{ij}) \geq \delta \qquad \forall\, i = 1,2,\ \forall\, j = 1, \ldots, J_i + K_i,\ \forall\, m = 1, \ldots, M'_i,\ m \neq m_{ij},$$

$$\int_{\Omega_{ij}} f_i \mathrm{d}V_g \geq \delta \qquad \forall\, i = 1,2,\ \forall\, j = 1, \ldots, J_i + K_i.$$

Then, there exist $\delta' > 0$, independent of f_1, f_2, and $\{\Omega_j\}_{j=1,\ldots,\max_i\{J_i+K_i\}}$ such that

$$d(\Omega_j, \Omega_{j'}) \geq \delta \qquad \forall\, j, j' = 1, \ldots, \max_{i=1,2}\{J_i + K_i\},\ j \neq j',$$

$$d(p'_{im}, \Omega_j) \geq \delta \qquad \forall\, i = 1,2,\ \forall\, j = 1, \ldots, \max_{i=1,2}\{J_i + K_i\},\ \forall\, m = 1, \ldots, M'_i,\ m \neq m_{ij},$$

$$\int_{\Omega_j} f_i \mathrm{d}V_g \geq \delta \qquad \forall\, i = 1,2,\ \forall\, j = 1, \ldots, \max_{i=1,2}\{J_i + K_i\}.$$

Proof.
It is not restrictive to suppose $J_1 + K_1 \geq J_2 + K_2$.
We choose $\delta_1 = \frac{\delta}{8}$ and we consider the open cover $\{B_{\delta_1}(x)\}_{x\in\Sigma}$ of Σ. By compactness, we can extract a finite sub-cover $\{B_{\delta_1}(x_l)\}_{l=1}^L$ with $L = L_{\delta_1,\Sigma}$.
We then take, for i and m as in the statement of the Lemma, $x_{ij} \in \{x_l\}_{l=1}^L$ such that

$$\int_{B_{\delta_1}(x_{ij})} f_i \mathrm{d}V_g = \max\left\{\int_{B_{\delta_1}(x_l)} f_i \mathrm{d}V_g : B_{\delta_1}(x_l)\cap\Omega_{ij} \neq \emptyset\right\}.$$

Since we have, for any $j \neq j'$,

$$d(x_{ij}, x_{ij'}) \geq d(\Omega_{ij}, \Omega_{ij'}) - d(x_{ij}, \Omega_{ij}) - d(x_{ij'}, \Omega_{ij'}) \geq \delta - 2\delta_1 = 6\delta_1,$$

then for a given $j = 1, \dots, J_1 + K_1$ there exists at most one $j'(j)$ satisfying $d(x_{1j}, x_{2j'}) < 3\delta_1$, with $j \mapsto j'(j)$ being injective as long as it is defined.
We can then reorder the indices j and j' so that $j'(j) = j$. Now, we define

$$\Omega_j := \begin{cases} B_{\delta'}(x_{1j}) \cup B_{\delta'}(x_{2j}) & \text{if } j = 1, \dots, J_2 + K_2 \\ B_{\delta'}(x_{1j}) & \text{if } j = J_2 + K_2 + 1, \dots, J_1 + K_1 \end{cases}.$$

Basically, we built these sets by joining two balls $B_{\delta_1}(x_{1j}), B_{\delta_1}(x_{2j})$ if they are close to each other or, if there are no disks sufficiently close to $B_{\delta_1}(x_{1j})$, making arbitrary unions.
Let us now check that the theses of the Lemmas are verified. The sets Ω_j's are distant at least δ_1 one to each other since

$$d(\Omega_j, \Omega_{j'}) = \inf_{i,i'=1,2} d\left(B_{\delta_1}(x_{ij}), B_{\delta_1}(x_{i'j'})\right) \geq \inf_{i,i'=1,2} d(x_{ij}, x_{i'j'}) - 2\delta_1 \geq \delta_1.$$

Moreover, for $m \neq m_{ij}$,

$$d(p'_{im}, \Omega_j) \geq \inf_{i=1,2} d\left(p'_{im}, B_{\delta_1}(x_{ij})\right) \geq \inf_{i=1,2}\left(d(p'_{im}, \Omega_{ij}) - d\left(\Omega_{ij}, B_{\delta_1}(x_{ij})\right)\right) \geq \delta - 2\delta_1 \geq \delta_1.$$

Finally, from the choice of the points x_{ij},

$$\int_{\Omega_j} f_i \mathrm{d}V_g \geq \int_{B_{\delta'}(x_{ij})} f_i \mathrm{d}V_g \geq \frac{1}{L}\int_{\Omega_{ij}} f_i \mathrm{d}V_g \geq \frac{\delta}{L} =: \delta_2.$$

The lemma follows by choosing $\delta' := \min\{\delta_1, \delta_2\}$ ☐

The next lemma is what is usually called an improved Moser-Trudinger inequality.
It essentially states that if both u_1 and u_2 are spread in sets which contain at most one singular point, then the constant 4π in (3.1) can be multiplied by a number depending on how many these sets are and on the singular points they contain. Such a phenomenon was first pointed out by Moser [63] and Aubin [2] for the scalar case.
It will be the most important step towards the proof of Theorem 3.21.

Lemma 3.23.
Let $\delta > 0, M_1, K_1, M_2, K_2 \in \mathbb{N}$ be given numbers, $\{m_{i1}, \dots, m_{iJ_i}\} \subset \{1, \dots, M'_i\}$ selections of indices and $\{\Omega_{ij}\}_{i=1,2}^{j=1,\dots,J_i+K_i}$ be measurable subsets of Σ such that

$$d(\Omega_{ij}, \Omega_{ij'}) \geq \delta \qquad \forall\, i = 1, 2,\ \forall\, j, j' = 1, \dots, J_i + K_i,\ j \neq j',$$

$$d(p'_{im}, \Omega_{ij}) \geq \delta \qquad \forall\, i = 1, 2,\ \forall\, j = 1, \dots, J_i + K_i,\ \forall\, m = 1, \dots, M'_i,\ m \neq m_{ij},$$

Then, for any $\varepsilon > 0$ there exists $C > 0$, not depending on u, such that any $u = (u_1, u_2) \in H^1(\Sigma)^2$ satisfying

$$\int_{\Omega_{ij}} f_{i,u} \mathrm{d}V_g \geq \delta \qquad \forall\, i = 1, 2,\ \forall\, j = 1, \dots, J_i + K_i$$

verifies

$$\sum_{i=1}^{2}\left(K_i+\sum_{j=1}^{J_i}\left(1+\alpha'_{im_{ij}}\right)\right)\left(\log\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g-\int_\Sigma u_i\mathrm{d}V_g\right)\le\frac{1+\varepsilon}{4\pi}\int_\Sigma Q(u)\mathrm{d}V_g+C.$$

Proof.
It will not be restrictive to suppose $J_1+K_1\ge J_2+K_2$.
We apply Lemma 3.22 with $f_i=f_{i,u}$ and we get a family of sets $\{\Omega_j\}_{j=1}^{J_1+K_1}$ satisfying

$$d(\Omega_j,\Omega_{j'})\ge\delta'>0\quad\forall\, j\ne j',\qquad\qquad\int_{\Omega_j}f_{i,u}\ge\delta'>0\quad\forall\, j=1,\dots,J_i+K_i.$$

Let us now consider, for any $j=1,\dots,J_1+K_1$, a cut-off function satisfying

$$0\le\chi_j\le1\qquad\quad \chi_j|_{\Omega_j}\equiv1\qquad\quad \chi_j|_{\Sigma\setminus\Omega'_j}\equiv0\quad\text{with }\Omega'_j=B_{\frac{\delta'}{2}}(\Omega_j)\qquad\quad |\nabla\chi_j|\le C_{\delta'}.\tag{3.32}$$

We now take $v_i\in L^\infty(\Sigma)$ with $\int_\Sigma v_i\mathrm{d}V_g=0$ and we set $w_i:=u_i-v_i-\int_\Sigma u_i\mathrm{d}V_g$ (which will also have null average). Therefore, we find

$$\begin{aligned}
&\log\int_\Sigma\widetilde{h}_ie^{u_i}\mathrm{d}V_g-\int_\Sigma u_i\mathrm{d}V_g\\
\le\;&\log\left(\frac{1}{\delta'}\int_{\Omega_j}\widetilde{h}_ie^{u_i}\mathrm{d}V_g-\int_\Sigma u_i\mathrm{d}V_g\right)\\
=\;&\log\left(\frac{1}{\delta'}\int_{\Omega_j}\widetilde{h}_ie^{v_i+w_i}\mathrm{d}V_g\right)\\
\le\;&\log\int_{\Omega_j}\widetilde{h}_ie^{w_i}\mathrm{d}V_g+\|v_i\|_{L^\infty(\Omega_j)}+\log\frac{1}{\delta'},\\
\le\;&\log\int_\Sigma\widetilde{h}_ie^{\chi_jw_i}\mathrm{d}V_g+\|v_i\|_{L^\infty(\Sigma)}+C.
\end{aligned}\tag{3.33}$$

Since $\chi_j\in\mathrm{Lip}(\Sigma)$, then $\chi_jw_i\in H^1(\Sigma)$, so we can apply a Moser-Trudinger inequality on it.
To this purpose, we notice that, for any $\varepsilon>0$,

$$\begin{aligned}
&\int_\Sigma|\nabla(\chi_jw_1)|^2\mathrm{d}V_g\\
=\;&\int_\Sigma|\chi_j\nabla w_1+w_1\nabla\chi_j|^2\mathrm{d}V_g\\
=\;&\int_\Sigma\left(\chi_j^2|\nabla w_1|^2+2(\chi_j\nabla w_1)\cdot(w_1\nabla\chi_j)+w_1^2|\nabla\chi_j|^2\right)\mathrm{d}V_g\\
\le\;&\int_\Sigma\left((1+\varepsilon)\chi_j^2|\nabla w_1|^2+\left(1+\frac{1}{\varepsilon}\right)w_1^2|\nabla\chi_j|^2\right)\mathrm{d}V_g\\
\le\;&(1+\varepsilon)\int_{\Omega'_j}|\nabla w_1|^2\mathrm{d}V_g+C_{\varepsilon,\delta',\Sigma}\int_{\Omega'_j}w_1^2\mathrm{d}V_g.
\end{aligned}$$

In the same way, writing

$$\frac{1}{3}\left(|x|^2+x\cdot y+|y|^2\right)=\frac{1}{4}|x|^2+\frac{1}{12}|x-2y|^2,\tag{3.34}$$

we get

$$\int_\Sigma Q(\chi_jw)\mathrm{d}V_g\le(1+\varepsilon)\int_{\Omega'_j}Q(w)\mathrm{d}V_g+C_{\varepsilon,\delta',\Sigma}\int_{\Omega'_j}\frac{1}{3}\left(w_1^2+w_1w_2+w_2^2\right)\mathrm{d}V_g.\tag{3.35}$$

At this point, we choose properly w_i (hence v_i) in such a way to have a control of its L^2 norm. Taking an orthonormal frame $\{\varphi^n\}_{n=1}^{\infty}$ of eigenfunctions for $-\Delta$ on $\overline{H}^1(\Sigma)$ with a non-decreasing sequence of associated positive eigenvalues $\{\lambda^n\}_{n=1}^{\infty}$ and writing $u_i = \int_\Sigma u_i \mathrm{d}V_g + \sum_{n=1}^{\infty} u_i^n \varphi^n$, we set $v_i = \sum_{n=1}^{N} u_i^n \varphi^n$ for

$$N = N_{\varepsilon,\delta',\Sigma} := \max\left\{n \in \mathbb{N} : \lambda^n < \frac{C_{\varepsilon,\delta',\Sigma}}{\varepsilon}\right\}.$$

This choice gives

$$C_{\varepsilon,\delta',\Sigma} \int_\Sigma w_1^2 \mathrm{d}V_g \leq \varepsilon \int_\Sigma |\nabla w_1|^2 \mathrm{d}V_g \leq \varepsilon \int_\Sigma |\nabla u_1|^2 \mathrm{d}V_g$$

and, through (3.34),

$$C_{\varepsilon,\delta',\Sigma} \int_\Sigma \frac{1}{3}\left(w_1^2 + w_1 w_2 + w_2^2\right) \mathrm{d}V_g \leq \varepsilon \int_\Sigma Q(w) \mathrm{d}V_g \leq \varepsilon \int_\Sigma Q(u) \mathrm{d}V_g. \tag{3.36}$$

Moreover, we get

$$\int_\Sigma |w_i| \mathrm{d}V_g \leq C_\Sigma \|\nabla w_i\|_{L^2(\Sigma)} \leq \varepsilon \int_\Sigma Q(w) \mathrm{d}V_g + C \leq \varepsilon \int_\Sigma Q(u) \mathrm{d}V_g + C \tag{3.37}$$

and, since v_i belongs to a finite-dimensional space of smooth function,

$$\|v_i\|_{L^\infty(\Sigma)} \leq C_N \|\nabla \varphi_i\|_{L^2(\Sigma)} \leq \varepsilon \int_\Sigma Q(v) \mathrm{d}V_g + C \leq \varepsilon \int_\Sigma Q(u) \mathrm{d}V_g + C. \tag{3.38}$$

Now, if $m = 1, \ldots, J_2 + K_2$, we apply the Moser-Trudinger inequality (3.1) to $\chi_j w$. Since these functions are supported on Ω'_j, we can replace $\widetilde{h}_i$ by a smooth interpolation which is constant outside a neighborhood of Ω'_j: we take $\eta_j \in C^\infty(\Sigma)$ satisfying

$$\eta_j(x) := \begin{cases} 1 & \text{if } x \in \Omega'_j \\ 0 & \text{if } x \notin B_{\frac{\delta'}{4}}(\Omega'_j) \end{cases} \qquad \widetilde{h}_{ij} := \eta_j \widetilde{h}_i + 1 - \eta_j = \begin{cases} \widetilde{h}_i & \text{if } x \in \Omega'_j \\ 1 & \text{if } x \notin B_{\frac{\delta'}{4}}(\Omega'_j) \end{cases}.$$

In this way, we can consider only the singularities $p'_{1m_{1j}}, p'_{2m_{1j}}$ which lie inside Ω_j (if there are any); from (3.35) and (3.37) we get

$$\begin{aligned}
&\sum_{i=1}^{2}\left(1+\alpha'_{im_{ij}}\right) \log \int_\Sigma \widetilde{h}_i e^{\chi_j w_i} \mathrm{d}V_g \\
= \;& \sum_{i=1}^{2}\left(1+\alpha'_{im_{ij}}\right) \log \int_\Sigma \widetilde{h}_{ij} e^{\chi_j w_i} \mathrm{d}V_g \\
\leq \;& \sum_{i=1}^{2}\left(1+\alpha'_{im_{ij}}\right) \int_\Sigma \chi_j w_i \mathrm{d}V_g + \frac{1}{4\pi} \int_\Sigma Q(\chi_j w) \mathrm{d}V_g + C \\
\leq \;& \sum_{i=1}^{2}\left(1+\alpha'_{im_{ij}}\right) \left(\|\chi_j\|_{L^\infty(\Sigma)} \int_\Sigma |w_i| \mathrm{d}V_g\right) + \frac{1+\varepsilon}{4\pi} \int_{\Omega'_j} Q(w) \mathrm{d}V_g \\
+ \;& \frac{C_{\varepsilon,\delta',\Sigma}}{4\pi} \int_{\Omega'_j} \frac{1}{3}\left(w_1^2 + w_1 w_2 + w_2^2\right) \mathrm{d}V_g + C \\
\leq \;& \int_\Sigma |w_1| \mathrm{d}V_g + \int_\Sigma |w_2| \mathrm{d}V_g + \frac{1+\varepsilon}{4\pi} \int_{\Omega'_j} Q(w) \mathrm{d}V_g + \frac{C_{\varepsilon,\delta',\Sigma}}{4\pi} \int_{\Omega'_j} \frac{1}{3}\left(w_1^2 + w_1 w_2 + w_2^2\right) \mathrm{d}V_g + C \\
\leq \;& 2\varepsilon \int_\Sigma Q(u) \mathrm{d}V_g + \frac{1+\varepsilon}{4\pi} \int_{\Omega'_j} Q(w) \mathrm{d}V_g + \frac{C_{\varepsilon,\delta',\Sigma}}{4\pi} \int_{\Omega'_j} \frac{1}{3}\left(w_1^2 + w_1 w_2 + w_2^2\right) \mathrm{d}V_g + C.
\end{aligned}$$

Therefore, from (3.33) and (3.38) we deduce

$$\begin{aligned}
& \sum_{i=1}^{2}\left(1+\alpha'_{im_{ij}}\right)\left(\log\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g - \int_\Sigma u_i\mathrm{d}V_g\right) \\
\leq\ & \sum_{i=1}^{2}\left(1+\alpha'_{im_{ij}}\right)\left(\log\int_\Sigma \widetilde{h}_i e^{\chi_j w_i}\mathrm{d}V_g + \|v_i\|_{L^\infty(\Sigma)} + C\right) \\
\leq\ & 3\varepsilon\int_\Sigma Q(u)\mathrm{d}V_g + \frac{1+\varepsilon}{4\pi}\int_{\Omega'_j} Q(w)\mathrm{d}V_g + \frac{C_{\varepsilon,\delta',\Sigma}}{4\pi}\int_{\Omega'_j}\frac{1}{3}\left(w_1^2+w_1w_2+w_2^2\right)\mathrm{d}V_g + C. \quad (3.39)
\end{aligned}$$

For $m = J_2 + K_2 + 1, \dots, J_1 + K_1$ we have estimates only for u_1 on Ω_j, so we apply the scalar Moser-Trudinger inequality (1.6). By (3.34) we get the integral of $Q(\chi_j w)$, then we argue as before. Notice that if $j > M_i$, then p'_{im_j} is not defined so these calculations would not make sense, but in this case both the previous and the following calculations hold replacing α'_{im_j} with 0.

$$\begin{aligned}
& \left(1+\alpha'_{1m_{1j}}\right)\log\int_\Sigma \widetilde{h}_1 e^{\chi_j w_1}\mathrm{d}V_g \\
=\ & \left(1+\alpha'_{1m_{1j}}\right)\log\int_\Sigma \widetilde{h}_{1m} e^{\chi_j w_1}\mathrm{d}V_g \\
\leq\ & \left(1+\alpha'_{1m_{1j}}\right)\int_\Sigma \chi_j w_1\mathrm{d}V_g + \frac{1}{16\pi}\int_\Sigma |\nabla(\chi_j w_1)|^2\mathrm{d}V_g + C \\
\leq\ & \int_\Sigma |w_1|\mathrm{d}V_g + \frac{1}{4\pi}\int_\Sigma Q(\chi_j w)\mathrm{d}V_g + C \\
\leq\ & \varepsilon\int_\Sigma Q(u)\mathrm{d}V_g + \frac{1+\varepsilon}{4\pi}\int_{\Omega'_j} Q(w)\mathrm{d}V_g + \frac{C_{\varepsilon,\delta',\Sigma}}{4\pi}\int_{\Omega'_j}\frac{1}{3}\left(w_1^2+w_1w_2+w_2^2\right)\mathrm{d}V_g + C.
\end{aligned}$$

Then in this case we deduce

$$\begin{aligned}
& \left(1+\alpha'_{1m_{1j}}\right)\left(\log\int_\Sigma \widetilde{h}_1 e^{u_1}\mathrm{d}V_g - \int_\Sigma u_1\mathrm{d}V_g\right) \\
\leq\ & \left(1+\alpha'_{1m_{1j}}\right)\left(\log\int_\Sigma \widetilde{h}_i e^{\chi_j w_1}\mathrm{d}V_g + \|v_1\|_{L^\infty(\Sigma)} + C\right) \\
\leq\ & 2\varepsilon\int_\Sigma Q(u)\mathrm{d}V_g + \frac{1+\varepsilon}{4\pi}\int_{\Omega'_j} Q(w)\mathrm{d}V_g + \frac{C_{\varepsilon,\delta',\Sigma}}{4\pi}\int_{\Omega'_j}\frac{1}{3}\left(w_1^2+w_1w_2+w_2^2\right)\mathrm{d}V_g + C. \quad (3.40)
\end{aligned}$$

Finally, we sum up together (3.39) and (3.40) for all the m's, exploiting (3.36) and the disjointness of the Ω'_j:

$$\begin{aligned}
& \sum_{i=1}^{2}\left(K_i + \sum_{j=1}^{J_i}\left(1+\alpha'_{im_{ij}}\right)\right)\left(\log\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g - \int_\Sigma u_i\mathrm{d}V_g\right) \\
=\ & \sum_{i=1}^{2}\sum_{m=1}^{J_2+K_2}\left(1+\alpha'_{im_{ij}}\right)\left(\log\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g - \int_\Sigma u_i\mathrm{d}V_g\right) \\
+\ & \sum_{m=J_2+K_2+1}^{J_1+K_1}\left(1+\alpha'_{1m_{ij}}\right)\left(\log\int_\Sigma \widetilde{h}_1 e^{u_1}\mathrm{d}V_g - \int_\Sigma u_1\mathrm{d}V_g\right) \\
\leq\ & (2J_1+2K_1+J_2+K_2)\varepsilon\int_\Sigma Q(u)\mathrm{d}V_g + \frac{1+\varepsilon}{4\pi}\int_\Sigma Q(w)\mathrm{d}V_g \\
+\ & \frac{C_{\varepsilon,\delta',\Sigma}}{4\pi}\int_\Sigma\frac{1}{3}\left(w_1^2+w_1w_2+w_2^2\right)\mathrm{d}V_g + C \\
\leq\ & (2J_1+2K_1+J_2+K_2)\varepsilon\int_\Sigma Q(u)\mathrm{d}V_g + \frac{1+2\varepsilon}{4\pi}\int_\Sigma Q(u)\mathrm{d}V_g + C
\end{aligned}$$

which is, renaming ε properly, what we desired. □

Now we need another technical lemma, which relates the condition of spreading, needed for Lemma 3.23, and of concentration around a finite number of points.
Through this lemma, we can then use the improved Moser-Trudinger inequality to get information about the concentration which occurs on sub-levels J_ρ^{-L}.
The following results will be extensions of the ones contained in [36, 59, 61] with suitable changes to take into account the singularities.

Lemma 3.24.
Let $i = 1, 2$, $\omega_0 > 0$, $\varepsilon > 0$ small enough, be such that any $\mathcal{J} \subset \Sigma$ satisfying $\omega_{\underline{\alpha}'_i}(\mathcal{J}) \le \omega_0$ verifies

$$\int_{\bigcup_{x_k \in \mathcal{J}} B_\varepsilon(x_k)} f_{i,u} \mathrm{d}V_g < 1 - \varepsilon.$$

Then, there exist $\varepsilon', r' > 0$, not depending on u_i, $J, K \in \mathbb{N}$, $\{m_1, \dots, m_J\} \subset \{1, \dots, M'_i\}$ and $\{x'_j\}_{j=1}^{J+K}$ satisfying

$$K + \sum_{j=1}^{J} \left(1 + \alpha'_{im_j}\right) > \omega_0, \qquad d\left(x'_j, p'_{im_j}\right) \le 2r' \le d\left(x'_j, p'_{im}\right) \quad \forall\, j = 1, \dots, J+K,\ m \ne m_j,$$

$$B_{2r'}\left(x'_j\right) \cap B_{2r'}\left(x'_{j'}\right) = \emptyset \quad \forall\, j \ne j', \qquad \int_{B_{r'}\left(x'_j\right)} f_{i,u} \mathrm{d}V_g \ge \varepsilon' \quad \forall\, j = 1, \dots, J+K. \tag{3.41}$$

Proof.
We fix $r' := \dfrac{\varepsilon}{10}$ and we cover Σ with a finite number of disks $\{B_{r'}(y_l)\}_{l=1}^{L}$ with $L = L_{r',\Sigma}$. Up to relabeling, there exists a $L' \le L$ such that

$$\int_{B_{r'}(y_l)} f_{i,u} \mathrm{d}V_g \ge \frac{\varepsilon}{L} := \varepsilon' \qquad \Longleftrightarrow \qquad l \le L'.$$

In fact, if none of the y_l's satisfied the above condition, this would imply

$$\int_\Sigma f_{i,u} \mathrm{d}V_g \le \sum_{l=1}^{L} \int_{B_{r'}(y_l)} f_{i,u} \mathrm{d}V_g \le L\varepsilon' = \varepsilon,$$

that is a contradiction if $\varepsilon < 1$.
Now, select inductively the points $\{x'_j\} \subset \{y_l\}_{l=1}^{L'}$: we set $x'_1 := y_1$ and define

$$\Omega_1 := \bigcup_{l=1}^{L'} \{B_{r'}(y_l) :\ B_{2r'}(y_l) \cap B_{2r'}(x'_1) \ne \emptyset\} \subset B_{5r'}(x'_1).$$

If there exists y_{l_0} such that $B_{2r'}\left(y_{l_0}\right) \cap B_{2r'}(x'_1) = \emptyset$, then we set $x'_2 := y_{l_0}$ and define

$$\Omega_2 := \bigcup_{l=1}^{L'} \{B_{r'}(y_l) :\ B_{2r'}(y_l) \cap B_{2r'}(x'_2) \ne \emptyset\} \subset B_{5r'}(x'_2).$$

In the same way, we find a finite number of points x'_j and sets Ω_j.
If $\varepsilon \le \min\limits_{m,m' \in \{1,\dots,M_i\}, m \ne m'} d(p'_{im}, p'_{im'})$, each x'_j can have at most one of the p'_{im} at a distance strictly smaller than $2r' \le \dfrac{\varepsilon}{2}$, and each p'_{im} can have at most one x'_j closer than $2r'$; if it exists we call this point p'_{im_j}. Denote the number of points for which the $2r'$-close point exists as J and the number of points for which it does not as K.
Therefore, we get $d\left(x'_j, p'_{im}\right) \ge 2r'$ for any $m \ne m_j$, whereas condition (3.41) holds by construction,

so the thesis will follow by showing $K + \sum_{j=1}^{J} \left(1 + \alpha'_{im_j}\right) > \omega_0$.

Take now $x_j = p'_{im_j}$ for $j \in \{1, \dots, J\}$ and $x_j = x'_j$ for $j \in \{J+1, \dots, J+K\}$ and suppose by contradiction $K + \sum_{j=1}^{J} \left(1 + \alpha'_{im_j}\right) = \omega_{\underline{\alpha}'_i}(\mathcal{J}) \le \omega_0$. By hypothesis, this would imply

$$\int_{\Sigma \setminus \bigcup_{x_k \in \mathcal{J}} B_\varepsilon(x_k)} f_{i,u} \mathrm{d}V_g \ge \varepsilon$$

However, since

$$\bigcup_{l=1}^{L'} B_{r'}(y_l) \subset \bigcup_{j=1}^{J+K} \Omega_j \subset \bigcup_{j=1}^{J+K} B_{5r'}\left(x'_j\right) \subset \bigcup_{j=1}^{J+K} B_{\frac{\varepsilon}{2}}\left(x'_j\right) \subset \bigcup_{x_k \in \mathcal{J}} B_\varepsilon(x_k),$$

then we find

$$\int_{\Sigma \setminus \bigcup_{x_k \in \mathcal{J}} B_\varepsilon(x_k)} f_{i,u} \mathrm{d}V_g \le \int_{\Sigma \setminus \bigcup_{l=1}^{L'} B_{r'}(y_l)} f_{i,u} \mathrm{d}V_g \le \int_{\bigcup_{l=L'+1}^{L} B_{r'}(y_l)} f_{i,u} \mathrm{d}V_g \le (L - L')\varepsilon' < \varepsilon,$$

that is we get a contradiction, hence the proof is complete. □

This is the last step needed to prove Theorem 3.21. We see that, if $J_\rho(u)$ is very very low, then either $f_{1,u}$ or $f_{2,u}$ has all its mass concentrated around a finite number of points, depending on $\rho_i, \underline{\alpha}'_i$.

