GOLDSTINE LEMMA | $J(\overline{B_1(0)})$ IS WEAKLY* DENSE IN $\overline{B_1(0)}$

$\overline{B_1(0)} \subset X$, $\quad \overline{B_1(0)} \subset X^{**}$

$(\Rightarrow J(X)$ IS WEAKLY* DENSE IN $X^{**})$

PROOF FIX $\Lambda_0 \in X^{**}$, $\|\Lambda_0\| \le 1$ AND $U = U_{L_1 \dots L_N;\varepsilon}(\Lambda_0)$, WE NEED TO FIND $x \in B := \overline{B_1(0)}$ SUCH THAT $J(x) \in U$, THAT IS $|L_i x - \Lambda_0 L_i| < \varepsilon$ $\forall i = 1, \dots, N$

ASSUME THIS IS FALSE, THAT IS $(\Lambda_0 L_1, \dots, \Lambda_0 L_N) =: y_0 \notin \overline{A(B)}$

WHERE $A : X \to \mathbb{R}^N$
$x \to (L_1 x, \dots, L_N x)$, THEN WE CAN STRICTLY SEPARATE $\{y_0\}$ AND $\overline{A(B)}$

$\exists (c_1, \dots, c_N) \in (\mathbb{R}^N)^* = \mathbb{R}^N$
$\exists \alpha \in \mathbb{R}$ SUCH THAT $\underbrace{c_1 (Ax)_1 + \dots + c_N (Ax)_N}_{= \sum_{i=1}^N c_i L_i x,\ \sum c_i L_i \in X^*} < \alpha < \underbrace{c_1 y_{0_1} + \dots + c_N y_{0_N}}_{= \sum_{i=1}^N c_i \Lambda_0 L_i}$ $\forall x \in B$

PASS TO $\sup_{x \in B}$ $\Rightarrow \left\| \sum_{i=1}^N c_i L_i \right\| \le \alpha < \sum c_i \Lambda_0 L_i \le \|\Lambda_0\| \left\| \sum c_i L_i \right\| \le \left\| \sum c_i L_i \right\|$ CONTRADICTION.

SEPARABLE SPACES | X IS SEPARABLE IF $\exists D \subset X$ D DENSE AND COUNTABLE

$\boxed{\text{PROP}}$ IF X IS A BANACH SPACE SUCH THAT X^* IS SEPARABLE, THEN X IS SEPARABLE.

$\boxed{\text{OSS}}$ IF X IS SEPARABLE, X^* MAY NOT BE SEPARABLE, FOR EXAMPLE $\ell_1, L^1([0,1])$ ARE SEPARABLE, $\ell_\infty, L^\infty([0,1])$ ARE NOT.

$\boxed{\text{COR}}$ IF X^* IS SEPARABLE, THEN $\sigma(X^*, X)$ IS LOCALLY METRIZABLE

PROOF TAKE $\{L_n\}_{n \in \mathbb{N}}$ DENSE IN X^*, FOR ANY L_n $\exists x_n \in X$ WITH $\|x_n\| = 1$ AND $L_n x_n \ge \frac{\|L_n\|}{2}$. LET US SHOW THAT $D := \mathrm{SPAN}_{\mathbb{Q}} \{x_n\} = \{c_1 x_1 + \dots + c_N x_N;\ c_i \in \mathbb{Q}\}$

D IS DENSE IN $Y := \mathrm{span}\{x_n\}$, SO I SUFFICE TO SHOW THAT Y IS DENSE IN X. BY A COROLLARY OF HAHN-BANACH THEOREM, I JUST NEED TO SHOW THAT IF $L|_Y \equiv 0$ THEN $L \equiv 0$ ON X.

ASSUME $L|_Y \equiv 0$, $\forall \varepsilon > 0$ TAKE L_N SUCH THAT $\|L_N - L\| \le \varepsilon$

$\|L\|_{X^*} \le \|L - L_N\| + \|L_N\| \le \varepsilon + 2 L_N x_N = \varepsilon + 2(L_N x_N - L x_N)$

$\|L\|_{X^*} \le \|L - L_N\| + \|L_N\| \le \varepsilon + 2 L_N x_N = \varepsilon + 2(L_N x_N - L x_N)$

$\le \varepsilon + 2 \underbrace{\|L_N - L\|}_{\le \varepsilon} \underbrace{\|x_N\|}_{=1} \le 3\varepsilon \implies L \equiv 0 \in X^* \implies Y$ IS DENSE.

