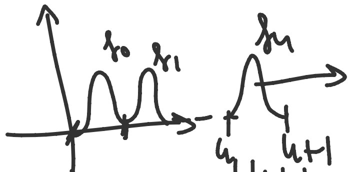
EXAMPLE | $\{f_n\}$ SEQUENCE IN $L^\infty(\mathbb{R})$ GIVEN BY

$f_n(x) = f_0(x-n)$ FOR SOME $f_0 \in C_0([0,1])$ $\int_0^1 f_0 = 1$



$f_n \to 0$ POINTWISE BUT $\|f_n\|_\infty = \|f_0\|_\infty \not\to 0$

LET US SHOW THAT $f_n \overset{*}{\rightharpoonup} 0$, THAT IS $\int f_n g \to 0$ $\forall g \in L^1(\mathbb{R})$

$$\left|\int f_n g\right| = \left|\int_n^{n+1} f_0(x-n)\, g(x)\,dx\right| \le \|f_0\|_\infty \int_n^{n+1} |g| \xrightarrow[n\to\infty]{} 0$$

BECAUSE $g \in L^1(\mathbb{R})$

SIMILARLY, $f_n \rightharpoonup 0$ IN $L^p(\mathbb{R})$, $1<p<+\infty$ $\left(\int f_n g \to 0 \ \forall g \in L^{p'}(\mathbb{R})\right)$

BUT f_n DOES NOT CONVERGE WEAKLY IN $L^1(\mathbb{R})$. IN FACT, TAKE $g \in L^\infty(\mathbb{R})$ WITH COMPACT SUPPORT, THEN

$\int f_n g = \underline{0}$ FOR n LARGE ENOUGH (BECAUSE $f_n(x) g(x) \equiv 0$)

BUT IF WE TAKE $g \equiv 1$, THEN $\int f_n g = \int f_n = \int f_0 = \underline{1}$,

SO IF $f_n \rightharpoonup f$ FOR SOME f, THEN $\int fg = 0$ $\forall g$ WITH COMPACT SUPPORT BUT $\int f = 1$, IMPOSSIBLE.

PROPOSITION | TAKE $\Lambda \in X^{**}$. THEN Λ IS CONTINUOUS IN $\sigma(X^*, X)$ IF AND ONLY IF IT HAS THE FORM $\Lambda: L \to Lx$ FOR SOME $x \in X$.
IN PARTICULAR, IF X IS NOT REFLEXIVE, THEN THERE ARE MAPS CONTINUOUS IN $\sigma(X^*, X^{**})$ BUT NOT IN $\sigma(X^*, X)$

PROOF ($\Leftarrow$) IF $\Lambda: L \to Lx$, IT IS WEAKLY* CONTINUOUS, ALREADY SEEN.

($\Rightarrow$) ASSUME Λ CONTINUOUS IN $\sigma(X^*, X)$, THEN $A := \{L \in X^* : |\Lambda L| < 1\}$ IS OPEN IN $\sigma(X^*, X)$

SO $A \supset U = U_{x_1, \dots, x_N; \varepsilon}(0)$ FUNDAMENTAL NEIGHBORHOOD OF 0.

WE RESCALE BY $t>0$ $\quad tA = \{|\Lambda| < t\}$ CONTAINS $tU = U_{x_1, \dots, x_N; t\varepsilon}(0)$

$t \to 0$ $\quad \downarrow \{\Lambda = 0\}$ $\quad \downarrow Lx_1 = \dots = Lx_N = 0$

WE HAVE $Lx_1 = \dots = Lx_N = 0 \Rightarrow \Lambda L = 0$.

$B: X^* \to \mathbb{R}^{N+1}$

$L \mapsto (\Lambda L, Lx_1, \dots, Lx_N)$ $\quad$ B IS NOT SURJECTIVE BECAUSE

$B : \Lambda \longrightarrow \mathbb{R}$

$L \to (\Lambda L, Lx_1 \dots, Lx_N)$

B IS NOT SURJECTIVE BECAUSE $(1,0 \dots, 0) \notin \operatorname{ran} A$, THEREFORE

$\operatorname{ran} A \subset \{ y \in \mathbb{R}^{N+1} ; \underline{c_0} y_0 + c_1 y_1 + \dots + c_N y_N = 0 \}$ $c_0 \neq 0$

$\Updownarrow$

$c_0 \Lambda L + c_1 L x_1 + \dots + c_N L x_N = 0 \quad \forall L \in X^*$

$\Updownarrow$

$$\Lambda L = L \underbrace{\left(-\frac{c_1}{c_0} x_1 \dots - \frac{c_N}{c_0} x_N \right)}_{x \in X} \Rightarrow \Lambda : L \longrightarrow Lx$$

$\boxed{\text{COROLLARY}}$ A HYPERPLANE IN X^* $H = \{ \Lambda = \alpha \}$ FOR SOME $\Lambda \in X^{**}$, $\alpha \in \mathbb{R}$ IS CLOSED IN $\sigma(X^*, X)$ IF AND ONLY IF $\Lambda : L \to Lx$.