Lemma 3.25.
For any $\varepsilon > 0$ small enough, there exists $L > 0$ such that, if $u \in J_{A_2,\rho}^{-L}$, then for at least one $i = 1, 2$ there exists $\mathcal{J} \subset \Sigma$ satisfying $4\pi\omega_{\underline{\alpha}'_i}(\mathcal{J}_i) \le \rho_i$ and

$$\int_{\bigcup_{x_k \in \mathcal{J}_i} B_\varepsilon(x_k)} f_{i,u} \mathrm{d}V_g \ge 1 - \varepsilon.$$

Proof.
Suppose by contradiction that the statement is not true. This means that there exist $\varepsilon > 0$ small enough and $\{u^n\}_{n \in \mathbb{N}} \subset H^1(\Sigma)^2$ such that $J_{A_2,\rho}(u^n) \underset{n \to +\infty}{\longrightarrow} -\infty$ and

$$\int_{\bigcup_{x_{ik} \in \mathcal{J}_i} B_\varepsilon(x_{ik})} f_{i,u^n} \mathrm{d}V_g < 1 - \varepsilon$$

for any $\mathcal{J}_1, \mathcal{J}_2 \subset \Sigma$ satisfying $4\pi\omega_{\underline{\alpha}'_i}(\mathcal{J}) \le \rho_i$. Then, we can apply Lemma 3.24 to f_{i,u^n} with $\omega_0 = \frac{\rho_i}{4\pi}$; we find $\varepsilon', r' > 0$, not depending on n, $K_i, M_i \in \mathbb{N}$, $\{m_{i1}, \dots, m_{iJ_i}\} \subset \{1, \dots, M'_i\}$ and $\{x'_{ij}\}_{j=1}^{J_i+K_i}$ satisfying

$$4\pi\left(K_i + \sum_{j=1}^{J_i}\left(1 + \alpha'_{im_{ij}}\right)\right) > \rho_i, \qquad d\left(x'_{ij}, p'_{im}\right) \ge 2r' \quad \forall\, m \ne m_{ij},$$

$$B_{2r'}\left(x'_{ij}\right) \cap B_{2r'}\left(x'_{ij'}\right) = \emptyset \quad \forall\, j \ne j', \qquad \int_{B_{r'}\left(x'_{ij}\right)} f_{i,u^n} \mathrm{d}V_g \ge \varepsilon' \quad \forall\, j.$$

We can then apply Lemma 3.23 with $\delta = \min\{\varepsilon', r'\}$ and $\Omega_{ij} = B_{r'}\left(x'_{ij}\right)$ to obtain an improved Moser-Trudinger for the u^n. Moreover, Jensen's inequality yields

$$\log \int_\Sigma \widetilde{h}_i e^{u_i^n} \mathrm{d}V_g - \int_\Sigma u_i^n \mathrm{d}V_g \ge -C.$$

Therefore, choosing

$$\varepsilon \in \left(0, \min_{i=1,2} \frac{4\pi}{\rho_i}\left(K_i + \sum_{j=1}^{J_i}\left(1+\alpha'_{im_{ij}}\right)\right) - 1\right),$$

we get, for both $i = 1, 2$,

$$\frac{4\pi}{1+\varepsilon}\left(K_i + \sum_{j=1}^{J_i}\left(1+\alpha'_{im_{ij}}\right)\right) - \rho_i > 0;$$

hence

$$\begin{aligned}
& -\infty \\
\underset{n\to+\infty}{\longleftarrow} \quad & J_{A_2,\rho}(u^n) \\
\geq \quad & \sum_{i=1}^{2}\left(\frac{4\pi}{1+\varepsilon}\left(K_i + \sum_{j=1}^{J_i}\left(1+\alpha'_{im_{ij}}\right)\right) - \rho_i\right)\left(\log\int_\Sigma \widetilde{h}_i e^{u_i^n}\mathrm{d}V_g - \int_\Sigma u_i^n \mathrm{d}V_g\right) - C \\
\geq \quad & -C\sum_{i=1}^{2}\left(\frac{4\pi}{1+\varepsilon}\left(K_i + \sum_{j=1}^{J_i}\left(1+\alpha'_{im_{ij}}\right)\right) - \rho_i\right) - C, \\
\geq \quad & -C
\end{aligned}$$

which is a contradiction. $\square$

Now we have all the tools to prove Theorem 3.21 in the case of A_2.
To treat the case of B_2, G_2 we need the following counter-part of Lemma 3.23. From this and Lemma 3.24 will follow the counterpart of Lemma 3.25.
Since the proof will be similar to the one of Lemma 3.23, we will be sketchy.

Lemma 3.26.
Let $\delta > 0, K_1, K_2 \in \mathbb{N}$, $\{\Omega_{1k}\}_{k=1}^{K_1}, \{\Omega_{2k}\}_{k=1}^{K_2}$ satisfy

$$\begin{aligned}
& d(\Omega_{ik}, \Omega_{ik'}) \geq \delta && \forall\, i = 1, 2,\ k, k' = 1, \dots, K_i,\ k \neq k', \\
& \int_{\Omega_{ik}} f_{i,u}\mathrm{d}V_g \geq \delta && \forall\, i = 1, 2,\ k = 1, \dots, K_i.
\end{aligned}$$

Then, for any $\varepsilon > 0$ there exists $C > 0$, not depending on u, such that

$$4\pi K_1\left(\log\int_\Sigma e^{u_1}\mathrm{d}V_g - \int_\Sigma u_1\mathrm{d}V_g\right) + 2\pi K_2\left(\log\int_\Sigma e^{u_2}\mathrm{d}V_g - \int_\Sigma u_2\mathrm{d}V_g\right) \leq (1+\varepsilon)\int_\Sigma Q_{B_2}(u)\mathrm{d}V_g + C,$$

$$4\pi K_1\left(\log\int_\Sigma e^{u_1}\mathrm{d}V_g - \int_\Sigma u_1\mathrm{d}V_g\right) + \frac{4}{3}\pi K_2\left(\log\int_\Sigma e^{u_2}\mathrm{d}V_g - \int_\Sigma u_2\mathrm{d}V_g\right) \leq (1+\varepsilon)\int_\Sigma Q_{G_2}(u)\mathrm{d}V_g + C.$$

Proof.
We apply Lemma 3.22 to $f_{1,u}, f_{2,u}$ and we get $\{\Omega_k\}_{k=1}^{K_1}$ such that

$$d(\Omega_k, \Omega_{k'}) \geq \delta' \qquad \int_{\Omega_k} f_{i,u}\mathrm{d}V_g \geq \delta'.$$

We then consider, for $k = 1, \dots, K_1$, some cut-off functions χ_k satisfying (3.32) and we split $u_i - \int_\Sigma u_i \mathrm{d}V_g = v_i + w_i$ as in the proof of Lemma 3.23. As before,

$$\log\int_\Sigma e^{u_i}\mathrm{d}V_g - \int_\Sigma u_i\mathrm{d}V_g \leq \log\int_\Sigma e^{\chi_k w_i}\mathrm{d}V_g + \|v_i\|_{L^\infty(\Sigma)} + C$$

and

$$\int_\Sigma |\nabla(\chi_k w_i)|^2 \mathrm{d}V_g \le (1+\varepsilon)\int_{\Omega'_k} |\nabla w_i|^2 \mathrm{d}V_g + C\int_{\Omega'_k} w_i^2 \mathrm{d}V_g.$$

Using the elementary equalities

$$\frac{|x|^2}{2}+\frac{x\cdot y}{2}+\frac{|y|^2}{4}=\frac{|x|^2}{4}+\frac{|x+y|^2}{4} \qquad |x|^2+x\cdot y+\frac{|y|^2}{3}=\frac{|x|^2}{4}+\frac{|3x+2y|^2}{12}, \tag{3.42}$$

we similarly get

$$\int_\Sigma Q_{B_2}(\chi_k w)\mathrm{d}V_g \le (1+\varepsilon)\int_{\Omega'_k} Q_{B_2}(w)\mathrm{d}V_g + C_{\varepsilon,\delta'}\int_{\Omega'_k}\left(\frac{w_1^2}{2}+\frac{w_1w_2}{2}+\frac{w_2^2}{4}\right)\mathrm{d}V_g$$

$$\int_\Sigma Q_{G_2}(\chi_k w)\mathrm{d}V_g \le (1+\varepsilon)\int_{\Omega'_k} Q_{G_2}(w)\mathrm{d}V_g + C_{\varepsilon,\delta'}\int_{\Omega'_k}\left(w_1^2+w_1w_2+\frac{w_2^2}{3}\right)\mathrm{d}V_g.$$

By the choice of v_i, w_i we have:

$$C_{\varepsilon,\delta'}\int_\Sigma\left(\frac{w_1^2}{2}+\frac{w_1w_2}{2}+\frac{w_2^2}{4}\right)\mathrm{d}V_g \le \varepsilon\int_\Sigma Q_{B_2}(w)\mathrm{d}V_g \le \varepsilon\int_\Sigma Q_{B_2}(u)\mathrm{d}V_g$$

$$C_{\varepsilon,\delta'}\int_\Sigma\left(w_1^2+w_1w_2+\frac{w_2^2}{3}\right)\mathrm{d}V_g \le \varepsilon\int_\Sigma Q_{G_2}(w)\mathrm{d}V_g \le \varepsilon\int_\Sigma Q_{G_2}(u)\mathrm{d}V_g,$$

$$\int_\Sigma |w_i|\mathrm{d}V_g \le \varepsilon\min\left\{\int_\Sigma Q_{B_2}(u)\mathrm{d}V_g, \int_\Sigma Q_{G_2}(u)\mathrm{d}V_g\right\}+C_\varepsilon,$$

$$\|v_i\|_{L^\infty(\Sigma)} \le \varepsilon\min\left\{\int_\Sigma Q_{B_2}(u)\mathrm{d}V_g, \int_\Sigma Q_{G_2}(u)\mathrm{d}V_g\right\}+C_\varepsilon.$$

At this point, for $k=1,\dots,K_2$ we apply Theorem 3.2 to $\chi_k w$:

$$\begin{aligned}
&4\pi\left(\log\int_\Sigma e^{u_1}\mathrm{d}V_g-\int_\Sigma u_1\mathrm{d}V_g\right)+2\pi\left(\log\int_\Sigma e^{u_2}\mathrm{d}V_g-\int_\Sigma u_2\mathrm{d}V_g\right)\\
\le\ &4\pi\log\int_\Sigma e^{\chi_k w_1}\mathrm{d}V_g+2\pi\log\int_\Sigma e^{\chi_k w_2}+4\pi\|v_1\|_{L^\infty(\Sigma)}+2\pi\|v_2\|_{L^\infty(\Sigma)}+C\\
\le\ &\int_{\Omega'_k} Q_{B_2}(w)\mathrm{d}V_g+\varepsilon'\int_\Sigma Q_{B_2}(u)\mathrm{d}V_g+C.
\end{aligned}\tag{3.43}$$

For $k=K_2+1,\dots,K_1$ we apply the scalar Moser-Trudinger inequality (1.6). By (3.34) we get again the integral of Q_{B_2}:

$$\begin{aligned}
&4\pi\left(\log\int_\Sigma e^{u_1}\mathrm{d}V_g-\int_\Sigma u_1\mathrm{d}V_g\right)\\
\le\ &4\pi\log\int_\Sigma e^{\chi_k w_1}\mathrm{d}V_g+4\pi\|v_1\|_{L^\infty(\Sigma)}+C\\
\le\ &\frac{1}{4}\int_{\Omega'_k}|\nabla(\chi_k w_1)|^2\mathrm{d}V_g+\varepsilon\int_\Sigma Q_{B_2}(u)\mathrm{d}V_g+C\\
\le\ &\int_{\Omega'_k} Q_{B_2}(w)\mathrm{d}V_g+\varepsilon'\int_\Sigma Q_{B_2}(u)\mathrm{d}V_g+C.
\end{aligned}\tag{3.44}$$

Putting together (3.43) and (3.44) we concluded the proof for $J_{B_2,\rho}$.
The improved inequality concerning $J_{G_2,\rho}$ can be proved in the very same way.
When $K_2 \ge K_1$ consider, in place of (3.42),

$$\frac{|x|^2}{2}+\frac{x\cdot y}{2}+\frac{|y|^2}{4}=\frac{|y|^2}{8}+\frac{|2x+y|^2}{8} \qquad |x|^2+x\cdot y+\frac{|y|^2}{3}=\frac{|y|^2}{12}+\frac{|2x+y|^2}{4}$$

$\square$

Proof of Theorem 3.21.
It suffices to prove the statement concerning $J^{-L}_{A_2,\rho}$ for small ε.
We apply Lemma 3.25 with $\frac{\varepsilon}{3}$. It is not restrictive to suppose that the thesis of the lemma holds for $i = 1$, since the case $i = 2$ can be treated in the same way. Therefore, we get $\mathcal{J} \subset \Sigma$, and we define

$$\sigma_u := \sum_{x_k \in \mathcal{J}} t_k \delta_{x_k}$$

where

$$t_k = \int_{B_{\frac{\varepsilon}{3}}(x_k) \setminus \bigcup_{k'=1}^{k-1} B_{\frac{\varepsilon}{3}}(x_{k'})} f_{1,u} \mathrm{d}V_g + \frac{1}{|\mathcal{J}|} \int_{\Sigma \setminus \bigcup_{x_{k'} \in \mathcal{J}} B_{r'}(x_{k'})} f_{1,u} \mathrm{d}V_g.$$

Notice that $\sigma_u \in \Sigma_{\rho_1, \underline{\alpha}'_1}$ because, from Lemma 3.25 we find $\omega_{\underline{\alpha}'_1}(\mathcal{J}) \le \rho_1$ and the last inequality is actually strict because we are supposing $\rho \notin \Gamma$.
To conclude the proof it would suffice to get

$$\left| \int_\Sigma (f_{1,u} - \sigma_u)\, \phi \mathrm{d}V_g \right| \le \varepsilon \|\phi\|_{\mathrm{Lip}(\Sigma)} \qquad \forall\, \phi \in \mathrm{Lip}(\Sigma). \tag{3.45}$$

In fact, following the definition of $d_{\mathrm{Lip}'}$, this would imply

$$d_{\mathrm{Lip}'}\left(f_{1,u}, \Sigma_{\rho_1,\underline{\alpha}'_1}\right) \le d_{\mathrm{Lip}'}(f_{1,u}, \sigma_u) = \sup_{\phi \in \mathrm{Lip}(\Sigma), \|\phi\|_{\mathrm{Lip}(\Sigma)} \le 1} \left| \int_\Sigma (f_{1,u} - \sigma_u)\, \phi \mathrm{d}V_g \right| < \varepsilon. \tag{3.46}$$

We will divide the integral in (3.45) into two points, studying separately what happens inside and outside the union of the $\frac{\varepsilon}{3}$-disks centered at the points x_m's.
Outside the disks, for any $\phi \in \mathrm{Lip}(\Sigma)$ we have

$$\begin{aligned}
& \left| \int_{\Sigma \setminus \bigcup_{x_k \in \mathcal{J}} B_{\frac{\varepsilon}{3}}(x_k)} (f_{1,u} - \sigma_u) \phi \mathrm{d}V_g \right| \\
= & \left| \int_{\Sigma \setminus \bigcup_{x_k \in \mathcal{J}} B_{\frac{\varepsilon}{3}}(x_k)} f_{1,u} \phi \mathrm{d}V_g \right| \\
\le & \|\phi\|_{L^\infty(\Sigma)} \int_{\Sigma \setminus \bigcup_{x_k \in \mathcal{J}} B_{\frac{\varepsilon}{3}}(x_k)} f_{1,u} \mathrm{d}V_g \\
< & \frac{\varepsilon}{3} \|\phi\|_{\mathrm{Lip}(\Sigma)}.
\end{aligned} \tag{3.47}$$

On the other hand, we also find

$$\begin{aligned}
& \left| \int_{\bigcup_{x_k \in \mathcal{J}} B_{\frac{\varepsilon}{3}}(x_k)} (f_{1,u} - \sigma_u) \phi \mathrm{d}V_g \right| \\
= & \left| \int_{\bigcup_{x_k \in \mathcal{J}} B_{\frac{\varepsilon}{3}}(x_k)} f_{1,u} \phi \mathrm{d}V_g - \sum_{x_k \in \mathcal{J}} \left(\int_{B_{\frac{\varepsilon}{3}}(x_k) \setminus \bigcup_{k'=1}^{k} B_{\frac{\varepsilon}{3}}(x_{k'})} f_{1,u} \mathrm{d}V_g + \right.\right. \\
+ & \left.\left. \frac{1}{|\mathcal{J}|} \int_{\Sigma \setminus \bigcup_{x_{k'} \in \mathcal{J}} B_{\frac{\varepsilon}{3}}(x_{k'})} f_{1,u} \mathrm{d}V_g \right) \phi(x_k) \right| \\
= & \left| \sum_{x_k \in \mathcal{J}} \left(\int_{B_{\frac{\varepsilon}{3}}(x_k) \setminus \bigcup_{k'=1}^{k} B_{\frac{\varepsilon}{3}}(x_{k'})} f_{1,u} (\phi - \phi(x_k)) \mathrm{d}V_g - \frac{1}{|\mathcal{J}|} \int_{\Sigma \setminus \bigcup_{x_{k'} \in \mathcal{J}} B_{\frac{\varepsilon}{3}}(x_{k'})} f_{1,u} \mathrm{d}V_g \phi(x_k) \right) \right| \\
\le & \|\nabla \phi\|_{L^\infty(\Sigma)} \sum_{x_k \in \mathcal{J}} \int_{B_{\frac{\varepsilon}{3}}(x_k) \setminus \bigcup_{k'=1}^{k} B_{\frac{\varepsilon}{3}}(x_{k'})} f_{1,u} d(\cdot, x_k) \mathrm{d}V_g \|\phi\|_{L^\infty(\Sigma)} + \int_{\Sigma \setminus \bigcup_{x_{k'} \in \mathcal{J}} B_{\frac{\varepsilon}{3}}(x_{k'})} f_{1,u} \mathrm{d}V_g \\
< & \frac{\varepsilon}{3} \|\nabla \phi\|_{L^\infty(\Sigma)} \int_{\bigcup_{x_{k'} \in \mathcal{J}} B_{\frac{\varepsilon}{3}}(x_{k'})} f_{1,u} \mathrm{d}V_g + \frac{\varepsilon}{3} \|\phi\|_{L^\infty(\Sigma)}
\end{aligned}$$

$$\leq \frac{\varepsilon}{3}\|\nabla\phi\|_{L^\infty(\Sigma)} + \frac{\varepsilon}{3}\|\phi\|_{L^\infty(\Sigma)}$$
$$\leq \frac{2}{3}\varepsilon\|\phi\|_{\mathrm{Lip}(\Sigma)}. \tag{3.48}$$

Therefore, from (3.47) and (3.48) we deduce (3.45), hence (3.46). □

3.5 Scale-invariant improved Moser-Trudinger inequalities

In this section we will prove a new kind of improved Moser-Trudinger inequality, which will be essential for the analysis of sub-levels of the energy functional J_ρ in the case considered by Theorem 3.5. Since we will only consider the A_2 Toda system, we will omit the subscript A_2 throughout all the section.

The main difference with respect to the results proved in the previous section is basically the following: in Section 3.4 we got improved inequalities under the assumption that $f_{i,u}$ attains some mass in a $B_\delta(x)$ for some $\delta > 0$, but we did not really need to know how large or small δ was; this led us to call *macroscopic* such inequalities. On the other hand, here we want to get new information on how fast $f_{i,u}$ concentrates around x.

To this purpose, we need a suitable definition of center of mass and scale of concentration. The idea is taken from [61] (Proposition 3.1), but with several modifications which take into account that we want to choose the center of mass in a given finite set $\mathcal{F} \subset \Sigma$ (which will be, in our application, the set of weighted barycenter $\Sigma_{\rho_i,\underline{\alpha}_i}$). As in [61], we map the space $\mathcal{A}$ of positive normalized L^1 functions (defined by (1.12)) on the topological cone based on $\mathcal{F}$ of height δ, namely

$$\mathcal{C}_\delta\mathcal{F} := \frac{\mathcal{F}\times[0,\delta]}{\sim}, \tag{3.49}$$

The meaning of such an identification is the following: if a function $f \in \mathcal{A}$ does not concentrate around any point $x \in \mathcal{F}$, then we cannot define a center of mass; in this case we set the scale equals δ, that is *large*.

Lemma 3.27.
Let $\mathcal{F} := \{x_1,\dots,x_K\} \subset \Sigma$ be a given finite set and $\mathcal{A}$, $\mathcal{C}_\delta$ be defined by (1.12) and (3.49).
Then, for $\delta > 0$ small enough there exists a map $\psi = (\beta,\varsigma) = (\beta_\mathcal{F},\varsigma_\mathcal{F}) : \mathcal{A} \to \mathcal{C}_\delta\mathcal{F}$ such that:

- *If $\varsigma(f) = \delta$, then either $\int_{\Sigma\setminus\bigcup_{x\in\mathcal{F}}B_\delta(x)} f\mathrm{d}V_g \geq \delta$ or there exists $x', x'' \in \mathcal{F}$ with $x' \neq x''$ and*
$$\int_{B_\delta(x')} f\mathrm{d}V_g \geq \delta \qquad \int_{B_\delta(x'')} f\mathrm{d}V_g \geq \delta$$
- *If $\varsigma(f) < \delta$, then*
$$\int_{B_{\varsigma(f)}(\beta(f))} f\mathrm{d}V_g \geq \delta \qquad \int_{\Sigma\setminus B_{\varsigma(f)}(\beta(f))} f\mathrm{d}V_g \geq \delta.$$

Moreover, if $f^n \underset{n\to+\infty}{\longrightarrow} \delta_x$ for some $x \in \mathcal{F}$, then $(\beta(f^n),\varsigma(f^n)) \underset{n\to+\infty}{\longrightarrow} (x,0)$.

Proof.
Fix $\tau \in \left(\frac{1}{2},1\right)$, take $\delta \leq \frac{\min_{x,x'\in\mathcal{F},\,x\neq x'} d(x,x')}{2}$ and define, for $k = 1,\dots,K$,

$$I_k(f) := \int_{B_\delta(x_k)} f\mathrm{d}V_g; \qquad I_0(f) := \int_{\Sigma\setminus\bigcup_{x\in\mathcal{F}}B_\delta(x)} f\mathrm{d}V_g = 1 - \sum_{k=1}^{k} I_k(f),$$

Choose now indices $\widetilde{k}, \widehat{k}$ such that

$$I_{\widetilde{k}}(f) := \max_{k\in\{0,\dots,K\}} I_k(f) \qquad\qquad I_{\widehat{k}}(f) := \max_{k\neq\widetilde{k}} I_k(f).$$

We will define the map ψ depending on $\widetilde{k}$ and $I_{\widetilde{k}}(f)$:

- $\widetilde{k} = 0$. Since f has little mass around each of the points x_k, we set $\varsigma(f) = \delta$ and do not define $\beta(f)$, as it would be irrelevant by the equivalence relation in (3.49). The assertion of the Lemma is verified, up to taking a smaller δ, because

$$\int_{\Sigma\setminus\bigcup_{x\in\mathcal{F}} B_\delta(x)} f\mathrm{d}V_g = I_0(f) \geq \frac{1}{K+1} \geq \delta$$

- $\widetilde{k} \geq 1$, $I_{\widetilde{k}}(f) \leq \dfrac{K\tau}{1-\tau} I_{\widehat{k}}(f)$. Here, f has still little mass around the point $x_{\widetilde{k}}$ (which could not be uniquely defined), so again we set $\varsigma(f) := \delta$. It is easy to see that $I_{\widehat{k}}(f) \geq \dfrac{1-\tau}{K}$, so

$$\int_{B_\delta\left(x_{\widetilde{k}}\right)} f\mathrm{d}V_g \geq \int_{B_\delta\left(x_{\widehat{k}}\right)} f\mathrm{d}V_g \geq \frac{1-\tau}{K} \geq \delta.$$

- $\widetilde{k} \geq 1$, $I_{\widetilde{k}}(f) \geq \dfrac{K\tau}{1-\tau} I_{\widehat{k}}(f)$. Now, $I_{\widetilde{k}}(f) > \tau$, so one can define a scale of concentration $s\left(x_{\widetilde{k}}, f\right) \in (0,\delta)$ of f around $x_{\widetilde{k}}$, uniquely determined by

$$\int_{B_{s\left(x_{\widetilde{k}},f\right)}\left(x_{\widetilde{k}}\right)} f\mathrm{d}V_g = \tau.$$

  We can also define a center of mass $\beta(f) = x_{\widetilde{k}}$ but we have to interpolate for the scale:

  - Case $I_{\widetilde{k}}(f) \leq \dfrac{2K\tau}{1-\tau} I_{\widehat{k}}(f)$: setting

$$\varsigma(f) = s\left(x_{\widetilde{k}}, f\right) + \frac{I_{\widetilde{k}}(f)}{\frac{K\tau}{1-\tau} I_{\widehat{k}}(f)}\left(\delta - s\left(x_{\widetilde{k}}, f\right)\right),$$

    we get $s\left(x_{\widetilde{k}}, f\right) < \varsigma(f) < \delta$; moreover, $I_{\widehat{k}}(f) \geq \dfrac{1-\tau}{K(1+\tau)}$, hence

$$\int_{B_{\varsigma(f)}(\beta(f))} f\mathrm{d}V_g \geq \int_{B_{s\left(x_{\widetilde{k}},f\right)}\left(x_{\widetilde{k}}\right)} f\mathrm{d}V_g = \tau \geq \delta$$

$$\int_{\Sigma\setminus B_{\varsigma(f)}(\beta(f))} f\mathrm{d}V_g \geq \int_{\Sigma\setminus B_\delta\left(x_{\widetilde{k}}\right)} f\mathrm{d}V_g \geq \frac{1-\tau}{K(1+\tau)} \geq \delta$$

  - Case $I_{\widetilde{k}}(f) \geq \dfrac{2K\tau}{1-\tau} I_{\widehat{k}}(f)$: we just set $\varsigma(f) : s\left(x_{\widetilde{k}}, f\right)$ and we get

$$\int_{B_{\varsigma(f)}(\beta(f))} f\mathrm{d}V_g = \tau \geq \delta \qquad\qquad \int_{\Sigma\setminus B_{\varsigma(f)}(\beta(f))} f\mathrm{d}V_g = 1-\tau \geq \delta.$$

To prove the final assertion, write (up to sub-sequences), $(\beta_\infty, \varsigma_\infty) = \lim_{n\to+\infty} (\beta(f^n), \varsigma(f^n))$. For large n we will have

$$\int_{\Sigma\setminus\bigcup_{x'\in\mathcal{F}} B_\delta(x')} f^n\mathrm{d}V_g \leq \frac{\delta}{2} \qquad\qquad \int_{B_\delta(x'')} f^n\mathrm{d}V_g \leq \frac{\delta}{2} \qquad\qquad \text{for any } x'' \in \mathcal{F}\setminus\{x\},$$

which excludes $\varsigma_\infty = \delta$.
We also exclude $\varsigma_\infty \in (0,\delta)$ because it would give

$$\int_{B_{\frac{3}{2}\varsigma_\infty}(\beta_\infty)} f^n \mathrm{d}V_g \geq \delta \qquad \int_{\Sigma \setminus B_{\frac{\varsigma_\infty}{2}}(\beta_\infty)} f^n \mathrm{d}V_g \geq \delta.$$

which is a contradictions since $\mathcal{F} \cap \left(\overline{A_{\frac{\varsigma_\infty}{2},\frac{3}{2}\varsigma_\infty}(\beta_\infty)}\right) = \emptyset$.
Finally, we exclude $\beta_\infty \neq x$ because we would get the following contradiction:

$$\int_{B_\delta(\beta_\infty)} f^n \mathrm{d}V_g \geq \delta.$$

The number τ in the proof of Lemma 3.27 will be chosen later in Section 3.6 in such a way that it verifies some good properties when evaluated on the test functions introduced in Section 3.3.

□

The main result from this section is the following: we will prove that if *both* $\beta_1 = \beta_2$ and $\varsigma_1 = \varsigma_2$, then $J_\rho(u)$ is bounded from below for a largest range of ρ.

Theorem 3.28.
Let δ, ψ be as in Lemma 3.27 and define, for $u \in H^1(\Sigma)^2$,

$$\beta_1(u) = \beta_{\Sigma_{\rho_1,\underline{\alpha}_1}}(f_{1,u}), \qquad \varsigma_1(u) = \varsigma_{\Sigma_{\rho_1,\underline{\alpha}_1}}(f_{2,u}),$$

$$\beta_2(u) = \beta_{\Sigma_{\rho_2,\underline{\alpha}_2}}(f_{2,u}), \qquad \varsigma_2(u) = \varsigma_{\Sigma_{\rho_2,\underline{\alpha}_2}}(f_{2,u}).$$

There exists $L \gg 0$ such that if

$$\begin{cases} \beta_1(u) = \beta_2(u) = p_m & \text{with } \rho_1,\rho_2 < 4\pi(2+\alpha_{1m}+\alpha_{2m}) \\ \varsigma_1(u) = \varsigma_2(u) \end{cases},$$

then $J_\rho(u) \geq -L$.

A first piece of information can be deduced from the macroscopic improved inequalities. In particular, if the scale of concentration is not too small, Lemma 3.23 gives an upper bound with few difficulties.

Corollary 3.29. *Let $\beta_i(u), \varsigma_i(u)$ be as in Theorem 3.28.*
Then for any $\delta' > 0$ there exists $L_{\delta'}$ such that if $\varsigma_i(u) \geq \delta'$ for both $i = 1,2$, then $J_\rho(u) \geq -L_{\delta'}$.

Proof.
Assume first $\varsigma_1(u) = \delta$: from the statement of Lemma 3.29, we get one of the following:

- $\displaystyle\int_{\Sigma \setminus \bigcup_{p=1}^M B_\delta(p_m)} f_{1,u} \mathrm{d}V_g \geq \frac{\delta}{2}$,
- $\displaystyle\int_{B_\delta(p_m)} f_{1,u} \mathrm{d}V_g \geq \frac{\delta}{2M}$ for some $p_m \notin \Sigma_{\rho_1,\underline{\alpha}_1}$,
- $\displaystyle\int_{B_\delta(p'_m)} f_{1,u} \mathrm{d}V_g \geq \delta, \int_{B_\delta(p_{m''})} f_{1,u} \mathrm{d}V_g \geq \delta$ for some $m' \neq m''$.

Depending on which possibility occurs, define respectively

- $\Omega_{11} := \Sigma \setminus \displaystyle\bigcup_{p=1}^M B_\delta(p_m)$,

- $\Omega_{11} := B_\delta(p_m)$,
- $\Omega_{11} := B_\delta(p_{m'})$, $\Omega_{12} := B_\delta(p_{m''})$.

