SPECTRAL THEORY

RUDIN - "FUNCTIONAL ANALYSIS" (INCLUDING EX. 12–15)

WE WILL ALWAYS CONSIDER COMPLEX BANACH SPACES/HILBERT SPACES

DEF LET X BE A COMPLEX VECTOR SPACE. A NORM ON X IS A FUNCTION $\|\cdot\|: X \to [0,+\infty)$ SUCH THAT:

1) $\|x+y\| \le \|x\|+\|y\| \quad \forall x,y\in X$
2) $\|\alpha x\| = |\alpha|\,\|x\| \quad \forall x\in X\ \forall \alpha\in\mathbb{C}$ ($|\alpha|$ → COMPLEX ABS. VALUE)
3) $\|x\|>0 \quad \forall x\in X\setminus\{0\}$

WE SAY THAT $(X,\|\cdot\|)$ IS A COMPLEX NORMED SPACE. IF IT IS COMPLETE WITH RESPECT TO $d(x,y)=\|x-y\|$, WE SAY IT IS A COMPLEX BANACH SPACE

DEF A HERMITIAN PRODUCT ON A COMPLEX VECTOR SPACE X IS A FUNCTION $(\ ,\ ): X\times X \to \mathbb{C}$ SUCH THAT:

1) $(x,y)=\overline{(y,x)} \quad \forall x,y\in X$
2) $(\alpha x+\beta y, z) = \alpha(x,z)+\beta(y,z) \quad \forall x,y,z\in X\ \forall \alpha,\beta\in\mathbb{C}$
3) $(x,x)\in\mathbb{R}_{>0} \quad \forall x\in X$

REMARK $(x,\alpha y+\beta z) = \bar\alpha(x,y)+\bar\beta(x,z) \quad \forall x,y,z\in X\ \forall\alpha,\beta\in\mathbb{C}$.

DEF IF X IS A COMPLEX BANACH SPACE WITH $\|x\|:=\sqrt{(x,x)}$ THEN WE SAY X IS A COMPLEX HILBERT SPACE.

REMARK ON COMPLEX HILBERT SPACES THE FOLLOWING ARE STILL TRUE: CAUCHY-SCHWARZ INEQUALITY, PARALLELOGRAM RULE, PYTHAGORAS'S THEOREM,

$$\underbrace{\text{CAUCHY-SCHWARZ INEQUALITY}}_{(x,y)\le\|x\|\,\|y\|},\quad \underbrace{\text{PARALLELOGRAM RULE}}_{\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2)},\quad \underbrace{\text{PYTHAGORAS'S THEOREM}}_{\|x\|^2+\|y\|^2=\|x+y\|^2 \text{ IF } (x,y)=0}$$

THE POLARIZATION IDENTITY CHANGES:

$$(x,y) = \frac{\|x+y\|^2-\|x-y\|^2}{4} + i\,\frac{\|x+iy\|^2-\|x-iy\|^2}{4}$$

$$\underbrace{(x,y)}_{\in \mathbb{C}} = \underbrace{\frac{\|x+y\|^2 - \|x-y\|^2}{4} + i \frac{\|x+iy\|^2 - \|x-iy\|^2}{4}}_{\in \mathbb{R}}$$

$\boxed{\text{EXAMPLE}}$ ALL THE (REAL) BANACH SPACES WE SAW, LIKE $L^p(\mu)$, C^k, $W^{1,p}$... ARE COMPLEX BANACH SPACES IF I TAKE FUNCTIONS WITH COMPLEX VALUES RATHER THAN REAL VALUES, AND IF I DEFINE NORMS USING COMPLEX ABSOLUTE VALUE.

FOR ISTANCE, $L^p(\mu) := \{ f : X \to \mathbb{C}\ ;\ f$ IS μ-MEASURABLE AND $\int_X |f|^p d\mu < +\infty \} / \sim$

$f^{-1}(A) \in \Sigma$ $\forall A$ OPEN IN $\mathbb{C}$

COMPLEX ABS. VALUE

$f \sim g$ IF $f(x) = g(x)$ FOR μ-A.E. x.

INCLUDING ℓ_2

$L^2(\mu)$ SPACES ARE COMPLEX HILBERT SPACES IF I TAKE $f : X \to \mathbb{C}$ AND $(f,g) := \int_X f \bar{g}\, d\mu$.

