EXAMPLES

① $A \in \mathcal{L}(\ell_2)$ $A: (x(1), x(2), x(3), \ldots) \to \left(x(1), \frac{x(2)}{2}, \frac{x(3)}{3}, \ldots\right)$

$(Ax)(k) = \frac{x(k)}{k}$

$0 \in \sigma_c(A)$

LET US FIND OTHER ELEMENTS IN THE SPECTRUM. ASSUME $\lambda \in \sigma_p(A)$

$\Rightarrow (A - \lambda I)x = 0$ $\quad x(1) - \lambda x(1) = 0$ $\quad x = 0$ IS THE ONLY SOLUTION

$\frac{x(2)}{2} - \lambda x(2) = 0$ $\quad$ IF $\lambda \notin \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

$\vdots$ $\quad$ BUT IF $\lambda = \frac{1}{k}$, $\lambda \in \sigma_p(A)$

$\frac{x(k)}{k} - \lambda x(k) = 0$ $\quad$ BECAUSE $(A - \frac{1}{k}) e_k = 0$

$\sigma(A)$ DOES NOT CONTAIN ANY OTHER ELEMENT: IF $\lambda \neq 0, \frac{1}{k}$ $(k \in \mathbb{N})$ THEN $A - \lambda I$ IS INVERTIBLE, BECAUSE $(A - \lambda I)x = y \iff \left(\frac{1}{k} - \lambda\right)x(k) = y(k)$

$\iff x(k) = \frac{k}{1 - \lambda k} y(k)$

$\frac{k}{1-\lambda k}$ → WELL-DEFINED BECAUSE $\lambda \neq \frac{1}{k}$

→ IN ℓ_∞ BECAUSE $\lambda \neq 0$

$\Rightarrow \sigma(A) = \underbrace{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}}_{\sigma_p(A)} \cup \underbrace{\{0\}}_{\sigma_c(A)}$

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② $A: (x(1), x(2), x(3) \ldots) \to (x(2), x(3), x(4), \ldots)$ LEFT SHIFT

LET US FIND $\lambda \in \sigma_p(A)$: $(A - \lambda I)x = 0 \iff x(2) - \lambda x(1) = 0 \Rightarrow x(2) = \lambda x(1)$

$x(3) - \lambda x(2) = 0 \Rightarrow x(3) = \lambda x(2) = \lambda^2 x(1)$

$\ldots$

$x(k+1) - \lambda x(k) = 0 \Rightarrow x(k+1) = \lambda^k x(1)$

$\Rightarrow (A - \lambda I)x = 0$ IF $x = x(1)(1, \lambda, \lambda^2, \ldots) \in \ell^2 \iff |\lambda| < 1$

WE SHOWED $\{|\lambda| < 1\} \subset \sigma(A)$, BUT SINCE $\sigma(A)$ IS CLOSED THEN $\{|\lambda| \leq 1\} \subset \sigma(A)$

IF $|\lambda| > \|A\| = 1$ THEN $\lambda \notin \sigma(A) \Rightarrow \sigma(A) = \{|\lambda| \leq 1\}$

IF $|\lambda| = 1$ THEN $A - \lambda I$ IS INJECTIVE BUT NOT SURJECTIVE, HOWEVER $\overline{\mathrm{ran}(A - \lambda I)} = \ell_2$

LET US SHOW $e_n \in \mathrm{ran}(A - \lambda I)$: $\quad x(2) - \lambda x(1) = 0$

$x(n) - \lambda x(n-1) = 0 \to x(n-1) = \frac{x(n)}{\lambda} = -\frac{1}{\lambda^2}$

$x(n+1) - \lambda x(n) = 1 \to x(n) = -\frac{1}{\lambda}$

$x(n+2) - \lambda x(n+1) = 0$ $\quad x(n+1) = x(n+2) = \ldots = 0$

$\Rightarrow x_n = \left(-\frac{1}{\lambda^n}, -\frac{1}{\lambda^{n-1}}, \ldots, -\frac{1}{\lambda}, 0 \ldots 0\right)$