It is easy to verify that such sets satisfy the hypotheses of Lemma 3.23, up to eventually redefining the map ψ with a smaller $\delta \le \dfrac{\min_{m\neq m'} d(p_m, p_{m'})}{4}$. Notice that Lemma 3.23 still holds under the assumptions of Theorem 3.5; all singular points are allowed ($\{p_{m_1}, \dots, p_{m_{iJ_i}}\} \subset \{p_1, \dots, p_M\}$) but the best constants are multiplied by $K_i + \sum_{j=1}^{J_i}\left(1 - \alpha^-_{im_{ij}}\right)$, since positive singularities do not affect Moser-Trudinger inequality (3.1) In the first case, we have $J_1 = 0, K_1 = 1$, in the second case either $J_1 = 0, K_1 = 1$ or $J_1 = 1, K_1 = 0$ but $\rho < 4\pi(1+\alpha_{1m})$, and in the third case we have $J_1 = 2, K_1 = 0$.
If $\delta' \le \varsigma_1(u) < \delta$, then $\int_{\Sigma\setminus B_{\delta'}(\beta_1(u))} f_{1,u}\mathrm{d}V_g \ge \delta$, so we have one between:

- $\int_{\Sigma\setminus\bigcup_{m=1}^M B_\delta(x)} f_{1,u}\mathrm{d}V_g \ge \dfrac{\delta}{2}$
- $\int_{B_\delta(\beta_1(u))} f_{1,u}\mathrm{d}V_g \ge \delta$, $\int_{B_\delta(p_m)} f_{1,u}\mathrm{d}V_g \ge \dfrac{\delta}{2M}$ for some $p_m \neq \beta_1(u)$.
- $\int_{A_{\delta',\delta}(\beta_1(u))} f_{1,u}\mathrm{d}V_g$.

Depending on which is the case, define:

- $\Omega_{11} := \Sigma \setminus \bigcup_{m=1}^{M} B_\delta(p_m)$.
- $\Omega_{11} := B_\delta(u)(\beta_1(u))$, $\Omega_{12} := B_\delta(p_m)$.
- $\Omega_{11} := A_{\delta',\delta}(\beta_1(u))$

Repeat the same argument for u_2 to get similarly Ω_{21}, and possibly Ω_{22}. Now apply Lemma 3.23 and you will get $J_\rho(u) \ge -L_{\delta'}$. □

To prove Theorem 3.28 we will study the behavior of u around a small neighborhood of the center of mass β, we will need a *localized* version of the Moser-Trudinger inequality.
It can be deduced by the standard Moser-Trudinger inequality by arguing via cut-off and Fourier decomposition, very similarly to the proof of Lemma 3.23.
We will give a version holding on the Euclidean unit disk, since we will use Theorem 1.18 which holds in a Euclidean setting.

Lemma 3.30.
For any $\varepsilon > 0$, $\alpha_1, \alpha_2 \in (-1, 0]$ there exists $C = C_\varepsilon$ such that for any $u \in H^1(B_1(0))^2$

$$4\pi\sum_{i=1}^{2}(1+\alpha_i)\left(\log\int_{B_{\frac12}(0)}|x|^{2\alpha_i}e^{u_i(x)}\mathrm{d}x - \fint_{B_1(0)} u_i(x)\mathrm{d}x\right) \le (1+\varepsilon)\int_{B_1(0)} Q(u(x))\mathrm{d}x + C, \tag{3.50}$$

$$\begin{aligned} &4\pi(1+\alpha_1)\left(\log\int_{B_{\frac18}(0)}|x|^{2\alpha_1}e^{u_1(x)}\mathrm{d}x - \fint_{B_1(0)} u_1(x)\mathrm{d}x\right) + 2\pi\left(\log\int_{A_{\frac14,1}(0)} e^{u_2(x)}\mathrm{d}x - \fint_{B_1(0)} u_2(x)\mathrm{d}x\right) \\ \le\ &(1+\varepsilon)\int_B Q(u(x))\mathrm{d}x + C. \end{aligned} \tag{3.51}$$

Proof.
Consider a closed surface Σ in which $B_1(0)$ is smoothly embedded and a cut-off function χ defined on Σ satisfying

$$0 \leq \chi \leq 1, \qquad \chi|_{B_{\frac{1}{2}}(0)} \equiv 1, \qquad \chi|_{\Sigma \setminus B_1(0)} \equiv 0, \qquad |\nabla \chi| \leq C.$$

We split u_i similarly as the proof of Lemma 3.23, but using truncations in Fourier modes on $\overline{H}^1(B_1(0))$: we take an orthonormal frame $\{\varphi^n\}_{n=1}^\infty$ of eigenfunctions for $-\Delta$ on $\overline{H}^1(B_1(0))$ with a non-decreasing sequence of positive eigenvalues $\{\lambda^n\}_{n=1}^\infty$. Writing $u_i = \int_{B_1(0)} u_i(x)\mathrm{d}x + \sum_{n=1}^\infty u_i^n \varphi^n$, we set $v_i = \sum_{n=1}^N u_i^n \varphi^n$ for $N := \max\left\{n \in \mathbb{N} : \lambda^n < \frac{C}{\varepsilon}\right\}$. Such a choice gives

$$\int_{B_1(0)} |w_i(x)|\mathrm{d}x \leq \varepsilon \int_{B_1(0)} Q(u(x))\mathrm{d}x + C, \qquad \|v_i\|_{L^\infty(B_1(0))} \leq \varepsilon \int_{B_1(0)} Q(u(x))\mathrm{d}x + C, \quad (3.52)$$

$$\int_\Sigma Q(\chi w)\mathrm{d}V_g \leq (1+\varepsilon) \int_{B_1(0)} Q(u(x))\mathrm{d}x + C.$$

Therefore, by arguing as in the proof of Lemma 3.23,

$$\begin{aligned}
&\sum_{i=1}^2 \left((1+\alpha_i) \log \int_{B_{\frac{1}{2}}(0)} |x|^{2\alpha_i} e^{u_i(x)}\mathrm{d}x - \fint_{B_1(0)} u_i(x)\mathrm{d}x \right) \\
\leq\ & \sum_{i=1}^2 \left((1+\alpha_i) \log \int_\Sigma d(\cdot,0)^{2\alpha_i} e^{\chi w_i}\mathrm{d}V_g + \|v_i\|_{L^\infty\left(B_{\frac{1}{2}}(0)\right)} \right) + C \\
\leq\ & \sum_{i=1}^2 \left((1+\alpha_i) \int_\Sigma \chi w_i \mathrm{d}V_g + \frac{1}{4\pi} \int_\Sigma Q(\chi w)\mathrm{d}V_g + \varepsilon \int_{B_1(0)} Q(u(x))dx \right) + C \\
\leq\ & \frac{1+\varepsilon'}{4\pi} \int_{B_1(0)} Q(u(x))dx + C,
\end{aligned}$$

which proves the first inequality.

For the second inequality, we consider two similar cut-off on $B_{\frac{1}{8}}(0)$ and $A_{\frac{1}{4},1}(0)$, respectively:

$$0 \leq \chi' \leq 1, \qquad \chi'|_{B_{\frac{1}{8}}(0)} \equiv 1, \qquad \chi'|_{\Sigma \setminus B_{\frac{3}{16}}(0)} \equiv 0 \qquad |\nabla \chi'| \leq C,$$

$$0 \leq \chi'' \leq 1, \qquad \chi''|_{A_{\frac{1}{4},1}(0)} \equiv 1, \qquad \chi''|_{\Sigma \setminus A_{\frac{3}{16},2}(0)} \equiv 0 \qquad |\nabla \chi''| \leq C.$$

We then argue as before, writing $u_1 - \fint_{B_{\frac{3}{16}}(0)} u_1(x)\mathrm{d}x = v_1 + w_1$ and $u_2 - \fint_{A_{\frac{3}{16},1}(0)} u_2(x)\mathrm{d}x = v_2 + w_2$, with v_1, v_2 obtained via Fourier decomposition of $-\Delta u$ on $\overline{H}^1\left(B_{\frac{3}{16}}(0)\right)$ and $\overline{H}^1\left(A_{\frac{3}{16},1}(0)\right)$, respectively. In such a way, v_1 and w_1 can be estimated by means of $\varepsilon \int_{B_1(0)} |\nabla u_1|^2$ as in (3.52). For the terms involving u_1, we suffice to apply to $\chi' w_1$ a scalar inequalities, much like in (3.40):

$$\begin{aligned}
& (1+\alpha_1) \log \int_{B_{\frac{1}{8}}(0)} |x|^{2\alpha_1} e^{u_1(x)}\mathrm{d}x - \fint_{B_{\frac{3}{16}}(0)} u_1(x)\mathrm{d}x \\
\leq\ & (1+\alpha_1) \log \int_\Sigma d(\cdot,0)^{2\alpha_1} e^{\chi' w_1}\mathrm{d}V_g + \|v_1\|_{L^\infty\left(B_{\frac{3}{16}}(0)\right)} + C \\
\leq\ & (1+\alpha_1) \int_\Sigma \chi' w_1 \mathrm{d}V_g + \frac{1}{16\pi} \int_\Sigma |\nabla(\chi' w_1)|^2 \mathrm{d}V_g + \varepsilon \int_{B_{\frac{3}{16}}(0)} Q(u(x))dx + C
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{16\pi}\int_{B_{\frac{3}{16}}(0)}|\nabla u_1(x)|^2\mathrm{d}x+\varepsilon\int_{B_{\frac{3}{16}}(0)}Q(u(x))dx+C\\
&\leq \frac{1+\varepsilon'}{4\pi}\int_{B_{\frac{3}{16}}(0)}Q(u(x))dx+C. \qquad (3.53)
\end{aligned}
$$

As for u_2, we want to *localize* the inequality (1.7).
Therefore, we study u_2 only on $B_1(0)$ and we apply (1.7) to $(\chi'' w_2)|_{B_1(0)}$:

$$
\begin{aligned}
&\log\int_{A_{\frac{1}{4},1}(0)}e^{u_2(x)}\mathrm{d}x-⨍_{A_{\frac{3}{16},1}(0)}u_2(x)\mathrm{d}x\\
&\log\int_{B_1(0)}e^{\chi''(x)w_2(x)}\mathrm{d}x+\|v_2\|_{L^\infty\left(A_{\frac{3}{16},1}(0)\right)}\\
\leq\;&\int_{B_1(0)}\chi''(x)w_2(x)\mathrm{d}x+\frac{1}{8\pi}\int_{B_1(0)}|\nabla(\chi''(x)w_2(x))|^2\mathrm{d}x+C\\
\leq\;&\frac{1}{8\pi}\int_{A_{\frac{3}{16},1}(0)}|\nabla u(x)|^2\mathrm{d}x+\varepsilon\int_{A_{\frac{3}{16},1}(0)}Q(u(x))dx+C\\
\leq\;&\frac{1+\varepsilon'}{4\pi}\int_{A_{\frac{3}{16},1}(0)}Q(u(x))dx+C. \qquad (3.54)
\end{aligned}
$$

The conclusion follows by summing (3.53) and (3.54) and by applying Lemma 1.19 to replace the two averages with the ones taken on $B_1(0)$. $\square$

The proof of Theorem 3.28 is based on the following two lemmas, inspired by [61].
Basically, we assume u_1 and u_2 to have the same center and scale of concentration and we provide local estimates in a ball which is roughly centered at the center of mass and whose radius is roughly the same as the scale of concentration. Inner estimates use a dilation argument, outer estimates use a Kelvin transform.
With respect to the above-cited paper, we also have to consider concentration around the boundary of the ball, hence we will combine those arguments with Theorem 1.18 and Lemma 3.30.

Lemma 3.31.
For any $\varepsilon>0$, $\alpha_1,\alpha_2\in(-1,0]$ there exists $C=C_\varepsilon$ such that for any $p\in\Sigma$, $s>0$ small enough and $u\in H^1(\Sigma)^2$ one has

$$
\begin{aligned}
&4\pi\sum_{i=1}^2(1+\alpha_i)\left(\log\int_{B_{\frac{s}{2}}(p)}d(\cdot,p)^{2\alpha_i}e^{u_i}\mathrm{d}V_g-⨍_{B_s(p)}u_i\mathrm{d}V_g\right)+8\pi\left((1+\alpha_1)^2+(1+\alpha_2)^2\right)\log s\\
\leq\;&(1+\varepsilon)\int_{B_s(p)}Q(u)\mathrm{d}V_g+C, \qquad (3.55)
\end{aligned}
$$

$$
\begin{aligned}
&4\pi(1+\alpha_1)\left(\log\int_{B_{\frac{s}{8}}(p)}d(\cdot,p)^{2\alpha_1}e^{u_1}\mathrm{d}V_g-⨍_{B_s(p)}u_1\mathrm{d}V_g\right)\\
+\;&2\pi\min\{1,2+\alpha_1+\alpha_2\}\left(\log\int_{A_{\frac{s}{4},s}(p)}d(\cdot,p)^{2\alpha_2}e^{u_2}\mathrm{d}V_g-⨍_{B_s(p)}u_2\mathrm{d}V_g\right)\\
+\;&4\pi\left(2(1+\alpha_1)^2+\min\{1,2+\alpha_1+\alpha_2\}(1+\alpha_2)\right)\log s\\
\leq\;&(1+\varepsilon)\int_{B_s(p)}Q(u)\mathrm{d}V_g+C, \qquad (3.56)
\end{aligned}
$$

$$
\begin{aligned}
&2\pi\sum_{i=1}^2\min\{1,2+\alpha_1+\alpha_2\}\left(\log\int_{A_{\frac{s}{2},s}(p)}d(\cdot,p)^{2\alpha_i}e^{u_i}\mathrm{d}V_g-⨍_{B_s(p)}u_i\mathrm{d}V_g\right)\\
+\;&4\pi\min\left\{2+\alpha_1+\alpha_2,(2+\alpha_1+\alpha_2)^2\right\}\log s
\end{aligned}
$$

$$\leq \quad (1+\varepsilon)\int_{B_s(p)} Q(u)\mathrm{d}V_g + C. \tag{3.57}$$

The last statement holds true if $B_s(p)$ is replaced by Ω_s simply connected belonging to $\mathfrak{A}_{\delta s}$ (see (1.8)) and such that $B_{(\frac{1}{2}+\delta)s}(p) \subset \Omega_s \subset B_{\frac{s}{\delta}}(p)$ for some $\delta > 0$, with C replaced with some $C_\delta > 0$.

Proof.
By assuming s small enough, we can suppose the metric to be flat on $B_s(p)$, up to negligible remainder terms.
Therefore, we will assume to work on a Euclidean ball centered at the origin: we will indicate such balls simply as B_s, omitting their center, and we will use a similar convention for annuli. Moreover, we will write $|x|$ for $d(x,p)$.
Consider the dilation $v_i(z) = u_i(sz)$ for $z \in B_1$. It verifies, for $r \in \left\{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}\right\}$

$$\int_{B_r} |z|^{2\alpha_i} e^{v_i(z)}\mathrm{d}z = s^{-2-2\alpha_i}\int_{B_{rs}} |x|^{2\alpha_i} e^{u_i(x)}\mathrm{d}x,$$

$$\int_{A_{r,1}} |z|^{2\alpha_i} e^{v_i(z)}\mathrm{d}z = s^{-2-2\alpha_i}\int_{A_{rs,s}} |x|^{2\alpha_i} e^{u_i(x)}\mathrm{d}x.$$

$$\int_{B_1} Q(v(z))\mathrm{d}z = \int_{B_s} Q(u(x))\mathrm{d}x, \qquad \fint_{B_1} v(z)\mathrm{d}z = \fint_{B_s} u(x)\mathrm{d}x,$$

To get (3.55), it suffices to apply (3.50) to $v = (v_1, v_2)$:

$$\begin{aligned}
& 4\pi\sum_{i=1}^{2}(1+\alpha_i)\left(\log\int_{B_{\frac{s}{2}}} |x|^{2\alpha_i} e^{u_i(x)}\mathrm{d}x - \fint_{B_s} u_i(x)\mathrm{d}x + 2(1+\alpha_i)\log s\right) \\
\leq \quad & 4\pi\sum_{i=1}^{2}(1+\alpha_i)\left(\log\int_{B_{\frac{1}{2}}} |z|^{2\alpha_i} e^{v_i(z)}\mathrm{d}z - \fint_{B_1} v_i(z)\mathrm{d}z\right) \\
\leq \quad & (1+\varepsilon)\int_{B_1} Q(v(z))\mathrm{d}z + C \\
= \quad & (1+\varepsilon)\int_{B_s} Q(u(x))\mathrm{d}x + C.
\end{aligned}$$

For (3.56), one has to use (3.51) on v, and the elementary fact that $\frac{1}{C} \leq |z|^{2\alpha_2} \leq C$ on $A_{\frac{1}{4},1}$:

$$\begin{aligned}
& 4\pi(1+\alpha_1)\left(\log\int_{B_{\frac{s}{8}}} |x|^{2\alpha_1} e^{u_1(x)}\mathrm{d}x - \fint_{B_s} u_1(x)\mathrm{d}x + 2(1+\alpha_1)\log s\right) \\
+ \quad & 2\pi\min\{1, 2+\alpha_1+\alpha_2\}\left(\log\int_{A_{\frac{s}{4},s}} |x|^{2\alpha_2} e^{u_2(x)}\mathrm{d}V_g(x) - \fint_{B_s} u_2(x)\mathrm{d}x + 2(1+\alpha_2)\log s\right) \\
\leq \quad & 4\pi(1+\alpha_1)\left(\log\int_{B_{\frac{1}{8}}} |z|^{2\alpha_1} e^{v_1(z)}\mathrm{d}z - \fint_{B_1} v_1(z)\mathrm{d}x\right) \\
+ \quad & 2\pi\min\{1, 2+\alpha_1+\alpha_2\}\left(\log\int_{A_{\frac{1}{4},1}} e^{v_2(z)}\mathrm{d}V_g(z) - \fint_{B_1} v_2(z)\mathrm{d}z\right) + C \\
\leq \quad & (1+\varepsilon)\int_{B_1} Q(v(z))\mathrm{d}z + C \\
= \quad & (1+\varepsilon)\int_{B_s} Q(u(x))\mathrm{d}x + C.
\end{aligned}$$

Finally, (3.57) follows by Theorem 1.18:

$$\begin{aligned}
& 2\pi\sum_{i=1}^{2}\min\{1,2+\alpha_1+\alpha_2\}\left(\log\int_{A_{\frac{s}{2},s}}|x|^{2\alpha_i}e^{u_i(x)}\mathrm{d}x-\fint_{B_s}u_i(x)\mathrm{d}x+2(1+\alpha_i)\log s\right)\\
\leq\ & 2\pi\sum_{i=1}^{2}\min\{1,2+\alpha_1+\alpha_2\}\left(\log\int_{A_{\frac{1}{2},1}}e^{v_i(z)}\mathrm{d}z-\fint_{B_1}v_i(z)\mathrm{d}z\right)+C\\
\leq\ & (1+\varepsilon)\int_{B_1}Q(v(z))\mathrm{d}z+C\\
=\ & (1+\varepsilon)\int_{B_s}Q(u(x))\mathrm{d}x+C.
\end{aligned}$$

The final remark holds true because of the final remarks in Lemmas 1.18 and 1.19. □

Lemma 3.32.
For any $\varepsilon>0$, $\alpha_1,\alpha_2\in(-1,0]$, $d>0$ small there exists $C=C_\varepsilon$ such that for any $p\in\Sigma$, $s\in\left(0,\frac{d}{8}\right)$ and $u\in H^1(\Sigma)^2$ with $u_i|_{\partial B_d(p)}\equiv 0$ one has

$$\begin{aligned}
& 4\pi\sum_{i=1}^{2}(1+\alpha_{3-i})\log\int_{A_{2s,d}(p)}d(\cdot,p)^{2\alpha_i}e^{u_i}\mathrm{d}V_g+4\pi(1+\varepsilon)\sum_{i=1}^{2}(1+\alpha_i)\fint_{B_s(p)}u_i\mathrm{d}V_g\\
-\ & 8\pi(1+\varepsilon)\left((1+\alpha_1)^2+(1+\alpha_2)^2\right)\log s\\
\leq\ & \int_{A_{s,d}(p)}Q(u)\mathrm{d}V_g+\varepsilon\int_{B_d(p)}Q(u)\mathrm{d}V_g+C,
\end{aligned}\tag{3.58}$$

$$\begin{aligned}
& 4\pi(1+\alpha_2)\log\int_{A_{8s,d}(p)}d(\cdot,p)^{2\alpha_1}e^{u_1}\mathrm{d}V_g+4\pi(1+\varepsilon)(1+\alpha_1)\fint_{B_s(p)}u_1\mathrm{d}V_g\\
+\ & 2\pi\min\{1,2+\alpha_1+\alpha_2\}\left(\log\int_{A_{s,4s}(p)}d(\cdot,p)^{2\alpha_2}e^{u_2}\mathrm{d}V_g+(1+\varepsilon)\fint_{B_s(p)}u_2\mathrm{d}V_g\right)\\
-\ & 4\pi(1+\varepsilon)\left(2(1+\alpha_1)^2+\min\{1,2+\alpha_1+\alpha_2\}(1+\alpha_2)\right)\log s\\
\leq\ & \int_{A_{s,d}(p)}Q(u)\mathrm{d}V_g+\varepsilon\int_{B_d(p)}Q(u)\mathrm{d}V_g+C,
\end{aligned}\tag{3.59}$$

$$\begin{aligned}
& 2\pi\sum_{i=1}^{2}\min\{1,2+\alpha_1+\alpha_2\}\left(\log\int_{A_{s,2s}(p)}d(\cdot,p)^{2\alpha_i}e^{u_i}\mathrm{d}V_g+(1+\varepsilon)\fint_{B_s(p)}u_i\mathrm{d}V_g\right)\\
-\ & 4\pi(1+\varepsilon)\min\left\{2+\alpha_1+\alpha_2,(2+\alpha_1+\alpha_2)^2\right\}\log s\\
\leq\ & \int_{A_{s,d}(p)}Q(u)\mathrm{d}V_g+\varepsilon\int_{B_d(p)}Q(u)\mathrm{d}V_g+C.
\end{aligned}\tag{3.60}$$

The last statement holds true if $B_s(p)$ is replaced by a simply connected domain Ω_s belonging to $\mathfrak{A}_{\delta s}$ and such that $B_{\delta s}(p)\subset\Omega_s\subset B_{\left(2+\frac{1}{\delta}\right)s}(p)$ for some $\delta>0$, with the constant C is replaced with some $C_\delta>0$.

Proof.
Just like Lemma 3.32, we will work with flat Euclidean balls, whose centers will be omitted. Moreover, it will not be restrictive to assume $\fint_{B_d}u_i(x)\mathrm{d}x=0$ for both i's.
Define, for $z\in B_d$ and $c_1,c_2\leq-2(2+\alpha_1+\alpha_2)$,

$$v_i(z):=\begin{cases}(2c_i-c_{3-i})\log s & \text{if } z\in B_s\\ u_i\left(ds\frac{z}{|z|^2}\right)+(2c_i-c_{3-i})\log|z| & \text{if } z\in A_{s,d}\end{cases}$$

By a change of variable, we find, for $r \in \left\{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}\right\}$,

$$\int_{A_{s,rd}} |z|^{-4-2\alpha_i-2c_i+c_{3-i}} e^{v_i(z)} \mathrm{d}z = \int_{B_{s,rd}} |z|^{-4-2\alpha_i} e^{u_i\left(ds\frac{z}{|z|^2}\right)} \mathrm{d}z = (ds)^{-2-2\alpha_i} \int_{A_{\frac{s}{r},d}} |x|^{2\alpha_i} e^{u_i(x)} \mathrm{d}x$$

$$\int_{A_{rd,d}} e^{v_i(z)} \mathrm{d}z \sim \int_{A_{rd,d}} |z|^{-4-2\alpha_i-2c_i+c_{3-i}} e^{v_i(z)} \mathrm{d}z = (ds)^{-2-2\alpha_i} \int_{A_{s,\frac{s}{r}}} |x|^{2\alpha_i} e^{u_i(x)} \mathrm{d}x.$$

Moreover, by Lemma 1.19, we get

$$\begin{aligned}
& \left| \fint_{B_s} u_i(x) \mathrm{d}x - \fint_{B_d} v_i(z) \mathrm{d}z \right| \\
\leq\ & \left| \fint_{B_s} u_i(x) \mathrm{d}x - \fint_{\partial B_s} u_i(x) \mathrm{d}x \right| + \left| \fint_{\partial B_s} u_i(x) \mathrm{d}x - \fint_{\partial B_d} v_i(z) \mathrm{d}z \right| + \left| \fint_{B_d} v_i(z) \mathrm{d}z - \fint_{B_d} v_i(z) \mathrm{d}z \right| \\
\leq\ & C \sqrt{\int_{B_s} |\nabla u(x)|^2 \mathrm{d}x} + |(2c_i - c_{3-i}) \log d| + C \sqrt{\int_{B_d} |\nabla v(z)|^2 \mathrm{d}z} \leq \\
\leq\ & \varepsilon' \int_{B_s} Q(u(x)) \mathrm{d}x + \varepsilon' \int_{B_d} Q(v(z)) \mathrm{d}z + C_d.
\end{aligned}$$

Concerning the Dirichlet integral, we can write

$$\begin{aligned}
& \int_{B_d} \nabla v_i(z) \cdot \nabla v_j(z) \mathrm{d}z \\
=\ & \int_{A_{s,d}} \left(\frac{d^2 s^2}{|z|^4} \nabla u_i \left(ds \frac{z}{|z|^2} \right) \cdot \nabla u_j \left(ds \frac{z}{|z|^2} \right) - (2c_i - c_{3-i}) ds \frac{z}{|z|^2} \cdot \nabla u_j \left(ds \frac{z}{|z|^2} \right) \right. \\
-\ & \left. (2c_j - c_{3-j}) ds \frac{z}{|z|^2} \cdot \nabla u_i \left(ds \frac{z}{|z|^2} \right) + \frac{(2c_i - c_{3-i})(2c_j - c_{3-j})}{|z|^2} \right) \mathrm{d}z \\
=\ & \int_{A_{s,d}} \nabla u_i(x) \cdot \nabla u_j(x) \mathrm{d}x - (2c_i - c_{3-i}) \int_{A_{s,d}} \frac{x}{|x|^2} \cdot \nabla u_j(x) \mathrm{d}x \\
-\ & (2c_j - c_{3-j}) \int_{A_{s,d}} \frac{x}{|x|^2} \cdot \nabla u_i(x) \mathrm{d}x - 2\pi (2c_i - c_{3-i})(2c_j - c_{3-j}) \log s \\
+\ & 2\pi (2c_i - c_{3-i})(2c_j - c_{3-j}) \log d \\
=\ & \int_{A_{s,d}} \nabla u_i(x) \cdot \nabla u_j(x) \mathrm{d}x + 2\pi (2c_i - c_{3-i}) \fint_{\partial B_s} u_j(x) \mathrm{d}x + 2\pi (2c_j - c_{3-j}) \fint_{\partial B_s} u_i(x) \mathrm{d}x \\
-\ & 2\pi (2c_i - c_{3-i})(2c_j - c_{3-j}) \log s + C_d,
\end{aligned}$$

therefore, since v has constant components in B_s,

$$\int_{B_d} Q(v(z)) \mathrm{d}z = \int_{A_{s,d}} Q(u(x)) \mathrm{d}x + 2\pi \sum_{i=1}^{2} c_i \fint_{\partial B_s} u_i(x) \mathrm{d}x - 2\pi \left(c_1^2 - c_1 c_2 + c_2^2 \right) \log s + C.$$

The assertion of the Lemma follows by applying Lemma 3.30 on B_d to v with different choices of c_1, c_2.
If we take $c_1 = c_2 = -2(2 + \alpha_1 + \alpha_2)$, then we get

$$\begin{aligned}
& 4\pi \sum_{i=1}^{2} (1 + \alpha_{3-i}) \log \int_{A_{2s,d}} |x|^{2\alpha_i} e^{u_i(x)} \mathrm{d}x \\
\leq\ & 4\pi \sum_{i=1}^{2} (1 + \alpha_{3-i}) \left(\log \int_{B_{\frac{d}{2}}} |z|^{2\alpha_{3-i}} e^{v_i(z)} \mathrm{d}z + 2(1 + \alpha_i) \log s \right) + C \\
\leq\ & (1 + \varepsilon') \int_{B_d} Q(v(z)) \mathrm{d}z + 4\pi \sum_{i=1}^{2} (1 + \alpha_{3-i}) \fint_{B_d} v_i(z) \mathrm{d}z + 16\pi (1 + \alpha_1)(1 + \alpha_2) \log s + C
\end{aligned}$$

$$
\begin{aligned}
\leq\ & (1+\varepsilon'')\int_{A_{s,d}} Q(u(x))\mathrm{d}x+\varepsilon''\int_{B_d} Q(u(x))\mathrm{d}x\\
+\ & 4\pi\sum_{i=1}^{2}(1+\alpha_{3-i})\fint_{B_s} u_i(x)\mathrm{d}x-4\pi(1+\varepsilon'')(2+\alpha_1+\alpha_2)\sum_{i=2}^{2}\fint_{\partial B_s} u_i(x)\mathrm{d}x\\
+\ & 8\pi\left(2(1+\alpha_1)(1+\alpha_2)-(1+\varepsilon'')(2+\alpha_1+\alpha_2)^2\right)\log s+C\\
\leq\ & (1+\varepsilon''')\int_{B_s} Q(u(x))\mathrm{d}x+\varepsilon'''\int_{B_d} Q(u(x))\mathrm{d}x\\
+\ & 4\pi\sum_{i=1}^{2}((1+\alpha_{3-i})-(1+\varepsilon''')(2+\alpha_1+\alpha_2))\fint_{B_s} u_i(x)\mathrm{d}x\\
+\ & 8\pi\left(2(1+\alpha_1)(1+\alpha_2)-(1+\varepsilon''')(2+\alpha_1+\alpha_2)^2\right)\log s+C,
\end{aligned}
$$

that is, re-naming ε properly, (3.58).