IN PARTICULAR, IF X IS NOT REFLEXIVE THEN $\sigma(X^*, X)$ IS <u>STRICTLY</u> LESS FINE THAN $\sigma(X, X^*)$.

$\boxed{\text{PROOF}}$ $\Leftarrow$ OBVIOUS

$\Rightarrow$ ASSUME H CLOSED IN $\sigma(X^*, X)$, THEN $X \setminus H$ CONTAINS WEAK* NEIGHBORHOODS OF ANY $L_0 \in X \setminus H$ $U = U_{x_1 \dots x_N, \varepsilon}(L_0) \subset \{ \Lambda \neq \alpha \}$ SINCE U IS CONNECTED, THEN $U \subset \{ \Lambda < \alpha \}$. IF I TAKE $L \in U$ THEN $L' := L - L_0$ VERIFIES $|L' x_1| < \varepsilon, \dots, |L' x_N| < \varepsilon \Leftrightarrow L \in U$, SO $|L' x_1| < \varepsilon \Rightarrow \Lambda L' < \alpha - \Lambda L_0$ AND BY CHANGING SIGN $|\Lambda L'| < \alpha - \Lambda L_0$ THEREFORE $\{ |\Lambda| < \underline{\alpha - \Lambda L_0} \}$ CONTAINS WEAK* NEIGHBORHOODS OF 0, BY HOMOGENEITY $\{ |\Lambda| < \delta \}$ CONTAINS WEAK* NEIGHBORHOODS OF 0 $\forall \delta > 0$, SO Λ IS CONTINUOUS AT 0, BY LINEARITY Λ IS CONTINUOUS AT ANY POINT, SO BY THE PROPOSITION $\Lambda : L \to Lx$.

BANACH-ALAOGLU THEOREM

THE CLOSED UNIT BALL OF $\overline{B_1(0)} = \{ L \in X^* : \|L\| \leq 1 \} \subset X^*$ IS COMPACT IN $\sigma(X^*, X)$.

$\boxed{\text{TYCHONOFF THEOREM}}$ LET X BE ANY SET, WE CONSIDER

TYCHONOFF THEOREM LET X BE ANY SET, WE CONSIDER $\mathbb{R}^X = \prod_{x\in X} \mathbb{R}_x = \{ f: X \to \mathbb{R} \}$ WITH THE PRODUCT TOPOLOGY HAVING AS NEIGHBORHOODS OF f_0 $\underbrace{\{ f \in \mathbb{R}^X : |f_0(x_1) - f(x_1)| < \varepsilon, \dots, |f_0(x_N) - f(x_N)| < \varepsilon \}}_{U_{x_1,\dots,x_N;\varepsilon}(f_0)}$!!

FOR $\varepsilon > 0$, $x_1, \dots, x_N \in X$

IF $F \subset \mathbb{R}^X$ VERIFIES $\sup_{f\in F} |f(x)| < +\infty$ $\forall x \in X$ THEN F IS RELATIVELY COMPACT IN $\mathbb{R}^X$.

PROOF OF BANACH-ALAOGLU'S THEOREM

WE OBSERVE THAT $\sigma(X^*, X)$ IS THE RESTRICTION TO THE PRODUCT TOPOLOGY ON $X^* \subset \mathbb{R}^X$. WE WANT TO APPLY TYCHONOFF THEOREM TO $\overline{B_1(0)}$: IT IS POINTWISE BOUNDED IN THE PRODUCT TOPOLOGY BECAUSE IF $L \in \overline{B_1(0)}$ THEN $|Lx| \leq \|L\| \|x\| \leq \|x\| < +\infty$ $\forall x \in X$. $\Rightarrow \overline{B_1(0)}$ IS REL. COMPACT

WE NEED TO SHOW $\overline{B_1(0)} \subset \mathbb{R}^X$ IS CLOSED.

WE WRITE

$$\overline{B_1(0)} = \underbrace{\{ f \in \mathbb{R}^X : \alpha f(x) + \beta f(y) = f(\alpha x + \beta y)\ \forall x,y \in X,\ \forall \alpha,\beta \in \mathbb{R} \}}_{\text{LINEAR}}$$
$$\cap \{ f \in \mathbb{R}^X : -\|x\| \leq f(x) \leq \|x\|\ \forall x \in X \} \to \text{CONTINUI CON NORMA} \leq 1$$

$$= \bigcap_{\alpha,\beta,x,y} \underbrace{\{ f : \alpha f(x) + \beta f(y) - f(\alpha x + \beta y) = 0 \}}_{\text{PRE-IMAGE OF 0 W.R.T. A CONTINUOUS MAP}} \cap \bigcap_{x} \underbrace{\{ f : -\|x\| \leq f(x) \leq \|x\| \}}_{\text{PRE-IMAGE OF } [-\|x\|, \|x\|]}$$

$\forall x \in X$, $\begin{array}{l}\mathbb{R}^X \to \mathbb{R} \\ f \to f(x)\end{array}$ IS CONTINUOUS, SO IS $f \to \alpha f(x) + \beta f(y) - f(\alpha x + \beta y)$

$\Rightarrow \overline{B_1(0)}$ IS AN INTERSECTION OF CLOSED SET, SO IT IS CLOSED

COROLLARY IF X IS REFLEXIVE, THEN $\overline{B_1(0)} \subset X^*$ IS WEAKLY COMPACT.