$\boxed{\text{DEF}}$ LET X BE A COMPLEX BANACH SPACES, $A \in \mathcal{L}(X)$ AND $\lambda \in \mathbb{C}$. WE SAY λ IS IN THE <u>SPECTRUM</u> OF A IF $A - \lambda I$ IS <u>NOT</u> INVERTIBLE (I → IDENTITY)

- IF $A - \lambda I$ IS NOT INJECTIVE, WE SAY λ IS IN THE <u>POINT SPECTRUM</u>. WE SAY λ IS AN <u>EIGENVALUE</u> AND ANY $x \in \ker(A - \lambda I)$, $x \neq 0$, IS AN <u>EIGENVECTOR</u>
- IF $A - \lambda I$ IS INJECTIVE BUT NOT SURJECTIVE AND $\overline{\operatorname{ran}(A - \lambda I)} = X$ WE SAY λ IS IN THE <u>CONTINUE SPECTRUM</u>
- IF $A - \lambda I$ IS INJECTIVE AND $\overline{\operatorname{ran}(A - \lambda I)} \subsetneq X$. WE SAY λ IS IN THE <u>RESIDUAL SPECTRUM</u>

NOTATION: THE SPECTRUM IS DENOTED AS $\sigma(A)$.

POINT SPECTRUM $= \sigma_p(A)$ CONTINUOUS SPECTRUM $= \sigma_c(A)$, RESIDUE SPECTRUM $= \sigma_R(A)$

$\boxed{\text{EXAMPLES}}$ $A = \ell_2$

① $A : (x(1), x(2), x(3) \ldots) \to (x(2), x(3), x(4), \ldots)$ //LEFT SHIFT// A IS SURJECTIVE BUT A IS <u>NOT</u> INJECTIVE BECAUSE $A(e_1) = 0$. $\Rightarrow \lambda = 0 \in \sigma_p(A)$ (POINT SPECTRUM) e_1 IS THE RELATIVE EIGENVALUE

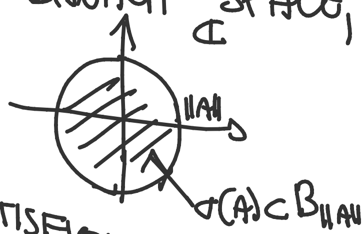
... INJECTIVE BECAUSE $A(e_1)=0$. $\Rightarrow \lambda=0 \in \sigma_p(A)$ (POINT SPECTRUM) e_1 IS THE RELATIVE EIGENVECTOR.

② $A:(x(1),x(2),x(3),\dots) \to \underbrace{(0,x(1),x(2),\dots)}_{=\text{RIGHT SHIFT}}$ A IS INJECTIVE BUT NOT SURJECTIVE $\Rightarrow \lambda=0\in\sigma(A)$. $\operatorname{ran}(A)=\{x(1)=0\}=e_1^{\perp} \Rightarrow \operatorname{ran} A$ IS <u>NOT</u> DENSE $\Rightarrow \lambda=0\in\sigma_r(A)$ (RESIDUAL SPECTRUM)

③ $A:(x(1),x(2),x(3),\dots)\to\left(x(1),\frac{x(2)}{2},\frac{x(3)}{3},\dots\right)$ A IS INJECTIVE BUT NOT SURJECTIVE: $\left(1,\frac{1}{2},\frac{1}{3},\dots\right)\in \ell_2 \setminus \operatorname{ran}(A)$. $\operatorname{ran}(A)$ IS DENSE, SO $\lambda=0\in\sigma_c(A)$ (CONTINUOUS SPECTRUM). IN FACT, $\operatorname{span}\{e_n\}_{n\in\mathbb{N}}\subset\operatorname{ran}(A)$, BECAUSE $e_n=A(ne_n)$ $\left(A:(0,\dots,n,0\dots)\to(0,\dots,1,0\dots)\right)$

PROP LET X BE A COMPLEX BANACH SPACE, $A\in\mathcal{L}(X)$, $\lambda\in\mathbb{C}$. THEN:

① IF $|\lambda|>\|A\|$, THEN $\lambda\notin\sigma(A)$



② IF A IS INVERTIBLE AND $B\in\mathcal{L}(X)$ SATISFIES $\|B-A\|<\frac{1}{\|A^{-1}\|}$ THEN B IS INVERTIBLE

③ IF $\lambda\notin\sigma(A)$ AND $\mu\in\mathbb{C}$ SATISFIES $|\mu-\lambda|<\frac{1}{\|(A-\lambda I)^{-1}\|}$ THEN $\mu\notin\sigma(A)$.