Choosing $c_1=-2(2+\alpha_1+\alpha_2)$ and $c_2=-2\min\{1,2+\alpha_1+\alpha_2\}=:-2m$, we get

$$
\begin{aligned}
& 4\pi(1+\alpha_2)\log\int_{A_{8s,d}}|x|^{2\alpha_1}e^{u_1(x)}\mathrm{d}x+2\pi m\log\int_{A_{s,4s}}|x|^{2\alpha_2}e^{u_2(x)}\mathrm{d}x\\
\leq\ & 4\pi(1+\alpha_2)\log\int_{B_{\frac{d}{8}}}|z|^{\max\{2+2\alpha_1+4\alpha_2,2\alpha_2\}}e^{v_1(z)}\mathrm{d}z+2\pi m\log\int_{A_{\frac{d}{4},d}} e^{v_2(z)}\mathrm{d}z\\
+\ & 4\pi(2(1+\alpha_1)(1+\alpha_2)+m(1+\alpha_2))\log s+C\\
\leq\ & (1+\varepsilon')\int_{B_d} Q(v(z))\mathrm{d}z+4\pi(1+\alpha_2)\fint_{B_d} v_1(z)\mathrm{d}z+2\pi m\fint_{B_d} v_2(z)\mathrm{d}z\\
+\ & 4\pi(1+\alpha_2)(2(1+\alpha_1)+m)\log s+C\\
\leq\ & (1+\varepsilon'')\int_{A_{s,d}} Q(u(x))\mathrm{d}x+\varepsilon''\int_{B_d} Q(u(x))\mathrm{d}x+4\pi(1+\alpha_2)\fint_{B_s} u_1(x)\mathrm{d}x+2\pi m\fint_{B_s} u_2(x)\mathrm{d}x\\
-\ & 4\pi(1+\varepsilon'')(2+\alpha_1+\alpha_2)\fint_{\partial B_s} u_1(x)\mathrm{d}x-4\pi(1+\varepsilon'')m\fint_{\partial B_s} u_2(x)\mathrm{d}x\\
+\ & 4\pi\left((1+\alpha_2)(2(1+\alpha_1)+m)-2(1+\varepsilon'')\left((2+\alpha_1+\alpha_2)^2-m(2+\alpha_1+\alpha_2)+m^2\right)\right)\log s+C\\
\leq\ & (1+\varepsilon''')\int_{A_{s,d}} Q(u(x))\mathrm{d}x+\varepsilon'''\int_{B_d} Q(u(x))\mathrm{d}x\\
+\ & 4\pi((1+\alpha_2)-(1+\varepsilon''')(2+\alpha_1+\alpha_2))\fint_{B_s} u_1(x)\mathrm{d}x-2\pi(1+2\varepsilon''')m\fint_{B_s} u_2(x)\mathrm{d}x\\
+\ & 4\pi((1+\alpha_2)(2(1+\alpha_1)+m)-2(1+\varepsilon''')((1+\alpha_1)(2+\alpha_1+\alpha_2)+(1+\alpha_2)m))\log s+C,
\end{aligned}
$$

namely (3.59).

Finally, taking $c_1=c_2=-2m$ one finds (3.60):

$$
\begin{aligned}
& 2\pi\sum_{i=1}^{2}\min\{1,2+\alpha_1+\alpha_2\}\log\int_{A_{s,2s}}|x|^{2\alpha_i}e^{u_i(x)}\mathrm{d}x\\
\leq\ & 2\pi\sum_{i=1}^{2}m\left(\log\int_{A_{\frac{d}{2},d}} e^{v_i(z)}\mathrm{d}z+2(1+\alpha_i)\log s\right)+C\\
\leq\ & (1+\varepsilon')\int_{B_d} Q(v(z))\mathrm{d}z+2\pi m\sum_{i=1}^{2}\fint_{B_d} v_i(z)\mathrm{d}z+4\pi m(2+\alpha_1+\alpha_2)\log s+C\\
\leq\ & (1+\varepsilon'')\int_{B_d} Q(u(x))\mathrm{d}x+\varepsilon''\int_{B_d} Q(u(x))\mathrm{d}x+2\pi m\sum_{i=1}^{2}\fint_{B_s} u_i(x)\mathrm{d}x\\
-\ & 4\pi(1+\varepsilon'')m\sum_{i=1}^{2}\fint_{\partial B_s} u_i(x)\mathrm{d}x+4\pi\left(m(2+\alpha_1+\alpha_2)-2(1+\varepsilon'')m^2\right)\log s+C
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & (1+\varepsilon'')\int_{B_d} Q(u(x))\mathrm{d}x + \varepsilon'''\int_{B_d} Q(u(x))\mathrm{d}x - 2\pi(1+2\varepsilon''')m\sum_{i=1}^{2}\fint_{B_s} u_i(x)\mathrm{d}x \\
-\ & 4\pi(1+2\varepsilon''')m(2+\alpha_1+\alpha_2)\log s + C.
\end{aligned}
$$

The final remark holds true, like in Lemma 3.31, because of the final remarks in Lemmas 1.18 and 1.19.
In particular, when integrating by parts, one gets

$$
\int_{B_\delta\setminus\Omega_s}\frac{x}{|x|^2}\cdot\nabla u_i(x)\mathrm{d}x = \int_{\partial\Omega_s} u_i(x)\underbrace{\frac{x}{|x|^2}\cdot\nu(x)}_{=:f(x)}\mathrm{d}x,
$$

with $\int_{\partial\Omega_s} f(x)\mathrm{d}x = 2\pi$ and $|f(x)| \leq \frac{C}{s} \leq \frac{C}{|\Omega_s|}$, therefore, by the Poincaré-Wirtinger inequality

$$
\begin{aligned}
& \left|\int_{\partial\Omega_s} u_i(x)f(x)\mathrm{d}x - 2\pi\int_{\partial\Omega_s} u_i(y)\mathrm{d}y\right| \\
=\ & \left|\int_{\partial\Omega_s} f(x)\left(u_i(x) - \fint_{\partial\Omega_s} u_i(y)\mathrm{d}y\right)\mathrm{d}y\right| \\
\leq\ & C\fint_{\partial\Omega_s}\left|u_i(x) - \fint_{\partial\Omega_s} u(y)\mathrm{d}y\right|\mathrm{d}x \\
\leq\ & C\int_{\Omega_s}|\nabla u_i(x)|^2\mathrm{d}x \\
\leq\ & \varepsilon\int_{\Omega_s} Q(u(x))\mathrm{d}x + C_\varepsilon,
\end{aligned}
$$

and $\left|\fint_{\partial\Omega_s} u_i(x)\mathrm{d}x - \fint_{\partial B_s} u_i(x)\mathrm{d}x\right| \leq \varepsilon\int_{\Omega_s} Q(u(x))\mathrm{d}x + C_\varepsilon$ by Lemma 1.19. $\square$

To prove Theorem 3.28 we also need the following lemma.
It basically allows us to divide a disk in two domains in such a way that the integrals of two given functions are both split exactly in two.

Lemma 3.33.
Consider $B := B_1(0) \subset \mathbb{R}^2$ and $f_1, f_2 \in L^1(B)$ such that $f_i > 0$ a.e. $x \in B$ for both $i = 1,2$ and $\int_B f_1(x)\mathrm{d}x = \int_B f_2(x)\mathrm{d}x = 1$.
Then, there exists $\theta \in \mathbb{S}^1$ and $a \in (-1,1)$ such that

$$
\int_{\{x\in B:\, x\cdot\theta<a\}} f_1(x)\mathrm{d}x = \int_{\{x\in B:\, x\cdot\theta>a\}} f_2(x)\mathrm{d}x = \frac{1}{2}
$$

Proof.
Define, for $(\theta, a) \in \mathbb{S}^1 \times (-1,1)$,

$$
I_1(\theta,a) := \int_{\{x\in B:\, x\cdot\theta<a_1(\theta)\}} f_1(x)\mathrm{d}x.
$$

For any given θ there exists a unique $a_1(\theta)$, smoothly depending on θ such that $I_1(\theta, a_1(\theta)) = \frac{1}{2}$.
Define similarly $I_2(\theta,a)$ and $a_2(\theta)$.
Let us now show the existence of θ such that $a_1(\theta) = a_2(\theta) := a$, hence the proof of the Lemma will follow. Suppose, by contradiction, that $a_1(\theta) < a_2(\theta)$ for any θ. Then, by definition, we get

$$
a_1(-\theta) = -a_1(\theta) > -a_2(\theta) = a_2(-\theta),
$$

which is a contradiction. One similarly excludes the case $a_1(\theta) > a_2(\theta)$. $\square$

Proof of Theorem 3.28. From Lemma 3.27 we have $\beta \in \Sigma_{\rho_1,\underline{\alpha}_1} \cap \Sigma_{\rho_1,\underline{\alpha}_2}, \varsigma \in (0,\delta)$ such that

$$\int_{B_\varsigma(\beta)} f_{1,u} \mathrm{d}V_g \geq \delta, \qquad \int_{\Sigma\setminus B_\varsigma(\beta)} f_{1,u} \mathrm{d}V_g \geq \delta,$$

$$\int_{B_\varsigma(\beta)} f_{2,u} \mathrm{d}V_g \geq \delta, \qquad \int_{\Sigma\setminus B_\varsigma(\beta)} f_{2,u} \mathrm{d}V_g \geq \delta.$$

Moreover, from Corollary 3.29, we will suffice to prove the Theorem for $\varsigma \leq 2^{-\frac{6}{\varepsilon}-4}\delta$.
We have to consider several cases, roughly following the proof of Proposition 3.2 in [61].

Case 1 : $\displaystyle\int_{A_{\varsigma,\delta'}(\beta)} f_{i,u} \mathrm{d}V_g \geq \frac{\delta}{2}$ for both $i = 1, 2$, where $\delta' := 2^{-\frac{3}{\varepsilon}}\delta$.

As a first thing, we modify u so that it vanishes outside $B_\delta(\beta)$: we take $n \in \left[1, \frac{2}{\varepsilon}\right]$ such that

$$\int_{A_{2^{n-1}\delta', 2^{n+1}\delta'}(\beta)} Q(u)\mathrm{d}V_g \leq \varepsilon \int_\Sigma Q(u)\mathrm{d}V_g$$

and we define u_i' as the solution of

$$\begin{cases} -\Delta u_i' = 0 & \text{in } A_{2^{n-1}\delta', 2^{n+1}\delta'}(\beta) \\ u_i' = u_i - \displaystyle⨍_{B_{2^n\delta'}(\beta)} u_i \mathrm{d}V_g & \text{on } \partial B_{2^n\delta'}(\beta) \\ u_i' = 0 & \text{on } \partial B_{2^{n+1}\delta'}(\beta) \end{cases}$$

u_i' verifies, by Lemma 1.20,

$$\int_{A_{2^{n-1}\delta', 2^{n+1}\delta'}(\beta)} Q(u')\mathrm{d}V_g \leq C \int_{A_{2^{n-1}\delta', 2^{n+1}\delta'}(\beta)} Q(u)\mathrm{d}V_g \leq C\varepsilon \int_\Sigma Q(u)\mathrm{d}V_g.$$

We obtained a function for which Lemma 3.32 can be applied on $B_\delta(\beta)$. This was done at a little *price*, since the Dirichlet integral only increased by ε; moreover, u' and u coincide (up to an additive constant) on $B_{\delta'}(\beta)$, which is where both $f_{i,u}$'s attain most of their mass.

Case 1.a : $\displaystyle\int_{B_{\frac{\varsigma}{4}}(\beta)} f_{i,u} \mathrm{d}V_g \geq \frac{\delta}{2}$ for both $i = 1, 2$.

We apply Lemma 3.31 to u on $B_{\frac{\varsigma}{2}(\beta)}$, with $\alpha_i := \alpha_{im}$ for $i = 1, 2$. From (3.55) we get

$$\begin{aligned}
& 4\pi \sum_{i=1}^2 (1+\alpha_{im}) \left(\log \int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g - ⨍_{B_{\frac{\varsigma}{2}}(\beta)} u_i \mathrm{d}V_g \right) + 8\pi \left((1+\alpha_{1m})^2 + (1+\alpha_{2m})^2 \right) \log \frac{\varsigma}{2} \\
\leq\ & 4\pi \sum_{i=1}^2 (1+\alpha_{im}) \left(\log \int_{B_{\frac{\varsigma}{4}}(\beta)} \widetilde{h}_i e^{u_i} \mathrm{d}V_g - ⨍_{B_{\frac{\varsigma}{2}}(\beta)} u_i \mathrm{d}V_g \right) \\
+\ & 8\pi \left((1+\alpha_{1m})^2 + (1+\alpha_{2m})^2 \right) \log \frac{\varsigma}{2} + 4\pi(2+\alpha_{1m}+\alpha_{2m}) \log \frac{2}{\delta} \\
\leq\ & 4\pi \sum_{i=1}^2 (1+\alpha_{im}) \left(\log \int_{B_{\frac{\varsigma}{4}}(\beta)} d(\cdot,\beta)^{2\alpha_{im}} e^{u_i} \mathrm{d}V_g - ⨍_{B_{\frac{\varsigma}{2}}(\beta)} u_i \mathrm{d}V_g \right) \\
+\ & 8\pi \left((1+\alpha_{1m})^2 + (1+\alpha_{2m})^2 \right) \log \frac{\varsigma}{2} + C \\
\leq\ & (1+\varepsilon) \int_{B_{\frac{\varsigma}{2}}(\beta)} Q(u) \mathrm{d}V_g + C.
\end{aligned} \tag{3.61}$$

We then apply Lemma 3.32 to u' on $B_\delta(\beta)\setminus B_{\frac{\varsigma}{2}}(\beta)$.

$$
\begin{aligned}
& 4\pi\sum_{i=1}^{2}(1+\alpha_{3-i,m})\log\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g + 4\pi(1+\varepsilon)\sum_{i=1}^{2}(1+\alpha_{im})\fint_{B_{\frac{\varsigma}{2}}(\beta)} u_i \mathrm{d}V_g \\
- \;& 8\pi(1+\varepsilon)\left((1+\alpha_{1m})^2+(1+\alpha_{2m})^2\right)\log\frac{\varsigma}{2} \\
\le \;& 4\pi\sum_{i=1}^{2}(1+\alpha_{3-i,m})\log\int_{A_{\varsigma,\delta'}(\beta)} \widetilde{h}_i e^{u_i'}\mathrm{d}V_g + 4\pi(1+\varepsilon)\sum_{i=1}^{2}(1+\alpha_{im})\fint_{B_{\frac{\varsigma}{2}}(\beta)} u_i' \mathrm{d}V_g \\
- \;& 8\pi(1+\varepsilon)\left((1+\alpha_{1m})^2+(1+\alpha_{2m})^2\right)\log\frac{\varsigma}{2} + 4\pi(2+\alpha_{1m}+\alpha_{2m})\log\frac{2}{\delta} \\
\le \;& 4\pi\sum_{i=1}^{2}(1+\alpha_{3-i,m})\log\int_{A_{\varsigma,\delta'}(\beta)} d(\cdot,\beta)^{2\alpha_{im}} e^{u_i'}\mathrm{d}V_g + 4\pi(1+\varepsilon)\sum_{i=1}^{2}(1+\alpha_{im})\fint_{B_{\frac{\varsigma}{2}}(\beta)} u_i' \mathrm{d}V_g \\
- \;& 8\pi(1+\varepsilon)\left((1+\alpha_{1m})^2+(1+\alpha_{2m})^2\right)\log\frac{\varsigma}{2} + C \\
\le \;& \int_{A_{\frac{\varsigma}{2},\delta'}(\beta)} Q(u')\mathrm{d}V_g + \varepsilon\int_{B_{\delta'}(\beta)} Q(u')\mathrm{d}V_g + C \\
\le \;& \int_{A_{\frac{\varsigma}{2},\delta'}(\beta)} Q(u)\mathrm{d}V_g + C\varepsilon\int_{\Sigma} Q(u)\mathrm{d}V_g + C.
\end{aligned}
\tag{3.62}
$$

By summing (3.61) and (3.62) and re-naming properly ε we get $J_{\rho_\varepsilon,\rho_\varepsilon}(u)\ge -L$ for $\rho_\varepsilon := 4\pi(2+\alpha_{1m}+\alpha_{2m})-\varepsilon$, which means, being ε arbitrary, $J_\rho(u)\ge -L$.

Case 1.b : $\displaystyle\int_{A_{4\varsigma,\delta'}(\beta)} f_{i,u}\mathrm{d}V_g \ge \frac{\delta}{4}$ for both $i=1,2$.
The result follows arguing as before, still applying Lemmas 3.31, 3.32, but this time on $B_{2\varsigma}(\beta)$ and $A_{2\varsigma,\delta'}(\beta)$.

Case 1.c :

$$
\int_{B_{\frac{\varsigma}{8}}(\beta)} f_{1,u}\mathrm{d}V_g \ge \frac{\delta}{2}, \qquad \int_{A_{8\varsigma,\delta'}(\beta)} f_{1,u}\mathrm{d}V_g \ge \frac{\delta}{4},
$$
$$
\int_{A_{\frac{\varsigma}{4},\varsigma}(\beta)} f_{2,u}\mathrm{d}V_g \ge \frac{\delta}{2}, \qquad \int_{A_{\varsigma,4\varsigma}(\beta)} f_{2,u}\mathrm{d}V_g \ge \frac{\delta}{4}.
$$

We still apply Lemmas 3.31 and 3.32, respectively on $B_\varsigma(\beta)$ and $A_{\varsigma,\delta'}(\beta)$, but this time we will exploit (3.56) and (3.59): we get

$$
\begin{aligned}
& 4\pi(1+\alpha_{1m})\left(\log\int_\Sigma \widetilde{h}_1 e^{u_1}\mathrm{d}V_g - \fint_{B_\varsigma(\beta)} u_1\mathrm{d}V_g\right) \\
+ \;& 2\pi\min\{1,2+\alpha_{1m}+\alpha_{2m}\}\left(\log\int_\Sigma \widetilde{h}_2 e^{u_2}\mathrm{d}V_g - \fint_{B_\varsigma(\beta)} u_2\mathrm{d}V_g\right) \\
+ \;& 4\pi\left(2(1+\alpha_{1m})^2+\min\{1,2+\alpha_{1m}+\alpha_{2m}\}(1+\alpha_{2m})\right)\log\varsigma + C \\
\le \;& 4\pi(1+\alpha_{1m})\left(\log\int_{B_{\frac{\varsigma}{8}}(\beta)} d(\cdot,\beta)^{2\alpha_{1m}} e^{u_1}\mathrm{d}V_g - \fint_{B_\varsigma(\beta)} u_1\mathrm{d}V_g\right) \\
+ \;& 2\pi\min\{1,2+\alpha_{1m}+\alpha_{2m}\}\left(\log\int_{A_{\frac{\varsigma}{4},\varsigma}(\beta)} d(\cdot,\beta)^{2\alpha_{2m}} e^{u_2}\mathrm{d}V_g - \fint_{B_\varsigma(\beta)} u_2\mathrm{d}V_g\right) \\
+ \;& 4\pi\left(2(1+\alpha_{1m})^2+\min\{1,2+\alpha_{1m}+\alpha_{2m}\}(1+\alpha_{2m})\right)\log\varsigma + C \\
\le \;& (1+\varepsilon)\int_{B_\varsigma(\beta)} Q(u)\mathrm{d}V_g + C,
\end{aligned}
\tag{3.63}
$$

and

$$
\begin{aligned}
& 4\pi(1+\alpha_{2m})\log\int_\Sigma \widetilde{h}_1 e^{u_1}\mathrm{d}V_g + 4\pi(1+\varepsilon)(1+\alpha_{1m})\fint_{B_\varsigma(p)} u_1\mathrm{d}V_g \\
+ \quad & 2\pi\min\{1,2+\alpha_{1m}+\alpha_{2m}\}\left(\log\int_\Sigma \widetilde{h}_2 e^{u_2}\mathrm{d}V_g + (1+\varepsilon)\fint_{B_\varsigma(p)} u_1\mathrm{d}V_g\right) \\
- \quad & 4\pi(1+\varepsilon)\left(2(1+\alpha_{1m})^2+\min\{1,2+\alpha_{1m}+\alpha_{2m}\}(1+\alpha_{2m})\right)\log\varsigma \\
\leq \quad & 4\pi(1+\alpha_{2m})\log\int_{A_{8\varsigma,d}(p)} d(\cdot,p)^{2\alpha_{1m}}e^{u_1'}\mathrm{d}V_g + 4\pi(1+\varepsilon)(1+\alpha_{1m})\fint_{B_\varsigma(p)} u_1'\mathrm{d}V_g \\
+ \quad & 2\pi\min\{1,2+\alpha_{1m}+\alpha_{2m}\}\left(\log\int_{A_{\varsigma,4\varsigma}(p)} d(\cdot,p)^{2\alpha_{2m}}e^{u_2'}\mathrm{d}V_g + (1+\varepsilon)\fint_{B_\varsigma(p)} u_2'\mathrm{d}V_g\right) \\
- \quad & 4\pi(1+\varepsilon)\left(2(1+\alpha_{1m})^2+\min\{1,2+\alpha_{1m}+\alpha_{2m}\}(1+\alpha_{2m})\right)\log\varsigma + C \\
\leq \quad & \int_{A_{\varsigma,\delta'}(\beta)} Q(u')\mathrm{d}V_g + \varepsilon\int_{B_{\delta'}(\beta)} Q(u')\mathrm{d}V_g + C, \\
\leq \quad & \int_{A_{\varsigma,\delta'}(\beta)} Q(u)\mathrm{d}V_g + C\varepsilon\int_\Sigma Q(u)\mathrm{d}V_g + C,
\end{aligned}
\tag{3.64}
$$

As before, $J_\rho(u) \geq -L$ follows from (3.63), (3.64) and a suitable redefinition of ε.

Case $1.d$:

$$
\int_{A_{\frac{\varsigma}{4},\varsigma}(\beta)} f_{1,u}\mathrm{d}V_g \geq \frac{\delta}{2}, \qquad \int_{A_{\varsigma,4\varsigma}(\beta)} f_{1,u}\mathrm{d}V_g \geq \frac{\delta}{4},
$$

$$
\int_{B_{\frac{\varsigma}{8}}(\beta)} f_{2,u}\mathrm{d}V_g \geq \frac{\delta}{2}, \qquad \int_{A_{8\varsigma,\delta'}(\beta)} f_{2,u}\mathrm{d}V_g \geq \frac{\delta}{4}.
$$

Here we argue as in case $1.b$, just exchanging the roles of u_1 and u_2.

Case $1.e$: $\displaystyle\int_{A_{\frac{\varsigma}{8},8\varsigma}(\beta)} f_{i,u}\mathrm{d}V_g \geq \frac{\delta}{4}$ for both $i=1,2$.

We would like to apply (3.57) and (3.60) and argue as in the previous cases. Anyway, we first need to define Ω_ς such that both components have some mass in both sets.

We cover $A_{\frac{\varsigma}{8},8\varsigma}(\beta)$ with balls of radius $\frac{\varsigma}{64}$; by compactness, we have $A_{\frac{\varsigma}{8},8\varsigma}(\beta) = \bigcup_{l=1}^{L} B_{\frac{\varsigma}{64}}(x_l)$, with L not depending on ς, therefore there will be x_{l_1}, x_{l_2} such that $\displaystyle\int_{B_{\frac{\varsigma}{64}}(x_{l_i})} f_{i,u}\mathrm{d}V_g \geq \frac{\delta}{4L}$.

We will proceed differently depending whether x_{l_1} and x_{l_2} are close or not.

Case $1.e'$: $d(x_{l_1},x_{l_2}) \geq \frac{\varsigma}{16}$.

We divide each of the balls $B_{\frac{\varsigma}{64}}(x_{l_1}), B_{\frac{\varsigma}{64}}(x_{l_2})$ with a segment $\{x : (x-x_{l_i})\cdot\theta_i = a_i\}$, with $\theta_i \in \mathbb{S}^1$ and $a_i \in \left(-\frac{\varsigma}{64},\frac{\varsigma}{64}\right)$, in such a way that

$$
\int_{\left\{x\in B_{\frac{\varsigma}{64}}(x_{l_i}),(x-x_{l_i})\cdot\theta_i<a_i\right\}} f_{i,u}\mathrm{d}V_g \geq \frac{\delta}{8L} \quad \int_{\left\{x\in B_{\frac{\varsigma}{64}}(x_{l_i}),(x-x_{l_i})\cdot\theta_i>a_i\right\}} f_{i,u}\mathrm{d}V_g \geq \frac{\delta}{8L}.
$$

We can define Ω_ς as the region of $B_{\delta'}(\beta)$ delimited by the curve defined in the following way:

Since $d\left(B_{\frac{\varsigma}{32}}(x_{l_1}), B_{\frac{\varsigma}{32}}(x_{l_2})\right) \geq \frac{\varsigma}{32}$, we can attach smoothly one endpoint of each segment without intersecting the two balls. We then join the other endpoint of each

segment winding around β.
Since $B_{\frac{\varsigma}{64}}(x_{l_1}) \subset A_{\frac{\varsigma}{16},9\varsigma}(\beta)$, we can build Ω_ς in such a way that $\partial\Omega_\varsigma \subset A_{\frac{\varsigma}{32},10\varsigma(\beta)}$ and $\Omega_\varsigma \in \mathfrak{A}_{\delta_\varsigma}$ (see (1.8) pictures below). Moreover, by construction,

$$\int_{B_{\delta'}(\beta)\setminus\Omega_\varsigma} f_{i,u} \mathrm{d}V_g \geq \frac{\delta}{8L} \qquad \int_{\Omega_\varsigma} f_{i,u} \mathrm{d}V_g \geq \frac{\delta}{8L},$$

hence Lemmas 3.31 and 3.32 still yield the proof.

Case $1.e''$: $d(x_{l_1}, x_{l_2}) \leq \frac{\varsigma}{16}$.

Since $B_{\frac{\varsigma}{64}}(x_{l_1}) \cup B_{\frac{\varsigma}{64}}(x_{l_2}) \subset B_{\frac{5}{64}\varsigma}(x_{l_1})$, we apply Lemma 3.33 to $f_i := \dfrac{\widetilde{h}_i e^{u_i}}{\int_{B_{\frac{5}{64}\varsigma}(x_{l_1})} \widetilde{h}_i e^{u_i} \mathrm{d}V_g}$ to find $\theta \in \mathbb{S}^1, a \in \left(-\frac{5}{64}\varsigma, \frac{5}{64}\varsigma\right)$ such that

$$\int_{\left\{x\in B_{\frac{5}{64}\varsigma}(x_{l_1}),(x-x_{l_1})\cdot\theta<a\right\}} f_{i,u} \mathrm{d}V_g \geq \frac{\delta}{8L} \qquad \int_{\left\{x\in B_{\frac{5}{64}\varsigma}(x_{l_1}),(x-x_{l_1})\cdot\theta>a\right\}} f_{i,u} \mathrm{d}V_g \geq \frac{\delta}{8L}.$$

We now join smoothly (and without intersecting the balls) the endpoints of the segment $\{x : (x - x_{l_2}) \cdot \theta = a\}$ with an arc winding around β. Then, we define Ω_ς as the region of $B_{\delta'}(\beta)$ delimited by the curve made by such an arc and that segment. Since $B_{\frac{5}{64}\varsigma}(x_{l_1}) \subset A_{\frac{3}{64}\varsigma,9\varsigma}(\beta)$, as before we will have $B_{\frac{\varsigma}{32}}(\beta) \subset \Omega_\varsigma \subset B_{10\varsigma}(\beta)$ and $\Omega_\varsigma \in \mathfrak{A}_{\delta_\varsigma}$, and we can argue again as before because clearly

$$\int_{B_{\delta'}(\beta)\setminus\Omega_\varsigma} f_{i,u} \mathrm{d}V_g \geq \frac{\delta}{8L} \qquad \int_{\Omega_\varsigma} f_{i,u} \mathrm{d}V_g \geq \frac{\delta}{8L}.$$

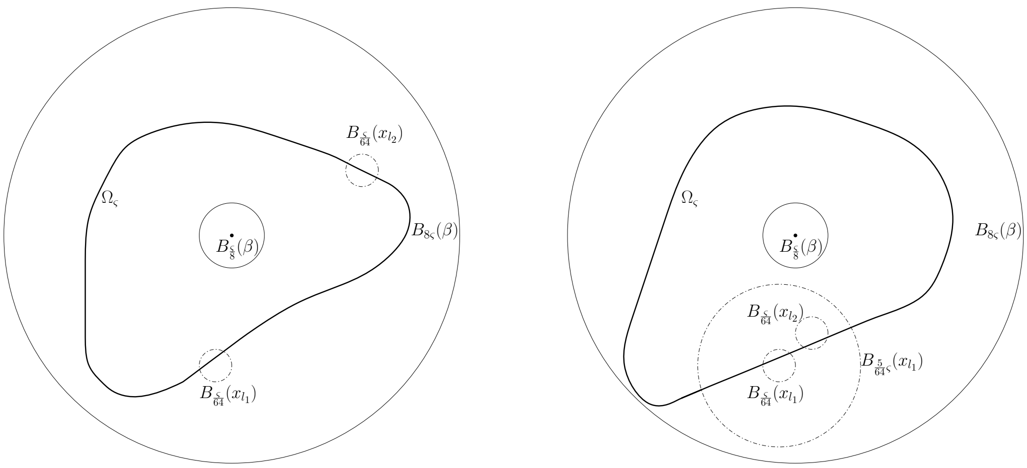

Figure 3.3: The set Ω_ς, respectively in the cases $1.e'$ and $1.e''$.

