Esercizio 1.
Calcolare la primitiva

$$\int \frac{1}{3x^3 - x^2} \mathrm{d}x.$$

$$\frac{1}{x^2(3x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3x-1}$$

(oppure $\frac{a}{x} + \frac{b}{3x-1} + \frac{d}{dx} \frac{c}{x}$ è uguale!)

$$\frac{1}{x^2(3x-1)} = \frac{(3x-1)(Ax+B) + Cx^2}{x^2(3x-1)} =$$

$$= \frac{3Ax^2 + (3B-A)x + Cx^2 - B}{x^2(3x-1)}$$

$$\begin{cases} 3A + C = 0 \\ 3B - A = 0 \\ -B = 1 \end{cases} \qquad \begin{cases} B = -1 \\ A = -3 \\ C = 9 \end{cases}$$

$$\int \frac{dx}{3x^3 - x^2} = -3\int \frac{dx}{x} - \int \frac{dx}{x^2} + 9\int \frac{dx}{3x-1}$$

$$= -3\ln(|x|) + \frac{1}{x} + 3\int \frac{3\,dx}{3x-1} \underset{y=3x-1}{=} 3\int \frac{dy}{y}\Big|_{3x-1}$$

$$= -3\ln(|x|) + \frac{1}{x} + 3\ln|3x-1| \quad (+\text{ cost})$$

$$\int \frac{x^2+2x}{x^3+1}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3+1}\,dx + 2\int \frac{x\,dx}{x^3+1}$$

nel primo termine il num. è la derivata del denom.

$$\frac{1}{3}\int \frac{3x^2}{x^3+1} + 2\int \frac{x\,dx}{x^3+1} = \frac{1}{3}\int \frac{dy}{y}\Big|_{y=x^3+1} + 2\int \frac{x}{x^3+1}\,dx$$

$$= \frac{1}{3}\ln(|x^3+1|) + 2\int \frac{x}{x^3+1}\,dx$$

$$\frac{x}{x^3+1} = \frac{x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

(Nota BENE x^2-x+1 ha il $\Delta < 0$!)

$$\frac{Ax^2 - Ax + A + Bx^2 + Bx + Cx + C}{(x+1)(x^2-x+1)} = \frac{x}{(x+1)(x^2-x+1)}$$

$$\begin{cases} A+B = 0 \\ B-A+C = 1 \\ A+C = 0 \end{cases} \Rightarrow \begin{cases} B = C = -A \\ 3B = 1 \end{cases}$$

$$B = \frac{1}{3} = C \quad ; \quad A = -\frac{1}{3}$$

$$\int \frac{x^2+2x}{x^3+1}\,dx = \frac{1}{3}\ln(|x^3+1|) + \frac{2}{3}\left(-\int\frac{dx}{x+1} + \int\frac{x+1}{x^2-x+1}dx\right)$$

$$= \frac{1}{3}\ln(|x^3+1|) - \frac{2}{3}\ln(|x+1|) + \frac{1}{3}\int\frac{2x-1}{x^2-x+1} + \int\frac{dx}{x^2-x+1}$$

$$= \frac{1}{3}\ln(|x^3+1|) - \frac{2}{3}\ln(|x+1|) + \frac{1}{3}\ln(|x^2-x+1|) + \int\frac{dx}{x^2-x+1}$$

guardiamo solo l'ultimo termine

$$x^2 - x + \frac{1}{4} - \frac{1}{4} + 1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)^2 + 1\right]$$

$$\int \frac{dx}{x^2-x+1} = \frac{4}{3} \int \frac{dx}{\Big(\underbrace{\tfrac{2}{\sqrt{3}}\left(x-\tfrac{1}{2}\right)}_{=\,y}\Big)^2+1}$$

$$= \frac{2}{\sqrt{3}} \int \frac{dy}{y^2+1} \Bigg|_{y=\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)}$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)$$

$$\int \frac{1}{x^4+1}$$

$$x^4+1 = (x^2+1)^2 - 2x^2 = \left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)$$

entrambe i fattori quadratici hanno $\Delta<0$

$$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

$$\text{Num} = Ax^3 - \sqrt{2}Ax^2 + Ax + Bx^2 - \sqrt{2}Bx + B + Cx^3 + \sqrt{2}Cx^2 + Cx$$
$$+ Dx^2 + \sqrt{2}Dx + D$$

$$= (A+C)x^3 + \left(B+D+\sqrt{2}(C-A)\right)x^2$$
$$+ \left(A+C-\sqrt{2}(B-D)\right)x + B+D \equiv 1$$

$$\begin{cases} A+C = 0 \\ B+D+\sqrt{2}(C-A) = 0 \\ A+C-\sqrt{2}(B-D) = 0 \end{cases} \Rightarrow \begin{cases} \sqrt{2}(C-A)+1 = 0 \\ \sqrt{2}(B-D) = 0 \\ B+D = 1 \end{cases}$$
$$B+D \equiv 1 \qquad A+C = 0$$

$$\begin{cases} -2\sqrt{2}A + 1 = 0 \\ B = D = \frac{1}{2} \\ C = -A \end{cases} \qquad A = \frac{1}{2\sqrt{2}} = -C$$
$$B = D = \frac{1}{2}$$

$$\int \frac{dx}{x^4+1} = \frac{1}{2}\left[\int \frac{\frac{\sqrt{2}}{2}x+1}{x^2+\sqrt{2}x+1} - \frac{\frac{\sqrt{2}}{2}x-1}{x^2+\sqrt{2}x+1}\right]$$

etc...