$(A - \lambda I)x_n = e_n$ $\quad \Rightarrow \sigma_r(A) = \{|\lambda| = 1\}$ $\quad \sigma_p(A) = \{|\lambda| < 1\}$ $\quad \sigma_c(A) = \emptyset$

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③ $A: (x(1), x(2), x(3) \ldots) \to (0, x(1), x(2), \ldots)$

IF $|\lambda| < 1$, THEN $\lambda \in \sigma_r(A)$. LET US SHOW THAT $e_1 \notin \overline{\mathrm{ran}(A - \lambda I)}$

IN PARTICULAR, IF $\|y\| \leq 1 - |\lambda|$ THEN $e_1 + y \notin \mathrm{ran}(A - \lambda I)$, BY CONTRADICTION

ASSUME $(A - \lambda I)x = e_1 + y \Rightarrow -\lambda x(1) = 1 + y(1) \Rightarrow x(1) = -\frac{1 + y(1)}{\lambda}$

... $\notin \operatorname{ran}(A-\lambda I)$, BY CONTRADICTION

ASSUME $(A-\lambda I)x = e_1 + y \Rightarrow$

$$-\lambda x(1) = 1 + y(1) \Rightarrow x(1) = -\frac{1+y(1)}{\lambda}$$
$$x(1) - \lambda x(2) = y(2) \qquad x(2) = \frac{x(1)-y(2)}{\lambda} = -\frac{1}{\lambda^2} - \frac{y(1)}{\lambda^2} - \frac{y(2)}{\lambda}$$
$$x(2) - \lambda x(3) = y(3) \qquad x(3) = \frac{x(2)-y(3)}{\lambda} = -\frac{1}{\lambda^3} - \frac{y(1)}{\lambda^3} - \frac{y(2)}{\lambda^2} - \frac{y(3)}{\lambda}$$

$$x(k) = -\frac{1}{\lambda^k} - \frac{y(1)}{\lambda^k} - \frac{y(2)}{\lambda^{k-1}} \cdots - \frac{y(k)}{\lambda}$$

$$|x(k)| = \frac{|1 + y(1) + \lambda y(2) \dots|}{|\lambda^k|} \geq \frac{1 - |y(1)| - |\lambda||y(2)| \dots}{|\lambda|^k}$$

$$\geq \frac{1 - \|y\|_{\ell^2} \underbrace{(1 + |\lambda| + |\lambda|^2) \dots}_{\text{GEOMETRIC SERIES}}}{|\lambda|^k} \geq \frac{1 - \frac{\|y\|}{1-\lambda}}{|\lambda|^k} \to +\infty$$

($1 - \frac{\|y\|}{1-\lambda} > 0$)

$\Rightarrow x \notin \ell_2 \Rightarrow e_1 + y \notin \operatorname{ran}(A-\lambda I)$

$\Rightarrow \sigma(A) \supset \{|\lambda| < 1\}$, SINCE IT IS CLOSED $\sigma(A) \supset \{|\lambda| \leq 1\}$

AS BEFORE, IF $|\lambda| > \|A\| = 1$ THEN $\lambda \notin \sigma(A)$

$\Rightarrow \sigma(A) = \{|\lambda| \leq 1\}$

IF $|\lambda| = 1$ THEN $\lambda \in \sigma_c(A)$: LET US SHOW FIRST $A-\lambda I$ IS INJECTIVE:

$(A-\lambda I)x = 0 \Rightarrow$
$$-\lambda x(1) = 0 \to x(1) = 0$$
$$x(1) - \lambda x(2) = 0 \to x(2) = 0$$
$$x(2) - \lambda x(3) = 0 \to x(3) = 0$$
$$x \equiv 0$$