Case 2 : $\displaystyle\int_{\Sigma\setminus B_{\delta'}(\beta)} f_{i,u} \mathrm{d}V_g \geq \frac{\delta}{2}$ for some i.

It will be not restrictive to assume $i = 1$. If we also have $\displaystyle\int_{\Sigma\setminus B_{\delta''}(\beta)} f_{2,u} \mathrm{d}V_g \geq \frac{\delta}{2}$, with $\delta'' : 2^{-\frac{3}{\varepsilon}}\delta'$, then we get $J_\rho(u) \geq -L$ by applying Lemma 3.23, as in the proof of Corollary 3.29. Therefore will will assume

$$\int_{A_{\varsigma,\delta''}(\beta)} f_{2,u} \mathrm{d}V_g \geq \frac{\delta}{2}.$$

The idea is to combine the previous arguments with a *macroscopic* improved Moser-Trudinger inequality.
As a first thing, define u'' as the solution of

$$\begin{cases} -\Delta u_i'' = 0 & \text{in } A_{2^{n-1}\delta'',2^{n+1}\delta''}(\beta) \\ u_i'' = u_i - \fint_{B_{2^n\delta''}(\beta)} u_i \mathrm{d}V_g & \text{on } \partial B_{2^n\delta''}(\beta) \\ u_i'' = 0 & \text{on } \partial B_{2^{n+1}\delta''}(\beta) \end{cases}$$

with $n \in \left[1, \frac{2}{\varepsilon}\right]$ such that

$$\int_{A_{2^{n-1}\delta'',2^{n+1}\delta'}(\beta)} Q(u'')\mathrm{d}V_g \leq C \int_{A_{2^{n-1}\delta'',2^{n+1}\delta''}(\beta)} Q(u)\mathrm{d}V_g \leq C\varepsilon \int_\Sigma Q(u)\mathrm{d}V_g.$$

Suppose u satisfies the hypotheses of Case $1.a$, that is $\int_{A_{\varsigma,\delta''}(\beta)} f_{i,u}\mathrm{d}V_g \geq \frac{\delta}{2}$ for both $i = 1,2$. Then, clearly (3.61) still holds, whereas (3.62) does not because we cannot estimate the integral of $\int_\Sigma \widetilde{h}_1 e^{u_1}\mathrm{d}V_g$ with the same integral evaluated over $A_{\varsigma,\delta''}$.
Anyway, by Jensen's inequality and Lemma 1.19 we get

$$\begin{aligned}
&\log \int_{A_{\varsigma,\delta''}(\beta)} \widetilde{h}_1 e^{u_1'}\mathrm{d}V_g \\
\geq\ &\log \int_{A_{\frac{\delta''}{2},\delta''}(\beta)} \widetilde{h}_1 e^{u_1}\mathrm{d}V_g - \fint_{B_{2^n\delta''}(\beta)} u_1\mathrm{d}V_g \\
\geq\ &\fint_{A_{\frac{\delta''}{2},\delta''}(\beta)} u_1\mathrm{d}V_g + \log\left|A_{\frac{\delta''}{2},\delta''}(\beta)\right| + \fint_{A_{\frac{\delta''}{2},\delta''}(\beta)} \log\widetilde{h}_1\mathrm{d}V_g - \fint_{B_{2^n\delta''}(\beta)} u_1\mathrm{d}V_g \\
\geq\ &-\varepsilon\int_\Sigma Q(u)\mathrm{d}V_g - C,
\end{aligned}$$

hence we obtain

$$\begin{aligned}
&4\pi(1+\alpha_{2m})\fint_{B_{2^n\delta''}(\beta)} u_1\mathrm{d}V_g + 4\pi(1+\alpha_{1m})\log\int_\Sigma \widetilde{h}_2 e^{u_2}\mathrm{d}V_g \\
+\ &4\pi(1+\varepsilon)\sum_{i=1}^2 (1+\alpha_{im})\fint_{B_{\frac{\varsigma}{2}}(\beta)} u_i\mathrm{d}V_g - 8\pi(1+\varepsilon)\left((1+\alpha_{1m})^2 + (1+\alpha_{2m})^2\right)\log\frac{\varsigma}{2} \\
\leq\ &\int_{A_{\frac{\varsigma}{2},\delta''}(\beta)} Q(u)\mathrm{d}V_g + C\varepsilon\int_\Sigma Q(u)\mathrm{d}V_g + C.
\end{aligned} \tag{3.65}$$

Now, by Jensen's inequality and a variation of the *localized* Moser-Trudinger inequality (3.50),

$$\begin{aligned}
&4\pi\left(1+\min_{m'\neq m}\alpha_{1m'}\right)\left(\log\int_{\Sigma\setminus B_{\delta'}(\beta)} \widetilde{h}_1 e^{u_1}\mathrm{d}V_g - \fint_\Sigma u_1\mathrm{d}V_g\right) \\
\leq\ &4\pi\sum_{i=1}^2\left(1, 1+\min_{m'\neq m}\alpha_{im'}\right)\left(\log\int_{\Sigma\setminus B_{\delta'}(\beta)} \widetilde{h}_i e^{u_i}\mathrm{d}V_g - \fint_\Sigma u_i\mathrm{d}V_g\right) + C \\
\leq\ &(1+\varepsilon)\int_{\Sigma\setminus B_{\frac{\delta'}{2}}(\beta)} Q(u)\mathrm{d}V_g + C.
\end{aligned} \tag{3.66}$$

By summing (3.61), (3.65) and (3.66) we get $J_{\rho_{1\varepsilon},\rho_{2\varepsilon}}(u) \geq -L$, with

$$\rho_{1\varepsilon} := 4\pi\min\left\{2+\alpha_{1m}+\alpha_{2m}, 1+\alpha_{1m}+\min_{m'\neq m}(1+\alpha_{1m'})\right\} - \varepsilon \qquad \rho_{2\varepsilon} := 4\pi(2+\alpha_{1m}+\alpha_{2m}) - \varepsilon,$$

therefore $J_\rho(u) \geq -L$.

We argue similarly if we are under the condition of Cases $1.b$, $1.b$, $1.d$, $1.e$.

The proof is thereby concluded. $\square$

3.6 Proof of Theorems 3.2, 3.3, 3.5, 3.6

We are finally in position to prove all the theorems stated at the beginning of this chapter. All such proofs will follow by showing that low energy sub-levels are dominated (see [40], page 528) by the space $\mathcal{X}$ $\mathcal{X}'$ or $\mathcal{X}''$ which is not contractible.

Lemma 3.34.
Let $\mathcal{X}$ be defined by (3.5).
Then, there exists $L > 0$ and two maps $\Phi : \mathcal{X}' \to J_{A_2,\rho}^{-L}$ and $\Psi : J_{A_2,\rho}^{-L} \to \mathcal{X}'$ such that $\Psi \circ \Phi$ is homotopically equivalent to $\mathrm{Id}_{\mathcal{X}}$.

Lemma 3.35.
Let $\mathcal{X}'$ be defined by (3.10).
Then, there exists $L > 0$ and two maps $\Phi' : \mathcal{X}' \to J_{A_2,\rho}^{-L}$ and $\Psi' : J_{A_2,\rho}^{-L} \to \mathcal{X}'$ such that $\Psi' \circ \Phi'$ is homotopically equivalent to $\mathrm{Id}_{\mathcal{X}'}$.

Lemma 3.36.
Let $\mathcal{X}''$ be defined by (3.31).
Then, there exists $L > 0$ and maps

$$\Phi_{B_2} : \mathcal{X}'' \to J_{B_2,\rho}^{-L}, \qquad \Psi_{B_2} : J_{B_2,\rho}^{-L} \to \mathcal{X}'', \qquad \Phi_{G_2} : \mathcal{X}'' \to J_{G_2,\rho}^{-L}, \qquad \Psi_{G_2} : J_{G_2,\rho}^{-L} \to \mathcal{X}''$$

such that $\Psi_{B_2} \circ \Phi_{B_2}$ and $\Psi_{G_2} \circ \Phi_{G_2}$ are homotopically equivalent to $\mathrm{Id}_{\mathcal{X}''}$.

Proof of Theorems 3.2, 3.5, 3.6 (first part).
Suppose by contradiction that the system (9) has no solutions under the hypotheses of Theorem 3.2.
By Corollary 2.17, J_ρ^{-L} is a deformation retract of $J_{A_2,\rho}^{L}$, which is contractible for large L.
Let $F(\zeta, s) : \mathcal{X} \times [0,1] \to \mathcal{X}$ be the homotopy equivalence defined in Lemma 3.34 and let F' be a homotopy equivalence between a constant map and $\mathrm{Id}_{J_{A_2,\rho}^{-L}}$.
Then $F''(\zeta, s) = \Psi(F'(\Phi(\zeta), s)) : \mathcal{X} \times [0,1] \to \mathcal{X}$ is an equivalence between the maps $\Psi \circ \Phi$ and a constant and $F'' * F$ is an equivalence between $\mathrm{Id}_{\mathcal{X}}$ and a constant map. This means that $\mathcal{X}$ is contractible, in contradiction with Corollary 3.10.
The same argument proves Theorems 3.5 and the part of 3.6 concerning existence of solutions. It suffices to use the homotopy equivalence from Lemmas 3.35, 3.36 and the fact that $\mathcal{X}'$, $\mathcal{X}''$ are not contractible, by Proposition 1.25, Remark 1.29 and Theorem 3.11, respectively. $\square$

Such Lemmas give also easily the proof of multiplicity results.

Proof of Theorems 3.3, 3.6 (second part).
Under the assumptions of theorem 3.3, we can decompose each $(\gamma_i)_{\rho_i,\underline{\alpha}_i'}$ in maximal strata

$$(\gamma_i)_{\rho_i,\underline{\alpha}_i'} = \bigcup_{l_i=1}^{L_i} (\gamma)^{K_{l_i},\mathcal{M}_{l_i}} \cup \bigcup_{l_i'=1}^{L_i'} (\gamma)^{K'_{l_i'},\mathcal{M}'_{l_i'}}.$$

Take the set of initial data such that J_ρ is a Morse function, which by Theorem 1.32 is a dense open set.
By the functorial properties of the homology, then $H_q(\mathcal{X}) \overset{\Phi_{*,q}}{\hookrightarrow} H_q\left(J_\rho^{-L}\right)$. Therefore, applying Lemma 1.31 and Theorem 3.7 we get:

$$\begin{aligned}
&\#\text{Solutions of } (9) \\
\geq\; & \sum_{q=0}^{+\infty} \widetilde{b}_q\left(J_{A_2,\rho}^{-L}\right) \\
\geq\; & \sum_{q=0}^{+\infty} \widetilde{b}_q(\mathcal{X}) \\
\geq\; & \sum_{l_1,l_2} \begin{pmatrix} K_{l_1} + |\mathcal{M}_{l_1}| + \left[\frac{-\chi(\Sigma)}{2}\right] \\ |\mathcal{M}_{l_1}| + \left[\frac{-\chi(\Sigma)}{2}\right] \end{pmatrix} \begin{pmatrix} K_{l_2} + |\mathcal{M}_{l_2}| + \left[\frac{-\chi(\Sigma)}{2}\right] \\ |\mathcal{M}_{l_2}| + \left[\frac{-\chi(\Sigma)}{2}\right] \end{pmatrix},
\end{aligned}$$

that is the thesis of Theorem 3.3.
As for theorem 3.6, we use Proposition 1.25 and Remark 1.29:

$$\#\text{Solutions of } (10) \geq \sum_{q=0}^{+\infty} \widetilde{b}_q\left(J_{B_2,\rho}^{-L}\right) \geq \sum_{q=0}^{+\infty} \widetilde{b}_q(\mathcal{X}'') = \begin{pmatrix} K_1 + \left[\frac{-\chi(\Sigma)}{2}\right] \\ \left[\frac{-\chi(\Sigma)}{2}\right] \end{pmatrix} \begin{pmatrix} K_2 + \left[\frac{-\chi(\Sigma)}{2}\right] \\ \left[\frac{-\chi(\Sigma)}{2}\right] \end{pmatrix},$$

$$\#\text{Solutions of } (11) \geq \sum_{q=0}^{+\infty} \widetilde{b}_q\left(J_{G_2,\rho}^{-L}\right) \geq \sum_{q=0}^{+\infty} \widetilde{b}_q(\mathcal{X}'') = \begin{pmatrix} K_1 + \left[\frac{-\chi(\Sigma)}{2}\right] \\ \left[\frac{-\chi(\Sigma)}{2}\right] \end{pmatrix} \begin{pmatrix} K_2 + \left[\frac{-\chi(\Sigma)}{2}\right] \\ \left[\frac{-\chi(\Sigma)}{2}\right] \end{pmatrix}.$$

□

To prove Lemmas 3.34, 3.35, 3.36 we will need a few technical estimates.
In the case of Lemmas 3.34 and 3.36, we need to consider the distance between $f_{i,u}$ and the space of (weighted) barycenters. The following Lemma gives some information in this direction.

Lemma 3.37.
Let $\sigma_i, \zeta, \Phi^\lambda(\zeta), \beta_{ik}$ be as in Theorem 3.12.
Then, there exists $C > 0$ such that, for any $i = 1, 2$, $\lambda > 2$, $\zeta \in \mathcal{X}$,

$$\frac{1}{C} H(\sigma_1, \lambda(1-t)) \leq d_{\mathrm{Lip}'}\left(f_{1,\Phi^\lambda(\zeta)}, \Sigma_{\rho_1,\underline{\alpha}_1'}\right) \leq C H(\sigma_1, \lambda(1-t)),$$

$$\frac{1}{C} H(\sigma_2, \lambda t) \leq d_{\mathrm{Lip}'}\left(f_{2,\Phi^\lambda(\zeta)}, \Sigma_{\rho_2,\underline{\alpha}_2'}\right) \leq C H(\sigma_2, \lambda t),$$

with

$$H(\sigma_i, \lambda') := \sum_{x_{ik} \in \mathcal{J}_i} \frac{t_{ik}}{\max\{1, \lambda'\}^{\min\left\{2, \frac{1}{1+\beta_{ik}}\right\}}}.$$

Moreover, if $t < 1$ we have

$$f_{1,\Phi^\lambda(\zeta)} \underset{\lambda\to\infty}{\rightharpoonup} \sigma_1' := \sum_{x_{1k} \in \mathcal{J}} t_{1k}' \delta_{x_{1k}},$$

and if $t > 0$ we have

$$f_{2,\Phi^\lambda(\zeta)} \underset{\lambda\to\infty}{\rightharpoonup} \sigma_2 := \sum_{x_{2k} \in \mathcal{J}} t_{2k}' \delta_{x_{2k}},$$

for some t_{ik} verifying $\frac{t_{ik}}{C} \leq t_{ik}' \leq C t_{ik}$.

Proof.
It clearly suffices to give the proof for $i=1$ and for large $\lambda(1-t)$.
For the upper bound, we will show that

$$d_{\mathrm{Lip}'}\left(f_{1,\Phi^\lambda(\zeta)},\sigma_1^\lambda\right) \le C \sum_{x_{1k}\in\mathcal{J}_1} \frac{t_{1k}}{(\lambda(1-t))^{\min\left\{2,\frac{1}{1+\beta_{1k}}\right\}}}$$

with

$$\sigma_1^\lambda := \sum_{x_{1k}\in\mathcal{J}_1} t_{1k}^\lambda \delta_{x_{1k}}, \qquad t_{1k}^\lambda = t_{1k}\frac{\int_\Sigma \frac{\widetilde{h}_1}{\left(1+(\lambda(1-t))^2 d(\cdot,x_{1k})^{2(1+\beta_{1k})}\right)^2} e^{-\frac{\varphi_2}{2}}\mathrm{d}V_g.}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1-\frac{\varphi_2}{2}}\mathrm{d}V_g}.$$

From Lemma 3.15, given any $\phi\in\mathrm{Lip}(\Sigma)$ with $\|\phi\|_{\mathrm{Lip}(\Sigma)}\le 1$ we find

$$\begin{aligned}
&\left|\int_\Sigma \left(f_{1,\Phi^\lambda(\zeta)}-\sigma_1^\lambda\right)\phi\mathrm{d}V_g\right| \\
= \ & \frac{1}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1-\frac{\varphi_2}{2}}\mathrm{d}V_g}\left|\int_\Sigma\left(\widetilde{h}_1 e^{\varphi_1-\frac{\varphi_2}{2}}-\int_\Sigma \widetilde{h}_1 e^{\varphi_1-\frac{\varphi_2}{2}}\mathrm{d}V_g\sigma_1^\lambda\right)\phi\mathrm{d}V_g\right| \\
\le \ & C\frac{(\lambda(1-t))^2}{\max\{1,\lambda t\}^2}\int_\Sigma \widetilde{h}_1 e^{\varphi_1-\frac{\varphi_2}{2}}\left|\phi-\sum_{x_{1k}\in\mathcal{J}_1} t_{1k}^\lambda\phi(x_{1k})\right|\mathrm{d}V_g \\
= \ & \frac{(\lambda(1-t))^2}{\max\{1,\lambda t\}^2}\left|\int_\Sigma\left(\sum_{x_{1k}\in\mathcal{J}_1} t_{1k}\frac{\widetilde{h}_1}{\left(1+(\lambda(1-t))^2 d(\cdot,x_{1k})^{2(1+\beta_{1k'})}\right)^2}e^{-\frac{\varphi_2}{2}}\right)(\phi-\phi(x_{1k}))\mathrm{d}V_g\right| \\
\le \ & C(\lambda(1-t))^2\left|\int_\Sigma\sum_k t_{1k}\frac{\widetilde{h}_1}{\left(1+(\lambda(1-t))^2 d(\cdot,x_{1k})^{2(1+\beta_{1k'})}\right)^2}(\phi-\phi(x_{1k}))\mathrm{d}V_g\right| \\
\le \ & C(\lambda(1-t))^2\sum_k t_{1k}\int_\Sigma\frac{\widetilde{h}_1 d(\cdot,x_{1k})}{\left(1+(\lambda(1-t))^2 d(\cdot,x_{1k})^{2(1+\beta_{1k'})}\right)^2}\mathrm{d}V_g,
\end{aligned}$$

hence the estimate will follow if we show

$$(\lambda(1-t))^2\int_\Sigma\frac{\widetilde{h}_1 d(\cdot,x)}{\left(1+(\lambda(1-t))^2 d(\cdot,x)^{2(1+\beta)}\right)^2}\mathrm{d}V_g \le \frac{C}{(\lambda(1-t))^{\min\left\{2,\frac{1}{1+\beta}\right\}}}$$

for any $x=x_{1k}$, $\beta=\beta_{1k}$.
We easily find

$$(\lambda(1-t))^2\int_{\Sigma\setminus B_\delta(x)}\frac{\widetilde{h}_1 d(\cdot,x)}{\left(1+(\lambda(1-t))^2 d(\cdot,x)^{2(1+\beta)}\right)^2}\mathrm{d}V_g \le \frac{C}{(\lambda(1-t))^2};$$

on the other hand, using normal coordinates and a change of variable we get

$$\begin{aligned}
&(\lambda(1-t))^2\int_{B_\delta(x)}\frac{\widetilde{h}_1 d(\cdot,x)}{\left(1+(\lambda(1-t))^2 d(\cdot,x)^{2(1+\beta)}\right)^2}\mathrm{d}V_g \\
\le \ & \frac{C}{(\lambda(1-t))^{\frac{1}{1+\beta}}}\int_{B_{(\lambda(1-t))^{\frac{1}{1+\beta}}\delta}(0)}\frac{\left|(\lambda(1-t))^{\frac{\beta-\alpha}{(1+\beta)\alpha}}y-(\lambda(1-t))^{\frac{\beta}{(1+\beta)\alpha}}p\right|^{2\alpha}|y|}{\left(1+|y|^{2(1+\beta)}\right)^2}dy,
\end{aligned}$$

where $p=p'_{1m}$, $\alpha=\alpha'_{1m}$ is the closest to x.
The last integral can be verified to be bounded from above by arguing as in the proof of Lemma 3.15. This concludes the proof of the upper bound.

To give a lower bound, it suffices to prove that, however we take $\sigma=\sigma^\lambda$, there exists a $1-Lip$ function ϕ_σ which satisfies

$$\left|\int_\Sigma\left(f_{1,\Phi^\lambda(\zeta)}-\sigma\right)\phi_\sigma\mathrm{d}V_g\right| \ge \frac{1}{C}\sum_{x_{1k}\in\mathcal{J}_1}\frac{t_{1k}}{(\lambda(1-t))^{\min\left\{2,\frac{1}{1+\beta_{1k}}\right\}}}.$$

Precisely, we choose

$$\phi_\sigma = \min_{x_{k'} \in \mathcal{J}'} d(\cdot, x_{k'}) \qquad \text{if } \sigma = \sum_{x_{k'} \in \mathcal{J}'} t_{k'} \delta_{x_{k'}}.$$

It holds

$$\begin{aligned}
&\left| \int_\Sigma \left(f_{1,\Phi^\lambda(\zeta)} - \sigma \right) \phi_\sigma \mathrm{d}V_g \right| \\
= \quad & \frac{1}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g} \left| \int_\Sigma \left(\widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} - \int_\Sigma \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g \sigma \right) \phi_\sigma \mathrm{d}V_g \right| \\
= \quad & \frac{1}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g} \int_\Sigma \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \min_{k'} d(\cdot, x_{k'}) \mathrm{d}V_g \\
\geq \quad & C \frac{(\lambda(1-t))^2}{\max\{1, \lambda t\}^2} \int_\Sigma \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \min_{k'} d(\cdot, x_{k'}) \mathrm{d}V_g \\
\geq \quad & C (\lambda(1-t))^2 \sum_{x_k \in \mathcal{J}_1} t_{1k} \int_\Sigma \frac{\widetilde{h}_1 \min_{k'} d(\cdot, x_{k'})}{\left(1 + d(\cdot, x_{1k})^{2(1+\beta_{1k'})}\right)^2} \mathrm{d}V_g.
\end{aligned}$$

Again, it is easy to see that any single integral outside a ball $B_\delta(x_{1,k})$ is greater or equal to constant times $(\lambda(1-t))^{-2}$.
Therefore, since the number of k' is at most $K = K(\rho_1, \underline{\alpha}'_1)$, we will suffice to show that any integral on the same ball can be estimated from below with constant times $(\lambda(1-t))^{-\frac{1}{1+\beta_{1k}}}$. Arguing as before,

$$\begin{aligned}
&(\lambda(1-t))^2 \int_{B_\delta(x)} \frac{\widetilde{h}_1 \min_{k'} d(\cdot, x_{k'})}{\left(1 + (\lambda(1-t))^2 d(\cdot, x)^{2(1+\beta)}\right)^2} \mathrm{d}V_g \\
\geq \quad & \frac{1}{C(\lambda(1-t))^{\frac{1}{1+\beta}}} \int_{B_{(\lambda(1-t))^{\frac{1}{1+\beta}}\delta}(0)} \frac{\left| (\lambda(1-t))^{\frac{\beta-\alpha}{(1+\beta)\alpha}} y - (\lambda(1-t))^{\frac{\beta}{(1+\beta)\alpha}} p \right|^{2\alpha} \min_{k'} \left| y - (\lambda(1-t))^{\frac{1}{1+\beta}} x_{k'} \right|}{\left(1 + |y|^{2(1+\beta)}\right)^2} dy.
\end{aligned}$$

To see that the last integral is bounded from above, we restrict ourselves to a portion of a ball where the minimum is attained by $x' = x_{k'}$. Since the number of k' is uniformly bounded, for at least one index the portion we are considering measures at least $\frac{1}{K}$ of the measure of the whole ball.
If we take $x' = x'^\lambda$ so that $(\lambda(1-t))^{\frac{1}{1+\beta}} x'^\lambda$ goes to infinity, the integral will tend to $+\infty$ as well.
If instead the last quantity converges, we will get the integral of a function which is uniformly bounded from both above and below, as in the proof of the upper estimates a few lines before.
To get the last claim, just set $t'_{ik} := \lim_{\lambda \to \infty} t^\lambda_{ik}$. We have $t^\lambda_{ik} \sim t_{ik}$ because of the estimates proved in Lemma 3.15. □

With the same argument we can also prove:

Lemma 3.38.
Let $\Phi^\lambda_{B_2}$, $\Phi^\lambda_{G_2}$ be as in Theorem 3.20 and suppose $\rho \in (4K_1\pi, 4(K_1+1)\pi) \times (4K_2\pi, 4(K_2+1)\pi)$. Then, there exists $C > 0$ such that for any $\zeta \in (\gamma_1)_{K_1} \star (\gamma_2)_{K_2}$ one has

$$\frac{1}{C \max\{1, \lambda(1-t)\}} \leq d\left(f_{1,\Phi^\lambda_{B_2}(\zeta)}, (\Sigma)_{K_1}\right) \leq \frac{C}{\max\{1, \lambda(1-t)\}},$$

$$\frac{1}{C \max\{1, \lambda(1-t)\}} \leq d\left(f_{1,\Phi^\lambda_{G_2}(\zeta)}, (\Sigma)_{K_1}\right) \leq \frac{C}{\max\{1, \lambda(1-t)\}},$$

$$\frac{1}{C \max\{1, \lambda t\}} \leq d\left(f_{2,\Phi^\lambda_{B_2}(\zeta)}, (\Sigma)_{K_2}\right) \leq \frac{C}{\max\{1, \lambda t\}},$$

$$\frac{1}{C\max\{1,\lambda t\}} \le d\left(f_{2,\Phi^\lambda_{G_2}(\zeta)},(\Sigma)_{K_2}\right) \le \frac{C}{\max\{1,\lambda t\}}.$$

Moreover, if $t<1$ one has

$$f_{1,\Phi^\lambda_{B_2}(\zeta)} \underset{\lambda\to+\infty}{\rightharpoonup} \sigma_1' := \sum_{k=1}^{K_1} t_{1k}'\delta_{x_{1k}} \qquad f_{1,\Phi^\lambda_{G_2}(\zeta)} \underset{\lambda\to+\infty}{\rightharpoonup} \sigma_1',$$

whereas if $t>0$ one has

$$f_{2,\Phi^\lambda_{B_2}(\zeta)} \underset{\lambda\to+\infty}{\rightharpoonup} \sigma_2' := \sum_{k=1}^{K_2} t_{2k}'\delta_{x_{2k}} \qquad f_{2,\Phi^\lambda_{G_2}(\zeta)} \underset{\lambda\to+\infty}{\rightharpoonup} \sigma_2',$$

for some t_{ik} verifying $\frac{t_{ik}}{C} \le t_{ik}' \le C t_{ik}$.

We are now in a position to prove Lemmas 3.34, 3.36.

Proof of Lemmas 3.34, 3.36.
Take C as in Lemma 3.37, ε_0 and $\psi_i := \psi_{\rho_i,\underline{\alpha}_i'}$ as in Lemma 1.27 and apply Theorem 3.21 with $\varepsilon := \frac{\varepsilon_0}{C^2}$; take $L = L_\varepsilon > 0$ as in Theorem 3.21. Define $\Phi := \Phi^{\lambda_0}$ as in Theorem 3.12, with λ_0 such that $\Phi^\lambda(\mathcal{X}) \subset J_\rho^{-L}$ for any $\lambda \ge \lambda_0$.
As for $\Psi : J_\rho^{-L} \to \mathcal{X}$, consider the push-forward $(\Pi_i)_*$ of the maps $\Pi_i = \Sigma \to \gamma_i$ defined by Lemma 1.22 and

$$t'(d_1,d_2) := \begin{cases} 0 & \text{if } d_2 \ge \varepsilon \\ \dfrac{\varepsilon - d_2}{2\varepsilon - d_1 - d_2} & \text{if } d_1, d_2 < \varepsilon \\ 1 & \text{if } d_1 \ge \varepsilon \end{cases} \qquad \text{where } d_i = d_{\mathrm{Lip}'}\left(f_{i,u},(\Sigma)_{\rho_i,\underline{\alpha}_i'}\right);$$

then, define

$$\Psi(u) := \left((\Pi_1)_*(\psi_1(f_{1,u})),(\Pi_2)_*(\psi_2(f_{2,u}),t'(d_1,d_2)\right).$$

First of all, t' is well-defined and continuous, because on J_ρ^{-L} at least one of d_1 and d_2 is less than ε.
This map is well-defined as well because, from the construction of t', when ψ_1 is not defined one has $d_1 \ge \varepsilon_0 > \varepsilon$, hence $t' = 1$, and similarly $t' = 0$ when ψ_2 is not defined.

