BANACH-STEINHAUS THEOREM

LET $\{A_\alpha\}$ BE LINEAR CONTINUOUS OPERATORS $A_\alpha \in \mathcal{L}(X,Y)$, WITH X BANACH SPACE
Y NORMED SPACE

IF $\sup_\alpha \|A_\alpha x\| < +\infty \quad \forall x \in X$, THEN $\sup_\alpha \|A_\alpha\|_{\mathcal{L}(X,Y)} < +\infty$

(*) THE SAME IS TRUE IF $\sup_\alpha \|A_\alpha x\| < +\infty \quad \forall x \in B$ WITH $B \subset X$ OF II CATEGORY

$\boxed{\text{REMARK}}$ BANACH-STEINHAUS THEOREM HELPS IN UNDERSTANDING IF A SUBSET IS I OR II CATEGORY. FOR EXAMPLE, $Y := \bigcup_{p>1} L^p([0,1])$

$X := L^1([0,1])$, $Y \subset X$ IS OF I CATEGORY: $L_n : f \to (\log n)\int_0^{1/n} f$

$L_n \in X^*$, IF $f \in Y$ THEN $|L_n f| \leq \log n \, \|f\|_p \, \frac{1}{n^{1-\frac{1}{p}}} \xrightarrow[n\to\infty]{} 0$

IN PARTICULAR $\sup_n |L_n f| < +\infty \quad \forall f \in Y$. BUT $\|L_n\|_{X^*} = \log n \xrightarrow[n\to+\infty]{} +\infty$

THEREFORE, Y CANNOT BE II CATEGORY, BECAUSE THIS WOULD CONTRADICT BANACH-STEINHAUS THEOREM.

$\boxed{\text{PROP}}$ "WHO IS THE POINTWISE LIMIT OF AN OPERATOR?"

LET $\{A_n\} \subset \mathcal{L}(X,Y)$ BE A SEQUENCE OF LINEAR CONTINUOUS OPERATORS BETWEEN ~~BANACH SPACES X, Y~~ SUCH THAT
X BANACH SPACE
Y NORMED SPACE

$A_n x \xrightarrow[n\to\infty]{} A(x) \quad \forall x \in X$.

THEN, $\sup_n \|A_n x\| < +\infty$ AND $A \in \mathcal{L}(X,Y)$ WITH $\|A\| \leq \liminf_{n\to\infty} \|A_n\|$

PROOF SINCE $A_n x$ CONVERGES $\forall x \in X$, IT IS BOUNDED,

$\sup_n |A_n x| < +\infty \quad \forall x \underset{\text{BANACH-STEINHAUS}}{\Longrightarrow} \sup_n \|A_n\| < +\infty$

LET US SHOW A IS LINEAR:

$$A_n(\alpha x + \beta y) = \alpha A_n x + \beta A_n y \longrightarrow \alpha A(x) + \beta A(y) \quad \forall x, y \in X, \ \forall \alpha, \beta \in \mathbb{R}$$

$$\downarrow$$

$$A(\alpha x + \beta y) \quad \Rightarrow A \text{ IS LINEAR}$$

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|x\| \leq 1} \lim_{n\to\infty} \|A_n x\| \leq \sup_{\|x\| \leq 1} \liminf_{n\to\infty} \|A_n\| \|x\|$$

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{\|x\|\leq 1} \lim_{n\to\infty} \|A_n x\| \leq \sup_{\|x\|\leq 1} \liminf_{n\to\infty} \|A_n\| \|x\| = \liminf_{n\to\infty} \|A_n\|.$$