LET US SHOW THAT $\overline{\operatorname{ran}(A-\lambda I)} = \ell_2$

WE SHOW THAT THE ONLY FUNCTIONAL IN $(\ell_2)^*$ THAT VANISHES ON $\overline{\operatorname{ran}(A-\lambda I)}$ IS THE ZERO FUNCTIONAL, THAT IS $(y, (A-\lambda I)x) = 0 \Rightarrow y \equiv 0$

$$(y, (A-\lambda I)x) = \sum y(k)\left(x(k-1) - \lambda x(k)\right)$$

TAKE $x = e_k \Rightarrow \lambda y(k) - y(k+1) = 0 \Rightarrow y(k+1) = \lambda y(k) = \dots = \lambda^k y(1)$

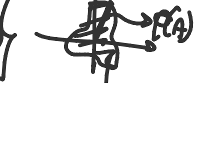
$\Rightarrow y = y(1)(1, \lambda, \lambda^2, \dots) \in \ell^2$

IMPOSSIBLE IF $y \not\equiv 0$.

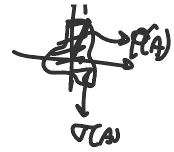
$$\Rightarrow \sigma_c(A) = \{|\lambda| = 1\}$$
$$\sigma_r(A) = \{|\lambda| < 1\}$$
$$\sigma_p(A) = \emptyset$$

SPECTRAL RADIUS

$\boxed{\text{DEF}}$ LET X BE A COMPLEX BANACH SPACE AND $A \in \mathcal{L}(X)$, THE **SPECTRAL RADIUS** OF A IS $\rho(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$



SPECTRAL RADIUS OF A IS $\rho(A) := \sup\{|\lambda| ; \lambda\in\sigma(A)\}$



$\boxed{\text{REMARK}}$ WE SAW THAT $|\lambda|>\|A\| \Rightarrow \lambda\notin\sigma(A) \Rightarrow \rho(A)\le\|A\|$.

$\boxed{\text{EXAMPLES}}$ ① IN THE THREE EXAMPLES ($Ax(k)=\frac{x(k)}{k}$ AND LEFT/RIGHT SHIFT) WE HAVE $\rho(A)=\|A\|=1$

② WE COULD HAVE $\rho(A)<\|A\|$ FOR EXAMPLE $A:\mathbb{R}^2\to\mathbb{R}^2$, $(x_1,x_2)\to(x_2,0)$ $\begin{pmatrix}0&1\\0&0\end{pmatrix}$

$\|A\|=1$, $\sigma(A)=\{0\}\Rightarrow\rho(A)=0$. $\quad A^2=0$

$\boxed{\text{THEOREM}}$ (CHARACTERIZATION OF THE SPECTRAL RADIUS)

① $\sigma(A)\neq\emptyset$ ② $\rho(A)=\lim_{n\to\infty}\|A^n\|^{\frac{1}{n}}$ (IN PARTICULAR, THE LIMIT EXISTS AND IT IS FINITE)

$\boxed{\text{REMARK}}$ IN GENERAL, $\|A^n\|\le\|A\|^n \Rightarrow \lim_{n\to\infty}\|A^n\|^{\frac{1}{n}}\le\|A\|$

$\boxed{\text{LEMMA}}$ FOR ANY POLYNOMIAL $P:\mathbb{C}\to\mathbb{C}$ AND $A\in\mathcal{L}(X)$ WE HAVE
$\sigma(\underbrace{P(A)}_{a_0I+a_1A+a_2A^2})=P(\sigma(A))=\{P(\lambda);\lambda\in\sigma(A)\}$, IN PARTICULAR $\rho(P(A))=P(\rho(A))$

PROOF BY FUNDAMENTAL THEOREM OF ALGEBRA $P(z)-\lambda=a_n(z-\alpha_1(\lambda))^{k_1(\lambda)}\cdots(z-\alpha_k(\lambda))^{k_k(\lambda)}$