$\boxed{\text{COR}}$ X IS REFLEXIVE AND SEPARABLE $\iff$ X^* IS REFLEXIVE AND SEPARABLE

PROOF X^* IS REFLEXIVE $\implies$ X IS REFLEXIVE (SEEN LAST TIME) $\Leftarrow$ ✓

X^* IS SEPARABLE $\implies$ X IS SEPARABLE (JUST SEEN)

$\Rightarrow$ X IS REFLEXIVE $\implies$ X^{**} IS A COPY OF X, SO X^{**} IS REFLEXIVE AND SEP.

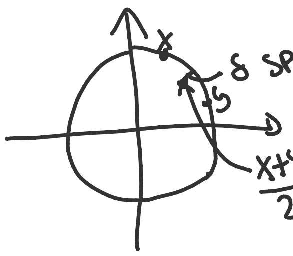
$\implies$ BY THE FIRST IMPLICATION X^* IS REFLEXIVE AND SEPARABLE.

$\boxed{\text{EXAMPLE}}$ $X = \ell_p$, $L^p([0,1])$ ARE:
- $1 < p < +\infty \to$ X AND X^* ARE REFLEXIVE AND SEP.
- $p = 1 \longrightarrow$ X IS SEPARABLE BUT X^* IS NOT SEP. X AND X^* ARE NOT REFLEXIVE
- $p = \infty \longrightarrow$ X IS NOT SEPARABLE, NOR X^* IS SEP. (OTHERWISE, X WOULD BE) AND X, X^* ARE NOT REFLEXIVE.

UNIFORMLY CONVEX SPACES

$\boxed{\text{DEF}}$ A BANACH SPACE X IS <u>UNIFORMLY CONVEX</u> IF $\forall \varepsilon > 0\ \exists \delta > 0$ (INDEPENDENT ON x, y)

SUCH THAT $\begin{matrix} \|x\| = \|y\| = 1 \\ \|x - y\| \ge \varepsilon \end{matrix} \implies \left\| \frac{x+y}{2} \right\| \le 1 - \delta$



ROUGHLY SPEAKING, BALLS ARE "UNIFORMLY" STRICTLY CONVEX

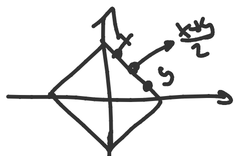
$\boxed{\text{EXAMPLE}}$ ① HILBERT SPACES ARE UNIFORMLY CONVEX.

BY PARALLELOGRAM'S RULE $\left\| \frac{x+y}{2} \right\| = \sqrt{\frac{\|x\|^2 + \|y\|^2}{2} - \frac{\|x-y\|^2}{4}} \le \sqrt{1 - \frac{\varepsilon^2}{2}} = 1 - \delta$

$\|x\| = \|y\| = 1$, $\|x - y\| \ge \varepsilon$

IF I SET $\delta := 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$

② $L^1(\mu)$



IS <u>NOT</u> UNIFORMLY CONVEX:

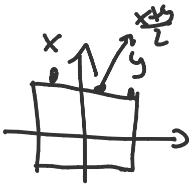
$f = \frac{\chi_A}{|\mu(A)|}$ $\quad g = \frac{\chi_B}{|\mu(B)|}$ WITH $A \cap B = \emptyset$, $\mu(A), \mu(B) > 0$

$\|f\| = \|g\| = 1$ ... $\|f + g\|$ ...

$f=\frac{\chi_A}{|\mu(A)|}$ $\quad g=\frac{\chi_B}{|\mu(B)|}$ WITH $A\cap B=\emptyset$, $\mu(A),\mu(B)>0$

$\|f\|=\|g\|=1$ BUT $\|\frac{f+g}{2}\|=1$

$\|f-g\|=2\geq\varepsilon$

(3) $L^\infty(\mu)$ 

IS NOT UNIFORMLY CONVEX:

$f=\chi_A+\chi_B$ $\quad A\cap B=\emptyset$ $\quad \|f\|_\infty=\|g\|_\infty=1$ BUT $\|\frac{f+g}{2}\|_\infty=1$

$g=\chi_A-\chi_B$ $\quad \mu(A)>0$, $\mu(B)>0$ $\quad \|f-g\|_\infty=2\geq\varepsilon$

REMARKS (1) ON UNIFORMLY CONVEX SPACES WE CAN PROJECT ON CLOSED CONVEX SUBSET. BY UNIFORM CONVEXITY WE GET EXISTENCE AND UNIQUENESS OF A POINT OF MINIMAL DISTANCE.