REMARK IT COULD BE $\|A\| < \liminf_{n\to\infty} \|A_n\|$

$$L_n : \ell_2 \to \mathbb{R}, \quad x = (x(1) \dots, x(n) \dots) \mapsto x(n), \qquad \|L_n\|_{\ell_2^*} = 1$$

$$L_n x \xrightarrow[n\to\infty]{} 0 \Rightarrow L = 0 \qquad \|L\| = 0 \neq 1$$

PROP LET $C(S^1)$ BE THE SPACE OF CONTINUOUS PERIODIC FUNCTIONS ON $[-\pi,\pi]$

$$C(S^1) := \{ f \in C([-\pi,\pi]) : f(-\pi) = f(\pi) \}$$

FOR ANY $f \in C(S^1)$ DEFINE THE PARTIAL FOURIER SUM:

$$S_N f(x) := \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos(kx) + b_k \sin(kx) \right) \qquad \begin{cases} a_k = \frac{1}{2\pi} \int f(x) \cos(kx)\,dx \\ b_k = \frac{1}{2\pi} \int f(x) \sin(kx)\,dx \end{cases}$$

THEN, $\exists f \in C(S^1)$ SUCH THAT $S_N f(0) \not\to f(0)$.

PROOF WE CAN WRITE

$$S_N f(x) = \sum_{k=-N}^{N} \left(\frac{1}{2\pi} \int f(t) e^{-ikt} dt \right) e^{ikx} = \int f(t) \underbrace{\left(\frac{1}{2\pi} \sum_{k=-N}^{N} e^{ik(x-t)} \right)}_{D_N(x-t)} dt$$

IN PARTICULAR, IF $x=0$, $S_N f(0) = \int f(t) D_N(-t)\,dt$

THEREFORE, $L_N : f \mapsto S_N f(0)$

$L_N \in C(S^1)^*$ WITH $\|L_N\| = \|D_N\|_{L^1(S^1)}$

IF $\underline{S_N f(0) \to f(0)}$ $\forall f$, THEN $\{\underset{S_N f(0)}{L_N f}\}$ IS BOUNDED, SO BY

BANACH-STEINHAUS THEOREM WE WOULD HAVE $\sup_N \|L_N\| = \sup_N \|D_N\|_{L^1} \leq C$

WE SUFFICE TO SHOW THAT $\|D_N\|_{L^1} \to +\infty$

WE CAN WRITE $D_N(t) = \frac{1}{2\pi} \frac{\sin\left((N+\frac{1}{2})t\right)}{\sin\left(\frac{t}{2}\right)}$ (EXERCISE: USE PROPERTIES OF SIN, COS AND THE SUM OF GEOMETRIC SERIES)

$$\int |D_N| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left|\sin\left((N+\frac{1}{2})t\right)\right|}{\left|\sin\frac{t}{2}\right|} dt \geq \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\left|\sin\left((N+\frac{1}{2})t\right)\right|}{|t|} dt \qquad \begin{pmatrix} s = (N+\frac{1}{2})t \\ ds = (N+\frac{1}{2})dt \end{pmatrix}$$

$\uparrow$ $\left|\sin\frac{t}{2}\right| \leq \frac{|t|}{2}$

$(N+\frac{1}{2})\pi$

$$|\sin \tfrac{t}{2}| \leq \frac{|t|}{2} \qquad ds = \left(N+\tfrac{1}{2}\right) dt$$

$$= \frac{1}{\pi}\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi} \frac{|\sin(s)|}{|s|}\, ds \xrightarrow[N\to+\infty]{} \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{|\sin(s)|}{|s|}\, ds = +\infty$$

$\Rightarrow \|D_N\|_{L^1} \longrightarrow +\infty$.

$\boxed{\text{DEF}}$ $f \in C(X)$ CONTINUOUS FUNCTION ON A NORMED SPACE X IS <u>HÖLDER CONTINUOUS</u> OF EXPONENT $\alpha \in (0,1]$ IF $\exists\, C>0$ SUCH THAT

$$|f(x) - f(y)| \leq C\,\|x-y\|^{\alpha} \quad \forall x,y \in X$$

WE DENOTE THE SPACE OF HÖLDER CONTINUOUS FUNCTIONS AS $C^{0,\alpha}(X)$

$\boxed{\text{COROLLARY}}$ HÖLDER CONTINUOUS FUNCTIONS ON S^1 ARE I CATEGORY IN THE SPACE OF CONTINUOUS FUNCTIONS

<u>PROOF</u> IF $f \in C^{0,\alpha}(S^1)$ THEN $S_N f(0) \to f(0)$ BECAUSE OF THE "DINI TEST", BECAUSE $\frac{f(t)-f(0)}{t} \in L^1(S^1)$, AS $\frac{|f(t)-f(0)|}{|t|} \leq \frac{1}{|t|^{1-\alpha}}$