Let us now compose the maps Φ and Ψ and see what happens if we let λ tend to $+\infty$.
From the previous corollary, $f_{i,\Phi^\lambda(\zeta)}$ converges weakly to a barycenter σ_i' centered at the same points as σ_i, and the same convergence still holds after applying ψ_i and $(\Pi_i)_*$, since both are retractions. However, the coefficients in σ_i' are different from the ones in σ_i, and moreover the parameter t in the join will be different in general from t'.
Following these considerations, we will construct the homotopy between $\Psi\circ\Phi$ and the identity in three steps: first letting λ to $+\infty$, then rescaling the coefficients in σ_i' and finally rescaling the parameter t in the join.
Writing $\Phi\left(\Psi^\lambda(\zeta)\right) = \left((\Pi_1)_*\left(\psi_1^\lambda(\zeta)\right),(\Pi_1)_*\left(\psi_2^\lambda(\zeta)\right),t'^\lambda(\zeta)\right)$, the homotopy map $H : \mathcal{X}\times[0,1]\to\mathcal{X}$ will be the composition $F := F_3 * (F_2 * F_1)$, where:

$$\begin{aligned} F_1 : (\zeta,s) = ((\sigma_1,\sigma_2,t),s) &\mapsto \left((\Pi_1)_*\left(\psi_1^{\frac{\lambda_0}{1-s}}(\zeta)\right),(\Pi_2)_*\left(\psi_2^{\frac{\lambda_0}{1-s}}(\zeta)\right),t'^{\lambda_0}\right) \\ F_2 : \left(\left(\sigma_1',\sigma_2',t'^{\lambda_0}\right),s\right) &\mapsto \left(((1-s)\sigma_1'+s\sigma_1),((1-s)\sigma_2'+s\sigma_2),t'^{\lambda_0}\right) \\ F_3 : \left(\left(\sigma_1,\sigma_2,t'^{\lambda_0}\right),s\right) &\mapsto \left(\sigma_1,\sigma_2,\left((1-s)t'^{\lambda_0}+st\right)\right). \end{aligned}$$

Let us now verify that the maps are well defined.
In the definition of the map F_1, Lemma 3.37 ensures that the retraction ψ_1 is defined if we have

$H\left(\sigma_1, \dfrac{\lambda_0(1-t)}{1-s}\right) < \dfrac{\varepsilon_0}{C}$. If the latter quantity is greater or equal to $\dfrac{\varepsilon_0}{C}$, then ψ_1 might not be defined, but in this case we have

$$d_1 \geq d_{\mathrm{Lip}'}\left(f_{1,\Phi^{\frac{\lambda_0}{1-s}}(\zeta)}, \Sigma_{\rho_1,\underline{\alpha}'_1}\right) \geq \frac{\varepsilon_0}{C^2} = \varepsilon,$$

hence $t' = 1$ and therefore everything makes sense. For the same reason we can compose ψ_2 and t'. In H_2, the convex combination of σ'_i and σ_i are allowed in $(\gamma_i)_{\rho_i,\underline{\alpha}'_i}$ because the centers of the Dirac masses which define them are the same.
Finally, it is immediate to see that the composition makes sense because the last assertion of Lemma 3.37 yields $F_1(\cdot,1) = F_2(\cdot,0)$; it is immediate to verify and $F_2(\cdot,1) = F_3(\cdot,0)$, that $F(\cdot,0) = \Psi \circ \Phi$ and $F(\cdot,1) = \mathrm{Id}_{\mathcal{X}}$, and that everything is continuous.
A very similar construction proves Lemma 3.36: we consider $\Phi^\lambda_{B_2}$, $\Phi^\lambda_{G_2}$ in place of Φ^λ and we use Lemma 3.38 rather than Lemma 3.37 to verify the well-posedness of the homotopy equivalence. $\square$

To get Lemma 3.35 we will need some estimates on the scales of concentration ς_1, ς_2.
Due to the different definition of the test functions Φ'^λ, we do not have uniform estimates like Lemma 3.37. Anyway, a suitable choice of τ will give a large first scale of Φ'^λ for t close to 1 and a similar result for ς_2.
Notice that no assumption have been made up to now on τ, except for being strictly between $\frac{1}{2}$ and 1 (see the proof of Lemma 3.27). Its value plays a role only in this lemma, hence it will be chosen here to let the following result be true.

Lemma 3.39.
Let δ be as in Lemma 3.27, $\beta_i(u), \varsigma_i(u)$ be as in Theorem 3.28 and Φ'^λ as in Theorem 3.16. Then, for a suitable choice of τ, there exists $C_0 > 0, \delta' \in (0,\delta)$ such that:

- *If either* $t \geq 1 - \dfrac{C_0}{\lambda}$ *or* $\begin{cases} t > \frac{1}{2} \\ x_1 = x_2 =: p_m \\ \rho_1, \rho_2 < 4\pi(2 + \alpha_{1m} + \alpha_{2m}) \end{cases}$, *then* $\varsigma_1\left(\Phi'^\lambda(\zeta)\right) \geq \delta'$;
otherwise, $\varsigma_1\left(\Phi'^\lambda(\zeta)\right) < \delta$ *and* $\beta_1\left(\Phi'^\lambda(\zeta)\right) = x_1$.
- *If either* $t \leq \dfrac{C_0}{\lambda}$ *or* $\begin{cases} t < \frac{1}{2} \\ x_1 = x_2 =: p_m \\ \rho_1, \rho_2 < 4\pi(2 + \alpha_{1m} + \alpha_{2m}) \end{cases}$, *then* $\varsigma_2\left(\Phi'^\lambda(\zeta)\right) \geq \delta'$;
otherwise, $\varsigma_2\left(\Phi'^\lambda(\zeta)\right) < \delta$ *and* $\beta_2\left(\Phi'^\lambda(\zeta)\right) = x_2$.

Proof.
We will only prove the statements involving ς_1 and $f_{1,\Phi^\lambda(\zeta)}$, since the same proof will work for the rest, up to switching indexes $i = 1, 2$.
We will show the proof only in the case $x_2 = p'_m$, $\rho_2 > 4\pi(2 + \alpha_{1m'} + \alpha_{2m'})$, which is somehow trickier because φ_1 does not vanish when $t \geq 1 - \dfrac{1}{\lambda}$.
Let us write

$$\begin{aligned}
\varphi_1 &= \left(\varphi^{\lambda(1-t)}_{1,p_m} + \varphi^{\lambda t}_{1,p_{m'}}\right) = \left(\varphi^{\lambda(1-t)}_{1,p_m} - 2\log\max\left\{1, (\lambda t)^{2(2+\alpha_{1m'}+\alpha_{2m'})} d(\cdot, p_{m'})^{2(1+\alpha_{1m'})}\right\}\right), \\
\varphi_2 &= \left(\varphi^{\lambda(1-t)}_{2,p_m} + \varphi^{\lambda t}_{2,p_{m'}}\right) = \left(\varphi^{\lambda(1-t)}_{2,p_m} - 2\log\max\left\{1, (\lambda t d(\cdot, p_{m'}))^{2(2+\alpha_{1m'}+\alpha_{2m'})}\right\}\right).
\end{aligned}$$

From the definition of ς_1, we have to show that, if $t \geq 1 - \dfrac{C_0}{\lambda}$, then

$$\int_{B_{\delta'}(p_{m''})} f_{1,\Phi'^\lambda(\zeta)} dV_g < \tau \qquad \forall\, m'' = 1, \ldots, M.$$

It is not hard to see that, for any $m'' \neq m, m'$,

$$\int_{B_{\delta'}(p_{m''})} f_{1,\Phi'^\lambda(\zeta)} \mathrm{d}V_g \leq C' \delta'^{2(1+\alpha_{1m''})},$$

which is smaller than any given τ if δ' is taken small enough.
Roughly speaking, $f_{1,\Phi'^\lambda(\zeta)}$ cannot attain mass too near p_m because its scale depends on $\lambda(1-t)$ which is bounded from above. Moreover, $\varphi_{1,p_m}^{\lambda(1-t)}$ is constant in $B_{(\lambda(1-t))^{-\frac{2+\alpha_{1m}+\alpha_{2m}}{1+\alpha_{1m}}}}(p_m)$, hence for large C_0

$$\int_{B_{C_0^{-1-\frac{(2+\alpha_{1m}+\alpha_{2m})}{1+\alpha_{1m}}}}(p_m)} f_{1,\Phi'^\lambda(\zeta)} \mathrm{d}V_g \leq C C_0^{2(1+\alpha_{1m})} \int_{B_{C_0^{-\frac{(2+\alpha_{1m}+\alpha_{2m})}{1+\alpha_{1m}}}}(p_m)} d(\cdot, p_m)^{2\alpha_{1m}} \mathrm{d}V_g \leq \frac{1}{2} < \tau.$$

On the other hand, a part of the mass of $f_{1,\Phi'^\lambda(\zeta)}$ could actually concentrate near p'_m, but not all of it. Here, we will have to take τ properly.
Since

$$\begin{aligned}
& \int_{B_{\frac{\delta}{2}}(p_{m'})} \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g \\
\leq \quad & C e^{\int_\Sigma \left(\varphi_{1,p_m}^{\lambda(1-t)} - \frac{\varphi_{2,p_m}^{\lambda(1-t)}}{2}\right) \mathrm{d}V_g} \left(\int_{B_{(\lambda t)^{-\frac{2+\alpha_{1m'}+\alpha_{2m'}}{1+\alpha_{1m'}}}}(p_{m'})} d(\cdot, p_{m'})^{2\alpha_{1m'}} \mathrm{d}V_g \right. \\
+ \quad & (\lambda t)^{-4(2+\alpha_{1m'}+\alpha_{2m'})} \int_{A_{(\lambda t)^{-\frac{2+\alpha_{1m'}+\alpha_{2m'}}{1+\alpha_{1m'}}}, \frac{1}{\lambda t}}(p_{m'})} d(\cdot, p_m)^{-2(2+\alpha_{1m'})} \mathrm{d}V_g \\
+ \quad & \left. (\lambda t)^{-4(2+\alpha_{1m'}+\alpha_{2m'})} \int_{A_{\frac{1}{\lambda t}, \frac{\delta}{2}}(p_{m'})} d(\cdot, p_{m'})^{2\alpha_{2m'}} \mathrm{d}V_g \right) \\
\leq \quad & C e^{\int_\Sigma \left(\varphi_{1,p_m}^{\lambda(1-t)} - \frac{\varphi_{2,p_m}^{\lambda(1-t)}}{2}\right) \mathrm{d}V_g} (\lambda t)^{-4(2+\alpha_{1m'}+\alpha_{2m'})},
\end{aligned}$$

and

$$\begin{aligned}
& \int_{A_{\frac{\delta}{2},\delta}(p_{m'})} \widetilde{h}_1 e^{\varphi_1 - \frac{\varphi_2}{2}} \mathrm{d}V_g \\
\geq \quad & \frac{1}{C} e^{\int_\Sigma \left(\varphi_{1,p_m}^{\lambda(1-t)} - \frac{\varphi_{2,p_m}^{\lambda(1-t)}}{2}\right) \mathrm{d}V_g} (\lambda t)^{-4(2+\alpha_{1m'}+\alpha_{2m'})} \int_{A_{\frac{\delta}{2},\delta}(p_{m'})} d(\cdot, p_{m'})^{2\alpha_{2m'}} \mathrm{d}V_g \\
\geq \quad & \frac{1}{C} e^{\int_\Sigma \left(\varphi_{1,p_m}^{\lambda(1-t)} - \frac{\varphi_{2,p_m}^{\lambda(1-t)}}{2}\right) \mathrm{d}V_g} (\lambda t)^{-4(2+\alpha_{1m'}+\alpha_{2m'})},
\end{aligned}$$

then

$$\int_{B_{\frac{\delta}{2}}(p_{m'})} f_{1,\Phi'^\lambda(\zeta)} \mathrm{d}V_g < \frac{\int_{B_{\frac{\delta}{2}}(p_{m'})} f_{1,\Phi'^\lambda(\zeta)} \mathrm{d}V_g}{\int_{B_\delta(p_{m'})} f_{1,\Phi'^\lambda(\zeta)} \mathrm{d}V_g} \leq \frac{C^2}{1+C^2}.$$

Therefore, setting $\tau := \dfrac{C^2}{1+C^2}$, we proved the first part of the Lemma.

Let us now assume $t \leq 1 - \dfrac{C_0}{\lambda}$.

From the proof of Lemma 3.19 (and of Lemma 2.23), we deduce that the ratio $\dfrac{\int_{B_\delta(p_m)} f_{1,\Phi'^\lambda(\zeta)} \mathrm{d}V_g}{\int_{B_\delta(p_{m''})} f_{1,\Phi'^\lambda(\zeta)} \mathrm{d}V_g}$ increases arbitrarily as $\lambda(1-t)$ increases. Therefore, for large C_0, most of the mass of $f_{1,\Phi'^\lambda(\zeta)}$ will be around p_m, hence by definition we will have and $\beta_1\left(\Phi'^\lambda(\zeta)\right) = p_m$ and $\varsigma_1\left(\Phi'^\lambda(\zeta)\right) < \delta$. $\square$

Proof of Lemma 3.35.
Take δ as in Lemma 3.27, $\beta_i(u), \varsigma_i(u)$ as in Theorem 3.28, δ' be as in Lemma 3.39 and L so large that Corollary 3.29 and Theorem 3.28 apply.
Define $\Phi' := \Phi'^{\lambda_0}$ as in Theorem 3.16, with λ_0 such that $\Phi'^{\lambda}(\mathcal{X}') \subset J_\rho^{-2L}$ for any $\lambda \geq \lambda_0$.
As for $\Psi' : J_\rho^{-2L} \to \mathcal{X}'$, define

$$t'(\varsigma_1(u), \varsigma_2(u)) := \begin{cases} 0 & \text{if } \varsigma_2(u) \geq \delta' \\ \dfrac{\delta' - \varsigma_2(u)}{2\delta' - \varsigma_1(u) - \varsigma_2(u)} & \text{if } \varsigma_1(u), \varsigma_2(u) \leq \delta' \\ 1 & \text{if } \varsigma_1(u) \geq \delta' \end{cases}$$

and

$$\Psi'(u) := (\beta_1(u), \beta_2(u), t'(\varsigma_1(u), \varsigma_2(u)))\,.$$

Let us verify the well-posedness of Ψ'.
The definition of t' makes sense because, from Corollary 3.29, $J_\rho(u) < -L$ implies $\min\{\varsigma_1(u), \varsigma_2(u)\} \leq \delta'$. Moreover, if $t' > 0$ (respectively, $t' < 1$), then $\varsigma_1 < \delta$ is well-defined (respectively, $\varsigma_2 < \delta$ is well-defined), hence β_1 (respectively, β_2) is also defined.
Finally, Ψ' is mapped on $\mathcal{X}'$ because, from Theorem 3.28, when $J_\rho(u) < -L$ we cannot have $(\beta_1(u), \beta_2(u), t'(\varsigma_1(u), \varsigma_2(u))) = \left(p_m, p_m, \frac{1}{2}\right)$ with $\rho_1, \rho_2 < 4\pi(2 + \alpha_{1m} + \alpha_{2m})$.

To get a homotopy between the two maps, we first let λ tend to $+\infty$, in order to get x_1 and x_2, then we apply a linear interpolation for the parameter t.
Writing $\Psi'\left(\Phi'^{\lambda}(\zeta)\right) = \left(\beta_1^\lambda(\zeta), \beta_2^\lambda(\zeta), t'^{\lambda}(\zeta)\right)$, we have $F = F_2 * F_1$, with

$$\begin{aligned} F_1 : (\zeta, s) = ((x_1, x_2, t), s) &\mapsto \left(\beta_1^{\frac{\lambda_0}{1-s}}(\zeta), \beta_2^{\frac{\lambda_0}{1-s}}(\zeta), t'^{\lambda_0}(\zeta)\right) \\ F_2 : \left(x_1, x_2, t'^{\lambda_0}(\zeta)\right) &\mapsto \left(x_1, x_2, (1-s)t'^{\lambda_0}(\zeta) + st\right). \end{aligned}$$

We have to verify that all is well-defined.
If we cannot define $\beta_1^{\frac{\lambda_0}{1-s}}(\zeta)$, then by Lemma 3.39 we either have $t \geq 1 - \frac{C_0(1-s)}{\lambda_0} \geq 1 - \frac{C_0}{\lambda_0}$ or we are on the first half of the punctured segment. By the same Lemma, we get $\varsigma_1\left(\Phi'^{\lambda_0}(\zeta)\right) \geq \delta'$,that is $t'^{\lambda_0}(\zeta) = 1$. For the same reason, if $\beta_2^{\frac{\lambda_0}{1-s}}(\zeta)$ is not defined, then $t'^{\lambda_0}(\zeta) = 0$, so $F_1 : \mathcal{X}' \times [0,1] \to \Sigma_{\rho_1, \underline{\alpha}_1} \star \Sigma_{\rho_2, \underline{\alpha}_2}$ makes sense.
Its image is actually contained in $\mathcal{X}'$ because, from Lemma 3.39, if $x_1 = x_2$ and $\rho < 4\pi\left(\omega_{\underline{\alpha}_1}(x) + \omega_{\underline{\alpha}_2}(x)\right)$, then either $t'^{\lambda_0}(\zeta) \in \{0, 1\}$, hence in particular it does not equal $\frac{1}{2}$.
Concerning F_2, the previous Lemma implies $\beta_1^{\frac{\lambda_0}{1-s}}(\zeta) = x_1$ if $t \leq 1 - \frac{C_0}{\lambda}(1-s)$, hence in particular passing to the limit as $s \to 1$, if $t < 1$. A similar condition holds for β_2, which gives $F_2(\cdot, 0) = F_1(\cdot, 1)$.
If x_1 is not defined then $t'^{\lambda_0}(\zeta) = 1$, hence $(1-s)t'^{\lambda_0}(\zeta) + st = 1$, and similarly there are no issues when x_2 cannot be defined. Finally, by the argument used before, if $x_1 = x_2 = p_m$ and $\rho_1, \rho_2 < 4\pi(2 + \alpha_{1m} + \alpha_{2m})$, then $(1-s)t'^{\lambda_0}(\zeta) + st \neq \frac{1}{2}$. □

We conclude by giving the proof of Theorem 3.4, which is mostly a variation of Theorem 3.2.

Proof of Theorem 3.4.
Due to the assumption $\rho_2 < 4\pi(1 + \alpha_{2\max})$, we can write

$$(\gamma_2)_{\rho_2, \underline{\alpha}'_2} := \left\{\sum_{m \in \mathcal{M}} t_{2m} \delta_{p'_{2m}} : t_{2m} \geq 0, \sum_{m \in \mathcal{M}} t_{2m} = 1, 4\pi \sum_{m \in \mathcal{M}} (1 + \alpha'_{2m}) < \rho\right\}.$$

If this set is not empty, that is if $\alpha_{2\max} > \widetilde{\alpha}_2$, we can still consider Φ^λ as in Theorem 3.12, since again, by construction, $d(\gamma_1, p'_{2m}) \geq \delta > 0$ for any $l \in \{0, \dots, L_2\}$.
Therefore we have, as in Theorem 3.12, a map $\Phi : \mathcal{X} \to J_\rho^{-L}$ and, as in Lemma 3.34, $\Psi : J_\rho^{-L} \to \mathcal{X}$ such that $\Psi \circ \Phi \simeq \mathrm{Id}_{\mathcal{X}}$. Hence, the sub-levels inherit the homology of the join, so existence and multiplicity of solutions follow by the estimating the Betti numbers as in Theorem 3.7.

On the other hand, if $\alpha_{2\max} = \min_m \alpha_{2m}$, then the set $(\gamma_2)_{\rho_2, \underline{\alpha}'_2}$ is empty. However, Φ^λ can still be defined on $(\gamma_1)_{\rho_1, \underline{\alpha}'_1}$ by restricting the map in Theorem 3.12 to the end $t = 0$ of the join. Since we are just considering a restriction of the map, the estimates of the theorem still hold.
Moreover, being ρ_2 small enough, Lemma 3.25 can only hold for $i = 1$, so in Theorem 3.21 we must have $f_{1,u}$ to be arbitrarily close to $\Sigma_{\rho_1, \underline{\alpha}'_1}$ as J_ρ is lower. Therefore, we can define $\Psi : J_\rho^{-L} \to (\gamma_1)_{\rho_1, \underline{\alpha}'_1}$ by $\Psi(u) = (\Pi_1)_* \psi_1(f_{1,u})$ (with $\psi_1 := \psi_{\rho_1, \underline{\alpha}'_1}$ as in Lemma 1.27).
A homotopy map between $\Psi \circ \Phi$ and $\mathrm{Id}_{(\gamma_1)_{\rho_1, \underline{\alpha}'_1}}$ is given by restricting to $t = 0$ the map F defined in the proof of Lemma 3.34. Therefore, we can again deduce existence and multiplicity of solution by estimating the number of solutions as in the proofs of Theorems 3.2 and 3.3. $\square$

Chapter 4

Non-existence of solutions

The last chapter of this thesis is devoted to proving three different non-existence result for systems (3). All these result are from the paper [13].

We begin by considering a simple situation: the unit disk of $\mathbb{R}^2$ with a singularity at the origin, and solutions satisfying Dirichlet boundary conditions. We find that, to ensure existence of solutions, ρ must satisfy an algebraic condition which involves the same quantities defined in (2.2).

Theorem 4.1.
Let $\left(\mathbb{B}^2, g_0\right)$ be the standard unit disk, suppose $h_i \equiv 1$, $M = 1$ and let $\alpha_1, \ldots, \alpha_N > -1$ be the singular weights of the point $p = 0 \in \mathbb{B}$.
If ρ satisfies

$$\Lambda_{\{1,\ldots,N\},p}(\rho) = 8\pi \sum_{i=1}^{N} (1+\alpha_i)\rho_i - \sum_{i,j=1}^{N} a_{ij}\rho_i\rho_j \leq 0,$$

then there are no solutions for the system

$$\begin{cases} -\Delta u_i = \displaystyle\sum_{j=1}^{N} a_{ij}\rho_j \frac{|\cdot|^{2\alpha_j} e^{u_j}}{\int_{\mathbb{B}} |x|^{2\alpha_j} e^{u_j}\,\mathrm{d}x} & \text{in } \mathbb{B} \\ u_i = 0 & \text{on } \partial\mathbb{B} \end{cases}. \tag{4.1}$$

This result is proved via a Pohožaev identity, and extends a scalar one from [4] (Proposition 5.7).

With a similar argument, one can find non-existence for (3) on the standard sphere with one singular point or two antipodal ones. We get again necessary algebraic conditions of ρ, similar as Theorem 4.1 but weaker.
It is still inspired by [4] (Proposition 5.8).

Theorem 4.2.
Let $(\Sigma, g) = \left(\mathbb{S}^2, g_0\right)$ be the standard sphere, suppose $h_1, h_2 \equiv 1$, $M = 2$, let $(\alpha_{11}, \ldots, \alpha_{N1}) \neq (\alpha_{12}, \ldots, \alpha_{N2})$ be the weights of the antipodal points $p_1, p_2 \in \mathbb{S}^2$, with $\alpha_{im} > -1$, and let $\Lambda_{\mathcal{I},x}$ be defined by (2.2).
If either

$$\Lambda_{\mathcal{I},p_1}(\rho) \geq \Lambda_{\{1,\ldots,N\}\setminus\mathcal{I},p_2}(\rho) \qquad \forall \mathcal{I} \subset \{1,\ldots,N\} \tag{4.2}$$

and at least one inequality is strict, or if all the opposite inequalities hold, then system (3) admits no solutions.

The last result we present makes no assumptions on the topology of Σ but it only works for the $SU(3)$ Toda system (9).

In fact, its proof will use a *localized* blow-up analysis around one singular point, which in turn uses the compactness theorem 2.16. This argument recall [20], Theorem 1.10.

Theorem 4.3.
Let $\Gamma_{\underline{\alpha}_{1\widehat{1}},\underline{\alpha}_{2\widehat{1}}} \subset \mathbb{R}^2_{>0}$ be as in (2.13), with $\underline{\alpha}_{i\widehat{1}} := (\alpha_{i2}, \dots, \alpha_{iM})$ and let $\rho \in \mathbb{R}^2_{>0} \setminus \Gamma_{\underline{\alpha}_{1\widehat{1}},\underline{\alpha}_{2\widehat{1}}}$ and $\alpha_{12}, \dots, \alpha_{1M}, \alpha_{22}, \dots, \alpha_{2M}$ be fixed.
Then, there exists $\alpha_* \in (-1, 0)$ such that the system (9) is not solvable for $\alpha_{11}, \alpha_{12} \leq \alpha_*$. Moreover, α_* can be chosen uniformly for ρ in a given $\mathcal{K} \Subset \mathbb{R}^2_{>0} \setminus \Gamma_{\underline{\alpha}_{1\widehat{1}},\underline{\alpha}_{2\widehat{1}}}$.

This result shows in particular that in Theorem 3.1 the assumption of having *all* the singularities to be non-negative is sharp. In fact, the statement still holds true if we allow all the coefficients $\alpha_{12}, \dots, \alpha_{1M}, \alpha_{22}, \dots, \alpha_{2M}$ to be positive and only $\alpha_{11}, \alpha_{12} < 0$.

This chapter is divided into two sections. The first contains the proof of Theorems 4.1 and 4.2, the second contains the proof of Theorem 4.3.

4.1 Proof of Theorems 4.1, 4.2

Before showing the proof of Theorems 4.1, 4.2, let us compare such results with the existence result proved in Chapters 2, 3.

We start by considering the case of the unit disk $(\mathbb{B}, g_0)$ with one singularity in its center.
Even though it is not a closed surface, most of the variational theory for the Liouville equations and systems can be applied in the very same way to Euclidean domains (or surfaces with boundary) with Dirichlet boundary conditions. This was explicitly pointed out in [4, 9] for the scalar equation, but still holds true for systems, in view of Remark 2.28.
We can extend both Theorem 2.1, to get minimizing solution for Liouville systems on any Euclidean domain, and Theorems 3.1, 3.2, 3.5, 3.6 can be extended, for the case of (9), (10), (11), to get min-max solutions.
Theorems 3.1, 3.2, 3.6, which require the surface Σ to have non-positive Euler characteristic, also give existence of solutions on any non-simply connected open domain of the plane, because such domains can be retracted on a *bouquet* of circles.
It is interesting to notice that multiplicity results cannot hold in the same form because, for instance, a twice-punctured disk retracts on a "figure-eight", but does not on two disjoint ones. Anyway, retracting on a single circle avoids issues for the purpose of existence of solutions.

From Theorem 4.1 we see that, whereas Theorem 2.1 gives existence of solutions for ρ in the bounded region $\{\Lambda > 0\}$ (colored in orange in Figure 4.1), solutions cannot exist outside the bigger bounded region $\left\{\Lambda_{\{1,\dots,N\},p} \geq 0\right\}$ (colored in blue).
For the case of the $SU(3)$ Toda system, Theorem 3.5 gives min-max solutions in the configuration $(M_1, M_2, M_3) = (1, 1, 0)$, namely on the green rectangle $(\rho_1, \rho_2) \in (4\pi(1+\alpha_1), \overline{\rho}_1) \times (4\pi(1+\alpha_2), \overline{\rho}_2)$.
Something similar also holds if $\alpha_1 = \alpha_2 = 0$, that is if we consider the regular Toda system. Here, arguing as in [61], we still have solutions in the second square $(4\pi, 8\pi)^2$, because we get low sub-levels being dominated by a space which is homeomorphic to $\mathbb{R}^6 \setminus \mathbb{R}^3 \simeq \mathbb{S}^2$. This was confirmed in [50], where the degree for the Toda system is computed and in this case it equals -1.
Figure 4.1 shows that there might not be solutions in each of all the other squared which are delimited by integer numbers of 4π. In particular, this shows that the degree is 0 in all these regions.

Proof of Theorem 4.1.
Let $u = (u_1, \dots, u_N)$ be a solution of (4.1). Since both components vanish on the boundary, then for any $x \in \partial\mathbb{B}$ one has $\nabla u_i(x) = (\nabla u_i(x) \cdot \nu(x))\nu(x) =: \partial_\nu u_i(x)\nu(x)$ for all i's.

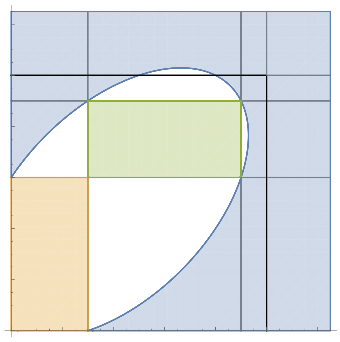

Figure 4.1: Values of ρ which yield existence and non-existence results for $\left(\mathbb{B}^2, g_0\right)$.