(2) UNIFORM CONVEXITY IS EQUIVALENT TO $\|x\|\leq 1, \|y\|\leq 1, \|x-y\|\geq\varepsilon \Rightarrow \|\frac{x+y}{2}\|\leq 1-\delta$

IN FACT, IF $\|y\|\leq 1-\eta \Rightarrow \|\frac{x+y}{2}\|\leq\frac{\|x\|+\|y\|}{2}\leq 1-\frac{\eta}{2}\leq 1-\delta$, THE SAME IF $\|x\|\leq 1-\eta$

IF INSTEAD $1-\eta\leq\|x\|,\|y\|\leq 1$, $\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\geq \underbrace{\|x-y\|}_{\geq\varepsilon}-\underbrace{\left\|x-\frac{x}{\|x\|}\right\|}_{\leq\eta}-\underbrace{\left\|\frac{y}{\|y\|}-y\right\|}_{\leq\eta}\geq\varepsilon-2\eta=\varepsilon'$

$$\Rightarrow \left\|\frac{x+y}{2}\right\|\leq\left\|\frac{\frac{x}{\|x\|}+\frac{y}{\|y\|}}{2}\right\|+\frac{\left\|x-\frac{x}{\|x\|}\right\|}{2}+\frac{\left\|\frac{y}{\|y\|}-y\right\|}{2}\leq 1-\delta'+\eta=1-\delta.$$

MILMAN-PETTIS THEOREM

UNIFORMLY CONVEX SPACES ARE REFLEXIVE

REMARK $(\mathbb{R}^n,\|\cdot\|_1)$ IS REFLEXIVE BUT NOT UNIFORMLY CONVEX, THE SAME $(\mathbb{R}^n,\|\cdot\|_\infty)$

PROOF WE NEED TO SHOW THAT J IS SURJECTIVE. BY HOMOGENEITY WE SUFFICE TO TAKE $\Lambda_0\in X^{**}$ WITH $\|\Lambda_0\|\leq 1$ AND SHOW $\Lambda_0\in J(X)$.

SINCE $J(X)$ IS DENSE WE SUFFICE TO SHOW THAT IT IS CLOSED, THAT IS $\forall\varepsilon>0$ $\exists x\in X$ SUCH THAT $\|J(x)-\Lambda_0\|\leq\varepsilon$.

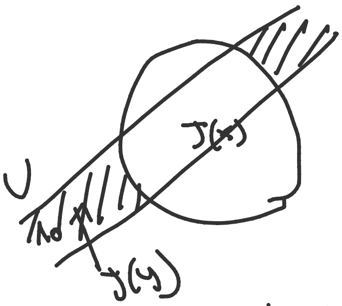
$\exists L\in X^*$ SUCH THAT $\|L\|=1$ AND $\Lambda_0 L\geq 1-\frac{\delta}{2}$ → AS IN THE UNIFORM CONVEXITY CONDITION

AND I TAKE $U=U_{L;\frac{\delta}{2}}(\Lambda_0)=\{\Lambda\in X^{**}:|\Lambda L-\Lambda_0 L|<\frac{\delta}{2}\}$. SINCE U IS OPEN IN $\sigma(X^{**},X^*)$ AND $J(X)$ IS DENSE IN $\sigma(X^{**},X)$, $\exists J(x)\in U$

LET US SHOW THAT $\|J(x)-\Lambda_0\|\leq\varepsilon$. ASSUME THIS IS FALSE, THAT IS $\Lambda_0\in X^{**}\setminus\overline{B_\varepsilon(J(x))}$. SINCE $\overline{B_\varepsilon(J(x))}$ IS CLOSED IN $\sigma(X^{**},X)$, THEN $U\setminus\overline{B_\varepsilon(J(x))}$ IS OPEN IN $\sigma(X^{**},X)$, BY THE DEFIN

$\ldots, B_\varepsilon(J(x))$. SINCE $\overline{B_\varepsilon(J(x))}$ IS CLOSED IN $\sigma(X^{**}, X)$

THEN $U \setminus \overline{B_\varepsilon(J(x))}$ IS OPEN IN $\sigma(X^{**}, X)$, BY THE DENSITY OF J $\exists\, J(y) \in U \setminus \overline{B_\varepsilon(J(x))}$.



$$J(y) \in U \Rightarrow \underbrace{|Ly - \Lambda_0 L| < \frac{\delta}{2}}_{\circledast}$$

$$J(y) \notin B_\varepsilon(J(x)) \Rightarrow \|x-y\| > \varepsilon$$

BUT ALSO $J(x) \in U \Rightarrow \underbrace{|Lx - \Lambda_0 L| < \frac{\delta}{2}}_{\circledast\circledast}$

$$\left\|\frac{x+y}{2}\right\| \underset{\|L\|=1}{\geq} \frac{Lx + Ly}{2} \underset{\circledast + \circledast\circledast}{\geq} \Lambda_0 L - \frac{\delta}{2} \geq 1 - \delta$$

, CONTRADICTION WITH $\|x-y\| > \varepsilon$.