$\Rightarrow P(A)-\lambda I=a_n(A-\alpha_1(\lambda)I)^{k_1(\lambda)}\cdots(A-\alpha_k(\lambda)I)^{k_k(\lambda)}$

$\lambda\in\sigma(P(A)) \iff P(A)-\lambda I$ NOT INVERTIBLE $\iff A-\alpha_j(\lambda)I$ NOT INVERTIBLE FOR SOME j

$\iff \alpha_j(\lambda)\in\sigma(A)$ FOR SOME $j \iff \lambda=P(\alpha_j(\lambda))\in P(\sigma(A))$

$\Rightarrow \sigma(P(A))=P(\sigma(A))$

PROOF OF THEOREM ① CONSIDER $f:\mathbb{C}\setminus\sigma(A)\to\mathcal{L}(X)$

$\lambda\longrightarrow(A-\lambda I)^{-1}$

LET US SHOW f IS HOLOMORPHIC!

$$\frac{f(\lambda+h)-f(\lambda)}{h}=\frac{(A-(\lambda+h)I)^{-1}-(A-\lambda I)^{-1}}{h}=$$

$$=(A-\lambda I)(A-(\lambda+h)I)^{-1}(A-\lambda I)^{-1}-(A-(\lambda+h)I)(A-(\lambda+h)I)^{-1}(A-\lambda I)^{-1}$$

$$= \frac{(A-\lambda I)(A-(\lambda+\mu)I)^{-1}(A-\lambda I)^{-1} - (A-(\lambda+\mu)I)(A-(\lambda+\mu)I)^{-1}(A-\lambda I)^{-1}}{\mu}$$

$$= \frac{\cancel{\mu} I (A-(\lambda+\mu)I)^{-1}(A-\lambda I)^{-1}}{\cancel{\mu}} \xrightarrow[\mu\to 0]{} (A-\lambda I)^{-2} \Rightarrow f \text{ HOLOMORPHIC}$$

MOREOVER, $f(\lambda) = \frac{1}{\lambda}\underbrace{\left(\frac{A}{\lambda} - I\right)^{-1}}_{\substack{\text{BOUNDED} \\ \text{IF } |\lambda| \text{ LARGE}}} \xrightarrow[|\lambda|\to\infty]{} 0$

IF $\sigma(A) = \emptyset$, f IS HOLOMORPHIC ON $\mathbb{C}$ AND BOUNDED $\Rightarrow$ BY LIOUVILLE'S THEOREM, f IS CONSTANT, IMPOSSIBLE.

2 WE WRITE f AS A LAURENT SERIES, WHICH CONVERGES IN THE $\underbrace{\text{BIGGEST ANNULUS}}_{= \mathbb{C} \setminus B_{\rho(A)}}$ WHERE f IS HOLOMORPHIC. BUT THE SERIES EXPRESSION OF f IS

$f(\lambda) = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k$. $|\lambda| > \rho(A) \Rightarrow \sum$ CONVERGES $\Rightarrow \limsup_{k\to\infty} \left\|\frac{A^k}{\lambda^k}\right\|^{\frac{1}{k}} < 1$

$\Leftrightarrow |\lambda| > \limsup_{k\to\infty} \|A^k\|^{\frac{1}{k}}$

$(*)$ $\Rightarrow \rho(A) \geq \limsup_{k\to\infty} \|A^k\|^{\frac{1}{k}}$. TO SHOW THE OTHER INEQUALITY WE USE THE LEMMA: $P(z) = z^k \Rightarrow \rho(A)^k = \rho(A^k) \leq \|A^k\|$

$\Rightarrow (**)\ \rho(A) \leq \liminf_{k\to\infty} \|A^k\|^{\frac{1}{k}}$

$(*) + (**) \Rightarrow \rho(A) = \lim_{k\to\infty} \|A\|^{\frac{1}{k}}$.