IN PARTICULAR, $\{L_N f\}$ IS BOUNDED FOR ANY $f \in \bigcup_{0<\alpha\leq 1} C^{0,\alpha}(S^1)$

$L_N : f \mapsto S_N f(0)$

BUT $\|L_N\| \xrightarrow[N\to\infty]{} +\infty$, SO IF $\bigcup_\alpha C^{0,\alpha}(S^1)$ WERE II CATEGORY IT WOULD CONTRADICT BANACH-STEINHAUS THEOREM.

$\boxed{\text{REMARK}}$ WE CAN SHOW $Y := \bigcup_\alpha C^{0,\alpha}(S^1)$ IS I CATEGORY IN $X = C(S^1)$ BY WRITING $Y := \bigcup_{n,m\in\mathbb{N}} A_{n,m}$

$A_{n,m} = \{ f \in C(S^1) : |f(x) - f(y)| \leq n\,|x-y|^{\frac{1}{m}} \ \forall x,y \in S^1 \}$

EXERCISE: SHOW $A_{n,m}$ IS CLOSED AND $\mathring{A}_{n,m} = \emptyset$ (SIMILAR TO $L^p \subset L^1$)

$\boxed{\text{OPEN MAP THEOREM}}$ LET X, Y BE BANACH SPACES

AND $A \in \mathcal{L}(X,Y)$ BE

OPEN MAP THEOREM LET X, Y BE BANACH SPACES AND $A \in \mathcal{L}(X,Y)$ BE <u>SURJECTIVE</u>. THEN, A IS <u>OPEN</u>, THAT IS $Z \subset X$ IS OPEN $\Rightarrow A(Z) \subset Y$ IS OPEN.

REMARK ① IN GENERAL, CONTINUOUS + SURJECTIVE $\not\Rightarrow$ OPEN

$f: \mathbb{R} \to \mathbb{R}$
$x \to x^3 - x$

$f((0,+\infty)) = \left[\frac{-2}{\sqrt{3}^3}, +\infty\right)$

$(0,+\infty)$ OPEN; $\left[\frac{-2}{\sqrt{3}^3}, +\infty\right)$ NOT OPEN

② IF $A \in \mathcal{L}(X,Y)$ IS NOT SURJECTIVE, THEN IT IS NOT OPEN AND <u>$\mathrm{ran}(A)$</u> HAS EMPTY INTERIOR

IN FACT, IF $B_\delta(y_0) \subset \mathrm{ran}(A)$ FOR SOME $\delta > 0$, $y_0 \in Y$, THEN BY LINEARITY $B_R(y_0) \subset \mathrm{ran}(A)$ $\forall R > 0 \Rightarrow \mathrm{ran}(A) = Y$

PROOF OF THE OPEN MAP THEOREM

STEP I: $\overline{A(B_1(0))} \supset B_{2\delta}(0)$

WE CAN WRITE $Y = \bigcup_{n \in \mathbb{N}} \overline{A(B_n(0))}$, BECAUSE A IS SURJECTIVE

COUNTABLE UNION OF CLOSED, BY BAIRE'S THEOREM, NOT ALL OF THEM HAVE EMPTY INTERIOR, THAT IS $\overline{A(B_{n_0}(0))} \supset B_r(y_0)$ FOR SOME $n_0 \in \mathbb{N}$, $y_0 \in Y$, $r > 0$. BY LINEARITY, $\overline{A(B_1(0))} \supset B_{\frac{r}{n_0}}(y_0)$

BY SYMMETRY, $\overline{A(B_1(0))} \supset B_{\frac{r}{n_0}}(-y_0)$.