Therefore, one can apply a standard Pohožaev identity:

$$\begin{aligned}
&\sum_{i,j=1}^{N} a^{ij} \int_{\partial\mathbb{B}} \partial_\nu u_i \partial_\nu u_j \mathrm{d}\sigma \\
= \;& 2 \sum_{i,j=1}^{N} a^{ij} \int_{\partial\mathbb{B}} \left(\partial_\nu u_i \partial_\nu u_j - \frac{\nabla u_i \cdot \nabla u_j}{2} \right) \mathrm{d}\sigma \\
= \;& 2 \sum_{i,j=1}^{N} a^{ij} \int_{\mathbb{B}} (x \cdot \nabla u_i(x)) \Delta u_j(x) \mathrm{d}x \\
= \;& -2 \sum_{i=1}^{N} \frac{\rho_i}{\int_{\mathbb{B}} |x|^{2\alpha_i} e^{u_i(x)} \mathrm{d}x} \int_{\mathbb{B}} (x \cdot \nabla u_i(x)) |x|^{2\alpha_i} e^{u_i(x)} \mathrm{d}x \\
= \;& 4 \sum_{i=1}^{N} \rho_i \left(\frac{\int_{\partial\mathbb{B}} |\cdot|^{2\alpha_i} e^{u_i} \mathrm{d}\sigma}{\int_{\mathbb{B}} |x|^{2\alpha_i} e^{u_i(x)} \mathrm{d}x} + 1 + \alpha_i \right).
\end{aligned}$$

For the boundary integral, take a orthogonal matrix $M = (m_{ij})_{i,j=1,\dots,N}$ which diagonalizes A^{-1}, namely such that $\sum_{i,j=1}^{N} a^{ij} x_i x_j = \sum_{i=1}^{N} \lambda_i \left(\sum_{j=1}^{N} m_{ij} x_j \right)^2$, for positive $\lambda_1, \dots, \lambda_N$.
By performing an algebraic manipulation, using Hölder's inequality and then integrating by parts we get

$$\begin{aligned}
&\sum_{i,j=1}^{N} a^{ij} \int_{\partial\mathbb{B}} \partial_\nu u_i \partial_\nu u_j \mathrm{d}\sigma \\
= \;& \sum_{i=1}^{N} \lambda_i \int_{\partial\mathbb{B}} \left(\sum_{j=1}^{N} m_{ij} \partial_\nu u_j \right)^2 \mathrm{d}\sigma \\
\geq \;& \frac{1}{2\pi} \sum_{i=1}^{N} \lambda_i \left(\int_{\partial\mathbb{B}} \sum_{j=1}^{N} m_{ij} \partial_\nu u_j \mathrm{d}\sigma \right)^2 \\
= \;& \frac{1}{2\pi} \sum_{i,j=1}^{N} a^{ij} \left(\int_{\partial\mathbb{B}} \partial_\nu u_i \mathrm{d}\sigma \right) \left(\int_{\partial\mathbb{B}} \partial_\nu u_j \mathrm{d}\sigma \right) \\
= \;& \frac{1}{2\pi} \sum_{i,j=1}^{N} a^{ij} \left(\int_{\mathbb{B}} \Delta u_i(x) \mathrm{d}x \right) \left(\int_{\mathbb{B}} \Delta u_j(x) \mathrm{d}x \right) \\
= \;& \frac{1}{2\pi} \sum_{i,j=1}^{N} a_{ij} \rho_i \rho_j.
\end{aligned}$$

Therefore, we get as a necessary condition for existence of solutions:

$$\sum_{i,j=1}^{N} a_{ij}\rho_i\rho_j \geq 8\pi\sum_{i=1}^{N}\rho_i\left(\frac{\int_{\partial\mathbb{B}}|\cdot|^{2\alpha_i}e^{u_i}\mathrm{d}\sigma}{\int_{\mathbb{B}}|x|^{2\alpha_i}e^{u_i(x)}\mathrm{d}x}+1+\alpha_i\right) > 8\pi\sum_{i=1}^{N}(1+\alpha_i)\rho_i.$$

This concludes the proof. □

Let us now consider the standard sphere $(\mathbb{S}^2, g_0)$ with two antipodal singularities.
In Theorem 4.2 we perform a stereographic projection which transforms the solutions of (9) on $\mathbb{S}^2$ on entire solutions on the plane, and then we use a Pohožaev identity for the latter problem (Theorem 1.21), getting necessary algebraic condition for the existence of solutions.

We get non-existence of solutions for the parameter ρ belonging to some regions of the positive orthant.
In particular, if we consider the $SU(3)$ Toda system and compare Theorems 4.2 and 3.1, we see that, to get such a general existence result, we need to assume that $\chi(\Sigma) \leq 0$, not only that $\alpha_{im} \geq 0$ for all i, m.
On the other hand, considering Theorem 3.5, we see that, in most of the regions where we the variational analysis gives no information, system (9) may actually have no solutions.
As before, regions where no solutions exist are blue, regions with minimizing solutions are orange and regions with min-max solutions are green.

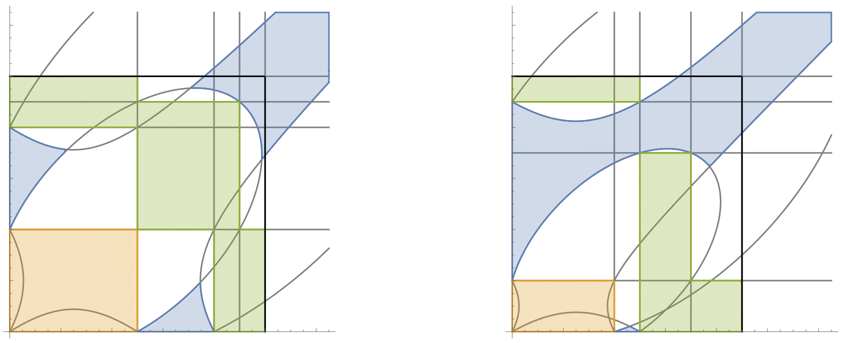

Figure 4.2: Values of ρ which yield existence and non-existence results for $(\mathbb{S}^2, g_0)$, in two different configurations of $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$.

Proof of Theorem 4.2.
Let $u = (u_1, \dots, u_N)$ be a solution of

$$-\Delta_{g_0}u_i = \sum_{j=1}^{N} a_{ij}\rho_j\left(\frac{e^{u_j}}{\int_{\mathbb{S}^2}e^{u_j}\mathrm{d}V_{g_0}}-\frac{1}{4\pi}\right)-4\pi\alpha_{i1}\left(\delta_{p_1}-\frac{1}{4\pi}\right)-4\pi\alpha_{i2}\left(\delta_{p_2}-\frac{1}{4\pi}\right) \qquad i=1,\dots,N,$$

and let $\Pi : \mathbb{S}^2 \setminus \{p_2\} \to \mathbb{R}^2$ the stereographic projection.
Consider now, for $x \in \mathbb{R}^2$,

$$U_i(x) := u_i\left(\Pi^{-1}(x)\right)+\log(4\rho_i)-\log\int_{\mathbb{S}^2}e^{u_i}\mathrm{d}V_{g_0}-2\alpha_{i1}\log|x|+\left(\frac{1}{4\pi}\sum_{j=1}^{N}a_{ij}\rho_j-\alpha_{i1}-\alpha_{i2}\right)\log\left(1+|x|^2\right).$$

$U=(U_1,\dots,U_N)$ solves

$$\begin{cases}-\Delta U_i=\sum_{j=1}^N a_{ij}H_je^{U_j}\\ \int_{\mathbb{R}^2}H_i(x)e^{U_i(x)}\mathrm{d}x=\rho_i\end{cases}\qquad\text{with}\qquad H_i(x):=\frac{|x|^{2\alpha_{i1}}}{(1+|x|^2)^{2+\alpha_{i1}+\alpha_{i2}-\frac{1}{4\pi}\sum_{j=1}^N a_{ij}\rho_j}}.$$

We are in position to apply Theorem 1.21, hence a necessary condition for existence of solutions is (1.10).
By the definition of $H_1,\dots,H_N$, we have

$$x\cdot\nabla H_i(x)=2\alpha_{i1}H_i(x)-2\left(2+\alpha_{i1}+\alpha_{i2}-\frac{1}{4\pi}\sum_{j=1}^N a_{ij}\rho_j\right)\frac{|x|^2}{1+|x|^2}H_i(x)$$

for both i's, hence we get

$$\tau_i=2\alpha_{i1}\rho_i-2\left(2+\alpha_{i1}+\alpha_{i2}-\frac{1}{4\pi}\sum_{j=1}^N a_{ij}\rho_j\right)\tau'_i\qquad\text{with}\qquad\tau'_i:=\int_{\mathbb{R}^2}\frac{|x|^2}{1+|x|^2}H_i(x)\mathrm{d}x.$$

Therefore the necessary condition (1.10) becomes

$$\sum_{i,j=1}^N a_{ij}\rho_i\rho_j+8\pi\sum_{i=1}^N\left(\left(2+\alpha_{i1}+\alpha_{i2}-\frac{1}{4\pi}\sum_{j=1}^N a_{ij}\rho_j\right)\tau'_i-(1+\alpha_{i1})\rho_i\right)=0. \tag{4.3}$$

Since $0<\tau'_i<\rho_i$, one can discuss the cases $2+\alpha_{i1}+\alpha_{i2}\lesseqqgtr\frac{1}{4\pi}\sum_{j=1}^N a_{ij}\rho_j$ and see, by tedious but not difficult algebraic computation, that (4.2) and their opposite inequalities are in contradiction with the aforementioned necessary condition.
Notice that if $2+\alpha_{i1}+\alpha_{i2}=\frac{1}{4\pi}\sum_{i=1}^N a_{ij}\rho_j$ for all i, then (4.3) just becomes $\Lambda_{\{1,\dots,N\},p_1}(\rho)=0$. Anyway, one can easily see that these two conditions are equivalent to $\Lambda_{\mathcal{I},p_1}(\rho)=\Lambda_{\{1,\dots,N\}\setminus\mathcal{I},p_2}(\rho)$ for all $\mathcal{I}$; this is the reason why we need to assume at least one inequality to be strict. $\square$

4.2 Proof of Theorem 4.3

We will prove Theorem 4.3 by arguing by contradiction, following [20] (Theorem 4.1).
Basically, we will assume that a solution exists for some $\alpha_{11}^n,\alpha_{12}^n\underset{n\to+\infty}{\longrightarrow}-1$. We will consider such a sequence of solutions u^n, we will perform a blow-up analysis, following Theorem 2.4 and we will reach a contradiction.

Proof of Theorem 4.3.
Assume the thesis is false. Then, for some given $\underline{\alpha}_{1\widehat{1}},\underline{\alpha}_{2\widehat{1}},\rho\notin\Gamma_{\underline{\alpha}_{1\widehat{1}},\underline{\alpha}_{2\widehat{1}}}$, there exist a sequence $(\alpha_{11}^n,\alpha_{21}^n)\underset{n\to+\infty}{\longrightarrow}(-1,-1)$ and a sequence $u^n=(u_1^n,u_2^n)$ of solutions of

$$\begin{cases}-\Delta u_1^n=2\rho_1\left(\frac{\widetilde{h}_1^ne^{u_1^n}}{\int_\Sigma\widetilde{h}_1^ne^{u_1^n}\mathrm{d}V_g}-1\right)-\rho_2\left(\frac{\widetilde{h}_2^ne^{u_2^n}}{\int_\Sigma\widetilde{h}_2^ne^{u_2^n}\mathrm{d}V_g}-1\right)\\ -\Delta u_2^n=2\rho_2\left(\frac{\widetilde{h}_2^ne^{u_2^n}}{\int_\Sigma\widetilde{h}_2^ne^{u_2^n}\mathrm{d}V_g}-1\right)-\rho_1\left(\frac{\widetilde{h}_1^ne^{u_1^n}}{\int_\Sigma\widetilde{h}_1^ne^{u_1^n}\mathrm{d}V_g}-1\right)\end{cases},$$

with $\widetilde{h}_1^n, \widetilde{h}_2^n$ such that $\widetilde{h}_i^n \sim d(\cdot,p_1)^{2\alpha_{i1}^n}$. It is not restrictive to assume

$$\int_\Sigma \widetilde{h}_1^n e^{u_1^n} \mathrm{d}V_g = \int_\Sigma \widetilde{h}_2^n e^{u_2^n} \mathrm{d}V_g = 1.$$

We would like to apply Theorem 2.4 to the sequence u^n.
Anyway, since the coefficients α_{i1}^n are not bounded away from -1, we cannot use such a Theorem on the whole Σ, but we have to remove a neighborhood of p_1. This can be done with suitable modifications, as pointed out in Remark 2.28. A first piece of information about blow-up is given by the following Lemma, inspired by [20], Lemma 4.3.

□

Lemma 4.4.
Let $\delta > 0$ small be given and u^n be as in the proof of Theorem 4.3.
Then, u_1^n, u_2^n cannot be both uniformly bounded from below on $\partial B_\delta(p_1)$.

Proof.
Assume by contradiction that $\inf_{\partial B_\delta(p_1)} u_i^n \geq -C$ for both i's and define $v^n := \dfrac{2u_1^n + u_2^n}{3}$.
Then

$$\begin{cases} -\Delta v^n = \rho_1 \left(\widetilde{h}_1^n e^{u_1^n} - 1\right) \geq -\rho_1 & \text{in } B_\delta(p_1) \\ v^n \geq -C & \text{on } \partial B_\delta(p_1) \end{cases}.$$

By the maximum principle, $v^n \geq -C$ on $B_\delta(p_1)$, therefore by the convexity of the exponential function we get the following contradiction:

$$\begin{aligned}
+\infty & \\
\underset{n\to+\infty}{\longleftarrow} \quad & \int_{B_\delta(p_1)} d(\cdot,p_1)^{2\max\{\alpha_{11}^n,\alpha_{21}^n\}} \mathrm{d}V_g \\
\leq \quad & C \int_{B_\delta(p_1)} d(\cdot,p_1)^{2\max\{\alpha_{11}^n,\alpha_{21}^n\}} e^{v^n} \mathrm{d}V_g \\
\leq \quad & C \left(\frac{2}{3} \int_{B_\delta(p_1)} d(\cdot,p_1)^{2\max\{\alpha_{11}^n,\alpha_{21}^n\}} e^{u_1^n} \mathrm{d}V_g + \frac{1}{3} \int_{B_\delta(p_1)} d(\cdot,p_1)^{2\max\{\alpha_{11}^n,\alpha_{21}^n\}} e^{u_2^n} \mathrm{d}V_g\right) \\
\leq \quad & C \left(\int_{B_\delta(p_1)} \widetilde{h}_1^n e^{u_1^n} \mathrm{d}V_g + \int_{B_\delta(p_1)} \widetilde{h}_2^n e^{u_2^n} \mathrm{d}V_g\right) \\
\leq \quad & C.
\end{aligned}$$

This concludes the proof. □

Proof of Theorem 4.3, continued.
Let us apply Theorem 2.4 to u^n on $\Omega := \Sigma \setminus B_{\frac{\delta}{2}}(p_1)$ for some given small $\delta > 0$.
By Lemma 4.4, boundedness from below cannot occur for both component, therefore we either have *Concentration* or (up to switching the indexes) $u_1^n \underset{n\to+\infty}{\longrightarrow} -\infty$ uniformly on $\Sigma \setminus B_\delta(p_1)$. In other words,

$$\rho_1 \frac{\widetilde{h}_1^n e^{u_1^n}}{\int_\Sigma \widetilde{h}_1^n e^{u_1^n} \mathrm{d}V_g} \Bigg\lfloor_{\Sigma\setminus B_\delta(p_1)} \underset{n\to+\infty}{\rightharpoonup} \sum_{x\in\mathcal{S}} \sigma_1(x)\delta_x \qquad \rho_2 \frac{\widetilde{h}_2^n e^{u_2^n}}{\int_\Sigma \widetilde{h}_2^n e^{u_2^n} \mathrm{d}V_g} \Bigg\lfloor_{\Sigma\setminus B_\delta(p_1)} \underset{n\to+\infty}{\rightharpoonup} f_2 + \sum_{x\in\mathcal{S}} \sigma_2(x)\delta_x,$$

where we set $\mathcal{S} = \emptyset$ if *Concentration* does not occur. Anyway, being δ arbitrary, a diagonal argument gives

$$\rho_1 \frac{\widetilde{h}_1^n e^{u_1^n}}{\int_\Sigma \widetilde{h}_1^n e^{u_1^n} \mathrm{d}V_g} \underset{n\to+\infty}{\rightharpoonup} \sum_{x\in\mathcal{S}} \sigma_1(x)\delta_x + \sigma_1(p_1)\delta_{p_1} \qquad \rho_2 \frac{\widetilde{h}_2^n e^{u_2^n}}{\int_\Sigma \widetilde{h}_2^n e^{u_2^n} \mathrm{d}V_g} \underset{n\to+\infty}{\rightharpoonup} f_2 + \sum_{x\in\mathcal{S}} \sigma_2(x)\delta_x + \sigma_2(p_1)\delta_{p_1},$$

with $\sigma_1(p_1) = \rho_1 - \sum_{x\in\mathcal{S}} \sigma_1(x)$ and $\sigma_2(p_1) = \rho_2 - \sum_{x\in\mathcal{S}} \sigma_2(x) - \int_\Sigma f_2 \mathrm{d}V_g$.
By arguing as in the proof of Theorem 2.9, we get

$$\sigma_1(p_1)^2 - \sigma_1(p_1)\sigma_2(p_1) + \sigma_2(p_1)^2 = 0,$$

that is $\sigma_1(p) = \sigma_2(p) = 0$.
In particular, we get $\rho_1 = \sum_{x\in\mathcal{S}} \sigma_1(x)$, which means either $\rho_1 = 0$ or $\rho \in \Gamma_{\underline{\alpha}_{1\hat{1}},\underline{\alpha}_{2\hat{1}}}$ This contradicts the assumptions and proves the Theorem. $\square$

Appendix A

Appendix

A.1 Proof of Theorem 1.32

We will prove here the density result stated in Chapter 1. It will be mostly and adaptation from the proof given in [32] for the scalar case.
The proof will consider only the $SU(3)$ Toda system with $\mathcal{M}^2(\Sigma)$, since all the other cases can be treated in the very same way.

Theorem 1.32 will be proved by applying to the suitable objects the following abstract transversality result. The same argument was used, other then in [32], also in [62] for a higher dimensional problem with polynomial nonlinearities.

Theorem A.1. *([66])*
Let X, Y, Z be Banach spaces, $\mathcal{U} \subset X$, $\mathcal{V} \subset Y$ be open subsets, $z_0 \in Z$ and $F : \mathcal{V} \times \mathcal{U} \to Z$ a C^k map, for $k \geq 1$, such that:

- *$\forall\, y \in \mathcal{V}$, $F(y, \cdot) : x \mapsto F(y, x)$ is a Fredholm map of index l, with $l \leq k$;*
- *The set $\{x \in \mathcal{U} : F(y, x) = 0,\, y \in \mathcal{K}\}$ is relatively compact in $\mathcal{U}$ for any $\mathcal{K} \Subset \mathcal{V}$;*
- *z_0 is a regular value of F, namely $F'(y_0, x_0) : Y \times X \to Z$ is onto at any point (y_0, x_0) such that $F(y_0, x_0) = z_0$.*

Then, the set

$$\mathcal{D} := \{y \in \mathcal{V} : z_0 \text{ is a regular value of } F(y, \cdot)\}$$

is a dense open subset of $\mathcal{V}$.

As a first thing, let us introduce the space $\mathcal{S}^2(\Sigma)$ of the C^2 symmetric matrices on Σ.
To define the norm of this space, take an open coordinate neighborhood $\{U_\alpha, \psi_\alpha\}_{\alpha\in A}$ and denote by $\{g_{ij}\}_{i,j=1,2}$ the components of any $g \in \mathcal{S}^2(\Sigma)$ with respect to the coordinates (x_1, x_2) on U_α; then, define

$$\|g\|_{\mathcal{S}^2} := \sum_{\alpha\in A, |\beta|\leq 2, i,j=1,2} \sup_{\psi_\alpha(U_\alpha)} \left| \frac{\partial^2 g_{ij}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \right|.$$

Such a space can be proved to be a Banach space, as well as the space of the symmetric C^k l-covariant tensors on a n-dimensional manifold, which can be defined in the same way.
We then define $\mathcal{M}^2(\Sigma) \subset \mathcal{S}^2(\Sigma)$ as the open subset containing all the positive definite matrices.
We similarly define $\mathcal{S}^2_1(\Sigma) \subset \mathcal{S}^2(\Sigma)$ as the closed affine subspace of the metrics g such that

$\displaystyle\int_\Sigma \mathrm{d}V_g = 1$ and $\mathcal{M}^2_1(\Sigma)$ as its open subset of positive definite matrices.

Before proving Theorem 1.32, we notice that the property of being a dense open set is *local*, namely $\mathcal{D}$ is dense and open if and only if, for any x, $\mathcal{D}\cap B_\delta(x)$ is dense in $B_\delta(x)$ for some δ.
In view of this, we fix $g_0\in\mathcal{M}^2(\Sigma)$, $h_0=(h_{1,0},h_{2,0})\in C^2_{>0}(\Sigma)$ and take δ so small that

$$\mathcal{G}_\delta := \left\{g\in\mathcal{S}^2(\Sigma): \|g-g_0\|_{\mathcal{S}^2(\Sigma)}<\delta\right\}\subset\mathcal{M}^2(\Sigma), \tag{A.1}$$

$$\mathcal{H}_\delta := \left\{h=(h_1,h_2)\in C^2(\Sigma)^2: \|h_1-h_{1,0}\|_{C^2(\Sigma)}+\|h_2-h_{2,0}\|_{C^2(\Sigma)}<\delta\right\}\subset C^2_{>0}(\Sigma). \tag{A.2}$$

Then, we just consider $(g,h)\in\mathcal{G}_\delta\times\mathcal{H}_\delta$.

We will now define the objects for which Theorem A.1 will be applied.
For any $g\in\mathcal{G}_\delta$ we will consider the spaces $H^1_g(\Sigma)$, $L^q_g(\Sigma)$, with the subscript underlining the dependence on the metric. Anyway, by the smallness of δ, they will coincide respectively with $H^1_{g_0}(\Sigma)$, $L^q_{g_0}(\Sigma)$.
Moreover, unlike in all the rest of the paper, such metrics will not give, in general, the surface area of Σ equal to 1; therefore, we will write $\displaystyle\fint_\Sigma f\mathrm{d}V_g$ to indicate the average of a function $f\in L^1_g(\Sigma)$ and we will write $\displaystyle\int_\Sigma \mathrm{d}V_g$ for the area of Σ.
We define, for $g\in\mathcal{G}_\delta$, the operator $\mathrm{A}_g: L^2_{g_0}(\Sigma)\to H^1_{g_0}(\Sigma)$ as the adjoint of the embedding $H^1_g(\Sigma)\hookrightarrow L^2_g(\Sigma)$, namely

$$\langle \mathrm{A}_g u, v\rangle_{H^1_g(\Sigma)} := \int_\Sigma \nabla_g(\mathrm{A}_g u)\cdot\nabla_g v\mathrm{d}V_g = \int_\Sigma uv\mathrm{d}V_g =: \langle u,v\rangle_{L^2_g(\Sigma)}, \tag{A.3}$$

for any $u\in L^2_{g_0}(\Sigma)$, $v\in H^1_{g_0}(\Sigma)$. From Sobolev embeddings, the domain of A_g can be extended to $L^q_{g_0}(\Sigma)$ for any $q>1$.
Such an operator depends regularly on g:

Lemma A.2. *([62], Lemma 2.3)*
Let $q>1$ be given, A_g be defined by (A.3) and $\mathcal{L}\left(L^q_{g_0}(\Sigma),H^1_{g_0}(\Sigma)\right)$ be the space of linear operators between $L^q_{g_0}(\Sigma)$ and $H^1_{g_0}(\Sigma)$.
Then, the map $\mathrm{A}: g\mapsto \mathrm{A}_g$ is of class C^1 from $\mathcal{G}_\delta$ to $\mathcal{L}\left(L^q_{g_0}(\Sigma),H^1_{g_0}(\Sigma)\right)$.

Concerning $\mathcal{H}_\delta$, it is convenient to observe that the presence of the singular points p_m does not really affect this analysis.
In fact, the map $h\mapsto\widetilde{h}$, defined by (4), is linear and continuous from $C^2(\Sigma)$ to $L^q(\Sigma)$, for a suitable $q>1$. Therefore, assuming $h\in\mathcal{H}_\delta$ will imply $\left\|\widetilde{h}_1-\widetilde{h}_{1,0}\right\|_{L^q(\Sigma)}+\left\|\widetilde{h}_2-\widetilde{h}_{2,0}\right\|_{L^q(\Sigma)}<\delta'$.

Take now $R>0$ such that all the solutions of (9) in $\overline{H}^1_{g_0}(\Sigma)^2$ are contained in $\mathcal{B}:=B_R(0)$.
Theorem A.1 will be applied to $X:=Z:=\overline{H}^1_{g_0}(\Sigma)$, $Y=\mathcal{S}^2(\Sigma)\times\mathcal{C}^2(\Sigma)$, $U:=\mathcal{B}$, $V=\mathcal{G}_\delta\times\mathcal{H}_\delta$, $z_0:=0$ and $F:\mathcal{G}_\delta\times\mathcal{H}_\delta\times\mathcal{B}\to\overline{H}^1_{g_0}(\Sigma)^2$ defined in the following way.
Consider the map $\mathrm{S}_g:\overline{H}^1_{g_0}(\Sigma)^2\to\overline{H}^1_g(\Sigma)^2$

$$\mathrm{S}_g(u_1,u_2) := \left(\mathrm{S}'_g(u_1),\mathrm{S}'_g(u_2)\right) = \left(u_1-\fint_\Sigma u_1\mathrm{d}V_g, u_2-\fint_\Sigma u_2\mathrm{d}V_g\right),$$

and his inverse $\mathrm{S}_g^{-1}:\overline{H}^1_g(\Sigma)^2\to\overline{H}^1_{g_0}(\Sigma)^2$:

$$\mathrm{S}_g^{-1}(v_1,v_2) = \left({\mathrm{S}'}_g^{-1}(v_1),{\mathrm{S}'}_g^{-1}(v_2)\right) = \left(v_1-\fint_\Sigma v_1\mathrm{d}V_{g_0}, v_2-\fint_\Sigma v_2\mathrm{d}V_{g_0}\right);$$

define $\widetilde{F}_g : \mathcal{H}_\delta \times \mathcal{B} \to \overline{H}^1_g(\Sigma)^2$ by

$$\widetilde{F}_g(h,v) := \left(\begin{array}{c} v_1 - \mathrm{A}_g\left(2\rho_1\left(\frac{\widetilde{h}_1 e^{v_1}}{\int_\Sigma \widetilde{h}_1 e^{v_1}\mathrm{d}V_g} - \frac{1}{\int_\Sigma \mathrm{d}V_g}\right) - \rho_2\left(\frac{\widetilde{h}_2 e^{v_2}}{\int_\Sigma \widetilde{h}_2 e^{v_2}\mathrm{d}V_g} - \frac{1}{\int_\Sigma \mathrm{d}V_g}\right) + u_1\right) \\ v_2 - \mathrm{A}_g\left(2\rho_2\left(\frac{\widetilde{h}_2 e^{v_2}}{\int_\Sigma \widetilde{h}_2 e^{v_2}\mathrm{d}V_g} - \frac{1}{\int_\Sigma \mathrm{d}V_g}\right) - \rho_1\left(\frac{\widetilde{h}_1 e^{v_1}}{\int_\Sigma \widetilde{h}_1 e^{v_1}\mathrm{d}V_g} - \frac{1}{\int_\Sigma \mathrm{d}V_g}\right) + u_2\right)\end{array}\right);$$

finally, set

$$F(g,h,u) := \mathrm{S}_g^{-1}\left(\widetilde{F}_g(h, \mathrm{S}_g(u))\right). \tag{A.4}$$

Such an application of Theorem A.1 would actually prove Theorem 1.32.
In fact, $\partial_u F(g,h,u) : \overline{H}^1_{g_0} \to \overline{H}^1_{g_0}$ is invertible as long as $F(g,h,u) = 0$ is equivalent to saying that all the solutions of (9) belonging to $\mathcal{B}$ are non-degenerate, and $\mathcal{B}$ is chosen so that it contains all the solutions. Theorem A.1 states that this condition holds on a dense open subset of $\mathcal{G}_\delta \times \mathcal{H}_\delta$. Therefore, we just suffice to show that the three hypotheses of Theorem A.1 are satisfied, that is to prove the following three Lemmas.

Lemma A.3.
Let $\mathcal{G}_\delta$ be as in (A.1), $\mathcal{H}_\delta$ be as in (A.2) and F_g as in (A.4).
Then, for any $(g,h) \in \mathcal{G}_\delta \times \mathcal{H}_\delta$ the map $u \mapsto F(g,h,u)$ is Fredholm of index 0.