LEMMA A BANACH SPACE X IS REFLEXIVE $\Leftrightarrow X^*$ IS REFLEXIVE

COR (1) $L^\infty([0,1])$ AND ℓ_∞ ARE NOT REFLEXIVE

(2) IF X IS REFLEXIVE, THEN $\overline{B_1(0)} \subset X$ IS WEAKLY COMPACT.

PROOF OF THE LEMMA

③ IF X IS REFLEXIVE, THEN $\overline{B_1(0)} \subset X$ IS WEAKLY COMPACT.

PROOF OF THE LEMMA ASSUME X^* TO BE REFLEXIVE AND ASSUME (BY CONTRADICTION) $\exists \Lambda_0 \in X^{**} \setminus J(X)$. THEN $\exists \alpha \in X^{***}$ SUCH THAT $\alpha|_{J(X)} \equiv 0$ BUT $\alpha \Lambda_0 \not\equiv 0$ (COROLLARY OF HAHN-BANACH), SINCE X^* IS REFLEXIVE, THEN $\alpha \Lambda = \Lambda L$ FOR SOME $L \in X^*$, SO $\forall x \in X$ I HAVE $Lx = J(x) L = \alpha J(x) = 0 \Rightarrow L \equiv 0 \Rightarrow \alpha \equiv 0$, CONTRADICTION WITH $\alpha \Lambda_0 \neq 0$.

NOW, ASSUME X REFLEXIVE. FOR ANY $\alpha \in X^{***}$ DEFINE $L_\alpha \in X^*$ $L_\alpha : x \to \alpha J(x)$. NOW, FOR ANY $\Lambda \in X^{**}$ I HAVE $\Lambda = J(x)$ FOR SOME $x \in X$, SO $\underline{\alpha \Lambda} = \alpha J(x) = L_\alpha x = \underline{\Lambda L_\alpha}$, SO $\alpha = J(L_\alpha)$ AND X^* IS REFLEXIVE.

$\boxed{\text{KAKUTANI THEOREM}}$ A BANACH SPACE X IS REFLEXIVE $\Leftrightarrow$ $\overline{B_1(0)} \subset X$ IS COMPACT IN $\sigma(X, X^*)$

$\boxed{\text{GOLDSTINE'S LEMMA}}$ LET X BE A NORMED SPACE AND $J: X \to X^{**}$ THE ISOMETRY IN THE BI-DUAL. THEN $J(\overline{B_1(0)})$ IS WEAKLY* DENSE IN $\overline{B_1(0)}$

$\boxed{\text{REMARK}}$ THE LEMMA IS FALSE IF WE CONSIDER DENSITY IN NORM. IN FACT, $J(\overline{B_1(0)})$ IS ALWAYS CLOSED IN X^{**}, SO IF IT IS DENSE, THEN $J(\overline{B_1(0)}) = \overline{B_1(0)}$, SO X IS REFLEXIVE.

PROOF OF KAKUTANI THEOREM ($\Rightarrow$) PROVED ALREADY.

($\Leftarrow$) ASSUME $\overline{B_1(0)}$ COMPACT IN $\sigma(X, X^*)$, LET US VERIFY THAT $J: (X, \sigma(X, X^*)) \to (X^{**}, \sigma(X^{**}, X^*))$ IS CONTINUOUS. TAKE A FUNDAMENTAL NEIGHB. $U = U_{L_1 \dots L_N, \varepsilon}(J(x_0))$ OF $J(x_0)$. THEN, ITS PRE-IMAGE IS

$J^{-1}(U) = U_{L_1 \dots L_N, \varepsilon}(x_0)$ IS A FUNDAMENTAL NEIGHBORHOOD, SO IT IS OPEN.

$\Rightarrow J(\overline{B_1(0)})$ IS COMPACT IN $\sigma(X^{**}, X^*)$, IN PARTICULAR $J(\overline{B_1(0)})$ IS CLOSED IN $\overline{B_1(0)}$ W.R.T. $\sigma(X^{**}, X^*)$. FROM GOLDSTINE'S LEMMA, $J(\overline{B_1(0)})$ IS ALSO DENSE   DENSE + CLOSED $\Rightarrow J(\overline{B_1(0)}) = \overline{B_1(0)}$

[illegible] FROM GOLDSTINE'S LEMMA,
$J(\overline{B_1(0)})$ IS ALSO DENSE. DENSE + CLOSED $\Rightarrow J(\overline{B_1(0)}) = \overline{B_1(0)}$
BY LINEARITY, $J(X) = X^{**}$, THAT IS X IS REFLEXIVE.