COROLLARY

$\sigma(A)$ IS <u>COMPACT</u> IN $\mathbb{C}$. MOREOVER, THE SET OF INVERTIBLE OPERATORS IS <u>OPEN</u> IN $\mathcal{L}(X)$

REMARK IF $\dim X<+\infty$, $\sigma(A)$ IS FINITE $\Rightarrow$ COMPACT. MOREOVER, $\{\text{INVERTIBLE OPERATORS}\}=\{\det\neq 0\}$ IS OPEN BECAUSE det IS CONTINUOUS.

<u>PROOF</u> ① ASSUME $\lambda>\|A\|$. WE DEFINE $(A-\lambda I)^{-1}:=-\frac{1}{\lambda}\sum_{n=0}^{+\infty}\frac{A^n}{\lambda^n}$

LET US VERIFY THAT THE SEQUENCE OF FINITE SUMS IS CAUCHY:

$$\left\|-\frac{1}{\lambda}\sum_{n=0}^{N}\frac{A^n}{\lambda^n}+\frac{1}{\lambda}\sum_{n=0}^{M}\frac{A^n}{\lambda^n}\right\|=\frac{1}{|\lambda|}\left\|\sum_{n=N+1}^{M}\frac{A^n}{\lambda^n}\right\|\leq\frac{1}{|\lambda|}\sum_{n=N+1}^{M}\frac{\|A^n\|}{|\lambda|^n}\leq\frac{1}{|\lambda|}\sum_{n=N+1}^{M}\Big(\underbrace{\frac{\|A\|}{|\lambda|}}_{<1}\Big)^n\xrightarrow[N,M\to+\infty]{}0$$

$\Rightarrow$ THE SERIES IS WELL-DEFINED, LET US VERIFY THAT IT INVERTS $A-\lambda I$:

$$(A-\lambda I)\left(-\frac{1}{\lambda}\sum_{n=0}^{+\infty}\frac{A^n}{\lambda^n}\right)=-\sum^{+\infty}\frac{A^{n+1}}{\lambda^{n+1}}+\sum^{+\infty}\frac{A^n}{\lambda^n}=\frac{A^0}{\dots}-I$$

$$(A-\lambda I)\left(-\frac{1}{\lambda}\sum_{k=0}^{+\infty}\frac{A^k}{\lambda^k}\right) = -\sum_{k=0}^{+\infty}\frac{A^{k+1}}{\lambda^{k+1}} + \sum_{k=0}^{+\infty}\frac{A^k}{\lambda^k} = \frac{A^0}{\lambda^0} = I.$$

$\Rightarrow A-\lambda I$ IS INVERTIBLE $\Rightarrow \lambda \notin \sigma(A)$.

(2) $\|B-A\| < \frac{1}{\|A^{-1}\|}$. CONSIDER $A^{-1}B \Rightarrow \|A^{-1}B - I\| = \|A^{-1}(B-A)\| \leq \|A^{-1}\|\,\|B-A\| < 1 = |-1| = |\lambda|$

$\Rightarrow$ FROM (1) IS INVERTIBLE $\underbrace{A^{-1}B - I - (-1)I = A^{-1}B}$

A IS ALSO INVERTIBLE $\Rightarrow B = A \circ A^{-1}B$ IS ALSO INVERTIBLE.

(3) WE APPLY (2) WITH $A-\lambda I$ IN PLACE OF A

$A-\mu I$ IN PLACE OF B.

$$\underbrace{\|(A-\mu I)-(A-\lambda I)\|}_{=\|\mu I-\lambda I\| = |\mu-\lambda|} < \frac{1}{\|(A-\lambda I)^{-1}\|}.$$

IF $\lambda \notin \sigma(A)$ THEN $A-\lambda I$ IS INVERTIBLE BY (2) ALSO $A-\mu I$ IS INVERTIBLE, THAT IS $\mu \notin \sigma(A)$.