ANY $y \in B_\delta(0)$ IS A MIDDLE POINT BETWEEN $y' \in B_{2\delta}(y_0)$
$y'' \in B_{2\delta}(-y_0)$

SO, $\underline{B_{2\delta}(0)} \subset \frac{1}{2} B_{2\delta}(y_0) + \frac{1}{2} B_{2\delta}(-y_0) \subset \frac{1}{2}\overline{A(B_1(0))} + \frac{1}{2}\overline{A(B_1(0))}$

$2\delta = \frac{r}{n_0}$

$\frac{1}{2}\overline{A(B_1(0))} + \frac{1}{2}\overline{A(B_1(0))} = \underline{\overline{A(B_1(0))}}$ ← CONVEX

<u>STEP II</u> $A(B_1(0)) \supset B_\delta(0)$

TAKE $y \in Y$ WITH $\|y\| < \delta$ AND LOOK FOR $x \in X$ WITH $\|x\| < 1$ AND $Ax = y$

BY STEP I, $y \in B_\delta(0) \subset \overline{A(B_{\frac{1}{2}}(0))}$, $\forall \varepsilon > 0$ $\exists x_\varepsilon$ WITH $\|x_\varepsilon\| < \frac{1}{2}$, $\|Ax_\varepsilon - y\| < \varepsilon$

$\varepsilon = \frac{1}{2} \rightsquigarrow \|x_1\| < \frac{1}{2}$, $\|Ax_1 - y\| < \frac{1}{2}$

$y - Ax_1 \in B_{\frac{\delta}{2}}(0) \subset \overline{A(B_{\frac{1}{4}}(0))}$ $\forall \varepsilon > 0$ $\exists x'_\varepsilon$ WITH $\|x'_\varepsilon\| < \frac{1}{4}$, $\|y - Ax_1 - Ax'_\varepsilon\|$

$y - Ax_1 \in B_{\frac{\delta}{2}}(0) \subset \overline{A(B_{\frac{1}{4}}(0))}$ $\forall \varepsilon > 0$ $\exists x'_\varepsilon$ WITH $\|x'_\varepsilon\| < \frac{1}{4}$, $\|y - Ax_1 - Ax'_\varepsilon\| < \varepsilon$

$\varepsilon = \frac{1}{4} \rightsquigarrow \|x_2\| < \frac{1}{4}$, $\|y - Ax_1 - Ax_2\| < \frac{1}{4}$

BY ITERATION, $\exists \{x_n\}$ WITH $\|x_n\| < \frac{1}{2^n}$, $\|y - A\underbrace{(x_1 + \dots + x_n)}\| < \frac{1}{2^n}$

$\tilde{x}_n : x_1 + \dots + x_n \in X$

$\tilde{x}_n$ IS CAUCHY : $\|\tilde{x}_m - \tilde{x}_n\| = \|\sum_{k=n+1}^{m} x_k\| \leq \sum_{k=n+1}^{m} \|x_k\| \leq \sum_{k=n+1}^{m} \frac{1}{2^k} \xrightarrow{n,m \to \infty} 0$

$\tilde{x}_n \longrightarrow x$ BECAUSE X IS BANACH

$$\|y - A\tilde{x}_n\| \leq \frac{1}{2^n} \xrightarrow[n \to \infty]{} 0 \implies y = Ax.$$

$\|y - A\tilde{x}_n\| \downarrow \|y - Ax\|$

STEP III : CONCLUSION

TAKE $Z \subset X$ OPEN, $y_0 \in A(Z)$, $y_0 = Ax_0$, $\exists r > 0$ SUCH THAT $B_r(x_0) \subset Z \Downarrow A(B_r(x_0)) \subset A(Z)$

BY LINEARITY, $y_0 + A(B_r(0)) = A(B_r(x_0)) \subset \underline{A(Z)}$

BY STEP II, $y_0 + B_{\delta r}(0) \subset y_0 + A(B_r(0))$

$y_0 + B_{\delta r}(0) = \underline{B_{\delta r}(y_0)}$ $\implies y_0$ IS IN THE INTERIOR OF $A(Z)$

$\forall y_0 \implies A(Z)$ IS OPEN.