Proof.
We will prove that $u \mapsto F(g,h,u)$ is a Fredholm map of index 0. In particular, we will show that $\partial_u F(g,h,u)$ can be written as $\mathrm{Id}_{\overline{H}^1_{g_0}(\Sigma)^2} - \mathrm{K}$ for some compact operator K.
We can write

$$\begin{aligned} & \partial_u F(g,h,u)[w] \\ = \ & \mathrm{S}_g^{-1}\left(\partial_v \widetilde{F}_g(h, \mathrm{S}_g(u))[\mathrm{S}_g(w)]\right) \\ = \ & \left(\begin{array}{c} \mathrm{S}'^{\,-1}_{\mathrm{g}}\left(\mathrm{S}'_g(u_1) - \mathrm{A}_g\left(2\rho_1 \mathrm{K}_{1,u}(w_1) - \rho_2 \mathrm{K}_{2,u}(w_2) + \mathrm{S}'_g(w_1)\right)\right) \\ \mathrm{S}'^{\,-1}_{\mathrm{g}}\left(\mathrm{S}'_g(u_2) - \mathrm{A}_g\left(2\rho_2 \mathrm{K}_{2,u}(w_2) - \rho_1 \mathrm{K}_{1,u}(w_1) + \mathrm{S}'_g(w_1)\right)\right) \end{array}\right), \\ = \ & \left(\begin{array}{c} u_1 - 2\rho_1 \mathrm{S}'^{\,-1}_{\mathrm{g}}(\mathrm{A}_g(\mathrm{K}_{1,u}(w_1))) - \rho_2 \mathrm{S}'^{\,-1}_{\mathrm{g}}(\mathrm{A}_g(\mathrm{K}_{2,u}(w_2))) + \mathrm{S}'^{\,-1}_{\mathrm{g}}(\mathrm{A}_g(\mathrm{S}'_g(w_1))) \\ u_2 - 2\rho_2 \mathrm{S}'^{\,-1}_{\mathrm{g}}(\mathrm{A}_g(\mathrm{K}_{2,u}(w_2))) - \rho_1 \mathrm{S}'^{\,-1}_{\mathrm{g}}(\mathrm{A}_g(\mathrm{K}_{1,u}(w_1))) + \mathrm{S}'^{\,-1}_{\mathrm{g}}(\mathrm{A}_g(\mathrm{S}'_g(w_2))) \end{array}\right), \end{aligned}$$

with $\mathrm{K}_{i,u} : \overline{H}^1_{g_0}(\Sigma) \to L^q_g(\Sigma)$ defined by

$$\mathrm{K}_{i,u}(w_i) := \frac{\widetilde{h}_i e^{u_i} w_i \int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g - \widetilde{h}_i e^{u_i}\int_\Sigma \widetilde{h}_i e^{u_i} w_i \mathrm{d}V_g}{\left(\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g\right)^2}.$$

Take now $\{w^n = (w_1^n, w_2^n)\}_{n\in\mathbb{N}}$ bounded in $\overline{H}^1_{g_0}(\Sigma)^2$ and converging to $w = (w_1, w_2)$ in $L^p_g(\Sigma)^2$ for any $p < +\infty$.
By the continuity of S_g, $\mathrm{S}_g(w)$ is bounded in $\overline{H}^1_{g_0}(\Sigma)^2$ and $\mathrm{S}_g(w^n) \underset{n\to+\infty}{\longrightarrow} \mathrm{S}_g(w)$ in any $L^p_{g_0}(\Sigma)^2$; therefore, $\mathrm{A}_g\left(\mathrm{S}'_g(w_i^n)\right) \underset{n\to+\infty}{\longrightarrow} \mathrm{A}_g\left(\mathrm{S}'_g(w_i)\right)$ in $H^1_{g_0}(\Sigma)^2$ for both i's, hence $\mathrm{S}'^{-1}_g\left(\mathrm{A}_g\left(\mathrm{S}'_g(w_i^n)\right)\right) \underset{n\to+\infty}{\longrightarrow} \mathrm{S}'^{-1}_g\left(\mathrm{A}_g\left(\mathrm{S}'_g(w_i)\right)\right)$ in $H^1_{g_0}(\Sigma)^2$.
Similarly, by the continuity of S_g^{-1} and A_g, we will suffice to show that $\mathrm{K}_{i,u}(w_i^n) \underset{n\to+\infty}{\longrightarrow} \mathrm{K}_{i,u}(w_i)$ in $L^{q'}_g(\Sigma)$ for some $q' > 1$:

$$\|\mathrm{K}_{i,u}(w_i^n - w_i)\|_{L^{q'}_g(\Sigma)}$$

$$\begin{aligned}
&\le \quad \frac{\left\|\widetilde{h}_i e^{u_i}(w_i - w_i^n)\right\|_{L_g^{q'}(\Sigma)}}{\left\|\widetilde{h}_i e^{u_i}\right\|_{L_g^1(\Sigma)}} + \frac{\left\|\widetilde{h}_i e^{u_i}\right\|_{L_g^{q'}(\Sigma)} \left\|\widetilde{h}_i e^{u_i}(w_i^n - w_i)\right\|_{L_g^1(\Sigma)}}{\left\|\widetilde{h}_i e^{u_i}\right\|^2_{L_g^1(\Sigma)}} \\
&\le \quad \frac{\left\|\widetilde{h}_i e^{u_i}\right\|_{L_g^{q}(\Sigma)} \|w_i^n - w_i\|_{L_g^{\frac{qq'}{q-q'}}(\Sigma)}}{\left\|\widetilde{h}_i e^{u_i}\right\|_{L_g^1(\Sigma)}} + \frac{\left\|\widetilde{h}_i e^{u_i}\right\|^2_{L_g^{q'}(\Sigma)} \|w_i^n - w_i\|_{L_g^{\frac{q'}{q'-1}}(\Sigma)}}{\left\|\widetilde{h}_i e^{u_i}\right\|^2_{L_g^1(\Sigma)}} \\
&\underset{n\to+\infty}{\longrightarrow} \quad 0.
\end{aligned}$$

□

Remark A.4.
The very same argument used in the proof of Lemma A.3 also shows that $v \to \partial_v \widetilde{F}_g(h, \mathrm{S}_g(u))[v]$ *is also a Fredholm map of index* 0.

Lemma A.5.
Let $\mathcal{G}_\delta$ *be as in* (A.1), $\mathcal{H}_\delta$ *be as in* (A.2), F_g *as in* (A.4) *and* $\mathcal{B}$ *as before.*
Then, the set

$$\{u \in \mathcal{B} : \ F(g, h, u) = 0, \ (g, h) \in \mathcal{K}\}$$

is relatively compact in $\mathcal{B}$ *for any* $\mathcal{K} \Subset \mathcal{G}_\delta \times \mathcal{H}_\delta$.

Proof.
Take $\{g^n, h^n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ and $u^n \in \mathcal{B}$. Up to subsequences, we may assume $g^n \underset{n\to+\infty}{\longrightarrow} g$ in $\mathcal{S}^2(\Sigma)$ and $h^n \underset{n\to+\infty}{\longrightarrow} h$ in $C^2(\Sigma)^2$ for some $(g, h) \in \mathcal{K}$, and $u^n \underset{n\to+\infty}{\longrightarrow} u$ in $L_g^q(\Sigma)^2$ and $L_{g_0}^q(\Sigma)^2$ for all $q < +\infty$. To prove the Lemma we will suffice to show that, for both $i = 1, 2$ and some $q' > 1$,

$$\widetilde{h}_i^n e^{u_i^n} \underset{n\to+\infty}{\longrightarrow} \widetilde{h}_i e^{u_i} \quad \text{in } L_{g_0}^{q'}(\Sigma), \qquad \int_\Sigma \widetilde{h}_i^n e^{u_i^n} \mathrm{d}V_{g^n} \underset{n\to+\infty}{\longrightarrow} \int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g. \tag{A.5}$$

In fact, from this we would get that

$$f^n := \begin{pmatrix} 2\rho_1 \left(\dfrac{\widetilde{h}_1^n e^{u_1^n}}{\int_\Sigma \widetilde{h}_1^n e^{u_1^n} \mathrm{d}V_{g^n}} - \dfrac{1}{\int_\Sigma \mathrm{d}V_{g^n}} \right) - \rho_2 \left(\dfrac{\widetilde{h}_2^n e^{u_2^n}}{\int_\Sigma \widetilde{h}_2^n e^{u_2^n} \mathrm{d}V_{g^n}} - \dfrac{1}{\int_\Sigma \mathrm{d}V_{g^n}} \right) + u_1^n \\ 2\rho_2 \left(\dfrac{\widetilde{h}_2^n e^{u_2^n}}{\int_\Sigma \widetilde{h}_2^n e^{u_2^n} \mathrm{d}V_{g^n}} - \dfrac{1}{\int_\Sigma \mathrm{d}V_{g^n}} \right) - \rho_1 \left(\dfrac{\widetilde{h}_1^n e^{u_1^n}}{\int_\Sigma \widetilde{h}_1^n e^{u_1^n} \mathrm{d}V_{g^n}} - \dfrac{1}{\int_\Sigma \mathrm{d}V_{g^n}} \right) + u_2^n \end{pmatrix}$$

converges in $L_{g_0}^{q'}(\Sigma)^2$ to

$$f := \begin{pmatrix} 2\rho_1 \left(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} \mathrm{d}V_g} - \dfrac{1}{\int_\Sigma \mathrm{d}V_g} \right) - \rho_2 \left(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} \mathrm{d}V_g} - \dfrac{1}{\int_\Sigma \mathrm{d}V_g} \right) + u_1 \\ 2\rho_2 \left(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} \mathrm{d}V_g} - \dfrac{1}{\int_\Sigma \mathrm{d}V_g} \right) - \rho_1 \left(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} \mathrm{d}V_g} - \dfrac{1}{\int_\Sigma \mathrm{d}V_g} \right) + u_2 \end{pmatrix}.$$

Since, by definition $u^n = \mathrm{A}_{g^n}(f^n)$, setting $u := \mathrm{A}_g(f)$, by the continuity of A_g and Lemma A.2 we get:

$$\begin{aligned}
& \quad \|u^n - u\|_{H_{g_0}^1(\Sigma)^2} \\
&\le \quad \|\mathrm{A}_{g^n}(f^n) - \mathrm{A}_g(f^n)\|_{H_{g_0}^1(\Sigma)^2} + \|\mathrm{A}_g(f^n) - \mathrm{A}_g(f)\|_{H_{g_0}^1(\Sigma)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \quad \|\mathrm{A}_{g^n}-\mathrm{A}_g\|_{\mathcal{L}\left(L^{q'}_{g_0}(\Sigma),H^1_{g_0}(\Sigma)\right)}\|f^n\|_{L^q_{g_0}(\Sigma)^2}+o(1)\\
&\leq \quad \|\mathrm{A}'\|_{\mathcal{L}\left(\mathcal{G}_\delta,\mathcal{L}\left(L^{q'}_{g_0}(\Sigma),H^1_{g_0}(\Sigma)\right)\right)}\|g^n-g\|_{\mathcal{S}^2(\Sigma)}\|f^n\|_{L^q_{g_0}(\Sigma)^2}+o(1)\\
&\underset{n\to+\infty}{\longrightarrow} \quad 0.
\end{aligned}
$$

To prove (A.5), notice that $\widetilde{h}^n \underset{n\to+\infty}{\longrightarrow} \widetilde{h}$ and $L^q_g(\Sigma)$ in $L^q_{g_0}(\Sigma)$.
Moreover, from Lemma 1.13 (see also the proof of Lemma 1.23), any $u\in H^1_g(\Sigma)$ verifies

$$
\int_\Sigma e^{p|u|}\mathrm{d}V_g \leq e^{p \fint_\Sigma |u|\mathrm{d}V_g+\frac{p^2}{16\pi}\int_\Sigma|\nabla_g u|^2\mathrm{d}V_g}<+\infty,
$$

and the same holds if it is replace by a sequence which is bounded in $\overline{H}^1_g(\Sigma)$. Therefore,

$$
\begin{aligned}
&\int_\Sigma\left|e^{u_i^n}-e^{u_i}\right|^p\mathrm{d}V_g\\
= \quad &\int_\Sigma e^{pu_i}\left|e^{u_i^n-u_i}-1\right|^p\mathrm{d}V_g\\
\leq \quad &\int_\Sigma e^{pu_i}\left|u_i^n-u_i\right|^p e^{p|u_i^n-u_i|}\mathrm{d}V_g\\
\leq \quad &\left(\int_\Sigma e^{3pu_i}\mathrm{d}V_g\right)^{\frac13}\left(\int_\Sigma|u_i^n-u_i|^{3p}\mathrm{d}V_g\right)^{\frac13}\left(\int_\Sigma e^{3p|u_i^n-u_i|}\mathrm{d}V_g\right)^{\frac13}\\
\underset{n\to+\infty}{\longrightarrow} \quad &0.
\end{aligned}
$$

From this estimate, we deduce:

$$
\begin{aligned}
&\left|\int_\Sigma \widetilde{h}_i^n e^{u_i^n}\mathrm{d}V_{g^n}-\int_\Sigma\widetilde{h}_i e^{u_i}\mathrm{d}V_g\right|\\
\leq \quad &\left|\int_\Sigma \widetilde{h}_i^n e^{u_i^n}\mathrm{d}V_{g^n}-\int_\Sigma\widetilde{h}_i^n e^{u_i^n}\mathrm{d}V_g\right|+\left|\int_\Sigma \widetilde{h}_i^n e^{u_i^n}\mathrm{d}V_g-\int_\Sigma\widetilde{h}_i e^{u_i^n}\mathrm{d}V_g\right|\\
+ \quad &\left|\int_\Sigma \widetilde{h}_i e^{u_i^n}\mathrm{d}V_g-\int_\Sigma\widetilde{h}_i e^{u_i}\mathrm{d}V_g\right|\\
\leq \quad &o(1)+\left\|\widetilde{h}_i^n-\widetilde{h}_i\right\|_{L^q_g(\Sigma)}\|e^{u_i}\|_{L_g^{\frac{q}{q-1}}(\Sigma)}+\left\|\widetilde{h}_i^n\right\|_{L^q_g(\Sigma)}\left\|e^{u_i}-e^{u_i^n}\right\|_{L_g^{\frac{q}{q-1}}(\Sigma)}\\
\underset{n\to+\infty}{\longrightarrow} \quad &0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\widetilde{h}_i^n e^{u_i^n}-\widetilde{h}_i e^{u_i}\right\|_{L^{q'}_{g_0}(\Sigma)}\\
\leq \quad &\left\|\widetilde{h}_i^n e^{u_i^n}-\widetilde{h}_i^n e^{u_i}\right\|_{L^{q'}_{g_0}(\Sigma)}+\left\|\widetilde{h}_i^n e^{u_i}-\widetilde{h}_i e^{u_i}\right\|_{L^{q'}_{g_0}(\Sigma)}\\
\leq \quad &\left\|\widetilde{h}_i^n-\widetilde{h}_i\right\|_{L^q_{g_0}(\Sigma)}\|e^{u_i}\|_{L_{g_0}^{\frac{qq'}{q-q'}}(\Sigma)}+\left\|\widetilde{h}_i^n\right\|_{L^q_{g_0}(\Sigma)}\left\|e^{u_i}-e^{u_i^n}\right\|_{L_{g_0}^{\frac{qq'}{q-q'}}(\Sigma)}\\
\underset{n\to+\infty}{\longrightarrow} \quad &0,
\end{aligned}
$$

hence (A.5). □

Lemma A.6.
Let $\mathcal{G}_\delta$ be as in (A.1), $\mathcal{H}_\delta$ be as in (A.2), F_g as in (A.4) and $\mathcal{S}^2(\Sigma)$, $\mathcal{B}$ as before.

Then, for any $(g,h,u) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B}$ such that $F(g,h,u)=0$ and for any $w \in \overline{H}^1_{g_0}(\Sigma)^2$ there exists $(g_w,h_w,u_w) \in \mathcal{S}^2(\Sigma) \times C^2(\Sigma)^2 \times \overline{H}^1_g(\Sigma)^2$ such that

$$\partial_{(g,h)}F(g,h,u)[g_w,h_w] + \partial_u F(g,h,u)[u_w] = w.$$

Proof.
As a first thing, we notice that, if $v \in \overline{H}^1_g(\Sigma)^2$ satisfies

$$\left\langle \mathrm{S}_g \partial_{(g,h)} F(g,h,u)[0,h'], v \right\rangle_{H^1_g(\Sigma)^2} = 0 \qquad \forall\, h' \in C^2(\Sigma)^2,$$

then $v \equiv 0$. This follow by writing

$$\begin{aligned} & 0 \\ = \; & \left\langle \mathrm{S}_g \partial_{(g,h)} F(g,h,u)[0,h'], v \right\rangle_{H^1_g(\Sigma)^2} \\ = \; & \left\langle \partial_h \widetilde{F}_g(h, \mathrm{S}_g(u))[h'], v \right\rangle_{H^1_g(\Sigma)^2} \\ = \; & \begin{pmatrix} 2\rho_1 C(h_1,h_1',v_1) - \rho_2 C(h_2,h_2',v_2) \\ 2\rho_2 C(h_2,h_2',v_2) - \rho_1 C(h_1,h_1',v_1) \end{pmatrix}, \end{aligned}$$

namely, for both $i=1,2$,

$$0 = C(h_i,h_i',v_i) := \frac{\int_\Sigma \widetilde{h}_i' e^{u_i} \left(v_i \int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g - \int_\Sigma \widetilde{h}_i e^{u_i} v_i \mathrm{d}V_g \right) \mathrm{d}V_g}{\left(\int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g \right)^2},$$

that is $v_i \equiv \dfrac{\int_\Sigma \widetilde{h}_i e^{u_i} v_i \mathrm{d}V_g}{\int_\Sigma \widetilde{h}_i e^{u_i} \mathrm{d}V_g}$, but since the only constant in $H^1_g(\Sigma)$, the claim follows.
Now, take g,h,u such that $F(g,h,u)=0$. From Remark A.4, $\partial_v \widetilde{F}_g(h,\mathrm{S}_g(u))$ is a Fredholm map of index 0, namely we can write

$$\overline{H}^1_g(\Sigma)^2 = \mathrm{Ker}\left(\partial_v \widetilde{F}_g(h,\mathrm{S}_g(u)) \right) \oplus \mathrm{Im}\left(\partial_v \widetilde{F}_g(h,\mathrm{S}_g(u)) \right),$$

and we indicate as $\mathrm{P}_{\mathrm{Ker}}$, P_{Im} the two orthogonal projections. In this way, any $w \in \overline{H}^1_{g_0}(\Sigma)^2$ can be written uniquely as a $w = \mathrm{S}_g^{-1}(\mathrm{P}_{\mathrm{Ker}}(\mathrm{S}_g(w))) + \mathrm{S}_g^{-1}(\mathrm{P}_{\mathrm{Im}}(\mathrm{S}_g(w)))$.
We claim that there exists $h_w \in C^2(\Sigma)^2$ such that

$$\mathrm{P}_{\mathrm{Ker}}(\mathrm{S}_g(w)) = \mathrm{P}_{\mathrm{Ker}}(\mathrm{S}_g(\partial_{(g,h)}F(g,h,u)[0,h_w])).$$

In fact, taking a orthonormal basis $\{v_1,\dots,v_D\}$ of $\mathrm{Ker}\left(\partial_v \widetilde{F}_g(h,\mathrm{S}_g(u))\right)$, the linear functionals $\mathrm{L}_1,\dots,\mathrm{L}_D$ on $C^2(\Sigma)^2$ defined by

$$\mathrm{L}_i[h'] := \left\langle \mathrm{S}_g \partial_{(g,h)} F(g,h,u)[0,h'], v_i \right\rangle_{H^1_g(\Sigma)^2}$$

are linearly independent, by what was shown at the beginning of this proof. Therefore, taking $h_1',\dots,h_D'$ such that $\mathrm{L}_i[h_j'] = \delta_{ij}$, if $\mathrm{P}_{Ker}(\mathrm{S}_g(w)) = \sum_{i=1}^N c_i v_i$, we will suffice to take $h_w := \sum_{i=1}^N c_i h_i'$.
Now, taking v_w defined by

$$\mathrm{P}_{\mathrm{Im}}\left(\mathrm{S}_g\left(w - \partial_{(g,h)}F(g,h,u)[0,h_w] \right) \right) = \partial_v \widetilde{F}_g(h,\mathrm{S}_g(u))[v_w]$$

one gets

$$w$$

$$
\begin{aligned}
&= \mathrm{S}_g^{-1}(\mathrm{P}_{\mathrm{Ker}}(\mathrm{S}_g(w))) + \mathrm{S}_g^{-1}(\mathrm{P}_{\mathrm{Im}}(\mathrm{S}_g(w))) \\
&= \mathrm{S}_g^{-1}\left(\mathrm{P}_{\mathrm{Ker}}\left(\mathrm{S}_g\left(\partial_{(g,h)}F(g,h,u)[0,h_w]\right)\right)\right) + \mathrm{S}_g^{-1}(\mathrm{P}_{\mathrm{Im}}(\mathrm{S}_g(w))) \\
&= \mathrm{S}_g^{-1}\left(\mathrm{S}_g\left(\partial_{(g,h)}F(g,h,u)\right)[0,h_w] - \mathrm{P}_{\mathrm{Im}}\left(\mathrm{S}_g\left(\partial_{(g,h)}F(g,h,u)\right)[0,h_w]\right)\right) + \mathrm{S}_g^{-1}(\mathrm{P}_{\mathrm{Im}}(\mathrm{S}_g(w))) \\
&= \partial_{(g,h)}F(g,h,u)[0,h_w] + \mathrm{S}_g^{-1}\left(\mathrm{P}_{\mathrm{Im}}\left(\mathrm{S}_g\left(w - \partial_{(g,h)}F(g,h,u)\right)\right)[0,h_w]\right) \\
&= \partial_{(g,h)}F(g,h,u)[0,h_w] + \mathrm{S}_g^{-1}\left(\partial_v \widetilde{F}_g(h,\mathrm{S}_g(u))[v_w]\right) \\
&= \partial_{(g,h)}F(g,h,u)[0,h_w] + \partial_u F(g,h,u)\left[\mathrm{S}_g^{-1}(v_w)\right].
\end{aligned}
$$

Therefore, setting $g_w := 0$, $u_w := \mathrm{S}_g^{-1}(v_w)$, the proof is complete. □

A.2 Proof of Theorem 1.21

Here we will prove the algebraic condition (1.10) which has to be satisfied by the masses of the entire solutions of (1.9).
Notice the similarities between (1.10) and (2.2): in particular, if $H_i(x) = |x|^{2\alpha_i}$, then (1.10) can be read as $\Lambda_{\{1,\dots,N\},0}(\rho) = 0$.
Theorem 1.21 is an extension of the results from [25] (Theorems 1, 2, 3) for the case $N = 1$. The proof we will show follows quite closely the one of such results. A similar result was also given in [54] for some regular Liouville systems.

As a first thing, we show that solutions of (1.9) are bounded from above. The following Lemma is inspired by [16], Theorem 2.

Lemma A.7.
Let $U = (U_1, \dots, U_N)$ be a solution of (1.9) and c be as in Theorem 1.21.
Then, $|\cdot|^c e^{U_i} \in L^\infty\left(\mathbb{R}^2\right)$ for all i's.

Proof.
We will suppose, at first, $c = 0$.
We fix $x_0 \in \mathbb{R}^2$ and we show $\sup\limits_{B_{\frac12}(x_0)} U_i \le C$ for all i's, with C not depending on x_0.

Write $H_i e^{U_i} = F_i + G_i$ with $F_i \in L^\infty\left(\mathbb{R}^2\right)$ and $\displaystyle\int_{\mathbb{R}^2} G_i(y)\mathrm{d}y \le \varepsilon$, with ε to be chosen later.
Consider now V_i, W_i defined by

$$
\left\{\begin{array}{ll} -\Delta V_i = F_i & \text{in } B_1(x_0) \\ V_i = 0 & \text{on } \partial B_1(x_0) \end{array}\right. \qquad \left\{\begin{array}{ll} -\Delta W_i = G_i & \text{in } B_1(x_0) \\ W_i = 0 & \text{on } \partial B_1(x_0) \end{array}\right.
$$

By Lemma 1.1, $\displaystyle\int_{B_1(x_0)} e^{\frac{|V_i(x)|}{\varepsilon}}\mathrm{d}x \le C$, and moreover $\|W_i\|_{L^\infty(B_1(x_0))} \le C$.

Consider now $Z_i := U_i - \sum\limits_{j=1}^N a_{ij}^+(V_j + W_j)$. Since $-\Delta Z_i = \sum\limits_{j=1}^N a_{ij}^- H_j e^{U_j} \ge 0$, the mean value theorem for subharmonic functions gives, for $x \in B_{\frac12}(x_0)$,

$$
\begin{aligned}
& Z_i(x) \\
\le\; & C \int_{B_{\frac12}(x)} Z_i(y)\mathrm{d}y \\
\le\; & C \int_{B_{\frac12}(x)} U_i^+(y)\mathrm{d}y + \sum_{j=1}^N a_{ij} \int_{B_{\frac12}(x)} |V_j(y)|\mathrm{d}y + \sum_{j=1}^N a_{ij} \int_{B_{\frac12}(x)} |W_j(y)|\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{C}{\theta}\int_{B_{\frac12}(x)} e^{\theta U_i(y)}\mathrm{d}y + C\sum_{j=1}^N\left(\int_{B_{\frac12}(x)} e^{|V_j(y)|}\mathrm{d}y + \|W_j\|_{L^\infty(B_1(x_0))}\right)\\
&\le C\left\|H_i^{-\theta}\right\|_{L^{\frac{1}{1-\theta}}(B_1(x_0))}\left(\int_{B_1(x_0)} H_i e^{U_i}\mathrm{d}V_g\right)^\theta + C\\
&\le C,
\end{aligned}
$$

where $\theta = \begin{cases} 1 & \text{if } a = 0 \\ \in \left(0, \dfrac{2}{2+a}\right) & \text{if } a > 0 \end{cases}$.

Take now $q \in \left(1, \dfrac{2}{b^-}\right)$, so that $\|H_i\|_{L^q(B_1(x_0))} \le C$, $q' \in (1,q)$ and $\varepsilon \le \dfrac{Nqq'}{q-q'}\max_{i,j} a_{ij}^+$:

$$
\begin{aligned}
&\left\|H_i e^{U_i}\right\|_{L^{q'}\left(B_{\frac12}(x_0)\right)}\\
=\ &\left\|H_i e^{Z_i + \sum_{j=1}^N a_{ij}^+(V_j+W_j)}\right\|_{L^{q'}\left(B_{\frac12}(x_0)\right)}\\
\le\ & e^{\|Z_i^+\|_{L^\infty\left(B_{\frac12}(x_0)\right)} + \sum_{j=1}^N a_{ij}^+\|W_j\|_{L^\infty\left(B_{\frac12}(x_0)\right)}}\left\|H_i e^{\sum_{j=1}^N a_{ij}^+ V_j}\right\|_{L^{q'}\left(B_{\frac12}(x_0)\right)}\\
\le\ & C\|H_i\|_{L^q\left(B_{\frac12}(x_0)\right)}\prod_{i=1}^N\left\|e^{a_{ij}^+|V_j|}\right\|_{L^{\frac{Nqq'}{q-q'}}\left(B_{\frac12}(x_0)\right)}\\
\le\ & C.
\end{aligned}
$$

Therefore, by elliptic regularity, V_j and W_j are uniformly bounded in $L^\infty\left(B_{\frac12}(x_0)\right)$, hence

$$
\sup_{B_{\frac12}(x_0)} U_i \le \sup_{B_{\frac12}(x_0)} Z_i^+ + \sum_{j=1}^N a_{ij}^+\left(\|V_j\|_{L^\infty\left(B_{\frac12}(x_0)\right)} + \|W_j\|_{L^\infty\left(B_{\frac12}(x_0)\right)}\right) \le C.
$$

For $c > 0$, we modify the argument as in [25], Lemma 1.2.
If $|x_0| \le 2$, then $|x|^c \le 3^c$ on $B_1(x_0)$, hence we can argue as before.
If $|x_0| \ge 2$, we consider $U_i'(x) = U_i(x) + a\log|x_0|$, which solves

$$
\begin{cases} -\Delta U_1' = \displaystyle\sum_{j=1}^N a_{ij} H_j' e^{U_j'} & \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} H_i'(x) e^{U_i'(x)}\mathrm{d}x < +\infty \end{cases}, \qquad \text{with } H_i' = \frac{H_i}{|x_0|^a}.
$$

Since, for $x \in B_1(x_0)$, one has

$$
0 < H'(x) \le C\left(\frac{|x|}{|x_0|}\right)^c \le C\left(1 + \frac{|x-x_0|}{|x_0|}\right)^c \le 2^c C,
$$

then by the previous argument we get $\sup_{B_{\frac12}(x_0)} U' \le C$ with C not depending on x_0., therefore for $x \in B_{\frac12}(x_0)$

$$
U_i(x) + c\log|x| \le U_i(x) + c\log|x_0| - c\log 2 \le C - c\log 2,
$$

which concludes the proof. □

Let us now define

$$
U_i^0(x) := \frac{1}{2\pi}\sum_{j=1}^N a_{ij}\int_{\mathbb{R}^2}\log\frac{|y|+1}{|x-y|} H_j(y) e^{U_j(y)} dy. \tag{A.6}
$$

Thanks to Lemma A.7, U_i^0 is well-defined in $\mathbb{R}^2$ and it verifies $-\Delta\left(U_i - U_i^0\right) = 0$ on $\mathbb{R}^2$. Since U_i is bounded from above and U_i^0 it has a sub-logarithmic growth, then by Liouville's Theorem their difference must be constant.
Therefore, the estimates of the derivatives on U_i can be done, equivalently, on U_i^0, which is easier because of its explicit expression.

All such considerations are summarized by the following lemma, which we do not prove explicitly because its can be done in the very same way as Lemmas 1.1, 1.2, 1.3 in [25].

Lemma A.8.
Let U_i^0 be defined by (A.6).
Then, the following estimates hold true:

$$\sup_{x\in\mathbb{R}^2\setminus B_r(0)}\left|\frac{U_i(x)}{\log|x|}+\frac{1}{2\pi}\sum_{j=1}^N a_{ij}\rho_j\right|\underset{r\to+\infty}{\longrightarrow}0.\qquad \sup_{x\in\mathbb{R}^2\setminus B_r(0)}\left||x|\nabla U_i(x)-\frac{1}{2\pi}\sum_{j=1}^N a_{ij}\rho_j\frac{x}{|x|}\right|\underset{r\to+\infty}{\longrightarrow}0.$$

Such estimates allow to argue similarly as Theorem 2.9, though integrating by parts on $B_r(0)$ and then letting r go to $+\infty$.
In fact, by the integrability condition in (1.9) implies that $\sum_{j=1}^N a_{ij}\rho_j > 2\pi(2+c)$, therefore

$$r\int_{\partial B_r(0)} H_i(x)e^{U_i(x)}\mathrm{d}\sigma(x)\underset{r\to+\infty}{\longrightarrow}0.$$

This and Lemma A.8 allow to perform the following calculations, which conclude the proof:

$$\begin{aligned}
& \frac{1}{4\pi}\sum_{i,j=1}^N a_{ij}\rho_i\rho_j \\
= & \lim_{r\to+\infty}\sum_{i,j=1}^N a^{ij}\int_{B_r(0)}(x\cdot\nabla U_i(x))\Delta U_j(x)\mathrm{d}x \\
= & \lim_{r\to+\infty}\sum_{i=1}^N\left(-\int_{B_r(0)}(x\cdot\nabla U_i(x))H_i(x)e^{U_i(x)}\mathrm{d}x\right) \\
= & \lim_{r\to+\infty}\sum_{i=1}^N\left(2\int_{B_r(0)}H_i(x)e^{U_i(x)}\mathrm{d}x+\int_{B_r(0)}(x\cdot\nabla H_i(x))e^{U_i(x)}\mathrm{d}x\right. \\
+ & \left. r\int_{\partial B_r(0)}H_i(x)e^{U_i(x)}\mathrm{d}\sigma(x)\right) \\
= & 2\rho_i+\tau_i.
\end{aligned}$$

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